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SKEW GROUP ALGEBRAS OF PATH ALGEBRAS 
AND PREPROJECTIVE ALGEBRAS

LAURENT DEMONET

ABSTRACT. We compute explicitly up to Morita-equivalence the skew group algebra of a finite group acting on the path algebra of a quiver and the skew group algebra of a finite group acting on a preprojective algebra. These results generalize previous results of Reiten and Riedtmann [RR] for a cyclic group acting on the path algebra of a quiver and of Reiten and Van den Bergh [RN] for a finite subgroup of $\mathbb{SL}(\mathbb{C} \oplus \mathbb{C}Y)$ acting on $\mathbb{C}[X, Y]$.

1. INTRODUCTION AND MAIN RESULTS

Let $k$ be an algebraically closed field and $G$ be a finite group such that the characteristic of $k$ does not divide the cardinality of $G$. If $\Lambda$ is a $k$-algebra and if $G$ acts on the right on $\Lambda$, the action being denoted exponentially, the skew group algebra of $\Lambda$ under the action of $G$ is by definition the $k$-algebra whose underlying $k$-vector space is $\Lambda \otimes_k k[G]$ and whose multiplication is linearly generated by $(a \otimes g)(a' \otimes g') = aa'g^{-1} \otimes gg'$ for all $a, a' \in \Lambda$ and $g, g' \in G$ (see [RR]). It will be denoted by $\Lambda G$. Identifying $k[G]$ and $\Lambda$ with subalgebras of $\Lambda G$, an alternative definition is

$$\Lambda G = \{\Lambda, k[G] \mid \forall (g, a) \in G \times \Lambda, g^{-1}ag = a^g\}_{k-\text{alg}}$$

Let now $Q = (I, A)$ be a quiver where $I$ denotes the set of vertices and $A$ the set of arrows. Consider an action of $G$ on the path algebra $kQ$ permuting the set of primitive idempotents $\{e_i \mid i \in I\}$ and stabilizing the vector space spanned by the arrows. Note that this is more general than an action coming from an action of $G$ on $Q$ since an arrow may be sent to a linear combination of arrows. We now define a new quiver $Q_G$. We first need some notation.

Let $\tilde{I}$ be a set of representatives of the classes of $I$ under the action of $G$. For $i \in I$, let $G_i$ denote the subgroup of $G$ stabilizing $e_i$, let $i_0 \in \tilde{I}$ be the representative of the class of $i$ and let $\kappa_i \in G$ be such that $\kappa_i^{e_i} = i$.

For $(i, j) \in \tilde{I}^2$, $G$ acts on $O_i \times O_j$ where $O_i$ and $O_j$ are the orbits of $i$ and $j$ under the action of $G$. A set of representatives of the classes of this action will be denoted by $F_{ij}$.

For $i, j \in \tilde{I}$, define $M_{ij} \subset kQ$ to be the vector space spanned by the arrows from $i$ to $j$. We regard $M_{ij}$ as a right $k[G_i \cap G_j]$-module by restricting the action of $G$.

The quiver $Q_G$ has vertex set

$$I_G = \bigcup_{i \in \tilde{I}} \{i\} \times \text{irr}(G_i)$$

where $\text{irr}(G_i)$ is a set of representatives of isomorphism classes of irreducible representations of $G_i$. The set of arrows of $Q_G$ from $(i, \rho)$ to $(j, \sigma)$ is a basis of

$$\bigoplus_{(i', j') \in F_{ij}} \text{Hom}_{k[G_{i'} \cap G_{j'}]} \left((\rho \cdot \kappa_{i'})_{G_{i'} \cap G_{j'}}, (\sigma \cdot \kappa_{j'})_{G_{i'} \cap G_{j'}} \otimes_k M_{i'j'}\right)$$

where the representation $\rho \cdot \kappa_{i'}$ of $G_{i'}$ is the same as $\rho$ as a vector space, and $(\rho \cdot \kappa_{i'})_g = \rho_{\kappa_{i'} g \kappa_{i'}^{-1}}$ for $g \in G_{i'} = \kappa_{i'}^{-1}G_i\kappa_{i'}$. Table 1 gives two examples of quivers $Q_G$. A detailed example is also computed in section 2.

We can now state the two main results of this paper and two corollaries :

**Theorem 1.** There is an equivalence of categories

$$\text{mod } k(Q_G) \simeq \text{mod } (kQ) G.$$
Theorem 1 was proved by Reiten and Riedtmann in [RR, §2] for cyclic groups.
The following theorem deals with the case of preprojective algebras. The definition of the preprojective algebra $Λ_Q$ of a quiver $Q$ is recalled in section 3.

**Theorem 2.** If $G$ acts on $kQ$, where $Q$ is the double quiver of $Q$, by permuting the primitive idempotents $e_i$ and stabilizing the linear subspace of $kQ$ spanned by the arrows, and if, for all $g ∈ G$, $r^g = r$ where $r$ is the preprojective relation of this quiver, then $(Q)_G$ is of the form $Q'$ for some quiver $Q'$ and $(Λ_Q)G$ is Morita equivalent to $Λ_{Q'}$.

One can always extend an action on $kQ$ to an action on $kQ$ and this yields :

**Corollary 3.** The action of $G$ on a path algebra $kQ$ permuting the primitive idempotents and stabilizing the linear subspace of $kQ$ spanned by the arrows induces naturally an action of $G$ on $kQ$ and $(Q)_G$ is isomorphic to the double quiver of $Q_G$. Moreover, there is an equivalence of categories

$$\text{mod } Λ_QG \simeq \text{mod } Λ_QG.$$

Theorem 2 and corollary 3 will be used in [Dem] for constructing 2-Calabi-Yau categorifications of skew-symmetric cluster algebras. Another corollary, which is an easy consequence of the definition of the McKay graph, is :

**Corollary 4.** Let $Q$ be the quiver

$$
\begin{array}{c}
\circlearrowleft \\
1 \\
\circlearrowright
\end{array}
$$

and $G$ a finite subgroup of $\text{SL}(\mathbb{C}α ⊕ \mathbb{C}α^*)$. Then

1. There is an identification of $Q_G$ with a double quiver $Q'$ such that the non oriented underlying graph of $Q'$ is isomorphic to the affine Dynkin diagram corresponding to $G$ through the McKay correspondence.
2. We have $kQ/(αα^* − α^*α) \simeq k[α, α^*]$ and there is an equivalence of categories

$$\text{mod } (k[α, α^*]G) \simeq \text{mod } Λ_{Q'}.$$

Corollary 4 was proved by a geometrical method in [RV, proof of proposition 2.13] (see also [CBH, theorem 0.1]).

**Table 1.** Examples of computations of $Q_G$

<table>
<thead>
<tr>
<th>$Q$</th>
<th>$G$</th>
<th>$Q_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \rightarrow 2 \rightarrow \cdots \rightarrow n - 1$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$1 \rightarrow 2 \rightarrow \cdots \rightarrow n - 1$</td>
</tr>
<tr>
<td>$1' \rightarrow 2' \rightarrow \cdots \rightarrow (n - 1)'$</td>
<td>$G \subset \text{SL}(\mathbb{C}α ⊕ \mathbb{C}β)$, type $A_n$</td>
<td></td>
</tr>
</tbody>
</table>
| $\begin{array}{c}
\circlearrowleft \\
1 \\
\circlearrowright
\end{array}$ | $\begin{array}{c}
1 \\
\circlearrowright
\end{array}$ | $\begin{array}{c}
0 \\
2 \cdots \rightarrow n - 1
\end{array}$ |
2. An example

Suppose that $k = \mathbb{C}$ and that $Q$ is the following quiver

```
1 ----> α* ----> γ*
|      |        |
|      |        |
β* ----> γ ----> β ----> δ ----> 0
```

Let also

$$G = (a, b | a^3 = b^2, b^4 = 1, aba = b)$$

be the binary dihedral group of order 12. One lets $G$ act on $kQ$ by :

<table>
<thead>
<tr>
<th>$e_0$</th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
<th>$\alpha$</th>
<th>$\alpha^*$</th>
<th>$\beta$</th>
<th>$\beta^*$</th>
<th>$\gamma$</th>
<th>$\gamma^*$</th>
<th>$\delta$</th>
<th>$\delta^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$e_0$</td>
<td>$e_2$</td>
<td>$e_3$</td>
<td>$\zeta\alpha$</td>
<td>$\zeta^*\alpha$</td>
<td>$\gamma$</td>
<td>$\gamma^*$</td>
<td>$\delta$</td>
<td>$\delta^*$</td>
<td>$\beta$</td>
<td>$\beta^*$</td>
</tr>
<tr>
<td>$b$</td>
<td>$e_0$</td>
<td>$e_1$</td>
<td>$e_2$</td>
<td>$\alpha^*$</td>
<td>$-\alpha$</td>
<td>$-\beta$</td>
<td>$-\beta^*$</td>
<td>$-\delta$</td>
<td>$-\delta^*$</td>
<td>$\gamma$</td>
<td>$\gamma^*$</td>
</tr>
</tbody>
</table>

where $\zeta$ is a primitive sixth root of unity.

Using the notation of the introduction, one can choose $\tilde{I} = \{0, 1\}$, $\kappa_0 = \kappa_1 = 1$, $\kappa_2 = a$, $\kappa_3 = a^2$. One has $G_0 = G$, $G_1 = \{b\} \simeq \mathbb{Z}/4\mathbb{Z}$, $G_2 = \{ba\} \simeq \mathbb{Z}/4\mathbb{Z}$, $G_3 = \langle ab \rangle \simeq \mathbb{Z}/4\mathbb{Z}$. One can also choose $F_{0,0} = \{(0,0)\}$, $F_{0,1} = \{(0,1)\}$, $F_{1,0} = \{(1,0)\}$ and $F_{1,1} = \{(1,1), (1,2), (2,1)\}.

The irreducible representations of $\mathbb{Z}/4\mathbb{Z}$ will be denoted by $\theta_\alpha$ where $\alpha \in \{i, -1, -i, 1\}$ is the scalar action of a specified generator ($h, ba$ or $ab$ when $\mathbb{Z}/4\mathbb{Z}$ is realized as $G_1$, $G_2$ or $G_3$). The group $G$ has six irreducible representations : four of degree 1 of the form $a \mapsto \alpha^2$, $b \mapsto \alpha$ for each $\alpha \in \{i, -1, -i, 1\}$, which will be denoted by $\lambda_\alpha$, and two of degree 2 :

$$\rho : a \mapsto \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix} \quad b \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\sigma : a \mapsto \begin{pmatrix} \zeta^{-2} & 0 \\ 0 & \zeta^2 \end{pmatrix} \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

One checks easily that $\lambda_i \otimes \rho \simeq \sigma$, $\rho \otimes \rho \simeq \sigma \oplus \lambda_1 \oplus \lambda_{-1}$, $\rho \otimes \sigma \simeq \rho \oplus \lambda_i \oplus \lambda_{-i}$, $\sigma \otimes \sigma \simeq \sigma \oplus \lambda_1 \oplus \lambda_{-1}$. The other product formulas are deduced from these. One computes $M_{0,0} = \rho$, $M_{0,1} = M_{1,0} = \lambda_1$ and $M_{1,1} = M_{1,2} = M_{2,1} = 0$. The vertices of $Q_G$ are then $0_i$, $0_{-i}$, $0_1$, $0_{-1}$, $0$, $\sigma$, $1$, $\lambda_1$ and $\lambda_{-1}$ where we write $0_\alpha = (0, \lambda_\alpha)$ and $1_\alpha = (1, \theta_\alpha)$ for simplicity. One has

$$\text{Hom}_G(\rho, \sigma \otimes \rho) \simeq \text{Hom}_G(\rho, \rho \oplus \lambda_i \oplus \lambda_{-i}) \simeq \mathbb{C}$$

and therefore there is one arrow from $0_{\mu}$ to $0_{\sigma}$,

$$\text{Hom}_G(\lambda_1, \sigma \otimes \rho) \simeq 0$$

and therefore there is no arrow from $0_1$ to $0_{\sigma}$,

$$\text{Hom}_G(\lambda_i, \sigma \otimes \rho) \simeq \mathbb{C}$$

and therefore there is one arrow from $0_i$ to $0_{\sigma}$,

$$\text{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\theta_i, \sigma_{\mathbb{Z}/4\mathbb{Z}} \otimes \theta_{-1}) \simeq \text{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\theta_i, \theta_{-1} \oplus \theta_1) \simeq 0$$

and therefore there is no arrow from $1_i$ to $0_{\sigma}$,

$$\text{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\theta_1, \sigma_{\mathbb{Z}/4\mathbb{Z}} \otimes \theta_{-1}) \simeq \text{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\theta_1, \theta_{-1} \oplus \theta_1) \simeq \mathbb{C}$$

and therefore there is one arrow from $1_i$ to $0_{\sigma}$,

$$\text{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\theta_1, \lambda_{-1} \sigma_{\mathbb{Z}/4\mathbb{Z}} \otimes \theta_{-1}) \simeq \mathbb{C}$$
and therefore there is one arrow from $1_1$ to $0_{-1}$. All the other computations can be done in the same way. Finally, $Q_G$ is the following quiver:

![Quiver Diagram]

where one can remark that the full subgraph having vertices $\{0, 0_{-1}, 0_{-1}, 0_1, 0_p, 0_p\}$ is the affine Dynkin diagram corresponding to $G$ in the McKay correspondence, as expected. Hence $\text{mod}(CQ)G \simeq \text{mod} CQ$. Moreover, it is easy to check that the preprojective relation $\alpha \alpha^* - \alpha^* \alpha + \beta \beta^* - \beta^* \beta + \gamma \gamma^* - \gamma^* \gamma + \delta \delta^* - \delta^* \delta$ is stable under the action of $G$ and therefore there is an equivalence of Morita between $\Lambda Q_1, G$ and $\Lambda Q_2$ where $Q_1 = Q, Q_2 = Q_G$, and $\Lambda Q_1, \Lambda Q_2$ are the preprojective algebras of $Q_1$ and $Q_2$.

3. Proofs of the main propositions

3.1. Proof of theorem \[.\] We retain the notation of section \[.\]. Thus $Q$ is a quiver, $G$ acts on $kQ$ by stabilizing the set of primitive idempotents corresponding to vertices and the vector space spanned by the arrows, $I$ is a fixed set of representatives of the $G$-orbits of $I$, etc.. Let $R$ be the subalgebra of $kQ$ generated by the primitive idempotents and $M \subset kQ$ be the linear subspace spanned by the arrows, seen as an $R$-bimodule. Define $T_0 = R$ and for every positive integer $n$, $T_n = T_{n-1} \otimes_R M$. Then, recall that the tensor algebra $T(R, M)$ is $\bigoplus_{i \geq 0} T_i$ endowed with the canonical product. It is clear that $kQ$ is canonically isomorphic to $T(R, M)$ on which the action of $G$ is graded.

As $G$ stabilizes $R$ and $M$, one can define the skew-group algebra $RG$ which is a subalgebra of $(kQ)G$. Thus, $MG = M \otimes_k k[G]$ is a sub-$RG$-bimodule of $(kQ)G$ and one gets easily a canonical isomorphism $(kQ)G \simeq T(R, MG)$ which maps $(m_1 \otimes m_2 \otimes \ldots \otimes m_n) \otimes g$ to $(m_1 \otimes 1) \otimes (m_2 \otimes 1) \otimes \ldots \otimes (m_{n-1} \otimes 1) \otimes (m_n \otimes g) = (m_1 \otimes 1) \otimes (m_2 \otimes 1) \otimes \ldots \otimes (m_{n-1} \otimes g) \otimes (m_n \otimes 1) = \cdots = (m_1 \otimes 1) \otimes (m_2 \otimes g) \otimes \ldots \otimes (m_{n-1} \otimes 1) \otimes (m_n \otimes 1) = (m_1 \otimes g) \otimes (m_2 \otimes 1) \otimes \ldots \otimes (m_{n-1} \otimes 1) \otimes (m_n \otimes 1).

Let now $e = \sum_{i \in I} e_i \in R \subset RG$ which is an idempotent. Then, according to [\[11\], proposition 1.6] and its proof,

$$\text{mod } RG \simeq \text{mod } e(RG)e$$

$$N \mapsto eN$$

$$(RG)e \otimes_{e(RG)e} N' \leftarrow N'$$

is a Morita equivalence between $RG$ and $e(RG)e$. Hence, it is a classical and easy fact that there is an equivalence of categories

$$\text{mod}(kQ)G \simeq \text{mod } T(RG, MG) \simeq \text{mod } T(e(RG)e, e(MG)e)$$

$$N \mapsto eN$$

$$(RG)e \otimes_{e(RG)e} N' \leftarrow N'$$
Moreover, using the isomorphism $MG \simeq (RG)e \otimes (RG)e (MG)e \otimes (RG)e e(RG)$. One has $e(RG)e \simeq \prod_{i \in I} k[G_i]$ where, for each $i \in I$, $G_i$ is the stabilizer of $e_i$. As $G_i$ is semi-simple, one can fix, for each $i \in I$ and $\rho \in \text{irr}(G_i)$, $\tilde{e}_{i\rho}$ to be a primitive idempotent of $k[G_i]$ corresponding to $\rho$. Let $\tilde{c} = \sum_{i \in I} \sum_{\rho \in \text{irr}(G_i)} \tilde{e}_{i\rho}$ which satisfies $\epsilon \epsilon \epsilon = \tilde{c}$. Then we have a Morita equivalence between $e(RG)e$ and $e \epsilon (RG) \epsilon e = \tilde{c}(RG) \epsilon$, and, as before, between $T(e(RG)e, e(MG)e)$ and $T(\epsilon(RG)\epsilon, \epsilon(MG)\epsilon)$. Moreover, using the notation $I_G$ of the introduction, $\epsilon(RG)\epsilon \simeq \prod_{(i, \rho) \in I_G} k e_{i\rho}$ and therefore, it is enough to compute $\tilde{e}_{i\sigma}(MG)\tilde{e}_{i\rho} = \tilde{e}_{i\sigma}\tilde{e}_{j}(MG)\tilde{e}_{i\rho}$ for each $(i, \rho)$ and $(j, \sigma)$ in $I_G$ to finish the proof of theorem $\square$. Remark now that

$$e_j(MG)e_i = \sum_{(i', j') \in O_i \times O_j} G_j \kappa_{j'} M_{i', j'} \kappa_{i'}^{-1} G_i = \bigoplus_{(i', j') \in F_{ij}} G_j \kappa_{j'} M_{i', j'} \kappa_{i'}^{-1} G_i$$

and therefore, if one denotes $G_{i', j'} = G_{i'} \cap G_{j'}$ for every $i', j' \in I$, $\tilde{e}_{i\sigma}e_j(MG)e_i \tilde{e}_{i\rho} \simeq \bigoplus_{(i', j') \in F_{ij}} \text{Hom}_{k} \left( k, \tilde{e}_{i\sigma}G_j \kappa_{j'} M_{i', j'} \kappa_{i'}^{-1} G_i \tilde{e}_{i\rho} \right)$

$$\simeq \bigoplus_{(i', j') \in F_{ij}} \text{Hom}_{k[G_i]} \left( \rho, \sigma \otimes k[G_j] G_j \kappa_{j'} M_{i', j'} \kappa_{i'}^{-1} G_i \right)$$

$$\simeq \bigoplus_{(i', j') \in F_{ij}} \text{Hom}_{k[G_i]} \left( \rho \cdot \kappa_{i'}, \left( \sigma \cdot \kappa_{j'} \right) \otimes k[G_j] G_j M_{i', j'} G_{j'} \right)$$

$$\simeq \bigoplus_{(i', j') \in F_{ij}} \text{Hom}_{k[G_i]} \left( \rho \cdot \kappa_{i'}, \left( \sigma \cdot \kappa_{j'} \right) \otimes k[G_{i'}] G_{i'} \otimes k[G_{j'}] G_{j'} \right)$$

and, because of the relations defining $(kQ)G$, the multiplication in $(kQ)G$ induces an isomorphism of $(G_{i', j'}, G_{i', j'})$-bimodules $k[G_{i', j'}] \otimes_k M_{i', j'} \simeq G_{i', j'} M_{i', j'} G_{j'}$. Note that the action of $(G_{i', j'}, G_{i', j'})$ on $k[G_{i', j'}] \otimes_k M_{i', j'}$ is defined here by $g(v \otimes m)h = gvh \otimes h^{-1} mh = gvh \otimes m^h$ (recall that we use the multiplication notation for the product in $(kQ)G$ and the exponential notation for the action of $G$ on $kQ$ fixed at the beginning). Therefore $\tilde{e}_{i\sigma}e_j(MG)e_i \tilde{e}_{i\rho} \simeq \bigoplus_{(i', j') \in F_{ij}} \text{Hom}_{k[G_{i'}]} \left( \rho \cdot \kappa_{i'}, \left( \sigma \cdot \kappa_{j'} \right) \otimes k[G_{i'}] G_{i'} \otimes k[G_{j'}] G_{j'} \right)$

$$\simeq \bigoplus_{(i', j') \in F_{ij}} \text{Hom}_{k[G_{i'}]} \left( \rho \cdot \kappa_{i'}, \left( \sigma \cdot \kappa_{j'} \right) \otimes k[G_{i'}] \otimes k[G_{j'}] \right)$$

where the representation $\rho \cdot \kappa_{i'}$ of $G_{i'}$ is the same as $\rho$ as a vector space and, if $g \in G_{i'} = \kappa_{i'}^{-1} G_{i'} \kappa_{i'}, \rho \cdot \kappa_{i'}g = \rho \kappa_{i'g \kappa_{i'}}^{-1}$. It concludes the proof of theorem $\square$. Note that we go from the penultimate line to the last one by the classical adjunction between induction and restriction of representations.

3.2. Proof of theorem $\square$. We retain the notation of section 3.1. Define the $R$-bimodule $M = M \oplus M^*$ (the $R$-bimodule structure on $M^*$ is the natural one : $(afb)(m) = f(bma)$ for $a, b \in R, f \in M^*$ and $m \in M$). The tensor algebra $T(R, M)$ is the path algebra of the double quiver $\overrightarrow{Q}$ of $Q$. The non-degenerate skew-symmetric bilinear form defined on $M$ by $\langle m + f, m' + f' \rangle = f'(m) - f(m')$ where $m, m' \in M$ and $f, f' \in M^*$ satisfies, for every $a, b \in R$ and $m, n \in M$, $\langle amb, n \rangle = \langle bma, n \rangle$. If $\{x_i\}_{i \in S}$ is a $k$-basis of $M$, then denote by $\{x_i^*\}_{i \in S}$ its (left) dual basis for the bilinear form $\langle -, - \rangle$ (that is the one satisfying $\langle x_i^*, x_j \rangle = \delta_{ij}$ for every
Then the element \( r = \sum_{i \in S} x_i \otimes x_i^* \in \overline{\mathcal{M}} \otimes_R \overline{\mathcal{M}} \) is independent of the choice of the basis. By definition, \( r \) is the preprojective relation corresponding to \( \mathcal{M} \) and the preprojective algebra \( \Lambda_Q \) of \( Q \) is \( T(R, \overline{\mathcal{M}})/(r) \). For more details about preprojective algebras, see for example [DR] or [Rin].

Suppose now that the group \( G \) acts on \( kQ \) by stabilizing the set of primitive idempotents corresponding to vertices and the \( k \)-subspace spanned by the arrows. Then it stabilizes \( r \) if and only if it stabilizes the bilinear form \( \langle -,- \rangle \) (indeed, for \( g \in G, r^g = r \) implies that \( \{ x_i^{*g} \}_{i \in S} \) is the left dual basis of \( \{ x_i^g \}_{i \in S} \) since for every \( i \in S \), \( (\text{Id}_{\overline{\mathcal{M}}} \otimes \langle -,- \rangle)(r^g \otimes x_i^g) = (\text{Id}_{\overline{\mathcal{M}}} \otimes \langle -,- \rangle)(r \otimes x_i^g) = x_i^g \)). Extend now \( \langle -,- \rangle \) to a bilinear form on \( \overline{\mathcal{M}} \mathcal{G} \) by setting

\[
\langle m \otimes g, n \otimes h \rangle = \begin{cases} \langle m, n^h \rangle & \text{if } gh = 1 \\ 0 & \text{else} \end{cases}
\]

which is clearly skew-symmetric and non-degenerate. Moreover, for \( a, b \in RG \) and \( m, n \in \overline{\mathcal{M}}G \), one has easily \( \langle amb, n \rangle = \langle m, bna \rangle \). If \( \{ x_i \}_{i \in S} \) is a basis of \( \overline{\mathcal{M}} \) and \( \{ x_i^* \}_{i \in S} \) is its left dual basis, then \( \{ x_i^{*g} \otimes g^{-1} \}_{(i,g) \in S \times G} \) is the left dual basis of the basis \( \{ x_i \otimes g \}_{(i,g) \in S \times G} \) of \( \overline{\mathcal{M}} \mathcal{G} \). Hence, the preprojective relation \( r_G \) corresponding to \( \langle -,- \rangle \) in \( \overline{\mathcal{M}} \mathcal{G} \) is

\[
r_G = \sum_{(i,g) \in S \times G} (x_i \otimes g) \otimes (x_i^{*g} \otimes g^{-1}) = \sum_{(i,g) \in S \times G} (x_i \otimes 1) (1 \otimes g) (x_i^{*g} \otimes g^{-1}) = \sum_{(i,g) \in S \times G} (x_i \otimes 1) (x_i^{*g} \otimes g^{-1}) = \sum_{(i,g) \in S \times G} (x_i \otimes 1) (x_i^g \otimes 1) = \#G \times r
\]

where the preprojective relation \( r \) of \( T(R, \overline{\mathcal{M}}) \) is mapped by the canonical inclusion from \( \overline{\mathcal{M}} \otimes_R \overline{\mathcal{M}} \) to \( \overline{\mathcal{M}} \mathcal{G} \otimes_{RG} \overline{\mathcal{M}} \mathcal{G} \). As \( \#G \) is invertible in \( k \), one gets

\[
(T(R, \overline{\mathcal{M}})/(r)) G \simeq (T(RG, \overline{\mathcal{M}}G)/(r)) = T(RG, \overline{\mathcal{M}}G)/(r_G).
\]

Moreover, for \( \langle -,- \rangle \), \( \tilde{e}(\overline{\mathcal{M}}G)e \) and \((1-\tilde{e})(\overline{\mathcal{M}}G) + (\overline{\mathcal{M}}G)(1-\tilde{e})\) are orthogonal supplementary subspaces. Thus, \( \langle -,- \rangle \) restricts to a skew-symmetric non-degenerate bilinear form on \( \tilde{e}(\overline{\mathcal{M}}G)e \) which satisfies, for every \( a, b \in \tilde{e}(RG)e \) and \( m, n \in \tilde{e}(\overline{\mathcal{M}}G)e \), \( \langle amb, n \rangle = \langle m, bna \rangle \). By taking a basis of \( \overline{\mathcal{M}}G \) which is the union of a basis of \( \tilde{e}(\overline{\mathcal{M}}G)e \) and a basis of \((1-\tilde{e})(\overline{\mathcal{M}}G) + (\overline{\mathcal{M}}G)(1-\tilde{e})\), it is clear that the equivalence of categories of section 3.1 restricts to an equivalence

\[
\text{mod} ( (kQG)/(r_G) ) \simeq \text{mod} ( (TRG, \overline{\mathcal{M}}G)/(r_G) ) \simeq \text{mod} ( (\tilde{e}(RG)e, \tilde{e}(\overline{\mathcal{M}}G)e)/(r_{\tilde{e}}) )
\]

\[
N \mapsto \tilde{e}N
\]

\[
(RG)e \otimes_{\tilde{e}(RG)e} N' \leftrightarrow N'
\]

where \( r_{\tilde{e}} \) is the preprojective relation in \( T(\tilde{e}(RG)e, \tilde{e}(\overline{\mathcal{M}}G)e) \simeq k(\overline{\mathcal{Q}})G \simeq kQ \). It completes the proof of theorem 3 (it is enough to take a maximal isotropic subspace of \( \tilde{e}(\overline{\mathcal{M}}G)e \) to find arrows of \( Q' \) which is of course non unique).

3.3. Proof of corollary 3. We retain the previous notation. If \( G \) acts on \( kQ \) by stabilizing the set of primitive idempotents and \( M \), then its action can be extended to an action on \( \overline{\mathcal{M}} = M \oplus M^* \) using the contragredient representation on \( M^* \). Then \( \overline{\mathcal{M}}G = MG \oplus M^*G \) as \( RG \)-bimodules. Moreover, \( MG \) is clearly maximal isotropic for the bilinear form \( \langle -,- \rangle \) extended on \( \overline{\mathcal{M}}G \) as before. Hence \( (Q')_G = Q^G \) and it proves corollary 3.

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