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SEMI-LINEAR PARABOLIC EQUATIONS ON THE HEISENBERG GROUP WITH A SINGULAR POTENTIAL

HOUDA MOKRANI
HOUDA.MOKRANI@ETU.UNIV-ROUEN.FR

Abstract. In this work, we discuss the asymptotic behavior of solutions for semi-linear parabolic equations on the Heisenberg group with a singular potential. The singularity is controlled by Hardy’s inequality, and the nonlinearity is controlled by Sobolev’s inequality. We also establish the existence of a global branch of the corresponding steady states via the classical Rabinowitz theorem.

Key words: Semi-linear parabolic equations, Heisenberg group, Hardy’s inequality, Sobolev’s inequality, Singular potential, Global bifurcation, Blow-up

A.M.S. Classification

1. Introduction

In this work, we study a class of parabolic equations on the Heisenberg group \( \mathbb{H}^d \). Let us recall that the Heisenberg group is the space \( \mathbb{R}^{2d+1} \) with the (non commutative) law of product

\[
(x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' + 2((y|x') - (y'|x))).
\]

The left invariant vector fields are

\[
X_j = \partial_{x_j} + 2y_j \partial_s, \quad Y_j = \partial_{y_j} - 2x_j \partial_s, \quad j = 1, \ldots, d \quad \text{and} \quad S = \partial_s = \frac{1}{4}[Y_j, X_j].
\]

In the sequel, we shall denote \( Z_j = X_j \) and \( Z_{j+d} = Y_j \) for \( j \in \{1, \ldots, d\} \). We fix here some notations :

\[
z = (x, y) \in \mathbb{R}^{2d}, \quad w = (z, s) \in \mathbb{H}^d, \quad \rho(z, s) = (|z|^4 + |s|^2)^{1/4}
\]

where \( \rho \) is the Heisenberg distance. Moreover, the Laplacian-Kohn operator on \( \mathbb{H}^d \) and Heisenberg gradient are given by

\[
\Delta_{\mathbb{H}^d} = \sum_{j=1}^n X_j^2 + Y_j^2; \quad \nabla_{\mathbb{H}^d} = (Z_1, \ldots, Z_{2d}).
\]

Let \( \Omega \) be an open and bounded domain of \( \mathbb{H}^d \), we define thus the associated Sobolev space by

\[
H^1(\Omega, \mathbb{H}^d) = \left\{ f \in L^2(\Omega) : \nabla_{\mathbb{H}^d} f \in L^2(\Omega) \right\}
\]

and \( H^1_0(\Omega, \mathbb{H}^d) \) is the closure of \( C_0^\infty(\Omega) \) in \( H^1(\Omega, \mathbb{H}^d) \).
We are concerned in the following semi-linear parabolic problem

\[
\begin{aligned}
\begin{cases}
\partial_t u - \Delta_{\mathbb{H}^d} u - \frac{|z|^2}{\rho^4} u = \lambda u + |u|^{p-2} u, & w \in \Omega, \ t > 0, \\
(u(0, w)) = u_0(w), & w \in \Omega, \\
\left. u \right|_{\partial \Omega} = 0, & t > 0,
\end{cases}
\end{aligned}
\]  

(1.1)

where \( \lambda \) is a real constant and \( 2 < p < 2^* \); the index \( 2^* = 2 + \frac{2}{d} \) is the critical index of Sobolev’s inequality on the Heisenberg group \([6, 9, 10, 18]\):

\[
\|u\|_{L^{2^*}(\Omega)} \leq C_\Omega \|u\|_{H^1(\Omega, \mathbb{H}^d)},
\]

(1.2)

for all \( u \in H^1_0(\Omega, \mathbb{H}^d) \).

The following Hardy inequality is first proved in \([11, 7]\):

\[
\bar{\mu} \int_{\Omega} \frac{|z|^2}{\rho(w)^2} |u(w)|^2 dw \leq \|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)}^2
\]

(1.3)

for all \( u \in H^1_0(\Omega, \mathbb{H}^d) \). By the work of Kombe \([19]\), we have the following improved Hardy inequality, for all \( u \in C^\infty_0(\mathbb{H}^d \setminus \{0\}) \):

\[
\frac{1}{C^{d^2}r^2(B)} \int_B u(w)^2 dw + \bar{\mu} \int_{\Omega} \frac{|z|^2}{\rho(w)^2} |u(w)|^2 dw \leq \|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)}^2,
\]

(1.4)

where \( \bar{\mu} = \left(\frac{2^* - 2}{2}\right)^2 \), \( C \) is a positive constant and \( r(B) \) is the radius of the ball \( B \). Moreover \( \bar{\mu} \) is optimal and it is not attained in \( H^1_0(\Omega, \mathbb{H}^d) \).

We recall the following compact embedding result:

**Lemma 1.1.** Let \( \Omega \subset \mathbb{H}^d \) be a bounded open domain. Then \( H^1_0(\Omega, \mathbb{H}^d) \) is compactly embedded in to \( L^p(\Omega), 2 \leq p < 2^* \).

In a remarkable paper, J. A. Goldstein and Q. S. Zhang \([14]\) considered the following particular case

\[
\begin{aligned}
\begin{cases}
\partial_t u - \Delta_{\mathbb{H}^d} u = \mu \frac{|z|^2}{\rho^4} u & t \in (0, T], \ T > 0, \\
(u(w, 0)) = u_0(w), & w \in \mathbb{H}^d.
\end{cases}
\end{aligned}
\]  

(1.5)

They found that if \( \mu > \bar{\mu} \), then the problem (1.5) has no negative solutions except \( u_0 = 0 \), and if \( \mu \leq \bar{\mu} \), then the problem (1.5) has a positive solution for some \( u_0 > 0 \).

On the Euclidian space \( \mathbb{R}^d \), problem (1.5) has been studied first by P. Barras and Goldstein \([3]\) for the potential \( V(x) = \frac{1}{|x|^2} \). Cabrel and Martel \([5, \text{Theorem 1, 2}]\), extend this result to some potential \( V(x) = \frac{1}{\delta(x)^2} \), where \( \delta(x) = \text{dist}(x, \partial \Omega) \), \( \Omega \subset \mathbb{R}^d \) is of class \( C^2 \). They show that the behavior of the solutions depends heavily on the critical value of the parameter \( \mu \) which is the best constant of the classical Hardy inequality.

The work \([3]\) generated a lot of activity on this topic and various questions have been investigated as, for example: general positive singular potentials, the asymptotic behavior of the solutions, semi-linear equations, etc. See, for example, \([15, 14, 27, 29]\).

Stimulated by the recent paper in the Euclidian space \( \mathbb{R}^d \) of Karachalios and Zographopoulos \([20]\) which studied the global bifurcation of nontrivial equilibrium solutions
on the bounded domain case for a reaction term $f(s) = \lambda s - |s|^2 s$, where $\lambda$ is a bifurcation parameter; our focus here is devoted to some results concerning the existence of a global attractor for the equation (1.1) and the existence of a global branch of the corresponding steady states

$$
\begin{cases}
-\Delta_{H^1}u - \mu \frac{|z|^2}{\rho(w)^4}u = \lambda u + |u|^{p-2}u & \text{in } \Omega, \\
u \big|_{\partial \Omega} = 0
\end{cases}
$$

with respect $\lambda$. Let us recall some definitions on semiflows :

**Definition 1.2.** Let $E$ be a complete metric space, a semiflow is a family of continuous maps $S(t) : E \to E$, $t \geq 0$, satisfying the semigroup identities

$$S(0) = I, \ S(t + t') = S(t)S(t').$$

For $B \subset E$ and $t \geq 0$, let

$$S(t)B := \{u(t) = S(t)u_0; u_0 \in B\}.$$

The positive orbit of $u$ through $u_0$ is the set

$$\gamma^+(u_0) = \{u(t) = S(t)u_0, t \geq 0\},$$

and the positive orbit of $B$ is the set $\gamma^+(B) = \cup_{t \geq 0} S(t)B$. The $W$-limit set of $u_0$ is

$$W(u_0) = \{\phi \in E : u(t_j) = S(t_j)u_0 \to \phi, \ t_j \to +\infty\}.$$

The $\alpha$-limit set of $u_0$ is

$$\alpha(u_0) = \{\phi \in E : u(t_j) \to \phi, \ t_j \to -\infty\}.$$

The subset $A$ attracts a set $B$ if $\text{dist}(S(t)B, A) \to 0, \ t \to +\infty$.

$A$ is invariant if $S(t)A = A, \ \forall t \geq 0$.

The functional $J : E \to \mathbb{R}$ is a Lyapunov functional for the semiflow $S(t)$ if

i) $J$ is continuous,

ii) $J(S(t)u_0) \leq J(S(t')u_0)$ for $0 \leq t' \leq t$.

iii) $J(S(t))$ is constant for some orbit $u$ and for all $t \in \mathbb{R}$.

And we have the following theorem from the papers of Ball [1, 2] :

**Theorem 1.3.** Let $S(t)$ be an asymptotically compact semiflow and suppose that there exists a Lyapunov functional $J$. Suppose further that the set $E$ is bounded. Then $S(t)$ is dissipative, so there exists a global attractor $A(t)$.

For each complete orbit $u$ containing $u_0$ lying in $A(t)$, the limit sets $\alpha(u_0)$ and $W(u_0)$ are connected subsets of $E$ on which $J$ is constant.

If $E$ is totally disconnected (in particular if it is countable), the limits

$$\phi_- = \lim_{t \to -\infty} u(t), \quad \phi_+ = \lim_{t \to +\infty} u(t)$$

exist and are equilibrium points. Furthermore, any solution $S(t)u_0$ tends to an equilibrium point as $t \to \pm \infty$.

The existence of a global branch of nonnegative solutions will be proved via the classical Rabinowitz theorem [25]:
Theorem 1.4. Assume that $X$ is a Banach space with norm $\|\cdot\|$ and let $G(\lambda, \cdot) = \lambda L + H(\lambda, \cdot)$, where $L$ is a compact linear map on $X$ and $H(\lambda, \cdot)$ is compact on $X$ and satisfies

$$\lim_{\|u\| \to 0} \frac{\|H(\lambda, u)\|}{\|u\|} = 0.$$  

If $\lambda$ is a simple eigenvalue of $L$, then the closure of the set

$$C = \{(\lambda, u) \in \mathbb{R} \times X : (\lambda, u) \text{ solves } u = G(\lambda, u), u \neq 0\},$$

possesses a maximal continuum (connected branch) of solutions $C_\lambda$, such that $(\lambda, 0) \in C_\lambda$ and $C_\lambda$ either

(i) meets infinity in $\mathbb{R} \times X$, or

(ii) meets $(\lambda^*, 0)$, where $\lambda^* \neq \lambda$ is also an eigenvalue of $L$.

The outline of the paper is as follows: In Section 2, we study the existence of global branch of nonnegative solutions of (1.6) with respect to the parameter $\lambda$. In Section 3, we describe the asymptotic behavior of solutions of (1.1) when $u_0$ has low energy smaller than the mountain pass level.

2. Existence of a global branch of the corresponding steady states

From the study of spectral decomposition of $H_0^1(\Omega, \mathbb{H}^d)$ with respect to the operator $-\Delta_{\mathbb{H}^d} - \mu \frac{|z|^2}{\rho(w)}$, where the singular potential $V$ satisfies Hardy’s inequality (1.3), we have:

Proposition 2.1. Let $0 < \mu_1 \leq \bar{\mu}$. Then there exist $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty$, such that for each $k \geq 1$, the following Dirichlet problem

$$\begin{cases}
-\Delta_{\mathbb{H}^d} \phi_k - \mu \frac{|z|^2}{\rho(w)} \phi_k = \lambda_k \phi_k, & \text{in } \Omega \\
\phi_k |_{\partial \Omega} = 0
\end{cases}$$  

admits a nontrivial solution in $H_0^1(\Omega, \mathbb{H}^d)$. Moreover, $\{\phi_k\}_{k \geq 1}$ constitutes an orthonormal basis of Hilbert space $H_0^1(\Omega, \mathbb{H}^d)$.

Remark that the first eigenvalue $\lambda_{1, \mu}$ characterized by

$$\lambda_{1, \mu} = \inf_{u \in H_0^1(\Omega, \mathbb{H}^d) \setminus \{0\}} \frac{\int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - \mu \frac{|z|^2}{\rho(w)} |u|^2 \right) dw}{\|u\|_{L^2(\Omega)}^2},$$

is simple with a positive associated eigenfunction $\phi_{1, \mu}$.

For the proof of this proposition, we refer to [21].

We discuss the behavior of $\lambda_{1, \mu}$ when $0 < \mu < \bar{\mu}$ and $\mu \uparrow \bar{\mu}$:

Proposition 2.2. Let $0 < \mu < \bar{\mu}$ and $\mu \uparrow \bar{\mu}$. Then,

(i) $(\lambda_{1, \mu})_\mu$ is a decreasing sequence, and there exist $\lambda_* > 0$ such that $\lambda_{1, \mu} \to \lambda_*$.

(ii) The corresponding normalized eigenfunction $\phi_{1, \mu}$ converges weakly to $0$ in $H_0^1(\Omega, \mathbb{H}^d)$.

Proof:

• Let $\mu_1 < \mu_2$. The characterization (2.10) of $\lambda_{1, \mu}$ implies that $\lambda_{1, \mu_1} > \lambda_{1, \mu_2}$. The improved Hardy inequality (1.4) implies that $\lambda_{1, \mu}$ is bounded from below by $\frac{1}{C_2^2 r^2(B)}$.

So, there exist $\lambda_* > 0$ such that $\lambda_{1, \mu} \to \lambda_*$. 
• The eigenfunction \( \phi_{1,\mu} \) satisfies, for any \( v \in C^\infty_0(\Omega) \):

\[
\int_\Omega \nabla H^s \phi_{1,\mu} \nabla \bar{H} \bar{v} \, dw - \mu \int_\Omega \frac{|z|^2}{\rho(w)^4} \phi_{1,\mu} \bar{v} \, dw = \lambda_{1,\mu} \int_\Omega \phi_{1,\mu} \bar{v} \, dw.
\]

We still denote by \( \phi_{1,\mu} \) the sequence of normalized eigenfunction, forming a bounded sequence in \( H^1_0(\Omega, \mathbb{H}^d) \). Then there exists \( u \in H^1_0(\Omega, \mathbb{H}^d) \) such that

\[
\phi_{1,\mu} \rightharpoonup u \text{ in } H^1_0(\Omega, \mathbb{H}^d),
\]

\[
\phi_{1,\mu} \to u \text{ in } L^q(\Omega), \text{ for any } 2 \leq q < 2^*.
\]

For some fixed small enough \( \varepsilon > 0 \) and any for \( v \in C^\infty_0(\Omega) \), we have

\[
\int_\Omega \frac{|z|^2}{\rho(w)^4} (\phi_{1,\mu} - u) \bar{v} \, dw \leq \|v\|_{L^\infty(\Omega)} \left( \int_\Omega |\phi_{1,\mu} - u|^\frac{Q-2-\varepsilon}{Q-2} \, dw \right)^\frac{Q-2}{Q-2-\varepsilon} \left( \int_\Omega \left( \frac{|z|}{\rho(w)^2} \right)^Q \, dw \right)^{\frac{Q-2}{Q-2-\varepsilon}}.
\]

Thus,

\[
\int_\Omega \frac{|z|^2}{\rho(w)^4} \phi_{1,\mu} \bar{v} \, dw \to \int_\Omega \frac{|z|^2}{\rho(w)^4} u \bar{v} \, dw, \text{ as } \mu \uparrow \bar{\mu}.
\]

We assume that \( u \neq 0 \), so passing to the limit in (2.11), we get that \( u \) is a nontrivial solution of the problem

\[-\Delta_{H^s} u - \bar{\mu} \frac{|z|^2}{\rho(w)^4} u = \bar{\mu} u, \ u \in H^1_0(\Omega, \mathbb{H}^d).\]

However, \( \bar{\mu} \) is not attained in \( H^1_0(\Omega, \mathbb{H}^d) \), so \( u = 0 \).

Thanks to Hardy inequality (1.3) and Poincaré inequality,

\[
\|u\|_{\bar{\mu}} = \left( \int_{\Omega} \left[ |\nabla H^s u|^2 + \frac{|z|^2}{\rho(w)^4} |u(z, s)|^2 \right] \, dzds \right)^\frac{1}{2}
\]

is equivalent to the norm on \( H^1_0(\Omega, \mathbb{H}^d) \) for all \( 0 < \mu < \bar{\mu} \), so that we will use \( \| \cdot \|_{\mu} \) as the norm of \( H^1_0(\Omega, \mathbb{H}^d) \).

**Theorem 2.3.** Let \( \Omega \subset \mathbb{H}^d \) be a bounded domain and assume that \( 0 < \mu < \bar{\mu} \). Then, the principal eigenvalue \( \lambda_{1,\mu} \) considered in \( H^1_0(\Omega, \mathbb{H}^d) \) with the norm \( \| \cdot \|_{\mu} \), is a bifurcation point of the problem (1.6) and \( \lambda_{1,\mu} \) is a global branch of nonnegative solutions of (1.6).

**Proof:** First we prove the existence of \( C_{\lambda_{1,\mu}} \):

We define the space \( X \) as a completion of \( C^\infty_0(\Omega) \) with respect to the norm induced by

\[
\langle u, v \rangle_X \equiv \int_\Omega \left[ \nabla H^s u \nabla \bar{H} \bar{v} - \mu \frac{|z|^2}{\rho(z)^4} u \bar{v} \right] \, dzds - \frac{\lambda_{1,\mu}}{2} \int_\Omega u \bar{v} \, dzds.
\]

We have

\[
\|u\|_X = \|u\|_\mu^2 - \frac{\lambda_{1,\mu}}{2} \|u\|_{L^2(\Omega)}^2 \leq \|u\|_\mu^2,
\]

and from the characterization of \( \lambda_{1,\mu} \), we have

\[
\|u\|_X \geq \|u\|_\mu^2 - \frac{\lambda_{1,\mu}}{2} \|u\|_{L^2(\Omega)}^2 \geq \|u\|_\mu^2 - \frac{1}{2} \|u\|_\mu^2 \geq \frac{1}{2} \|u\|_\mu^2.
\]

Since \( C^\infty(\Omega) \) is dense both in \( X \) and \( H^1_0(\Omega, \mathbb{H}^d) \), it follows that \( X = H^1_0(\Omega, \mathbb{H}^d) \), and the inner product in \( X \) is given by \( \langle u, v \rangle_X = \langle u, v \rangle_\mu \). Let

\[
a(u, v) = \int_\Omega u \bar{v} \, dzds, \text{ for all } u, v \in X.
\]
The bilinear form $a(u, v)$ is continuous in $X$, so the Riesz representation theorem implies that there exists a bounded linear operator $L$ such that

$$a(u, v) = \langle Lu, v \rangle,$$

for all $u, v \in X$.

The operator $L$ is self-adjoint and compact and its largest eigenvalue $\nu_1$ is characterized by

$$\nu_1 = \sup_{u \in X} \frac{\langle Lu, u \rangle}{\langle u, u \rangle_X} = \sup_{u \in X} \frac{\|u\|_{L^2(\Omega)}}{\|u\|_{X}} = \frac{1}{\lambda_{1, \mu}}.
$$

We define energy functional $I_{\mu, \lambda}$ on $H^1_0(\Omega, \mathbb{H}^d)$ by

$$I_{\mu, \lambda}(u) = \frac{1}{2} \int_\Omega \left[ |\nabla u|^2 - \mu \frac{|z|^2}{\rho(z, s)^4} |u|^2 \right] dzds - \frac{1}{p} \int_\Omega |u|^p dzds - \frac{\lambda}{2} \int_\Omega |u|^2 dzds.$$

Similarly to the classical case, $I_{\mu, \lambda}(\cdot)$ is well-defined on $H^1_0(\Omega, \mathbb{H}^d)$ and belongs to $C^1(H^1_0(\Omega, \mathbb{H}^d); \mathbb{R})$ and we have

$$\langle I_{\mu, \lambda}(u), v \rangle = \int_\Omega \left[ \nabla u \nabla \bar{v} - \mu \frac{|z|^2}{\rho(z, s)^4} u \bar{v} - |u|^{p-2} u \bar{v} - \lambda u \bar{v} \right] dzds$$

for any $v \in H^1_0(\Omega, \mathbb{H}^d)$. Let $N(\lambda, \cdot) : \mathbb{R} \times X \to X^*$ where $X^*$ is the dual space of $X$ be defined as by

$$N(\lambda, u) = \frac{1}{2} \int_\Omega \left[ |\nabla u|^2 - \mu \frac{|z|^2}{\rho(z, s)^4} |u|^2 \right] dzds - \frac{1}{p} \int_\Omega |u|^p dzds - \frac{\lambda}{2} \int_\Omega |u|^2 dzds.$$

for all $u \in X$. Since $I'_{\mu, \lambda}(u)$ is a bounded linear functional, $N(\lambda, \cdot)$ is well defined, and $N(\lambda, u) = u - G(\lambda, u)$ where $G(\lambda, u) = \lambda Lu + H(u)$,

$$\langle H(u), v \rangle = \int_\Omega |u|^{p-2} u \bar{v} dzds \forall v \in X.$$

Thanks to the compact embedding (1.1), the map $H$ is compact. On the other hand, we have

$$|\langle H(u), v \rangle| \leq \|u\|_{L^p(\Omega)} \|v\|_{L^p(\Omega)}.$$

Since $X = H^1_0(\Omega, \mathbb{H}^d)$ and thanks to the compact embedding (1.1), we have

$$\frac{1}{||u||_X} |\langle H(u), v \rangle| \leq ||u||_X^{-2} ||v||_X.$$

Thus

$$\lim_{||u||_X \to 0} \frac{||H(u)||_X}{||u||_X} = \lim_{||v||_X \to 0, ||v||_X \leq 1} \frac{1}{||u||_X} |\langle H(u), v \rangle| = 0.$$

It remains to prove that $C_{\lambda_{1, \mu}}$ is a global branch for nonnegative solutions of (1.6):

First, we prove that there exist $\varepsilon_0 > 0$ such that $u > 0$ for any $(\lambda, u) \in C_{\lambda_{1, \mu}} \cap B_{\varepsilon_0}(0)$ where $B_{\varepsilon_0}(0)$ is the open ball of $C_{\lambda_{1, \mu}}$ with center $(0, 0)$ and radius $\varepsilon_0$.

By contradiction, we assume that there exists $\lambda_n \to \lambda_{1, \mu}$, $u_n \to 0$ in $H^1_0(\Omega, \mathbb{H}^d)$ and that $(u_n)_n$ are changing sign in $\Omega$. 


Let \( u_n^- \equiv \min\{0, u_n\} \) and \( \mathcal{U}_n^- \equiv \{ x \in \Omega : u_n(x) < 0 \} \). Since \( u_n = u_n^+ - u_n^- \) is a weak solution of (1.6), \( u_n^- \) satisfies
\[
\begin{align*}
-\Delta_{\text{grad}} u_n^- - \frac{|z|^2}{p^2} u_n^- &= \lambda u_n^- + |u_n|^{p-2} u_n^- \text{ in } \Omega, \\
u_n^-|_{\partial\Omega} &= 0.
\end{align*}
\]
(2.21)

We thus have
\[
\int_{\mathcal{U}_n^-} \left[ \nabla |z|^2 \nabla u_n^- \right]^2 - \frac{|z|^2}{p} |u_n^-|^2 \, d\nu \, dzds = \lambda_n \int_{\mathcal{U}_n^-} |u_n^-|^2 \, d\nu \, dzds + \int_{\mathcal{U}_n^-} |u_n|^{p-2} |u_n^-|^2 \, d\nu \, dzds.
\]
(2.22)

But \( \lambda_n \) is bounded, so we get by Hölder inequality, Sobolev inequality and Sobolev embedding
\[
\|u_n^-\|^2_{H_0^1(\mathcal{U}_n^-)} \leq \lambda_n \|u_n^-\|^2_2 \left( \int_{\mathcal{U}_n^-} |u_n^-|^2 \, d\nu \, dzds \right)^{\frac{2}{p}} + \|u_n^-\|^p_{L^p(\mathcal{U}_n^-)}
\]
(2.23)
\[
\leq C_1 \|u_n^-\|^2_2 \|u_n^-\|^2_{H_0^1(\mathcal{U}_n^-)} + C_2 \|u_n^-\|^{p-2}_{H_0^1(\mathcal{U}_n^-)},
\]
(2.24)
thus
\[
1 \leq C_1 \|u_n^-\|^2_2 + C_2 \|u_n^-\|^{p-2}_{H_0^1(\mathcal{U}_n^-)}.
\]
(2.25)

Since \( \|u_n\|_{H_0^1(\Omega, \mathbb{H}^d)} \to 0 \) and \( p > 2 \), we derive that
\[
|\mathcal{U}_n^-| \geq C_3, \forall n,
\]
(2.26)

where the constant \( C_3 > 0 \) depends neither on \( \lambda_n \) nor \( u_n \).

Next we denote by \( v_n = \frac{u_n}{\|u_n\|_{H_0^1(\Omega, \mathbb{H}^d)}} \), then there exists a subsequence of \( v_n \), which we denote again by \( v_n \), such that
\[
v_n \rightharpoonup v_0 \text{ in } H_0^1(\Omega, \mathbb{H}^d),
v_n \to v_0 \text{ in } L^2(\Omega).
\]
Since \( u_n = G(\lambda_n, u_n) = \lambda_n L u_n + H(u_n), \)
\[
v_n = \lambda_n L v_n + \frac{H(u_n)}{\|u_n\|_{H_0^1(\Omega, \mathbb{H}^d)}}.
\]

As \( L \) is a compact linear operator and \( H(u_n) = 0 \|u_n\|_{H_0^1(\Omega, \mathbb{H}^d)} \), so \( v_0 = \lambda_{1, \mu} L v_0 \) and then \( v_0 = \phi_{1, \mu} > 0 \). Hence, by applying Egorov’s Theorem [4, Theorem IV.28] or [17], \( v_n \) converges uniformly to \( \phi_{1, \mu} \) in the exterior of a set of arbitrarily small measure. Then, there exists \( \Sigma \) a piece of \( \Omega \) of arbitrarily small measure in which \( v_n \) is positive outside \( \Sigma \) for \( n \) large enough, obtaining a contradiction with (2.26) and we conclude that the functions \( u_n \) are nonnegative, for \( n \) large enough. It then follows that \( u > 0 \) for any \((\lambda, u) \in C_{\lambda_1, \mu} \cap B_{\varepsilon_0}(\lambda_{1, \mu}, 0)\) with \( \varepsilon_0 > 0 \) small enough. Assume now that there exists \((\lambda, u) \in C_{\lambda_1, \mu}\) such that \( u(w_0) \leq 0 \) at some point \( w_0 \in \Omega \). From the previous part, we have \( u(w) > 0 \) for all \( w \in \Omega \) whenever \((\lambda, u) \in C_{\lambda_1, \mu}\) is close to \((\lambda_{1, \mu}, 0)\). Since \( C_{\lambda_1, \mu}\) is connected, there exists \( (\lambda^*, u^*) \in C_{\lambda_1, \mu}\) such that \( u^*(w) \geq 0 \) for all \( w \in \Omega \), except possibly some point \( w_0 \in \Omega \) where \( u^*(w_0) = 0 \), and in any neighbourhood of \((\lambda^*, u^*)\), we can find a point \((\lambda, u) \in C_{\lambda_1, \mu}\) with \( \bar{u}(w) < 0 \) for some \( w \in \Omega \). Then, the maximum principle implies that \( u^* = 0 \) on \( \Omega \). Thus we can construct a sequence \((\lambda_n, u_n) \in C_{\lambda_1, \mu}\) such that \( u_n > 0 \)
for all \( n, u_n \to 0 \) in \( H^1_0(\Omega; \mathbb{R}^d) \) and \( \lambda_n \to \lambda^* \).

Let \( v_n = \frac{u_n}{\|u_n\|_{H^1_0(\Omega; \mathbb{R}^d)}} \), then

\[
v_n = \lambda_n L v_n + \frac{H(u_n)}{\|u_n\|_{H^1_0(\Omega; \mathbb{R}^d)}}.
\]

So, the subsequence \((v_n)_n\) converges to \( v_0 = \lambda^* L v_0 \). Since \( v_n > 0 \), for all \( n \) and \( \|v_0\|_{H^1_0(\Omega; \mathbb{R}^d)} = 1 \), we have \( v_0 > 0 \). Thus \( \lambda^* \) is an eigenvalue of (1.6) corresponding to a positive eigenfunction. But \( \lambda_{1, \mu} \) is the only positive eigenvalue of (1.6) corresponding to a positive eigenfunction, so we deduce that \( \lambda^* = \lambda_{1, \mu} \), and that \( (\lambda^*, u^*) = (\lambda_{1, \mu}, 0) \). This contradicts the fact that every neighbourhood of \((\lambda^*, u^*)\) must contain a point \((\bar{\lambda}, \bar{u}) \in C_{\lambda_{1, \mu}}\) with \( \bar{u}(w) < 0 \) for some \( w \in \Omega \). Hence \( u(w) > 0 \) for all \( w \in \Omega \) whenever \((\lambda, u) \in C_{\lambda_{1, \mu}}\), and \( C_{\lambda_{1, \mu}} \) cannot cross points of the form \((\lambda, 0)\), where \( \lambda \neq \lambda_{1, \mu} \).

3. Asymptotic behavior of solutions for problem (1.1)

Similarly [22, 23], we are interested here in the description of the behavior of solutions of (1.1) when \( u_0 \) has low energy smaller than the mountain pass level

\[
c_{\mu, \lambda} = \inf_{h \in \Gamma} \max_{t \in [0, 1]} I_{\mu, \lambda}(h(t)), \quad \Gamma = \{ h \in C([0, 1]; H^1_0(\Omega; \mathbb{R}^d)) ; h(0) = 0 \text{ and } h(1) = e \}.
\]

In view of [21], since \( 2 < p < 2^* \), the functional \( I_{\mu, \lambda} \) satisfies the Palais-Smale condition and admits at least a positive solution (called mountain pass solution).

Lemma 3.1. For \( \lambda > 0 \), \( 0 < \mu < \bar{\mu} \) and \( 2 < p < 2^* \), the function \( f(t) = \lambda t^+ \) \( t |t|^{p-2} t \), \( t \in \mathbb{R} \) defines a locally Lipschitz map \( f : H^1_0(\Omega; \mathbb{R}^d) \to H^{-1}(\Omega; \mathbb{R}^d) \).

Proof: The function \( f_1(u) = \lambda u \), defines a locally Lipschitz map \( f_1 : L^2(\Omega) \to L^2(\Omega) \), so \( f_1 : H^1_0(\Omega; \mathbb{R}^d) \to H^{-1}(\Omega; \mathbb{R}^d) \) is locally Lipschitz. Let \( u \in L^p(\Omega) \) and \( f_2(u) = |u|^{p-2} u \). The function \( f_2 : L^p(\Omega) \to L^{p'}(\Omega) \) is locally Lipschitz, thanks to the following estimate:

\[
\|f_2(u) - f_2(v)\|_{L^{p'}(\Omega)} \leq (p-1) \left( \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \right)^{p-2} \|u - v\|_{L^p(\Omega)}.
\]

for all \( u, v \in L^p(\Omega) \). So thanks to compact embedding (1.1) and from \( L^{p'}(\Omega) \subset H^{-1}(\Omega; \mathbb{R}^d) \), the function \( f_2 : H^1_0(\Omega; \mathbb{R}^d) \to H^{-1}(\Omega; \mathbb{R}^d) \) is locally Lipschitz.

Proposition 3.2. Let \( u_0 \in H^1_0(\Omega; \mathbb{R}^d) \), \( \lambda > 0 \) and \( 0 < \mu < \bar{\mu} \), the problem (1.1) has a unique weak solution \( u \) such that

\[
u \in C([0, T); H^1_0(\Omega; \mathbb{R}^d)) \cap C^1([0, T); H^{-1}(\Omega; \mathbb{R}^d))
\]

and we have

\[
\frac{d}{dt} I_{\mu, \lambda}(u(t)) = -\|\partial_t u\|_{L^2(\Omega)}^2.
\]

Proof: By means of the Hille-Yosida theorem, \( \mathcal{T}(t) = \{ e^{-tL_{\mu}} \}_{t \geq 0} \) is the semigroup generated by the operator \( L_{\mu} = -\Delta_{\mathbb{H}^d} - \mu \frac{|z|^2}{\rho(z, s)^4} \). Since \( f : H^1_0(\Omega; \mathbb{R}^d) \to H^{-1}(\Omega) \) is locally Lipschitz, so by Pazy [24, Theorem 1.4] or Haraux [16, Theorem 6.2.2] or Goldstein [12, Theorem 2.4]; there exists a unique solution of (1.1) defined on a maximal interval \([0, T_{\text{max}}]\), where \( 0 < T_{\text{max}} \leq +\infty \) and

\[
u \in C([0, T); H^1_0(\Omega; \mathbb{R}^d)) \cap C^1([0, T); H^{-1}(\Omega))
\]
satisfying the variation of constants formula

\[ u(t) = T(t)u_0 + \int_0^t T(t - \tau) f(u(\tau)) \, d\tau. \]  

Moreover, if \( T_{\max} < +\infty \), we say that \( T_{\max} \) is a blow-up time, whereas if \( T_{\max} = +\infty \), we say that \( u \) is global solution.

We will show that \( u \) satisfies (3.29): Let \( u \in D(L_\mu) \) (\( D(L_\mu) \) be the domain of definition of \( L_\mu \)) and \( t \in [0, T), T < T_{\max} \). Since \( I_{\mu,\lambda} \in C^1(H^1_0(\Omega; \mathbb{H}^d); \mathbb{R}) \), we have

\[ \langle I_{\mu,\lambda}'(u), \Delta_{\mathbb{H}^d}u + \mu \frac{|z|^2}{\rho(w)} u + f(u) \rangle = -\int_\Omega | \Delta_{\mathbb{H}^d}u + \mu \frac{|z|^2}{\rho(w)} u + f(u) |^2 \, dw \]

(3.31)

Set \( g(t) = f(u(t)) \) and let \( g_n \in C^1([0, T]; H^1_0(\Omega; \mathbb{H}^d)) \), \( u_{0n} \in D(L_\mu) \) such that

\[ g_n \to g \quad \text{in} \quad C^1([0, T]; H^1_0(\Omega; \mathbb{H}^d)), \]

\[ u_{0n} \to u_0 \quad \text{in} \quad H^1_0(\Omega; \mathbb{H}^d). \]

Define \( u_n(t) = T(t)u_{0n} + \int_0^t T(t - \tau) g_n(\tau) \, d\tau \), then \( u_n \in C^1([0, T]; H^1_0(\Omega; \mathbb{H}^d)) \) satisfies

\[ \partial_t u_n - \Delta_{\mathbb{H}^d}u_n - \mu V u_n = g_n \]

and

\[ u_n \to u \quad \text{in} \quad H^1_0(\Omega; \mathbb{H}^d). \]

Thus, from (3.31),

\[ I_{\mu,\lambda}(u_n(t)) - I_{\mu,\lambda}(u_{0n}) = \int_0^t \langle I_{\mu,\lambda}'(u_n(\tau)), \Delta_{\mathbb{H}^d}u_n + \mu \frac{|z|^2}{\rho(w)} u_n + g_n(\tau) \rangle \, d\tau \]

\[ = -\int_0^t \| \partial_t u_n(\tau) \|_{L^2(\Omega)}^2 \, d\tau + \int_0^t \langle I_{\mu,\lambda}'(u_n(\tau)), g_n(\tau) - f(u_n(\tau)) \rangle \, d\tau. \]

Passing to the limit, we deduce (3.29).

Next, we introduce the following sets:

\[ \mathcal{O}^+ \equiv \{ u \in H^1_0(\Omega; \mathbb{H}^d) : I_{\mu,\lambda}(u) < c_{\mu,\lambda}; \langle I_{\mu,\lambda}'(u), u \rangle > 0 \}, \]

\[ \mathcal{O}^- \equiv \{ u \in H^1_0(\Omega; \mathbb{H}^d) : I_{\mu,\lambda}(u) < c_{\mu,\lambda}; \langle I_{\mu,\lambda}'(u), u \rangle < 0 \}, \]

(3.32)

\[ \mathcal{N} \equiv \{ u \in H^1_0(\Omega; \mathbb{H}^d) : \langle I_{\mu,\lambda}'(u), u \rangle = 0 \}. \]

\( \mathcal{N} \) is named the Nehari manifold relative to \( I_{\mu,\lambda} \). The mountain-pass level \( c_{\mu,\lambda} \) defined in (3.27) may also be characterized as

\[ c_{\mu,\lambda} = \inf_{u \in \mathcal{N}} I_{\mu,\lambda}(u). \]

(3.33)

**Theorem 3.3.** If there exist \( t_0 \geq 0 \) such that \( I_{\mu,\lambda}(u(t_0)) \leq 0 \), then \( u(t) \) blows-up in finite time.

**Proof:** Let \( t_0 \geq 0 \) such that \( I_{\mu,\lambda}(u(t_0)) \leq 0 \) and we suppose that \( u(t) \) is a global solution to the problem (1.1). Since \( u(t) \) satisfies (3.29), we have

\[ I_{\mu,\lambda}(u(t_0)) = I_{\mu,\lambda}(u(t)) + \int_0^t \| \partial_t u(\tau) \|_{L^2(\Omega)}^2 \, d\tau. \]
Set \( g(t) \equiv \int_{\Omega} |u(t)|^2 \, dw \), then
\[
\frac{d}{dt} g(t) = 2 \int_{\Omega} u(t) \partial_t u(t) \, dw \\
= -2 \int_{\Omega} \left[ |\nabla_H u(t)|^2 - \mu \frac{|z|^2}{\rho(w)} |u(t)|^2 \right] \, dw \\
+ 2\lambda \int_{\Omega} |u(t)|^2 \, dw + 2 \int_{\Omega} |u(t)|^p \, dw \\
= 4 \int_{t_0}^t \| \partial_t u(\tau) \|^2_{L^2(\Omega)} \, d\tau - 4I_{\mu,\lambda}(u(t_0)) + 2 \left( 1 - \frac{2}{p} \right) \int_{\Omega} |u(t)|^p \, dw \\
\geq 2 \left( 1 - \frac{2}{p} \right) \int_{\Omega} |u(t)|^p \, dw > 0. \\
(3.34)
\]

Hence we get for any \( t \geq t_0 \), \( g(t) \geq g(t_0) = \int_{\Omega} |u(t_0)|^2 \, dw \). Let \( \epsilon \in (1, \frac{p}{2}) \), so we deduce by (3.34), that for any \( t \geq t_0 \):
\[
- \frac{1}{\epsilon - 1} \frac{d}{dt} g^{1-\epsilon}(t) = g^{-\epsilon}(t) \frac{d}{dt} g(t) \\
\geq 2 \left( 1 - \frac{2}{p} \right) g^{-\epsilon}(t) \int_{\Omega} |u(t)|^p \, dw \\
\geq C g^{-\epsilon}(t) \left( \int_{\Omega} |u(t)|^2 \, dw \right)^{\frac{\epsilon}{2}} \\
\geq C \left( \int_{\Omega} |u(t_0)|^2 \, dw \right)^{\frac{\epsilon}{2} - \epsilon}.
\]

Hence for \( t \geq t_0 \) sufficiently large, we have
\[
0 < \left( \int_{\Omega} |u(t)|^2 \, dw \right)^{1-\epsilon} = g^{1-\epsilon}(t) \\
\leq g^{1-\epsilon}(t_0) + C(\epsilon - 1) g^{\frac{\epsilon}{2} - \epsilon}(t_0)(t_0 - t).
\]

Then
\[
-1 < C(\epsilon - 1) g^{\frac{\epsilon}{2} - 1}(t_0)(t_0 - t)
\]
and so \( t < t_0 + \left[ C(\epsilon - 1) g^{\frac{\epsilon}{2} - 1}(t_0) \right]^{-1} \), which is a contradiction.

**Theorem 3.4.** Assume that \( u_0 \in \mathcal{O}^+ \) and \( \lambda < \lambda_{1, \mu} \), then the problem (1.1) admits a global solution \( u(t) \). Moreover, there exists a positive number \( \alpha \) such that
\[
||u(t)||_{\mu} = O(e^{-\alpha t}), \quad \text{as } t \to +\infty.
\]

**Proof:** Let \( u_0 \in \mathcal{O}^+ \) and \( u(t) = u(t, w) \) be the unique solution, the existence of which has been proved in Proposition 3.2. From (3.29), we have that \( t \mapsto I_{\mu, \lambda}(u(t)) \) is strictly decreasing, so
\[
I_{\mu, \lambda}(u(t)) \leq I_{\mu, \lambda}(u_0) \leq c_{\mu, \lambda}.
\]

Suppose there exists \( t^* \in (0, T_{\max}) \) such that \( u(t^*) \not\in \mathcal{O}^+ \). Then
\[
\langle I'_{\mu, \lambda}(u(t^*)), u(t^*) \rangle \leq 0.
\]

Moreover since \( t \mapsto \langle I'_{\mu, \lambda}(u(t)), u(t) \rangle \) is continuous, there exists \( t_0 \in (0, t^*] \) such that
\[
\langle I'_{\mu, \lambda}(u(t_0)), u(t_0) \rangle = 0.
\]
Hence $u(t_0) = 0$ in $\Omega$, or $u(t_0) \in \mathcal{N}$. If $u(t_0) = 0$ in $\Omega$, then by the uniqueness of $u(t)$, we conclude that $u(t) = 0$ for any $t \in [t_0, T_{\text{max}}]$. Thus $u(t)$ is global by extending to 0 for all $t \geq T_{\text{max}}$, and so $I_{\mu, \lambda}(u(t)) > 0$ for any $t \geq 0$ by Theorem 3.3. But $I_{\mu, \lambda}(u(t_0)) = 0$, which is a contradiction, and so $u(t_0) \in \mathcal{N}$. It is well known that $c_{\mu, \lambda} = \inf_{u \in \mathcal{N}} I_{\mu, \lambda}(u)$ [28, Theorem 4.2], thus $c_{\mu, \lambda} \leq I_{\mu, \lambda}(u(t_0))$, which is a contradiction with (3.36). So, we conclude that $u(t) \in \mathcal{O}^+$ for all $t \in [t_0, T_{\text{max}}]$.

On other hand, we can write

$$I_{\mu, \lambda}(u(t)) = \frac{1}{p} \langle I'_{\mu, \lambda}(u(t)), u(t) \rangle$$

$$+ \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega} \left( \frac{|z|^2}{\rho(w)^4} |u(t, w)|^2 \right) dw - \left( \frac{1}{2} - \frac{1}{p} \right) \lambda \int_{\Omega} |u(t, w)|^2 dw$$

$$\geq \left( \frac{1}{2} - \frac{1}{p} \right) \left( 1 - \frac{\lambda}{\lambda_{1, \mu}} \right) \|u(t, \cdot)\|_{\mu}^2 > 0.$$

Since $u(t)$ satisfies (3.29), we have

$$\int_{t_0}^{t} \|\partial_t u(\tau, \cdot)\|_{L^2(\Omega)}^2 d\tau + \left( \frac{1}{2} - \frac{1}{p} \right) \left( 1 - \frac{\lambda}{\lambda_{1, \mu}} \right) \|u(t, \cdot)\|_{\mu}^2 \leq I_{\mu, \lambda}(u(t_0, \cdot)) < c_{\mu, \lambda}.$$

Then we have

$$\int_{t_0}^{t} \|\partial_t u(\tau, \cdot)\|_{L^2(\Omega)}^2 d\tau < c_{\mu, \lambda}, \text{ and } \|u(t, \cdot)\|_{\mu} < \left[ \left( \frac{1}{2} - \frac{1}{p} \right) \left( 1 - \frac{\lambda}{\lambda_{1, \mu}} \right) \right]^{-1} c_{\mu, \lambda},$$

which implies that $u(t)$ is a global solution of the problem (1.1) and $\mathcal{O}^+$ is invariant set.

Letting $t \to +\infty$ in (3.39), the integral $\int_{t_0}^{+\infty} \|\partial_t u(\tau, \cdot)\|_{L^2(\Omega)}^2 d\tau$ is finitely determined.

Therefore, there exists a sequence $(t_n)_{n \geq 0}$ with $t_n \to +\infty$ as $n \to +\infty$, such that

$$\int_{\Omega} |\partial_t u(t_n, \cdot)|^2 dw \to 0, \text{ and } u(t_n) \rightharpoonup v \text{ in } H_0^1(\Omega, \mathbb{H}^d).$$

Letting $t_n \to +\infty$, we obtain that $v \in H_0^1(\Omega, \mathbb{H}^d)$ is a solution of problem (1.6). So

$$\langle I'_{\mu, \lambda}(v), v \rangle = 0.$$

If $v \neq 0$, then $v \in \mathcal{N}$, and so

$$I_{\mu, \lambda}(v) \geq c_{\mu, \lambda}.$$

Since $u(t_n)$ satisfies (3.29), it follows by H"older inequality and from (3.40), that

$$|\langle I'_{\mu, \lambda}(u(t_n, \cdot)), u(t_n, \cdot) \rangle| \leq \int_{\Omega} u(t_n, w) \partial_t u(t_n, w) dw \leq \|u(t_n, \cdot)\|_{L^2(\Omega)} \|\partial_t u(t_n, \cdot)\|_{L^2(\Omega)} \leq \sqrt{\Lambda_{1, \mu}} \|u(t_n, \cdot)\|_{\mu} \|\partial_t u(t_n, \cdot)\|_{L^2(\Omega)} \leq C \|\partial_t u(t_n, \cdot)\|_{L^2(\Omega)}.$$
We deduce by (3.37), (3.41) and (3.44) that
\[
I_{\mu,\lambda}(v) = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega} \left[ |\nabla \mathbb{H} u(w)|^2 - \mu \frac{|z|^2}{\rho(w)^2} |v(w)|^2 \right] \, dw - \left( \frac{1}{2} - \frac{1}{p} \right) \lambda \int_{\Omega} |v(w)|^2 \, dw
\]
\[
\leq \lim_{n \to +\infty} \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega} \left[ |\nabla \mathbb{H} u(t_n, w)|^2 - \mu \frac{|z|^2}{\rho(w)^2} |u(t_n, w)|^2 \right] \, dw
\]
\[
- \left( \frac{1}{2} - \frac{1}{p} \right) \lambda \int_{\Omega} |u(t_n, w)|^2 \, dw + \lim_{n \to +\infty} \langle I'_{\mu,\lambda}(u(t_n)), u(t_n) \rangle
\]
\[
\leq \lim_{n \to +\infty} I_{\mu,\lambda}(u(t_n))
\]
\[
\leq I_{\mu,\lambda}(u_0) < c_{\mu,\lambda},
\]
which contradicts (3.42), and so \( v = 0 \) in \( \Omega \).

Hence by (3.40), we have
\[
u(t_n, \cdot) \to 0 \text{ in } L^q(\Omega), \ 2 \leq q < 2^*.
\]
Since
\[
\|u(t_n, \cdot)\|_\mu^2 = \langle I'_{\mu,\lambda}(u(t_n)), u(t_n) \rangle + \lambda \int_{\Omega} |u(t_n, w)|^2 \, dw + \int_{\Omega} |u(t_n, w)|^p \, dw
\]
\[
\to 0, \ \text{as } n \to +\infty,
\]
we have
\[
(3.45) \quad u(t_n, \cdot) \to 0 \text{ in } H_0^1(\Omega, \mathbb{H}^d), \ \text{as } n \to +\infty.
\]

For simplicity, let us denote by \( t \) the divergent sequence and by \( u(t) = u(t_n, w) \). We have from (3.44) that
\[
I_{\mu,\lambda}(u(t)) = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega} \left[ |\nabla \mathbb{H} u(t)|^2 - \mu \frac{|z|^2}{\rho(w)^2} |u(t)|^2 \right] \, dw - \left( \frac{1}{2} - \frac{1}{p} \right) \lambda \int_{\Omega} |u(t)|^2 \, dw
\]
\[
= \left( \frac{1}{2} - \frac{1}{p} \right) ||u(t)||_\mu^2 - \left( \frac{1}{2} - \frac{1}{p} \right) \|u(t)\|_{L^2(\Omega)}^2.
\]
So, due to (3.29) we have
\[
\|u(t)\|_\mu^2 = \frac{2p}{p - 2} I_{\mu,\lambda}(u(t)) + \|u(t)\|_{L^2(\Omega)}^2
\]
\[
\leq \frac{2p}{p - 2} I_{\mu,\lambda}(u_0) + \|u(t)\|_{L^2(\Omega)}^2
\]
\[
< \frac{2p}{p - 2} c_{\mu,\lambda} + o(1).
\]
Therefore there exists \( t_0 \) such that for all \( t \geq t_0 \),
\[
(3.47) \quad \|u(t)\|_\mu^2 \leq \frac{2p}{p - 2} c_{\mu,\lambda}.
\]
On the other hand,
\[
\int_{\Omega} |u(t)|^p \, dw \leq C_\Omega^p \left( \frac{\bar{\mu}}{\mu - \bar{\mu}} \right)^\frac{p}{2} \|u(t)\|_\mu^p
\]
\[
\leq C_\Omega^p \left( \frac{\bar{\mu}}{\mu - \bar{\mu}} \right)^\frac{p}{2} \left[ \frac{2p}{p - 2} c_{\mu,\lambda} \right]^{\frac{p - 2}{2}} \|u(t)\|_\mu^2.
\]
Let $C_1 = C_1^p \left( \frac{\bar{\mu}}{\bar{\mu} - \mu} \right)^\frac{p}{2} \left[ \frac{2p}{p - 2} c_{\mu, \lambda} \right]^{\frac{p - 2}{2}}$, we have

\[
(1 - C_1) \|u(t)\|_\mu^2 \leq \|u(t)\|_\mu^2 - \|u(t)\|^p_{L^p(\Omega)} \leq \langle I'_{\mu, \lambda}(u(t)), u(t) \rangle + \lambda \|u(t)\|_{L^2(\Omega)}^2.
\] (3.48)

Let us recall that if we set $g(t) = \int_\Omega |u(t)|^2 \, dw$, then

\[
\frac{d}{dt} g(t) = 2 \int_\Omega u(t) \partial_t u(t) \, dw
= -2 \int_\Omega \left[ \nabla H u(t) \right]^2 - \mu \frac{|z|^2}{\rho(w)} |u(t)|^2 \, dw
+ 2 \lambda \int_\Omega |u(t)|^2 \, dw + 2 \int_\Omega |u(t)|^p \, dw
= -2 \langle I'_{\mu, \lambda}(u(t)), u(t) \rangle.
\]

So we get from (3.37) that for any $t \geq t_0$, we have

\[
\int_t^\infty \langle I'_{\mu, \lambda}(u(\tau)), u(\tau) \rangle \, d\tau = \frac{1}{2} \|u(t)\|^2_{L^2(\Omega)} \leq \frac{1}{2 \lambda_1, \mu} \|u(t)\|^2_{\mu}
\] (3.49)

So from (3.48) and (3.49), we have for any $t \geq t_0$ that

\[
\int_t^\infty I_{\mu, \lambda}(u(\tau)) \, d\tau \leq \frac{1}{2 \lambda_1, \mu} \int_t^\infty \|u(\tau)\|^2_{\mu}
\leq (1 - C_1)^{-1} \frac{1}{2 \lambda_1, \mu} \left[ \int_t^\infty \langle I'_{\mu, \lambda}(u(\tau)), u(\tau) \rangle \, d\tau + \lambda \int_t^\infty \|u(\tau)\|^2_{L^2(\Omega)} \, d\tau \right]
\leq (1 - C_1)^{-1} \frac{p}{2 \lambda_1, \mu} I_{\mu, \lambda}(u(t)) + (1 - C_1)^{-1} \frac{\lambda}{2 \lambda_1, \mu} \int_t^\infty \|u(\tau)\|^2_{L^2(\Omega)} \, d\tau.
\] (3.50)

Since $\lim_{t \to +\infty} \|u(t)\|_{L^2(\Omega)} = 0$, there exists $t_1 > t_0$ such that for any $t \geq t_1$, we have

\[
\int_t^\infty I_{\mu, \lambda}(u(\tau)) \, d\tau \leq \frac{(1 - C_1)^{-1} p}{2 \lambda_1, \mu} I_{\mu, \lambda}(u(t)).
\] (3.51)

Thus

\[
\int_t^\infty I_{\mu, \lambda}(u(\tau)) \, d\tau \leq C(t_1) e^{-\alpha t},
\]

with $\alpha = \frac{(1 - C_1)^{-1} p}{2 \lambda_1, \mu}$. But we remark that

\[
I_{\mu, \lambda}(u(t + 1)) \leq \int_t^{t+1} I_{\mu, \lambda}(u(\tau)) \, d\tau < \int_t^\infty I_{\mu, \lambda}(u(\tau)) \, d\tau,
\]

hence we deduce that for any $t \geq t_1$, we have

\[
I_{\mu, \lambda}(u(t + 1)) < C(t_1) e^{-\alpha t},
\]

and we can conclude that for any $t \geq t_1$, we have

\[
\|u(t)\|_\mu = O(e^{-\alpha t}).
\] (3.55)
Theorem 3.5. Assume that $u_0 \in \mathcal{O}^-$. Then the solution $u(t)$ of the problem (1.1) blows up in finite time.

Proof: Let $u_0 \in \mathcal{O}^-$ and $u(t) = u(w, t, u_0)$ be the unique solution, the existence of which has been proved in Proposition 3.2. From the inequality (3.29), we have that $t \mapsto I_{\mu, \lambda}(u(t))$ is strictly decreasing, so

$$I_{\mu, \lambda}(u(t)) \leq I_{\mu, \lambda}(u_0) \leq c_{\mu, \lambda}.$$  

(3.56)

Suppose there exists $t \in (0, T_{\text{max}})$ such that $u(t) \notin \mathcal{O}^-$. Then

$$\langle I_{\mu, \lambda}'(u(t)), u(t) \rangle \geq 0.$$

And since the application $t \mapsto \langle I_{\mu, \lambda}'(u(t)), u(t) \rangle$ is continuous, there exists $\tilde{t} \in (0, \tilde{t}]$ such that

$$\langle I_{\mu, \lambda}'(u(\tilde{t})), u(\tilde{t}) \rangle = 0.$$

Hence $u(\tilde{t}) = 0$ in $\Omega$, or $u(\tilde{t}) \notin \mathcal{N}$. If $u(\tilde{t}) = 0$ in $\Omega$, then by the uniqueness of $u(t)$, we conclude that $u(t) = 0$ for any $t \in [\tilde{t}, T_{\text{max}})$. Thus $u(t)$ is global by extending to 0 for all $t \geq T_{\text{max}}$, and thanks to Theorem 3.3, $I_{\mu, \lambda}(u(t)) > 0$ for any $t \geq 0$. But $I_{\mu, \lambda}(u(\tilde{t})) = 0$, which is a contradiction, and so $u(\tilde{t}) \in \mathcal{N}$. But by [28],

$$c_{\mu, \lambda} = \inf_{u \in \mathcal{N}} I_{\mu, \lambda}(u),$$

then $c_{\mu, \lambda} \leq I_{\mu, \lambda}(u(\tilde{t}))$, which contradicts (3.56). So, we conclude that $u(t) \in \mathcal{O}^-$ for all $t \in [\tilde{t}, T_{\text{max}})$. We suppose by contradiction that $T_{\text{max}} = +\infty$, i.e. $u(t) = u(t, \cdot)$ exists for all $t \geq 0$. For $u \in \mathcal{O}^-$, we have

$$\frac{d}{dt}\|u(t, \cdot)\|_{L^2(\Omega)}^2 = -2 \langle I_{\mu, \lambda}'(u(t)), u(t) \rangle > 0.$$

Then $t \mapsto \|u(t, \cdot)\|_{L^2(\Omega)}$ is strictly increasing and so

$$\lim_{t \to +\infty} \|u(t, \cdot)\|_{L^2(\Omega)} = c \in (0, +\infty].$$

(3.57)

We suppose that $c < +\infty$. Following the same reasoning as in the proof of Theorem 3.4, we deduce that we can select a divergent subsequence, still denoted by $t$, such that when $t \to +\infty$,

$$u(t, \cdot) \to 0 \quad \text{in} \quad H^1_0(\Omega, \mathbb{H}^d).$$

Letting $t \to +\infty$ in the inequality

$$\sqrt{\lambda_{1, \mu}} \|u(t, \cdot)\|_{L^2(\Omega)} \leq \|u(t, \cdot)\|_{L^2(\Omega)}$$

we get that $0 < c \leq 0$, which is a contradiction. So we conclude that

$$\lim_{t \to +\infty} \|u(t, \cdot)\|_{L^2(\Omega)} = +\infty.$$

(3.58)

Set $g(t) = \|u(t, \cdot)\|_{L^2(\Omega)}^2$, so

$$-\frac{2}{p - 2} \frac{d}{dt} g^\frac{1}{2}(t) = g'(t)g^{-\frac{1}{2}}(t) = -2 \|u(t, \cdot)\|_{L^2(\Omega)}^{-p}(\|u(t, \cdot)\|_{L^2(\Omega)}^2 - \lambda \|u(t, \cdot)\|_{L^2(\Omega)}^2 \|u(t, \cdot)\|_{L^p(\Omega)}^2)$$

$$\geq -2 \|u(t, \cdot)\|_{L^2(\Omega)}^{-p} \|u(t, \cdot)\|_{L^2(\Omega)}^2 + 2 \|u(t, \cdot)\|_{L^2(\Omega)}^2 \|u(t, \cdot)\|_{L^p(\Omega)}^2.$$

(3.59)

By Hölder inequality, we have

$$\|u(t, \cdot)\|_{L^p(\Omega)}^2 \geq |\Omega|^{-\frac{1}{2}} \|u(t, \cdot)\|_{L^2(\Omega)}^2.$$
and by (3.58), there exist $t_1 > 0$ and a constant $C_1 > 0$, such that for $t \geq t_1$, we have
\begin{equation}
\|u(t, \cdot)\|_{L^2(\Omega)} \geq C_1.
\end{equation}
Then, there exist $t_1 > 0$ and a constant $C_2 > 0$, such that for $t \geq t_1$, we have
\begin{equation}
-\frac{2}{p-2} \frac{d}{dt} g^{1-\frac{p}{2}}(t) \geq -2\lambda_{1,\mu} C_1^{2-p} + 2|\Omega|^{1-\frac{p}{2}} \geq C_2.
\end{equation}
Hence, we have from (3.61), that for any $t \geq t_1$,
\[0 < g(t) \leq g(t_1) + \frac{p-2}{2} C_2 (t - t_1),\]
which is a contradiction if $t$ is sufficiently large. So we conclude that $T_{\text{max}} < +\infty$.

REFERENCES

[19] I. Kombe, Best constant for weighted Rellich and uncertainty principle inequalities on Carnot groups, submitted


