Understanding, Modeling and Managing Longevity Risk: Key Issues and Main Challenges

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In this article we investigate the latest developments on longevity risk modeling. We first introduce longevity risk and some key actuarial definitions as to allow for a better understanding of the related challenges in term of risk management from both a financial and insurance point of view. The article also provides a global view on the practical issues on longevity-linked insurance and pension funds products that arise mainly from the steady increase in life expectancy since 1960s. Those issues are leading the industry to adopt more effective regulations to better assess and efficiently manage the inherited risks. Simultaneously, the development on the longevity has enhanced the need of capital markets as to manage and transfer the risk throughout the so-called insurance-linked securities (ILS). Therefore, we also highlight future developments on longevity risk management from a financial point of view, bringing up practices from the banking industry in terms of modeling and pricing.

Key words: Longevity Risk, securitization, risk transfer, incomplete market, life insurance, stochastic mortality, pensions, long term interest rate, regulation, population dynamics.

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The observed constant improvements in longevity are bringing new issues and challenges at various levels: social, political, economic and regulatory to mention only a few. But one of the most publicized impacts of longevity improvements is certainly on pensions. In 2009, in most developed countries, many companies have closed the defined benefit retirement plans (such as the 401(K) plans in the United States) that they used to offer to their employees. Such a scheme represents indeed a risk transfer from both the industry and the insurers back to the policyholders, which, from a social point of view, is not satisfactory anymore. Similarly, in several countries, defined benefit pension plans have been continuously replaced with defined contribution plans, leading to the same result. In addition, some governments are about to increase the retirement age by 2 or 5 years to take into account longevity improvements, population ageing and the financing of pension.

The insurance industry is also facing some specific challenges related to longevity risk, i.e. the risk that the trend of longevity improvements significantly changes in the future. More and more capital has to be constituted to face this long-term risk, and new regulations in Europe, together with the recent financial crisis only amplify this phenomenon. Hence, it has become more and more important for insurance companies and pension funds to find a suitable and efficient way to cross-hedge or to transfer part of the longevity risk to reinsurers or to financial markets. Longevity risk is however not so easy to transfer, as it is hard to understand, and therefore to manage. In particular, because of its long-term nature, accurate longevity projections are delicate and modeling the embedded interest rate risk remains challenging.

As to better manage longevity risk, prospective life tables, containing longevity trend projections are used. They prove to be very helpful for reserving in life insurance in particular but the irregular updates of these tables can cause some problems. For instance, the French prospective life tables were updated in 2006. As longevity improvements were more important than expected according to the previous prospective tables established in 1993, French insurers had to increase their reserves by 8% on average to account for this phenomenon! Moreover, in addition to this risk of observing a significant change in the longevity trend, the insurance sector is facing some basis risk as the evolution of the policyholders mortality is usually different from that of the national population, because of some selection effects. This selection effect has different impacts on different insurance companies portfolios, as mortality levels, speeds of decrease and accelerations are very heterogeneous in the insurance industry. This makes it hard for insurance companies to rely on national indices or even on industry indices to manage their own longevity risk.

To better understand the longevity risk and avoid some managerial over-reactions due to short-term oscillations around the average trend, the dynamics of longevity improvements, their causes, and the above mentioned heterogeneity have to be studied carefully. Many standard stochastic models for mortality have been developed, some of them inspired by the classical credit risk and interest rates literature. In these models, mortality is mainly explained by age and time. An alternative approach consists of a microscopic modeling for a population where individuals are characterized not only by their age but also by other features reflecting their living conditions. Such models are very useful for the risk analysis of a given insurance portfolio, but also at a social and political level when combined with a study of other demographic rates, such as fertility and immigration rates: projective scenarii can guide the strategies of governments

concerning bills on immigration and on the age of retirement for example.
The European insurance industry will soon have to comply to some new solvency regulations, namely Solvency II. Those regulations and standards lay the emphasis on the way risks endorsed by an insurance company should be handled in order to face adverse economic and demographic situations. Those regulations will be effective by late 2012 and certainly enhance the development of alternative risk transfer solutions for insurance risk in general and for longevity risk in particular.
No doubt that the pricing methodologies for insurance related transactions, and in particular longevity linked securities will be impacted as more and more alternative solutions appear in the market. Today, the longevity market is an immature and incomplete market, with an evident lack of liquidity. Standard replication strategies are impossible, making the classical financial methodology not applicable. In this case, indifference pricing, involving utility maximization, seems to be a more appropriate point of view to adopt. Besides, due to the long maturities of the underlying risk, the modeling of long term interest rate becomes also unavoidable and adds to the complexity of the problem.
Our paper is organized as follows: we first describe the main characteristics of longevity risk, insisting on the classical and prospective life tables and mortality data, and some specific features such as the cohort effect. In Section 2, we present the key models for mortality risk and how they can be used to model longevity risk. Section 3 is dedicated to the new solvency regulations that will be enforced by 2012 and discuss in particular longevity risk management. In Section 4, we are concerned with longevity risk transfer issues and the convergence between the insurance industry and the capital markets. Finally, we look at the main modeling questions regarding the pricing of longevity risk, with a discussion on long term interest rates modeling.

1. Characteristics of longevity risk

In this first section, we introduce some fundamental notions related to longevity risk. In particular, we give some basics on life tables, including the standard notation used both in practice and in the literature, and detail some noticeable features such as the cohort effect.

1.1. Mortality and longevity data

1.1.1. Life tables for mortality risk To analyze variations of mortality across different age classes and to take into account various factors (e.g. infant mortality, ageing, accidents,...), actuaries have been using life tables, also called mortality tables. Classical life tables usually have two entries: the first one corresponds to the age $x \in \mathbb{N}$, the second one, denoted by l_x, seems to stand at first sight for the number of survivors at age x.
It is important however to understand that the population under study is in fact a fictitious one and that those survivors do not exist in real world. During a specific common observation period (from 1995 to 2000 inclusive, say) the probability to die between two ages (usually x and $x+1$, or x and $x+5$) is estimated. Let us give more details on this point: for each $x \in \mathbb{N}$ (up to the maximal age, for example 120 years), ignoring for the sake of clarity both censored data (i.e. when the time of death of individuals is not known precisely) and truncated observations, let us consider the

number l_x of individuals who turn age x between 1st of January 1995 and 1st of January 2000. Assume that d_x out of those l_x individuals will die between age x and $x+1$. The annual mortality rate q_x at age x is the probability for someone aged x to die within one year and may be estimated by $\frac{d_x}{l_x}$. Of course, in practice, an individual born in early January 1914 would only be "observed" during a few days between ages 80 and 81 during the period 1995-2000 (as this person would turn 81 in early January 1995). Some people may change country or purchase another policy and stop being observed before the end of the observation period and age $x+1$. Actuaries will take into account these types of reduced observations using some classical statistical tools such as the Kaplan-Meier estimator (see Klein and Moeschberger (2003))
Internal data in insurance companies usually enables actuaries to estimate the full but still fictitious survival function S defined by

$$S(x) = \mathbb{P}(\tau > x)$$

for $x \geq 0$ (but not necessarily in $\mathbb{N}$), where τ is the random lifetime of a member of this virtual population and $\mathbb{P}$ is the statistical probability measure.
In a similar way, national data only consists of pictures of the population every year. Starting from national life table which produce estimates for $S(x)$ for $x \in \mathbb{N}$, it is then necessary to make some additional arbitrary assumptions to reconstruct the full survival function. In practice, actuaries assume that the mortality force $\mu(x)$ (or the hazard rate of S) at any age $x \geq 0$, defined as

$$\mu(x) = -\frac{d(\ln S)}{dx}(x), \tag{1.1}$$

is either locally constant or admits a certain local parametric form (in accordance to Gompertz or Makeham survival functions), or that mortality rates follow some properties (see for example Denuit and Delwaerde (2005)).
In some cases, an alternative mortality rate, called the central mortality rate, is used instead of annual mortality rates in order to take into account the fact that after the first deaths, the size of the population has decreased, and so the following deaths will have a heavier weight in the estimation. As to define this adjusted mortality rate, we need to introduce first the notion of exposure to risk, which refers to the average number of individuals in the population over a calendar year adjusted for the length of time they are in the population. The exposure to risk is defined as:

$$ETR_x = \int_0^1 l_{x+u} du.$$

The central mortality rate is then defined as:

$$m_x = \frac{l_x - l_{x+1}}{ETR_x}.$$

Using some standard passage formulae, we can obtain a relationship between the various quantities q_x, m_x and μ_x, depending on the previous assumptions.
Note that classical life tables are well-suited to quantify short-term mortality risk (death insurance), for time horizons from 1 to 5 years provided that no exceptional event occurs (such as pandemic or heat wave). On the contrary, these tables are not relevant for reserving issues regarding long term longevity-based contracts like annuities or pensions, as mortality rates are changing over time and one must take this evolution into account.

1.1.2. Data for longevity risk When looking at longevity risk, it is necessary to quantify not only the level of mortality rates, but also their evolution in time. To be able to elaborate prospective life tables (as detailed in the subsequent Subsection 2.1) and take into account the evolution of mortality over time, one must first collect mortality data during different periods of time or regarding different cohorts (also called generations) to answer the following questions. What is exactly a life table for the generation t_0 (i.e. for individuals born in year t_0)? First, note that in year t, the death of survivors at ages larger than $t-t_0+1$ is not observed (as those events would occur after year $t+1$) and can only be estimated using some projections (see Subsection 2.1). Second, how should one understand the mortality rate at age x during calendar year t (denoted by $q_{x,t}$)? How to define mortality parameters?
To illustrate this last point, let us focus on a particular example. We would like to define and estimate $q_{65,2008}$. To do so, we start by estimating the probability $q_{65,2008;1942}$ that an individual born in 1942 (aged in average between 65.5 years on 1st of January 2008) dies during her 66th year in 2008 (being exposed ages 65.5 and 66 in average), and the probability $q_{65,2008;1943}$ that an individual born in 1943 (aged in average 64.5 on 1st of January 2008) dies during her 66th year in 2008 (being exposed between ages 65 and 65.5). Under some assumptions on mortality rates, one defines

$$q_{65,2008} = 1-(1-q_{65,2008;1943})(1-q_{65,2008;1942}),$$

from the so-called Lexis diagram (see Figure 1.1).

Note that

$$q_{65,2008;1942} = \frac{d(65,2008;1942)}{p(65,2008)} \text{ and } q_{65,2008;1943} = \frac{d(65,2008;1943)}{p(65,2009)+d(65,2008;1943)},$$

where $d(x,y;g)$ corresponds to the number of individuals born in year g and deceased during year y between age x and age $x+1$, and $p(x,y)$ corresponds to the number of survivors with age between x and $x+1$ on 1st of January of year y (born during year $y-x-1$). Using national data (as we only have at our disposal the size of the population for each age and the number of deaths), the central mortality rates can only be obtained after some assumptions on the distribution for the death times, because of the exposure to risk. Assuming that the mortality force $\mu(x,t)$ is constant on each box of the Lexis diagram, note that central mortality rates and mortality forces are identical[1]. It is then possible to define the probability to survive $T-t$ additional years (up to date $T \geq t$) for someone aged x at time t as:

$$S_t(x,T) = \mathbb{P}_t\left(\tau > x+(T-t) \mid \tau > x\right) = e^{-\int_t^T \mu(x+s-t,s)ds}. \tag{1.2}$$

These approximations are reasonable when studying average trends or making projections. Indeed, in practice, except for high ages, $\ln \mu_{x,t}$, $\ln m_{x,t}$ and logit $q_{x,t}$[2] are quite similar. Nevertheless, if one wants to quantify the size of oscillations around the

[1] This is the reason why the Lee-Carter model (presented in details in Section 2.1), which was initially formulated by Lee and Carter (1992) for central mortality rates, has been used by many actuaries and statisticians (e.g. Denuit and Delwaerde (2005)) for mortality forces.

[2] The *logit* function is defined as logit $(x) = \ln(\frac{x}{1-x})$.

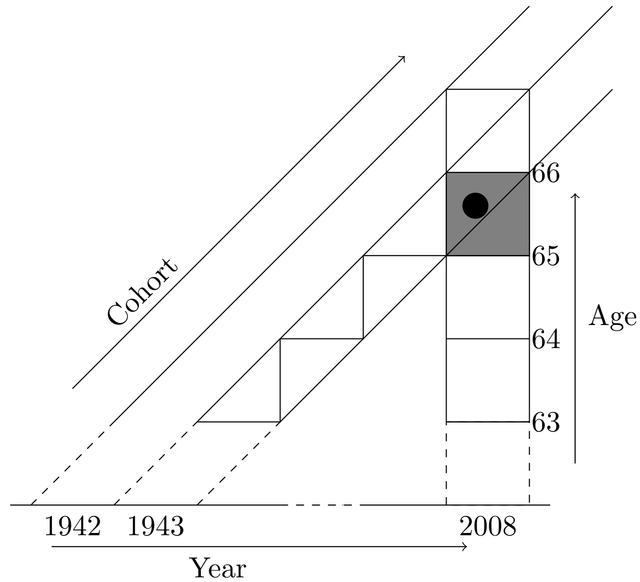

Figure 1.1 Lexis diagram: This is an age-year-cohort diagram representing the evolution of mortality over time. The real cohort mortality is followed on a diagonal manner and the fictitious (also called period) cohort could be read vertically. For example, the black circle corresponds to death of an individual, born in 1943, in 2008. The circle is situated on the upper triangle of the death year 2008 box (the grey box), meaning that the individual died in late 2008, say at age 64.61.

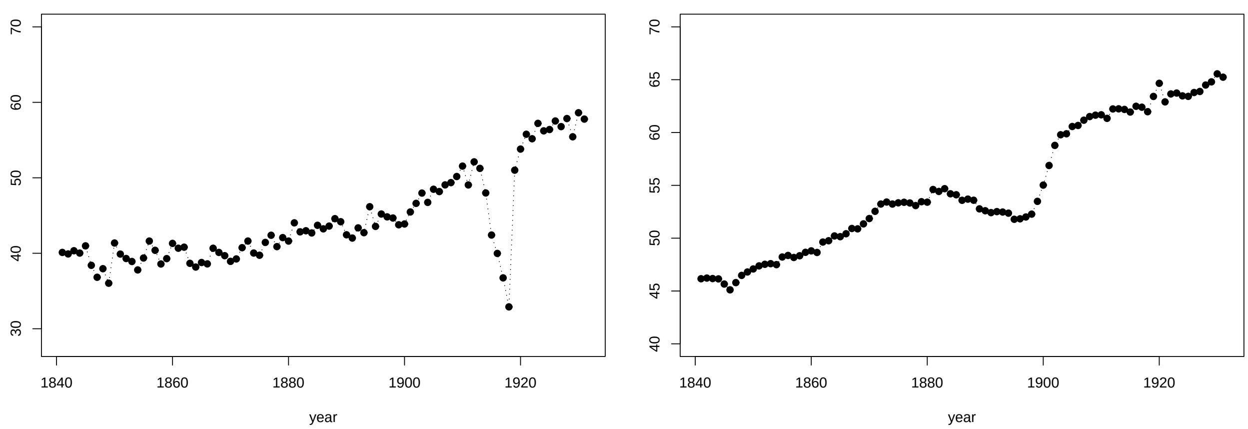

Figure 1.2 Periodic life expectancy (left) and generational life expectancy (right) at birth in the UK.

trend and try to detect changes in the trend, these approximations may deteriorate the results. Besides, these measures of the mortality rate $q_{x,t}$ are sensitive to some sources of randomness apart from the longevity risk: for example, if there are changes in natality in the year of birth, additional artificial oscillations may be observed. This phenomenon exists but can be neglected outside wars or periods of important economic difficulties.

Insurance companies have much more detailed information: they know the exact age of the policyholders and they observe (almost) exact death instants. This may allow them to test various assumptions, such as constant interest force on each box of the Lexis Diagram. However, the limited size of their portfolios (in comparison to national populations) is a clear drawback.

1.2. Period and cohort tables and effects

As briefly mentioned earlier, prospective life tables may be constructed and presented either by cohort or by period. To emphasize once more the fictitious nature of period life tables, let us compare retrospective life expectancy at birth for English and Welsh males obtained from period tables and cohort tables from the period 1840 to 1925 (see Figure 1.2): on the left-hand side, due to the first World War, mortality rates increase for all adult ages from 1914 to 1917, reaching a peak in 1918 due to the flu pandemic. The consequences on the life expectancy (computed from annual period tables) are very strong: life expectancy at birth is around 32 years in 1918, instead of 51 years in 1919! The reason for this extreme fluctuation is that period life expectancy at birth in 1918 is obtained from annual mortality rates observed at each age during year 1918: it would correspond to the life expectancy for individuals who would spend their whole life with "pandemic-style" mortality and with no longevity improvement. This has nothing to do with the cohort view (life expectancy at birth is around 60 years for 1918 and 1919 cohorts). The values of period life expectancy at birth for periods 1918 and 1919 point out the fact that one must be very cautious with period-based longevity indices to avoid over-reactions and as a consequence basis risk. Nevertheless, it may be adapted and it is sometimes preferable to make prospective tables by extrapolating period mortality tables, provided that the consistency of the deduced cohort-based tables is preserved. Alternate solutions are to extrapolate directly cohort tables using diagonal-wise projections, or to use age-period-cohort approaches to take both the cohort and the period effects into account.

The cohort effect refers to historical factors that are specific to a year of birth (such as the introduction of new drugs or vaccines), or to a group of birth years (such as smoking habits or women's professional activity level). It may be hard to perceive this effect as individuals are primarily involved in time and age dynamics, and at a lower level undergoing the consequences of their belonging to a certain generation, even if some cohort or cohort-group specific mortality patterns are striking, particularly in the United Kingdom. Only longevity improvements that are observed between two cohorts, for most periods and ages, correspond to the cohort effect. On the contrary, a heat wave and the fact that new medicine is available for adults correspond to period effects. In practice, the continuum of longevity improvements makes it difficult to isolate cohort effect and period effect, which can significantly intercept each other.

1.3. Smoothing and closing tables

Age profiles of empirical annual mortality rates (based on yearly published national statistics) are inconsistent for high ages (see Figure 1.3). The older, the more inconsistent the results are. This is the reason why actuaries usually close mortality tables, i.e. extrapolate the shape of the survival functions at high ages from some exogenous assumptions. Besides, in each country, annual population size estimations are done only up to a certain age. In France this maximal age is 100. In the past, mortality after age 100 was not a very important point and had a very small impact on residual life expectations (and so annuities) for workers or young retirees. With the recent longevity improvements, this is no longer the case, and it becomes important to have a better view on mortality and longevity risk for high ages, because this part of the population has a heavier weight but also because the mortality is now improving for those ages.

Up to now, after estimating bulk mortality rates, actuaries smooth empirical life tables up to age 90, and then close the table by using a local parametric shape and by taking some exogenous assumptions for parameter fitting (such as the central mortality rate at age 115 is 1, or the residual life expectation at age x is 2, etc...). This process is illustrated in Figure 1.3 for mortality rates of English males. For age-cohort or age-period models, the problem is more difficult as surfaces (instead of curves) have to be smoothed.

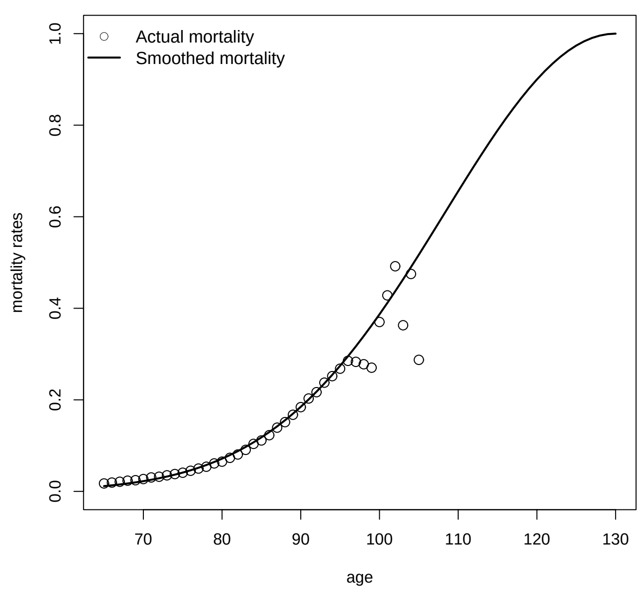

Figure 1.3 Smoothing and closing life tables

1.4. Heterogeneity, revisions, migrations, indices

Longevity patterns and longevity improvements are very different from one company portfolio to the other, and even for different countries. This variability is very important

for longevity risk transfer, as basis risk may be too important for insurers to accept to use financial instruments based on national indices to hedge their longevity risk, as this hedge would be too imperfect (see Section 4). Guarantees could also be based on national industry indices, but the problem is then that those indices and projections are revised too seldom.
To take into account national or entity specific mortality data, one would also need to collect more accurate information on migrations, which is not always easy. This is of course not always possible due some restrictions to access to those information depending on the migration politics of the underlying country.
The discrepancies between countries and portfolios (due to socio-economic factors, health, differences between males and females, migrations, ...) make every longevity study specific. Even if some common features may exist, this heterogeneity makes it difficult to jointly model longevity risks contained in different portfolios and to aggregate them without additional work.

2. Modeling longevity risk

Prospective life tables provide a view on the future evolution of mortality rates. Indeed, in most developed countries (apart from Russia), longevity has been improving for several decades and a simple look at the standard life tables, as we introduced above, is more restrictive and can underestimate the real evolution of future mortality. The prospective life tables offers a better view of the mortality evolution. Consequently, estimating mortality rate at age 70 from individuals who had this age in the past usually underestimates the probability of a person who is 50 years old today and who reaches age 70 to survive one more year. Prospective life tables may be defined for calendar years, or for cohorts (per year of birth). Every year, the various national institutes (INSEE in France, Bureau of Census in the US, CMI in the UK, ...) publish the level of national mortality through annual mortality rates. This is the data used by LifeMetrics and other providers of longevity indices as described in Section 4. It is also possible to obtain mortality data for most developed countries using the Human Mortality Database (HMD). The HMD is a free[3] database launched in 2002 by the Department of Demography at the University of California, Berkeley, USA and the Max Plank Institute for Demographic Research in Rostock, Germany. This database provides detailed mortality and population data to those interested in the history of human longevity.
Life tables and prospective life tables can also be built from entity-specific data, e.g. insurance companies or pension funds. For those entities, mortality may strongly differ from that of the national population due to the selection effect. Nevertheless, it is often very difficult to construct entity-specific prospective life tables without any reference to one or several national references. As an example, the last market-wide French prospective life tables, constructed in 2006, were produced on the basis of 700000 individuals from 19 different insurance companies and mutual companies. Even grouping these 19 insurance portfolios did not provide enough data to avoid the necessity to use a national reference. Actuaries usually try to find an appropriate link between the level of mortality and the longevity improvements of the portfolio, and the ones

[3] available at `http://www.mortality.org`

of the national population(s), thanks to so-called relational models (see Denuit and Delwaerde (2005)). But quantifying longevity basis risk remain very difficult.

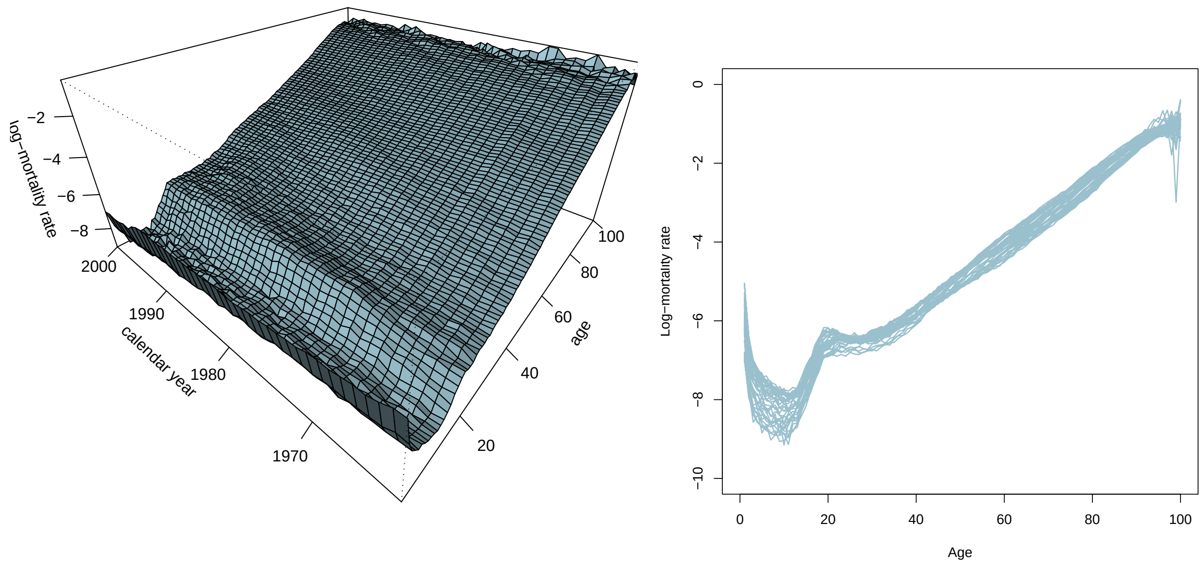

Figure 2.1 Log-mortality structure of French male, 1962-2000

2.1. Some standard models

2.1.1. Mortality models A variety of models have been introduced, starting with the famous Lee-Carter model (Lee and Carter (1992)), widely used by insurance practitioners. We can also name among others, the Renshaw-Haberman model (Renshaw and Haberman (2006, 2003)) that incorporates for the first time a cohort effect parameter to characterize the observed variations on mortality among individuals from a different cohort. A detailed survey on the classical mortality models has been carried by Pitacco (2004). More recently, many authors introduce stochastic models to capture the cohort effect (see e.g. Cairns et al. (2006, 2007)). In this subsection, we briefly present some of them.

The Lee-Carter model This model describes the central mortality rate $m_t(x)$ or the force of mortality, $\mu_{x,t}$ at age x and time t by three series of parameters namely α_x, β_x and κ_t as follows:

$$\log \mu_{x,t} = \alpha_x + \beta_x \cdot \kappa_t + \varepsilon_{x,t}, \qquad \varepsilon_{x,t} \sim \mathcal{N}(0,\sigma),$$

α_x gives the average level of mortality at each age over time; the time varying component κ_t is the general speed of mortality improvement over time and β_x is an age-specific component that characterizes the sensitivity to κ_t at different ages; the β_x also describes (on a logarithmic scale) the deviance of the mortality from the mean behavior, κ_t. The error term $\varepsilon_{x,t}$ captures the remaining variations.

To enforce the uniqueness of the parameters, some constraints are imposed on those parameters:

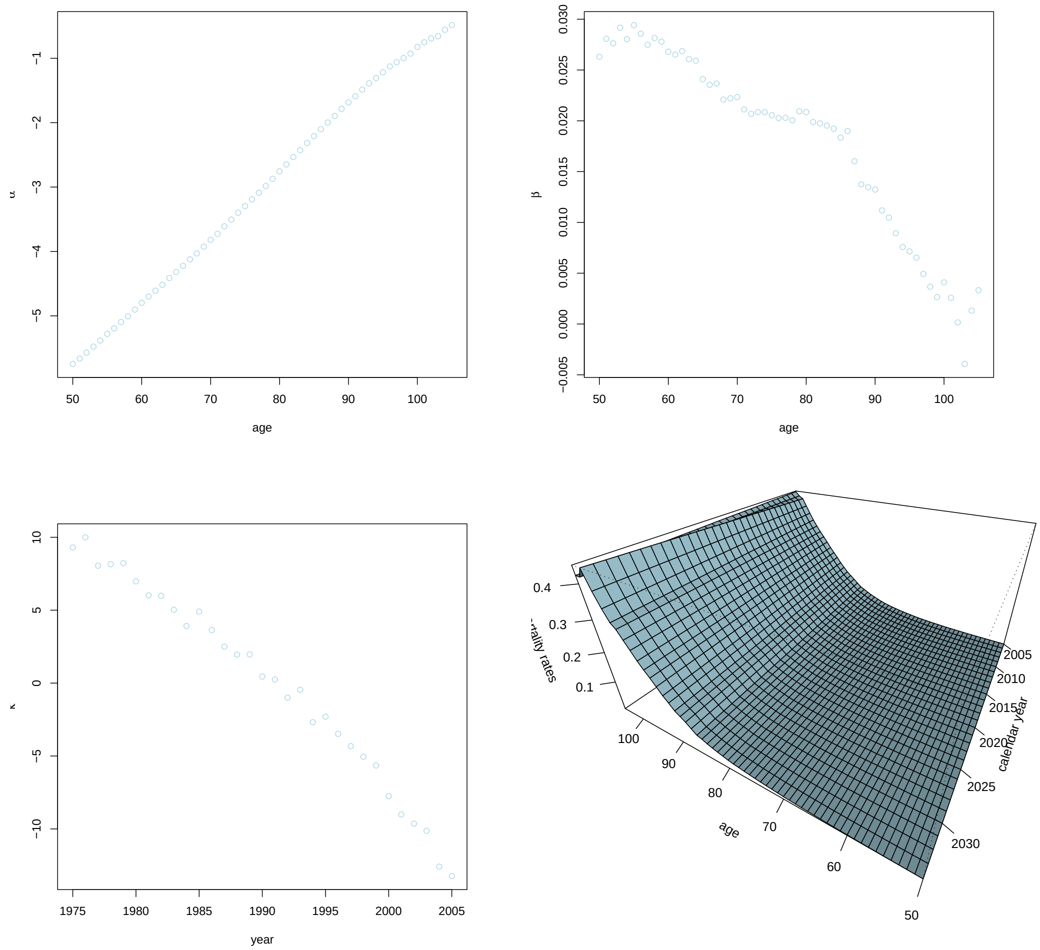

Figure 2.2 Parameters estimates for the England and Wales mortality table

$$\sum \beta_x = 1 \quad \text{and} \quad \sum \kappa_t = 0.$$

To calibrate the various parameters we can use standard likelihood methods and thus assume a Poisson distribution for the numbers of deaths at each age and over time. The estimated parameters are presented on Figure 2.2. In particular, note that estimated values for β_t are higher at lowest ages, meaning that at those ages the mortality improvements are faster and deviate considerably from the mean evolution.

The P-Spline model The P-spline model is widely used especially to model UK mortality rates. The model fits the mortality rates using penalized splines (P-splines), in order to derive future mortality pattern. This approach is used by Currie et al. (2004) to smooth the mortality rates and extracts ”shocks” as suggested by Kirkby and Currie (2007), which can be exploited to derive scenarii using stress tests. Generally, the P-spline model takes the form

$$\log m_t(x) = \sum_{i,j} \theta^{i,j} B_t^{i,j}(x),$$

where $B^{i,j}$ are the basis cubic functions used to fit the historical curve, and $\theta^{i,j}$ are the parameters to be estimated. The P-spline approach is being different from a basic cubic spline approach when introducing penalties on parameters $\theta^{i,j}$ to adjust the log-likelihood function. Since, to predict mortality, the parameters $\theta^{i,j}$ are to extrapolate using the given penalty.

The CDB model Cairns, Dowd and Blake (CDB) introduce a general form of models that could be stated depending on the purpose of the modeling but also on the underlying shape of mortality structure. The general model is given by:

$$\text{logit} q_t(x) = \kappa_t^1 \beta_x^1 \gamma_{t-x}^1 + \cdots + \kappa_t^n \beta_x^n \gamma_{t-x}^n. \tag{2.1}$$

As we can see there are three types of parameters starting with those specific to age β^i and calendar year κ^i and finally the cohort effect parameters γ^i. We should note that the Lee-Carter model is a particular case of this model. The authors also investigate the right criterion to decide upon a particular model (i.e. the parameters to keep or to remove). So, they underline the need for a tractable and a data consistent model and bring out statistical gauges to rank models and determine the better suited to forecast mortality. A particular example of a model derived from the general form (2.1) is the model below featuring both the cohort effect and the age-period effect, e.g. Cairns et al. (2008):

$$\text{logit}\mu_{x,t} = \kappa_t^1 + \kappa_t^2(x - \bar{x}) + \kappa_t^3 \left((x - \bar{x})^2 - \sigma_x^2 \right) + \gamma_{t-x},$$

where

$$\bar{x} = \frac{\sum_{x=x_0}^{x_n} x}{x_n - x_0 + 1}$$

is the mean age of the historical mortality rates to be fitted (x_0 to x_n), σ^2 is the standard deviation of ages, equal to

$$\frac{\sum_{x=x_0}^{x_n} (x - \bar{x})}{x_n - x_0 + 1},$$

the parameters κ_t^1, κ_t^2 and κ_t^3 correspond respectively to the general mortality improvement over time, the specific improvement for every age (taking into account the fact that mortality for high ages improves slower than for younger) and finally the age-period related coefficient, $\left((x - \bar{x})^2 - \sigma_x^2 \right)$ corresponds to the age-effect component. Similarly, γ_{t-x} represents the cohort-effect component.

2.1.2. From mortality to longevity models The close relationship between mortality and longevity modeling is particularly clear when considering the survival probability. Mathematically, life expectancy appears to be the product of some correlated mortality rates as it is underlined by the following expression for the survival probability until date $t + u$ of a person aged x at time t:

$$S_t(x, T) = \prod_{i=0}^{T-1} [1 - q(x + i, t + i)],$$

As a consequence, the models described above can be used for both mortality and longevity risks. However, the extreme events in both cases are different: for mortality

risks, extreme events would correspond to pandemic, terrorist attack, heat wave or other unusual events, whereas for longevity risk, an extreme scenario would correspond to an important change in the longevity improvement trend.
Indeed, on the one hand, mortality risk is a short-term risk (1 to 5-year maturity) with a catastrophic component (pandemic, heat wave, ...), which, from an actuarial point of view, looks very much like a natural catastrophe risk. On the other hand, longevity risk is a long-term risk with maturities ranging from 20 to 80 years and is mainly about changes in the trend (especially the risk that the longevity improves faster than predicted). Therefore, the influence of the trend parameters (such as κ_t in the Lee-Carter model) is more important for longevity modeling than the mortality modeling.
An interesting feature to note at this stage is the impact of financial markets on both mortality and longevity risk management: adjustments and re-evaluations have to be made more often than for other classical insurance portfolios. It is very difficult to distinguish a change in the longevity trend from the noise around the average trend. Besides, stakeholders may have access to different information, which might be in favor of the ones with privileged information (insurers in comparison to bankers have access to more detailed experienced mortality data). Changes of regimes are crucial to take into account as neglecting or misinterpreting them can potentially lead investors or risk managers to overreact to yearly oscillations. Changes in the longevity trends are studied in a recent work by Cox et al. (2009), while the specific issue of the early detection of these changes and the risk of false alarms are addressed in El Karoui et al. (2009).

The fact that actuaries use different tables for mortality and longevity risks could be seen at first sight as a pure safety process: in the past, male tables were used for mortality risk and female tables were used for longevity risk, irrespectively of the sex of the policyholder. But, mortality risk is quite different from longevity risk. Evolution of longevity patterns for males and females may be perfectly correlated in some developed countries, whereas they are almost uncorrelated in some others. There may also be short-term inter-age correlations coming from period effects, correlations arising from cohort effects, as well as long-term dependencies between longevity time series of different age classes or countries. Depending on the country or on the insurance portfolio, the relative importance of these various sources of correlation may vary a lot. Therefore, it is essential to have a bi-dimensional viewpoint (males and females) to study the aggregate risk associated to an insurance portfolio (see e.g. Bienvenue et al. (2009) and Lazar and Denuit (2009) for inter-age and inter-sex correlations). Similarly, there may be some correlation (sometimes arising from co-integration) between time series of national population mortality and of mortality of policyholders of the same country, and thus due to an existing common long term trend (see Salhi et al. (2009)).

2.1.3. Multiple death causes A significant part of longevity improvements is clearly due to medical progress, and changes in smoking and nutrition habits. Different factors have different consequences on frequency and lethality of illnesses. Therefore, it may be interesting to see the human body as a machine with components, and to try to model system failures like in reliability theory (see Gavrilov and Gavrilova (2001)). Using medical data, one could hope to get better mortality projections by considering

improvements in different mortality causes for each age class. Unfortunately, up to now, this promising approach is far from being applicable: indeed, according to US data, there is one unique cause of death in only 30 % of cases. The probability to have four different causes of death is still relatively significant (11%).
Improvements by cause of death are of course correlated, as some causes are positively correlated, but also because survivors to certain diseases (thanks to longevity improvements related to a particular cause of death) will anyway die from another cause.
To take advantage of these ideas, it is essential to generate discussions with medical doctors as to determine what kind of detailed data must be collected in order to be able to estimate the longevity improvements and the induced changes in longevity patterns generated by a given medical progress (such as a new treatment for lung cancer). This is also a necessity for insurers who have to be very careful about selection and counter-selection of policyholders.

2.2. Micro-Macro modeling for longevity risk

The current demographic situation suggests the need to develop an efficient model for population dynamics. Indeed, the debate about mortality and longevity is widely open as the estimations given by demographers generally underestimate the reality. Besides, some unsolved questions remain, such as knowing whether longevity is indefinitely elastic or whether there is a critical age that a human being will never exceed. From a probabilistic point of view, the evolution of longevity is not deterministic but stochastic and it is really difficult to estimate for long time horizons.

2.2.1. Mortality modeling The classical mathematical models for mortality, such as for the models presented in Subsection 2.1, consider that mortality rate is a stochastic process that only depends on age and time. As the variance of this mortality process exponentially grows in time, the long-term estimations become inaccurate and have to be improved.
A new approach is a microscopic modeling on an individual scale that accurately describes individuals of a population with their own characteristics (Bensusan and El Karoui (2009)). Inspired by the Cairns-Dowd-Black modeling presented in Subsection 2.1.1, this model suggests a mortality rate that depends on age but also on various individual characteristics and on the environment of the country where the studied population lives. Thus, it is a microscopic model used for picturing a macroscopic situation such as the mortality and the demography of a given population. The study consists of finding what individual characteristics (other than age) can explain mortality and taking them into account in a stochastic mortality model.
In fact, according to some demographers, the slump of the mortality in Europe would principally result from the evolution of the socio economic level, the evolution of education and the advances in medical research. As explained in Subsection 2.1.3, a model that describes mortality by causes such as diseases is not selected for the moment because of the lack of accurate and objective mortality data and the impossibility for identifying the cause of the death given that the correlation between the occurrence of diseases. However, a recent study, published by researchers from IRDES[4] and

[4] Institut de Recherche et Documentation en Economie de la Santé

INED[5], describes a relationship between the individual socio-economic level and the life expectancy (see Jusot (2004)). The researches of demographers suggest to analyze the influence of some individual characteristics on the mortality level like socio-professional group, individual income, matrimonial status. For example, regarding the life expectancy of a 35 years old French male, there is a gap of 6 years between a working-man and an executive manager (see Cambois et al. (2008)). Moreover, there exists a very strong correlation between individual income and mortality. The income impact on mortality significantly persists, even though being reduced by a socio-professional groups control.

Besides, the impact of social environment on the individual mortality can be underlined by the Wilkinson's hypothesis, according to which, health is strongly affected by the extent of social and economic differences within a population. This hypothesis is validated in France (see Jusot (2004)). Therefore, death risk appears as an increasing function of economic inequalities and a decreasing function of medicine environment.

This approach allows to reduce the variance of the mortality rate by taking into consideration specific information about the studied population. From a financial point of view, this model gives accurate information on the portfolio basis risk. By studying individual characteristics of the insured people, we could estimate the deviation of the "individual mortality" from the general mean mortality given by the mortality tables which only depend on age.

2.2.2. Population dynamic modeling Insurance, pension funds and governments are exposed to a huge financial risk concerning the longevity of the people they insure as well as its evolution over the years. The mortality information is not enough to understand the population dynamics in the future: it appears essential to have an access at any given time to some demographic information such as an indicator of fertility, mortality and immigration. Therefore, a model for other demographic rates such as general fertility rate (GFR) and immigration rates is needed in order to generate some demographic and population pyramid projections.

Inspired by recent probabilistic research works (Fournier and Méléard (2004), Tran (2006) and Tran and Méléard (2009)) and considering a model for fertility and mortality rates, a population dynamic modeling is proposed in Bensusan and El Karoui (2009) by taking into consideration the population pyramid and immigration concepts. This study is based on ecological phenomena and describes an adaptive dynamic for aged-structured populations. Moreover, by considering the limit when the size of the population goes to infinity, the microscopic birth and death process converges to the measure-valued solution of an equation that generalizes the McKendrick-Von Foerster and Gurtin-McCamy equations in demography (for more details see Webb (1985) and Tran (2007)). Therefore, taking the limit of this process allows to specify the micro-macro modeling given that it furnishes macroscopic information on the population evolution using individual characteristics.

This model takes into account the demographic situation of a country and provides projections of a population structure in the forthcoming year. A mean scenario of evolution can be deduced and analyzed from these simulations, but extreme scenarii with their probability of occurrence have also to be taken into consideration. As illustrated

[5] Institut National d'Etudes Démographiques

in Figure 2.3, some scenarii may lead to very different demographical situations. For more details, please refer to Bensusan and El Karoui (2009).

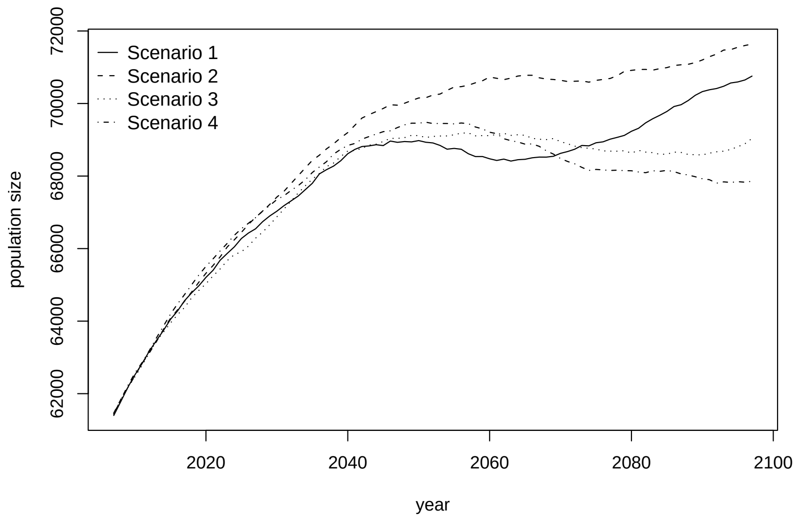

Figure 2.3 Evolution Scenario for French population until 2097

Moreover, the *potential support ratio* -i.e. the ratio between working people and inactive people- has to be taken into consideration. Indeed, the concept of retirement is possible if the working people can take care of the payment of retirees compensation. This potential support ratio is widely representative of the demographic situation.
Besides, most of the demographic institutes consider that immigration will be of great importance for the retirement issue in the future. Some demographers and geographers such as Monnier (2000) estimate that immigration would be the only one way to stop the actual demographic decline in Europe. According to these studies, Europe would need about more than 100 millions of immigrants by 2050 in order to maintain the actual size of the population. However, keeping the actual potential support ratio with immigration is impossible given that the number of immigrants would be huge, absurd and totally incompatible with the actual governments' policy. Therefore, the population ageing in the developed countries will have an impact on future bills concerning demography and political decisions have to be taken.

3. Longevity risk and new regulations

In most developed countries, life expectancy has increased by 25 to 30 years during the last century and from a human point of view, this is really a good news. However, financial institutions, such as pension fund, national governments and life insurance companies have to face this longevity risk. Indeed, in life insurance, the rate-making of annuities is strongly impacted by the recent improvement of longevity. Thus, the longevity risk inherited on retirement plans and lifetime benefits is very likely to make pension funds and life insurers paying out more than expected. This is due to the increasing life expectancy. Therefore, regulations have to be set up in order to maintain a balance and control the inherent risks in such plans and contracts. Moreover, the specific characteristic of such insurance products is their long term maturities. In contrast to mortality risk, which is for short run exposure, longevity risk implies maturities that can reach 50 to 80 year and thus involving other risks that are to be assessed carefully.

3.1. Longevity risk

The main financial characteristic of longevity risk is the long horizon of maturities up to 50 years. From a financial and economic viewpoint, the ageing of population leads to many reforms such as retirement bill and the setting up of long-term care insurance. In order to manage the longevity risk, it is important to analyse the influence of longevity on economy and dependence.

3.1.1. Financial differences concerning longevity and mortality risks Longevity appears as a trend risk whereas mortality is a variability risk. Is there an orthogonality between mortality and longevity? In other words, can we "buy" mortality risk in order to hedge longevity risk? How could we price a trend risk?
Long-term horizons have financial consequences: interest rate risk often becomes predominant. Oscillations around the average trend are also important because their size cannot be neglected and also because they can lead to over-reactions by insurance managers, regulators, policyholders and governments. Even if a certain mutualization between mortality and longevity risks obviously exists, it is very difficult to obtain a significant risk reduction between the two, because of their different natures.
Indeed, the replication of life annuities with death insurance contracts is not perfect because it does not concern the same group of people and mortality portfolios give a huge importance for the insured individuals who have a big share of the portfolio capital. Thus, the hedge is often bad because of the variability related to the death of the insured individuals whose death benefits will be high.
Moreover, the impact of a pandemic or a catastrophe on mortality is really different from the impact on longevity. Indeed, an abnormally high death rate at a given date has a qualified influence on the longevity trend as it was with the 1918 flu pandemic (see Figure 1.2).

3.1.2. Impact of longevity risk on the economy As noted before, many entities are concerned by longevity risk and have to hedge this long term risk. For example, the government really concerned by the retirement challenge and its associated longevity risk. An article of Antolin and Blommestein (2007) underlines that the longevity improvement of the people aged eighty or older has an important impact on the country gross domestic product and on the political decisions. Consequently, population ageing has macroeconomic consequences and is generally considered as a factor of economic slackening for some countries.
However, this issue is mitigated given that when life expectancy increases, consumption also increases. However, note that the ageing of the population does not inevitably correspond to an economic ageing but could, on the contrary, inspire an economy of ageing with inventions in many fields like medicine, home automation, the organization of cities and transport among other things. In many developed countries, a urban redevelopment is carried out in order to facilitate the free circulation for the elderly.

In 2005, the French Academy of Pharmacy published a report "Personnes Agées et Médicaments" (see FAP (2007)) that revealed an increase of medicine consumption by seniors as well as a whole medical economy of ageing. Indeed, with population ageing and the growth of demand, medicine consumption increases at a high rate. Moreover, the innovating pharmaceutical companies are looking for developing new medicines especially designed for the elderly.

3.1.3. Correlation between longevity and dependence Loss of autonomy and state of dependence, that generally concern elderly people, are major demographic issues. Taking France as an example, important means have been introduced in order to help these people they call "dependent people".
The question of long-term care insurance has also an economic issue insofar as a year of dependence costs four times more than a classical year of retirement. Consequently, a raising public awareness campaign is run worldwide in order to take preventive action about this phenomena and its consequences.
Moreover, the level of dependence accurately reflects the individual health capital. Indeed, some statistics from INSEE reveal that the entrance of a person in a state of dependence drops the life expectancy by 4 years, and this almost independently of the age at the entry into dependence. Those statistics are obtained by taking into account all dependency states such as the one linked to Alzheimer's disease including the less severe forms for which certain people stay dependent during many years. Thus, individual longevity is strongly correlated to dependency level. It is finally important to notice that the meaning of the word "dependence" differ from one country to another with the panel of regulations and consequently the correlation between longevity and dependence is really difficult to define.

3.2. New regulations

As far as longevity risk is involved in many economic and financial challenges, as we have mentioned in the last section, regulators bring more accurate standard to unify and homogenize practices in terms of solvency capitals computation and risk assessment. Since, regulations in life insurance in particular and in insurance in general, will soon enter a new era. The European project of new standards, namely Solvency II, comes to update the former regulation.
The actual practices in life insurance are based on a deterministic view of risk. Although those practices are very prudent as to ensure the solvency of the insurer, they exclude any unexpected deviation of the risk. Indeed, the amount of provisions and the value of products themselves are, most of the time, obtained via deterministic computation methods and calculation of provisions is reduced to a net present value of future cash flows discounted with risk-free rates. The new standards highlight the necessity of integrating the market price of risk into the calculation of provisions and evaluation of products so that we have "market consistent" values.
For this purpose, regulators differentiate two kinds of risks: Hedgeable risks and non-hedgeable risks. The later are widely discussed and treated independently of any market. For hedgeable risks, however, the hedging strategy is used to evaluate the underlying liabilities.
Another aim of solvency II is to define capital requirements for insurance firms which should be in line with the firm's real incurred risk.

In the following, we present in details the different calculations for both the technical provisions and the capital requirements. Note that for the reason previously discussed, we will focus on the technical provisions associated to non-hedgeable risks.

3.2.1. Technical provisions Technical provisions in insurance are future obligations, which will be probably faced by the insurer, i.e. "*claims related to an insurance contract that have not settled at the date on which the financial statements are finalized*". For example, it could correspond to future payments of annuities to policyholders. Thus, the technical provisions stand for the anticipated engagements, and they are reported on the liabilities side of the insurer's balance sheet. The Solvency II directive proposals and more precisely the quantitative impact studies (QIS) (see for more details Ceiops (2008)) are bringing in some standards in order to unify practices in term of provisions' calculation and product valuation. In particular, technical provisions will have to be calculated by taking into account the market available information. In other words, the provisions should be market consistent.
Technical provisions in insurance are based on realistic assumptions concerning the future evolution of the various risk factors. More precisely, the risk factors are first estimated and then their future patterns are derived under some prudential assumptions. In this case, the best estimated value of a liability is simply the mean over all future scenarii.
In practice and for longevity linked contracts, the best estimate assumptions are mainly derived from internal models or based on some relevant models allowing to identify the future pattern of the mortality, it could be for example based on a model among those presented in the previous section.
The fact that the best estimate does not replicate the actual value of the liability imposes on insurers the constraint to hold an excess of capital to cover the mismatch between the best estimate and the actual cash flows of the liability. Such a capital is referred to as the Solvency Capital Requirement. Note that similarly to the replicating portfolio, holding an extra capital beyond the best estimated provisions could be seen as a super-replicating strategy.

3.2.2. Capital requirements for a single risk As we have mentioned earlier, any insurer must constitute some reserves to ensuring its solvency. The required capital is divided into two parts: The first is the Minimum Capital Requirement (MCR), which is the minimum level of capital a firm must hold. The second is the Solvency Capital Requirement (SCR) is more important than the MCR. According to QIS4, the SCR will be determined so that the firm's solvency standing will be equivalent to a BBB rated firm, in other words, "*equivalent to the firm to hold a sufficient capital buffer to withstand 1 in 200 year event (the otherwise termed 99.5% level)*".
The calculation of the SCR could rely either on an internal model that captures the firm risk profile, or on the standard formula proposed by the QIS4, where the risk profile is obtained using a variety of 'modules'. For the approach, the capital calculation is computed separately for each modules and risk factor and then aggregated.
First of all, there is the module based framework that proposes pre-defined scenarii to compute solvency capitals, and concerning the longevity risk, capital requirements have to be added to the best estimate technical provisions in order to face unexpected deviations of the mortality trends, and allow the insurer to meet its obligations in adverse scenarii.

For this purpose, insurer should use a scenario-based method involving permanent changes in mortality rates (a yearly based evolution). For example, the proposed scenario associated with the mortality risk is a 10% increase in mortality rates for each age over years. Similarly, for contracts that provide benefits over the whole life of the policyholder (i.e. longevity risk), the scenario suggest to set an additive permanent 20% decrease on mortality rates each year. The SCR is then merely computed given formulae introduced in the fourth QIS.
Meanwhile, the regulators admit an existing 'natural' hedge between the mortality component and the longevity risk component. However, as we outlined in 3.1.1, there is no orthogonality between these risks but a partial hedge. This natural hedge is translated in term of correlation, which is assumed to be negative and equal to -25%. This correlation serves when we are aggregating the SCR to the whole life module.
The alternative to the standard formula for calculating the solvency capital requirements is the use of an internal model (or one or more partial models). In this case, the internal model should capture the risk profile of the insurer by identifying the various risks it faces. Therefore, the internal model should incorporate the identification, measurement and modeling of the insurer key risks. The Solvency II guidelines, in term of internal models, propose using the Value at Risk to compute the required capital when the insurer prefers developing its own framework to assess the incurred risks. The methodology considered here is very different from the one already in use in banking industry.
The Value at Risk measure is recently introduced in insurance and is based on a year available data. This is the main difference between the banking and insurance industry, where in banking we have access to high frequency data permitting computation of daily risk measure, in insurance the Value at Risk is computed over the whole year, and thus assessing the solvency. The required capital for the year SCR_i insuring the solvency during this given period and is set equal to the VaR at level of 1%

$$SCR_i = VaR_\alpha(M_i) - \mathbb{E}(M_i),$$

where M_i is the liabilities we aim to compute the associated solvency capitals. This framework is also outlined in the Swiss Solvency Test, which is detailed in the internal model of SCOR published recently.
As far as longevity risk is concerned the yearly-based VaR is computed by separating the fluctuation surrounding the losses and the long-term risk assumption incurred in the trend. Those two components are modeled using stochastic approaches or by considering scenarii. Most often, scenarii used to be the most used methods in insurance, for example one should perturb the best estimate mortality table by stressing the volatility, in order to assess the need of capital facing a short-term losses fluctuations. Similarly, scenarii are used to stress the long-term trend, and thus assuming a deviation of the best estimate trend.

3.2.3. Capital requirements for aggregate risks The capital requirement are determined separately for all risk factors, and the global SCR is computed by aggregating each single $(SCR^j)_j$ when stressing those risk factors. The dependency structure

$\Theta = (\theta_{i,j})_{i>0,j>0}$ allowing the aggregation is pre-defined by the regulator and summarized here below:
Finally, the whole solvency capital is aggregated given the equation:

$$\mathrm{SCR}_{\mathrm{global}} = \sqrt{\sum_{i>0}\sum_{j>0}\theta_{i,j}\mathrm{SCR}_i\mathrm{SCR}_j}.$$

Other risk are also to be incorporated in this framework such as market risk and default risk. The latter is to consider when the insurer is transferring risk to another entity or when it holds derivatives for risk mitigation purposes. Finally, those required capitals have to yield a return (it is not necessarily fixed, and it depends on the internal targets on capital) each year, because there are brought by shareholders and are risky. The whole margin to take into account to satisfy the shareholders return requirement is called the risk margin and is seen as the price of risk. The framework of computing the risk margin in such a way is known as the cost of capital approach and is to be add to the technical provisions. The risk margin stands for the market price of the risk. The regulator highlights the effectiveness of risk mitigation such as reinsurance and derivatives, that are to handle as a release of capital especially when the new regulations arise the need for capital due to the increasing solvency requirements. The capital markets, indeed, seems to be an attractive means to transfer the longevity risk because the traditional risk transfer through reinsurance has a limited capacity to fund and to absorb this risk. Therefore, the transfer through capital market should, in fact, funds releasing capital and thus at lower cost which may increase and maintain the profitability of the insurer and the risk margin and so enhance its competitiveness in the market.

4. Transferring longevity risk

As noted before, a steady increase in life expectancy in Europe and North America has been observed since 1960s. This represents an important risk for both the pension funds and the life insurers. Various risk mitigation techniques have been recently attempted to better manage this risk. Reinsurance and capital market solutions in particular have received an accrued interest.

4.1. Convergence between insurance and capital markets

Even if no Insurance-Linked Securitization (ILS) related to longevity risk has been completed yet, the development of this market for other insurance risks has been experiencing a continuous growth for several years, mainly encouraged by changes in the regulatory environment and need of additional capital from the insurance industry. Today, longevity risk securitization lies at the heart of many discussions and is widely seen as a potentiality for the future.
The convergence of the insurance industry with the capital markets has become more and more important over the recent years. Such convergence has taken many forms and of the many attempts some have been more successful than others. Academically, the first mention of the use of capital markets to transfer insurance risk was in a paper by Goshay and Sandor (1973), where the authors considered the feasibility of an organized market and how this could complement the reinsurance industry in catastrophic

risk management. In practice, while some attempts have been made to development an insurance future and option market, the results have been rather disappointing so far. In parallel to these attempts however, the ILS market has been growing very fast over the last 15 years. There are many different motivations for ILS including risk transfer, capital strain relief, acceleration of profits, speed of settlement, and duration. Different motives mean different solutions and structures, as the variety of instruments on the ILS market illustrate.

While the non-life part of the ILS market is the most visible with the famous and highly successful cat-bonds, the life part of the ILS market is the bigger in terms of volume of the transactions with an estimated outstanding of 35 to 40 billion USD[6]. Today's situation is very much mixed and there is a huge contrast between the non-life and the life ILS market, especially in terms of impact of the financial crisis and therefore development and success. While in the non-life sector, a very limited impact of the credit crisis can be noticed, partly due to the structuring of the products, a dedicated investors base and a market discipline in terms of modeling and structuring, the life sector has been very much affected by the recent crisis, mostly because of the structuring of the deals and the nature of the underlying risks, with more than half the transactions being wrapped or having embedded investment risks. Hence, the constitution and management of the collateral account and the assessment of the counterpart risk are at the heart of current debates to develop a sustainable and robust market.

4.2. Recent developments in the transfer of longevity risk

Coming back to longevity risk, we have observed some important developments over the past 2 years, with in particular an increased attention from US and UK pension and life insurance companies, and the estimation of a tremendous potential underlying public and private exposure over 20 trillions USD. Even if many private equity transactions have been completed, very few known/public capital markets transactions, which have mainly taken a derivative form (swaps) have been done.

Despite this limited activity, using the capital markets to transfer some of the longevity risk seems to be a natural move. Longevity seems to meet the basic requirements of a successful market innovation, but there are however some important questions to consider. To create liquidity and attract investors, annuity transfers need to move from an insurance format to a capital markets format.

As a consequence, one of the main obstacles to develop capital markets' solutions seems to be the one-way exposure of investors since there is almost no natural buyers of longevity risk, which creates a problem to generate demand, despite some potential as a new asset class if priced with the right risk premium which could interest hedge funds and specialized ILS investors.

But also the issue of basis risk can prevent a longevity market from being successful. Indeed, the full population mortality indices have basis risk to liabilities of individual pension funds and insurers. Age and gender are the main sources of basis risk, but also regional and socio-economic basis risk could be significant. Therefore, using standardized instruments based upon a longevity index to hedge a particular exposure would

[6] Note that, due to the nature of the market, with a limited number of participants, and many transactions not being "public", the size of the market can only be an estimation.

result in leaving the pension fund or life insurer with a remaining risk, sometimes difficult to understand and hence to manage. An important challenge lies in developing transparency and liquidity by standardization without neglecting the hedging purposes of the instruments.
Many different initiatives have been undertaken in the market recently, as to increase the transparency around longevity risk and contribute to the development of longevity risk transfer mechanisms.

4.3. Various longevity indices

Among the different initiatives to improve the visibility, transparency and understanding of the longevity risk, various indices have been created. A good longevity index should be based on national data (available and credible) to have some transparency but be flexible enough as to reduce the basis risk for the original longevity risk bearer. National statistical institutes can build up annual indices based on national data with projected mortality rates or life expectancies (for gender, age, socio-economic class...): this can potentially limit basis risk and help insurance companies to set up a weighted average index related to their specific exposure.
Today, the existing indices are:

- Credit Suisse Longevity Index, launched in December 2005. This index is based upon national statistics for the US population, with some gender and age specific sub-indices.
- JP Morgan Index with LifeMetrics, launched in March 2007. This index covers the US, England & Wales and the Netherlands and used national population data. The methodology and future longevity modeling are fully disclosed and open with a software including various stochastic mortality models.
- Goldman Sachs Mortality Index, launched in December 2007. This index is based on a sample of US insured population over 65 and targets the life settlement market.
- Xpect Data, launched in March 2008 by Deutsche Borse. This index initially delivered monthly data on life expectancy for Germany, but now covers the Netherlands.

4.4. q-forwards

JP Morgan has been particularly active in trying to establish a benchmark for a longevity market. Not only have they developed the longevity risk platform LifeMetrics but also some standardized longevity instruments called "q-forwards". These contracts are based upon an index, which can either be the mortality rate or the survival rate, as quoted in LifeMetrics. Very naturally, survivor swaps are more intuitive hedging instruments for pension funds and insurers. But as the survival rate is path-dependent and so the starting date of the contract is important, this may prevent the fungibility of the different contracts relating to the same cohort and time in the future and therefore mortality swaps are also likely instruments. The mechanisms of a q-forward can be summarized as follows:

The mechanisms of the q-forwards are quite simple: a pension fund hedging its longevity risk will expect to be paid by the counterpart of the forward if the mortality falls by more than expected. So typically, a pension fund is a q-forward seller, while an investor is a q-forward buyer.

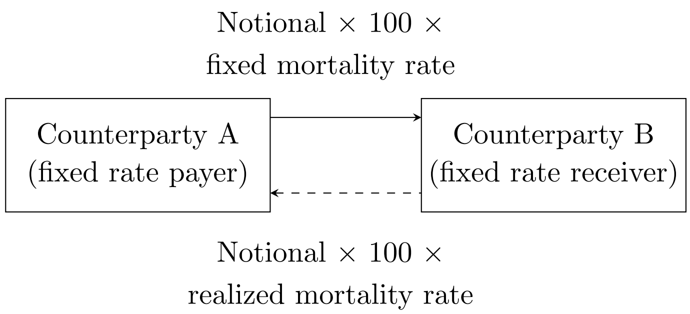

Figure 4.1 q-forward mechanism

Notional Amount	GBP 50,000,000
Trade Date	31 Dec 2006
Effective Date	31 Dec 2006
Maturity Date	31 Dec 2016
Reference Year	2015
Fixed Rate	1.2000%
Fixed Amount Payer	JPMorgan
Fixed Amount	Notional Amount × Fixed Rate × 100
Reference Rate	LifeMetrics graduated initial mortality rate for 65-year-old males in the reference year for England & Wales national population (Bloomberg ticker: LMQMEW65 index <GO>)
Floating Amount Payer	WYZ Pension
Floating Amount	Notional Amount × Reference Rate × 100
Settlement	Net settlement = Fixed amount - Floating amount

Figure 4.2 An example of a q-forward contract

4.5. Longevity swap transactions

Very recently, some longevity swap transactions have been completed. They are very private transactions and therefore their pricing remains confidential and subject to negotiation between the various parties involved in the deal. Some of these swaps were contracted between a life insurance company and a reinsurer as a particular reinsurance agreement. Others have involved counterparts outside the insurance industry. Most of these transactions have a very long maturity and incorporate an important counterpart risk, which is difficult to assess given the long term commitment. As a consequence, the legal discussions around these agreements make them particularly heavy to finalize.

Over the last year 2008, two particular longevity swaps have been arranged by JP Morgan. Both are very different in terms of basis risk as detailed below. More precisely:

A customized swap transaction In July 2008, JP Morgan executed a customized longevity swap with a UK life insurer for a notional amount of GBP 500 millions for 40 years. The life insurer has agreed to pay fixed payments and to receive floating payments which replicates the actual benefit payments made on a closed portfolio of retirement policies. The swap is before all a hedging instrument of cash flows for the life insurer, with no basis risk.
At the same time, JP Morgan entered into smaller swaps with several investors who take the longevity risk at the end. In this type of indemnity based transaction, the investors are provided with the relevant information regarding the underlying portfolio as to be able to assess their risk. The back-to-back swap structure of this transaction means that JP Morgan has no residual longevity exposure. The longevity risk is transferred from the insurer to the investors in return for a risk premium. The counterpart risk for this swap is important given the long term maturity of the transaction, but also the number of agents involved.

A standardized transaction: Lucida In January 2008, JP Morgan executed a standardized longevity swap with the pension insurer Lucida for a notional amount of GBP 100 millions for 10 years, and using LifeMetrics index for England and Wales as underlying index. This swap structure enables a value hedge for Lucida, which has accepted in this case to keep the basis risk. For more details on both transactions in particular and on longevity swaps, please refer to Barrieu and Albertini (2009).

Longevity risk is very specific as previously underlined but the market for longevity risk is also very particular, being strongly unbalanced in terms of exposures and needs. This makes the question of the pricing of risk transfer solutions particularly important but even before this, the problem of designing suitable, efficient and attractive structures for both risk bearers and risk takers absolutely essential, as underlined by the failure of the EIB-BNP Paribas longevity bond in 2005. The recent financial crisis has also emphasized the importance of assessing counterpart risk, and properly managing collateral accounts, which help to secure transactions. These questions are even more critical when considering longevity risk, due to the long-term maturity of the transactions, but also the social, political and ethical nature of this risk.

5. Modeling issues for pricing

Designing longevity securities brings together various modeling issues besides the pure longevity risk modeling challenges. Firstly, the pricing of any longevity "derivative" is not straightforward and it depends on the estimate of uncertain future mortality trends and the level of uncertainty. This risk induces a mortality risk premium that should be priced by the market. However because of the absence of any liquid traded longevity security, it is today impossible to rely on market data for pricing purposes. Long term interest rate should play a key role for the valuation of such derivatives with long maturities (up to 50 years), creating new challenges in terms of modeling.

5.1. Pricing methodologies

In this section, we investigate methodologies for pricing longevity-linked securities. Note first that the longevity market is an immature market based on a non-financial risk. Therefore, the classical methodology of risk neutral pricing cannot be used carelessly.

The lack of liquidity of the market induces incompleteness, which is typically the case when non-hedgeable and non-tradable claims exist. Thus, to price some financial contracts on longevity risk, the use of a classical arbitrage-free pricing methodology is far from being applicable as it relies upon the idea of risk replication. The replication technique is only possible for markets with high liquidity and for deeply traded assets. It induces a unique price of the contingent claim, which is the cost of the replicating portfolio hedging away the market risk. Hence, in a complete market the price of the contingent claim is the expectation of the future discounted cash-flows under the unique risk neutral probability measure. On the contrary, in a longevity-linked securities market, there will be no universal pricing probability measure, and then the choice of the pricing probability measure is crucial.

What will be a good pricing measure for longevity? The historical probability measure will naturally play a key role because the reliable data we have are given under this probability measure. Therefore, it seems natural to look for a pricing probability measure equivalent to this historical probability measure. A relevant pricing measure must be robust with respect to the statistical data but also coherent with the prices of the liquid assets quoted in the market. Therefore, a relevant probability measure should make the link between the historical vision and the market vision. Once the subset of all such probability measures capturing the whole information we wanted is specified, we can look for the optimal one by maximizing the likelihood or the entropic criterion.

5.1.1. Characterization of a pricing rule The question of the pricing rule is also essential when considering financial transactions in an incomplete market.

Classical pricing methods The change of probability measures, from the historical probability to the pricing one, introduces a longevity risk premium. This method is similar to those based upon actuarial arguments, where the price of some risky cash-flow F can be obtained as

$$\pi(F) = \mathbb{E}_{\mathbb{P}}(F) + \lambda \sigma_{\mathbb{P}}(F).$$

The risk premium λ is a measure of the Sharpe ratio of the risky cash-flow F. Different authors have studied the impact of the choice of the probability measure on the pricing (Jewson and Brix (2005)) for another type of financial contracts, namely the weather derivatives.

As recalled before, in a very liquid and complete market where risky derivatives can be replicated by a self-financing portfolio, the risk-neutral (universal) pricing rule is used:

$$\pi(F) = \mathbb{E}_{\mathbb{P}*}(F) = \mathbb{E}_{\mathbb{P}}(F) + \mathrm{cov}\big(F, \frac{d\mathbb{P}*}{d\mathbb{P}}\big).$$

This pricing rule is linear as the actuarial rule does not take into account the risk induced by large transactions. However, when hedging strategies can not be constructed, the nominal amount of the transactions becomes an important risk factor and this methodology is not accurate any more, especially when the market is highly illiquid. To face this problem, the utility based indifference pricing methodology that we present below seems to be more appropriate.

Indifference pricing In an incomplete market framework, where a perfect replication is no longer possible, a more appropriate point of view involves utility maximization. Following Hodges and Neuberger (1989), the maximum price that an agent is ready to pay is the price such that she is indifferent (from her preference point of view) between doing or not the transaction. More precisely, given a utility function u^b and an initial wealth W_0^b, the indifference buyer price of F is $\pi^b(F)$ determined by the non linear relationship:

$$\mathbb{E}_{\mathbb{P}}\big(u^b(W_0^b + F - \pi^b(F)\big) = \mathbb{E}_{\mathbb{P}}\big(u^b(W_0^b)\big).$$

This price, which theoretically depends on the initial wealth and on the utility function, is not necessarily the price at which the transaction will take place. This gives an upper bound to the price the agent is ready to pay. Similarly, the indifference seller price is related to the preference of the seller and characterized by

$$\mathbb{E}_{\mathbb{P}}\big(u^s(W_0^s - F + \pi^s(F)\big) = \mathbb{E}_{\mathbb{P}}\big(u^s(W_0^s)\big). \tag{5.1}$$

We should note that this pricing rule is non-linear and provides a price range (difficult to compute) instead of a single price.

Fair price for small transactions When the agents are aware of their sensitivity to the unhedgeable risk, they can try to transact for only a little amount in the risky contract. In this case, the buyer wants to transact at the buyer's "fair price", which corresponds to the zero marginal rate of substitution p^b.

$$\partial_\theta \mathbb{E}_{\mathbb{P}}\big(u^b(W_0^b + \theta F)\big)|_{\theta=0} = \partial_\theta \mathbb{E}_{\mathbb{P}}\big[u^b(W_0^b - \theta p(F))\big]|_{\theta=0}$$

$$\text{that implies} \quad p^b(F) = \mathbb{E}_{\mathbb{P}}\big(u_x^b(W_0^b)F\big)/\mathbb{E}_{\mathbb{P}}\big(u_x^b(W_0^b)\big). \tag{5.2}$$

The same formula holds when the random initial wealth W_0^b is in fact the value of the optimal portfolio of the classical optimization problem in incomplete market related to the utility function u^b. In this case, the normalized random variable $u_x^b(W_0^b)$ may be viewed as the optimal martingale measure.
When both agents have the same utility function, they can transact at this fair price. Therefore, with an exponential utility function, the fair price for a small transaction is equivalent to the one given by the expectation under an equivalent probability measure. This methodology can be compared to the Wang transform, which is a distortion of the historical probability. If we linearise the Wang transform, this also leads to an equivalent change in probability measures.

Economic point of view When adopting an economic point of view, the transaction price is an equilibrium price, either between the seller and the buyer, or between different players in the market, where the agents maximize their expected utility at the same time (Pareto-optimality). Obviously a transaction takes place if it is possible to find two agents for whom $\pi^b(F) \geqslant \pi^s(F)$.

5.1.2. A dynamic point of view The economic pricing methodologies described in the above subsection are static and correspond in this respect mainly to an insurance or accounting point of view. The standard financial approach to pricing is however different and relies on the so-called risk-neutral methodology. The main underlying assumption of this approach is the possibility to replicate dynamically the cash flows of a given transaction using the basic traded securities in a market with high liquidity.

Using a non-arbitrage argument, the price of the contract is then uniquely defined as the cost of this replicating strategy. It can also be proved that this cost is in fact the expected value of the discounted future cash flows using a risk neutral probability measure as reference. This approach is clearly dynamic since the construction of the replicating strategy is done dynamically. Note that the replicating portfolio is not only a tool to find the price of the contract, but can also be used to hedge dynamically the risks associated with the transaction. Adopting such an approach for the pricing of financial contracts based upon mortality or longevity risks requires therefore a dynamic modeling of the underlying risks (see subsection 5.2). Besides, the cash flow are actualized and longevity-linked securities have a long maturity, thus this approach raises also some specific issues related to dynamic long term interest rate (see subsection 5.3). It is however important to emphasize the need to have a very liquid underlying market, which is essential for the construction of the replicating strategy. But, as mentioned before, as it now stands, the longevity market is far from been liquid and this risk neutral methodology has been questioned in many research papers, such as Bauer et al. (2008). An appropriate point of view may be to extend the fair price approach to an illiquid and dynamic setting : since a perfect hedge does not exist, we can extend (5.2) as

$$\sup\big(\mathbb{E}_{\mathbb{P}}\big(u^s(W_0^s - F + \pi^s(F))\big)\big) = \mathbb{E}_{\mathbb{P}}\big(u^s(W_0^s)\big), \tag{5.3}$$

where $\pi^s(F)$ is not a static price anymore but a dynamic price strategy associated with a hedge and the supremum in (5.3) is taken over the strategies. Thus an optimal hedge given criteria (5.3) is derived (see Barrieu and El Karoui (2009)).

5.1.3. Design issues Due to the absence of liquidity and maturity of the longevity market, a dynamic replicating strategy cannot be constructed and therefore the various risks embedded in a longevity transaction will be difficult to hedge dynamically. As a consequence, investors will not only consider the pricing of a transaction but will also take into consideration its design to choose the one which seems to be the less risky from their own point of view. Therefore, the design of new securities appears as an extremely important feature in the transaction. It may be the difference between success and failure.

The problem is now to design a contract F and a price $\pi(F)$ such that both the seller and the buyer would like to do the transaction. Such a problem has been studied in details by Barrieu and El Karoui (2009). An interesting feature of the risk transfer appears when both agents have the same utility function but different risk tolerance: $u^b(x) = \gamma^b u(x/\gamma^b)$ and $u^s(x) = \gamma^s u(x/\gamma^s)$. Given an initial risk that the seller wants to partially transfer to the buyer, the best choice is to transfer $F^* = \frac{\gamma^b}{\gamma^s + \gamma^b} X$. Thanks to the optimality of F^*, it can be shown that there still exists a pricing probability measure common to both agents, but depending on the pay-out of the derivatives. Nevertheless, this feature is somehow simplistic because investors do not share usually the same utility function (because, for example, they do not have access at the same market). Thus the issue does not reduce to the transfer mentioned above, but becomes more crucial and complex.

5.2. Stochastic mortality modeling for financial transactions

For pricing longevity linked securities, we need to develop future mortality forecasting models that are more accurate than those used by insurance companies. Thus, we can enrich the models introduced in Section 2.1 by introducing more uncertainty in their parameters. As we have seen earlier, in a deterministic approach for the hazard rate $\mu(x,t)$, the survival probability of an individual aged x at time t is defined as:

$$S_t(x,T) = \mathbb{P}_t\left(\tau > x + (T-t) \mid \tau > x\right) = \exp(-\int_t^T \mu(x+s-t,s)ds). \tag{5.4}$$

This can also be extended to a stochastic hazard rate $\mu(x,t)$ using two different approaches. First, by analogy with periodic life tables, the period approach models the process $s \to \mu(x,s)$ for each age x. In this case, $\mu(x,s)$ is referred to as the spot mortality rate (see Milevsky and Promislow (2001)). On the other hand, by analogy with life tables, the second approach is cohort-based and models instead the process $s \to \mu(x+s,s)$, taking into account the whole future mortality evolution.

More precisely, let $(\Omega, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a probability space representing the uncertainty under the historical probability measure $\mathbb{P}$. The intensity process $\mu(x,t)$ is adapted to the information $\mathcal{F}_t$ available to the agents at any time t. The survival probability is now defined as:

$$S_t(x,T) = \mathbb{P}_t\big(\tau > x + (T-t) \mid \tau > x\big) = \mathbb{E}_t\Big(\exp(-\int_t^T \mu(x+s-t,s)ds)\Big). \tag{5.5}$$

Here $\mathbb{P}_t(.)$ stands for the conditional probability of $\mathbb{P}$ with respect to $\mathcal{F}_t$, and x for the age of the cohort at time t.
Using Equation (5.5), the hazard rate can be defined as:

$$\begin{aligned}\mu(x,T) &= -\lim_{t\uparrow T} \partial_T \ln S_t(x,T) \\ &= \lim_{t\uparrow T}\Big[-\frac{\partial}{\partial T}\ln \mathbb{E}_t(\exp(-\int_t^T \mu(x+s-t,s)ds))\Big]. \end{aligned} \tag{5.6}$$

Note that $\mu(x,T)$ is indeed $\mu(x_T,T)$, where x_T is the age of the individual at time T.
On the one hand, if we consider a cohort based mortality modeling, we can define the quantity $\mu_t(x,T)$ where x is the age at time t by

$$\mu_t(x,T) = -\frac{\partial}{\partial T}\ln \mathbb{E}_t\Big(\exp(-\int_t^T \mu(x+s-t,s)ds\Big). \tag{5.7}$$

By analogy with interest rates framework or credit risk modeling, $\mu_t(x,T)$ is called the instantaneous forward mortality rate. Note that $S_t(x,T)$ is equivalent here to the price at time t of a defaultable zero-coupon bond maturing at T. This definition is very useful when valuing insurance contracts because it takes into account the whole age-term structure of mortality, which is consistent with insurers practices.
On the other hand, in the periodic approach of mortality modeling (*spot* mortality rates), some authors (Schrager (2006) and Dahl (2004) or Biffis (2005)) introduce the so-called intensity framework to stochastic mortality modeling, which is very popular both in interest rates and credit risk modeling. By doing so, they draw an analogy between death time and the default time. Moreover, this methodology has great

underpins because it facilitates the derivation of closed-form formulae for survival probabilities, especially when assuming an affine diffusion for the mortality hazard rate process $\mu(x,t)$. However, this class of processes shows restrictive behaviors and do not ensure positive mortality intensities. Gouriéroux and Sufana (2003) have outlined these drawbacks and introduced a new class of processes: the so-called quadratic class. Following this approach, Gouriéroux and Monfort (2008) use a quadratic stochastic intensity model with a Gaussian autoregressive factor to model spot mortality rates. This proves to be more suitable and consistent with historical data. They also underline the tractability of such processes when used to evaluate insurance contracts using a risk neutral approach. Nevertheless, the intensity approach for mortality is more complex than in credit risk, because the hazard rate depends on the age x and this dependency is difficult to model and capture in such a framework.
Henceforth, it will be essential to understand on which data the various parameters of the models have to be estimate statistically. The most accurate and reliable data we get are the national data of mortality. However, as we have mentioned in Section 2.4, these mortality tables present significant differences from a country to another and are heterogeneous.
The statistical data will be used to construct pricing methodologies for longevity-linked securities. As for derivatives written on non-financial risk, in an immature market, there would be no financial benchmark to estimate the market risk premium.

5.3. Long-term interest rates

Financial contracts written on mortality-related risks typically have a maturity up to 20 years, while, on the other hand, longevity-linked securities are typically characterized by a much longer maturity (40 years and beyond). In most of these contracts, there is an embedded interest rate risk. For the shorter time horizon (up to 20 years), the standard financial point of view can be used to hedge this risk as the interest rate market with such maturities is quite liquid. However, this is not the case anymore for longer maturities as the interest rate market becomes highly illiquid and the standard financial point of view cannot be easily extended. Nevertheless, an abundant literature on the economic aspects of long-term policy-making (i.e. a time horizon between 50 to 200 years), has been developed.
The economic assumptions set to handle the problem concern, first of all, the agent preferences. As for the pricing problem discussed above, the agent is considered to be risk-averse and her utility function on consumption is assumed to be differentiable, increasing and concave. Moreover, she is assumed to behave as a price taker. In this framework the optimal interest rate for each maturity is derived from an equilibrium linking together the saving-consumption dilemma to the optimal utility over the given period. This approach is inspired from the well-known neoclassical economical theory and has been outlined by many authors (e.g. Gollier (2007, 2008), Hansen and Scheinkman (2009) and Breeden (1989)). It focuses on the aggregate behavior of all agents, and the economy is represented by the strategy of the representative agent. This is the same economic model introduced by Breeden (1989). The derivation of the yield curve for far-distant maturities is induced from the maximization of the representative agent's intertemporal utility function on the aggregate consumption :

$$\max_{c\geqslant 0}\int_{t\geqslant 0} e^{-\delta t}u(c_t)dt$$

where c_t is the aggregate consumption, u the agent's utility function and δ her pure time preference parameter (i.e. δ quantifies the agent preference of immediate goods versus future ones). In the setting of deterministic rate and consumption, the optimal consumption is given by:

$$u'(\widehat{c_t})e^{-\delta t} = u'(c_0)e^{-\int_0^t r_s ds}$$

where r_t is the spot rate and $e^{-\int_0^t r_s ds}$ the discount factor. This leads to the so-called "Ramsey rule"

$$\frac{1}{t}\int_0^t r_s ds = \delta - \frac{1}{t}\ln\frac{u'(\widehat{c_t})}{u'(c_0)}. \tag{5.8}$$

Adding uncertainty on the interest rate and the consumption, the maximization of the representative agent's utility function takes into account the budget constraint

$$\max_{c\geqslant 0}\int_{t\geqslant 0} e^{-\delta t}\mathbb{E}\left(u(\widetilde{c_t})\right)dt \quad \text{s.t. } \mathbb{E}\left(\int_{t\geqslant 0} e^{-\int_0^t \widetilde{r}_s ds}\widetilde{c_t}dt\right) \leqslant x_0.$$

The budget constraint expresses the initial wealth x_0 in the economy that allows to finance the consumption plan $\widetilde{c_t}$. The optimal consumption is given pathwise by:

$$\mathbb{E}(u'(\widehat{c_t}))e^{-\delta t} = u'(c_0)e^{-\int_0^t \widetilde{r}_s ds}.$$

The initial consumption c_0 is a function of the initial wealth x_0, given by the budget constraint. Note that the consumption is deterministic if and only if the interest rate is deterministic. The Ramsey rule can be extended in this stochastic framework :

$$R_0(t) := \frac{1}{t}\ln\mathbb{E}\left(e^{\int_0^t \widetilde{r}_s ds}\right) = \delta - \frac{1}{t}\ln\frac{\mathbb{E}(u'(\widehat{c_t}))}{u'(c_0)}, \tag{5.9}$$

Developing this solution using the second-order Taylor approximation leads to the following equation (see Gollier (2007)):

$$R_0(t) \simeq \delta + R(c_0)\frac{\mathbb{E}(\widehat{c_t}) - c_0}{tc_0} - \frac{1}{2}R(c_0)P(c_0)\frac{Var(\widehat{c_t}/c_0)}{t},$$

where $R(c) = -c\dfrac{u''(c)}{u'(c)}$ is the relative risk aversion parameter and $P(c) = -c\dfrac{u'''(c)}{u''(c)}$ is the relative prudence parameter. The yield curve is now governed by three main behaviors of the agent facing a consumption-saving problem (i.e. the three terms on right hand side of the equation above). First of all, the representative agent is interested in goods bringing an immediate satisfaction rather than those with the same effect later in the future. This effect works in an additive manner with the "wealth effect", i.e. the agent prefers to consume rather than to save in sight of potential better days in the future, which is incorporated in the deterministic trend on growth in the economic evolution (i.e. consumption). Finally, the "precautionary effect" appears. It increases as the future is uncertain and with the representative agent's willingness to save.

For example, Gollier (2007, 2008) studies the case of a consumption given by a geometric Brownian motion and Hansen and Scheinkman (2009) uses a pure Levy process to model the consumption and derive the optimal interest rates yield curve in a Markov environment. Nevertheless, in the economic literature on long-term policy-making, modeling processes in a dynamic way is not common and the usual approach

is deterministic with static random perturbations. Furthermore, the global approach with a representative agent never takes into account the existence of a market. To study financial problems involving long term interest rates, we need to look at this approach by putting also the financial market at the heart of our preoccupations.
A first way to achieve this in the case of a liquid market is to consider an agent, who optimizes the utility of her consumption under the budget constraint. The budget constraint evaluate the present value of the future cash flows under the pricing probability $\mathbb{P}^*$ (and not under the historical probability $\mathbb{P}$ as it was done before) :

$$\max_{c \geqslant 0} \int_{t \geqslant 0} e^{-\delta t} \mathbb{E}\left(u(\widetilde{c}_t)\right) dt \quad \text{s.t. } \mathbb{E}^* \left(\int_{t \geqslant 0} e^{-\int_0^t \widetilde{r}_s ds} \widetilde{c}_t dt \right) \leqslant x_0.$$

Denoting by L_t^* the density of $\mathbb{P}^*$ with respect to $\mathbb{P}$, we have

$$\exp(-\delta t) u'(\widehat{c}_t) = u'(c_0) \exp\left(- \int_0^t \widetilde{r}_s ds \right) L_t^*.$$

Thus

$$B(0,t) = \exp(-\delta t) \frac{\mathbb{E}[u'(\widehat{c}_t)]}{u'(c_0^*)},$$

where $B(0,t) = \mathbb{E}^* \left[\exp\left(-\int_0^t \widetilde{r}_s ds \right) \right]$ is the price at time 0 of a zero-coupon with maturity t. Therefore, the same relation as (Equation 5.9) - with $B(0,t) = \exp(-R_0(t)t)$ - is obtained from a financial point of view, in the case of a liquid market. In the case of an illiquid market, the pricing probability is not universal and might depend on the maturity and the utility function. Thus the budget constraint takes a more complex form. The consequences on Equation (5.9) are an important issue we are currently working on.
This model can be enriched by considering a stochastic utility function : the case where the agent preferences are unknown is treated by Lazrak and Zapatero (2004) using a stochastic utility function modeling the fact that the agent can change her beliefs during the considered period.

6. Conclusion

Having presented its main characteristics and the state of the art in its understanding and modeling, longevity risk appears to be a very complex risk, due to its specificities compared with other insurance risks - in particular the trend sensitivity, the geographical variability and the associated long-term maturities - and its potential correlations with other sources of risk, financial and non-financial.
As a recent move, the insurance industry and especially the life insurance sector are adopting new regulations not only to allow for a more accurate risk assessment but also to impose more effective solvency risk management rules. Those regulations and the increasing convergence between insurance and capital markets have opened the way for alternative risk management solutions and innovative risk transfers. To support the emergence of this new market, not only any asymmetry of information between the various agents involved in potential transactions has to be reduced but it is also necessary to develop specific pricing methods and partial hedging methodologies, well-suited to the particular features of longevity risk and to the immature and illiquid

state of the current market. Additional challenges appear naturally, especially related to the modeling of long term interest rate due to the long maturities of the potential transactions.
However, the longevity risk is far from being a concern for the insurance industry alone. It is indeed at the core of an open discussion for politicians, economists and strategists, who have to determine the "effective" retirement age and the "effective" pension scheme as the future generations will almost surely face one of the greatest challenges with an increasing life expectancy. The potential impacts of longevity risk at various levels of the global economy and society make a better management of this risk one of the key challenges of the coming decade. Interactions with other sources of risk, such as dependence, economy, or ecology, have not been investigated in-depth in this paper. But it would be certainly very interesting to study at a macro-level the impacts of longevity on the whole economy and the environment.

References

Antolin, P., H. Blommestein. 2007. Governments and the market for longevity-indexed bonds. *OECD Journal : Financial Market Trends* .

Barrieu, P., L. Albertini. 2009. *The handbook of insurance-linked securities*. Wiley Finance.

Barrieu, P., N. El Karoui. 2009. Pricing, hedging and optimally designing derivatives via minimization of risk measures. R. Carmona, ed., *Volume on Indifference Pricing*. Princeton University Press, 77–146.

Bauer, D., Boerger M., J. Russ. 2008. On the Pricing of Longevity-Linked Securities. Tech. rep., Working Paper. Available at: www.mortalityrisk.org.

Bensusan, H., N. El Karoui. 2009. Microscopic modelling of population dynamic : An analysis of longevity risk. *Preprint* .

Bienvenue, A., A. Illig, L. Loisel, D. Serant. 2009. On inter-age correlations in stochastic mortality models. *Working Paper* .

Biffis, E. 2005. Affine processes for dynamic mortality and actuarial valuations. *Insurance Mathematics and Economics* **37**(3) 443–468.

Breeden, D.T. 1989. An intertemporal asset pricing model with stochastic consumption and investment opportunities. *Theory of Valuation: Frontiers of Modern Financial Theory* **7** 53.

Cairns, A.J.G., D. Blake, K. Dowd. 2006. A two-factor model for stochastic mortality with parameter uncertainty: Theory and calibration. *Journal of Risk & Insurance* **73**(4) 687–718.

Cairns, A.J.G., D. Blake, K. Dowd. 2008. Modelling and management of mortality risk: a review. *Scandinavian Actuarial Journal* **108**(2) 79–113.

Cairns, A.J.G., D. Blake, K. Dowd, G.D. Coughlan, D. Epstein, A. Ong, I. Balevich. 2007. A quantitative comparison of stochastic mortality models using data from England & Wales and the United States. *Working paper, Heriot-Watt University, and Pension Institute Discussion Paper PI-0701* .

Cambois, E., Laborde C., J.M Robine. 2008. La "double peine" des ouvriers : plus d'années d'incapacité au sein d'une vie plus courte. *Population et Sociétés* (441).

Ceiops. 2008. Quantitative impact study. Tech. rep., European Commission, available at: `http://www.ceiops.eu/content/view/118/124/`.

Cox, S.H., Y. Lin, A. Milidonis. 2009. Mortality Regimes and Pricing. Tech. rep.

Currie, I.D., M. Durban, P.H.C. Eilers. 2004. Smoothing and forecasting mortality rates. *Statistical Modelling* **4**(4) 279.

Dahl, M. 2004. Stochastic mortality in life insurance: market reserves and mortality-linked insurance contracts. *Insurance Mathematics and Economics* **35**(1) 113–136.

Denuit, M., A. Delwaerde. 2005. *Construction de tables de mortalité périodiques et prospectives*. Economica.

El Karoui, N., S. Loisel, C. Mazza, A. Shirayev. 2009. Detection of changes in longevity trends by banks, insurers and reinsurers. *Working Paper* .

FAP. 2007. Personnes âgées et médicaments. *Activity report, available at: `http://www.acadpharm.org/medias/direct/Agees.pdf`* .

Fournier, N., S. Méléard. 2004. A microscopic probabilistic description of a locally regulated population and macroscopic approximations. *Annals of Applied Probability* 1880–1919.

Gavrilov, L.A., N.S. Gavrilova. 2001. The reliability theory of aging and longevity. *J. Theoret. Biol.* **213**(4) 527–545.

Gollier, C. 2007. The consumption-based determinants of the term structure of discount rates. *Mathematics and Financial Economics* **1**(2) 81–101.

Gollier, C. 2008. Discounting with fat-tailed economic growth. *Journal of Risk and Uncertainty* 1–16.

Goshay, R.C., R. Sandor. 1973. An inquiry into the feasibility of a reinsurance futures market. *Journal of Business Finance* **5**(2) 56–66.

Gouriéroux, C., A. Monfort. 2008. Quadratic stochastic intensity and prospective mortality tables. *Insurance Mathematics and Economics* .

Gouriéroux, C., R. Sufana. 2003. Wishart quadratic term structure models. Tech. rep., working paper, University of Toronto.

Hansen, L.P., J.A. Scheinkman. 2009. Long-term risk: An operator approach. *Econometrica* **77**(1) 177–234.

Hodges, S.D., A. Neuberger. 1989. Optimal replication of contingent claims under transaction costs. *Review of futures markets* **8**(2) 222–239.

Jewson, S., A. Brix. 2005. *Weather derivative valuation: the meteorological, statistical, financial and mathematical foundations*. Cambridge University Press.

Jusot, F. 2004. Mortalité, revenu et inégalités de revenu en france : une analyse multi-niveaux. *IRDES* .

Kirkby, J., I. Currie. 2007. Smooth models of mortality with period shocks. *Age* **20**(40) 60.

Klein, J.P., M.L. Moeschberger. 2003. *Survival analysis: techniques for censored and truncated data*. Springer.

Lazar, D., M. Denuit. 2009. A multivariate time series approach to projected life tables. *Working Paper* .

Lazrak, A., F. Zapatero. 2004. Efficient consumption set under recursive utility and unknown beliefs. *Journal of Mathematical Economics* **40**(1-2) 207–226.

Lee, R. D., L. Carter. 1992. Modelling and forecasting the time series of us mortality. *Journal of the American Statistical Association* **87** 659–671.

Milevsky, M.A., S.D. Promislow. 2001. Mortality derivatives and the option to annuitise. *Insurance Math. Econom.* **29**(3) 299–318. 4th IME Conference (Barcelona, 2000).

Monnier, A. 2000. La population de l'Europe : 1950-2050. *Population et Sociétés* (353).

Pitacco, E. 2004. Survival models in a dynamic context: a survey. *Insurance Mathematics and Economics* **35**(2) 279–298.

Renshaw, A.E., S. Haberman. 2003. On the forecasting of mortality reduction factors. *Insurance Mathematics and Economics* **32**(3) 379–401.

Renshaw, A.E., S. Haberman. 2006. A cohort-based extension to the Lee–Carter model for mortality reduction factors. *Insurance Mathematics and Economics* **38**(3) 556–570.

Salhi, Y., S. Loisel, M. Denuit. 2009. Joint modeling of portfolio experienced and national mortality: A co-integration based approach. *Working Paper* .

Schrager, D.F. 2006. Affine stochastic mortality. *Insurance Mathematics and Economics* **38**(1) 81–97.

Tran, V.C. 2006. Modèles particulaires stochastiques pour des problèmes d'évolution adaptative et pour l'approximation de solutions statistiques. *Ph.D. Thesis* .

Tran, V.C. 2007. Large population limit and time behaviour of a stochastic particle model describing an age-structured population. *ESAIM* .

Tran, V.C., S. Méléard. 2009. Age structured trait substitution sequence process and canonical equation. *Journal of Mathematical Biology* .

Webb, G.F. 1985. *Theory of nonlinear age-dependent population dynamics*. Marcel Dekker.