High-gain observer with uniform in the initial condition finite time convergence

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1 Introduction

In this short note, we show how the framework introduced in [1] allows to obtain an observer with finite time convergence for globally Lipchitz upper triangular systems.

2 Finite time observer

we introduce an observer for systems of the form :

$$\dot{x} \;=\; \mathcal{S}\,x \;+\; B\,u \;+\; \delta(x,t) \quad , \qquad y \;=\; x_1 \;, \tag{1}$$

where $x = (x_1, \dots, x_n)$ is in $\mathbb{R}^n$ and $\delta \;: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is a continuous function globally Lipchitz in its first argument (uniformly in t).

The domination approach has been used to design observer for systems of the form (1). This approach has been popularized by high-gain observer [2]. These observers are given as:

$$\dot{\hat{x}} = \mathcal{S}\,\hat{x} + Bu + \delta(\hat{x},t) \;+\; L\mathfrak{L}^{-1}K(\hat{x}_1 - y) \tag{2}$$

where L is the extra high-gain parameter, $\mathfrak{L} = \texttt{diag}\,(1, L^{-1}, L^{-2}, \dots, L^{1-n})$ and K is the output injection which have to be designed to ensure that the state of the error system:

$$\dot{\tilde{x}} = \mathcal{S}\,\tilde{x} + \delta(\tilde{x},t) - \delta(\hat{x} - \tilde{x},t) \;+\; L\mathfrak{L}^{-1}K(\tilde{x}_1) \tag{3}$$

converges to the origin.

The error system (3) has the structure of a chain of integrators disturbed by nonlinear terms which, assuming a global Lipschitz condition (as in [2]), is linearly bounded. In [2], the

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domination approach has been employed and a linear vector field K in the observer (2) was introduced to ensure global and asymptotic convergence of the error $\tilde{x}$ toward the origin.

Recently, this approach has been extended in [4] (see also [3]) to a homogeneous vector field K with negative degree to allow semi-global and finite time estimation.

The homogeneous in the bi-limit vector field K obtained from [1, Section 3] allows us to get a global observer with finite-time estimation and with an estimation time uniform in the initial condition:

Corollary 1 (Finite time observer) *If for $(x,\tilde{x})$ in $\mathbb{R}^{2n}$,*

$$|\delta_i(x+\tilde{x},t)-\delta_i(x,t)| \ \leq c\sum_{j=1}^{i}|\tilde{x}_j| \tag{4}$$

where c is a positive real numbers, then there exist a continuous vector field $K:\mathbb{R}\to\mathbb{R}^n$ and a real number $L^*>0$ such that for every L in $[L^*,+\infty)$, the estimate given by the system (2) converges to the state of system (1) in finite time uniformly in the initial condition, i.e., there exists a positive real number T such that for all initial state x_0 in $\mathbb{R}^n$, initial estimate $\hat{x}_0$ in $\mathbb{R}^n$ and all locally bounded continuous function $u:[0,T]\to\mathbb{R}$, we get:

$$x(T)=\hat{x}(T)$$

where $(x,\hat{x}):\mathbb{R}\to\mathbb{R}^{2n}$ is a C^1 functions solution of systems (1) and (2) such that $x(0)=x_0$ and $\hat{x}(0)=\hat{x}_0$.

Proof : To construct the vector field K we employ the homogeneous in the bi-limit framework and the procedure introduced in [1]. We introduce two real numbers $\mathfrak{d}_0$ and $\mathfrak{d}_\infty$ (the degree of the homogeneous in the bi-limit vector field K) such that

$$-1<\mathfrak{d}_0<0<\mathfrak{d}_\infty<\frac{1}{n-1}\ . \tag{5}$$

As in [1], we introduce the associated weights vector r_0 and r_∞ both in $\mathbb{R}^n_+$ defined as

$$r_{b,n}=1\ ,\quad r_{b,i}=r_{b,i+1}\ -\ \mathfrak{d}_b=1\ -\ \mathfrak{d}_b\,(n-i)\ , \tag{6}$$

where the letter "b" stand for "0" or "∞". Following the procedure [1, Section 3], we obtain a homogeneous in the bi-limit vector field $K:\mathbb{R}\to\mathbb{R}^n$ with associated triples $(r_0,\mathfrak{d}_0,K_0)$ and $(r_\infty,\mathfrak{d}_\infty,K_\infty)$ such that the origin of the systems with state $z=(z_1,\ldots,z_n)$ in $\mathbb{R}^n$:

$$\begin{aligned}\dot{z} &= \mathcal{S}z+K(z_1)\ ,\\ \dot{z} &= \mathcal{S}z+K_0(z_1)\ ,\\ \dot{z} &= \mathcal{S}z+K_\infty(z_1)\ ,\end{aligned}$$

is globally and asymptotically stable. Hence, we can employ [1, Corollary 2.22] to get a positive real number c_G such that for all continuous function $R:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$, satisfying

$$R_i(z,t)\leq c_G\left(\sum_{j=1}^{i}|z_j|^{\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)}}+\sum_{j=1}^{i}|z_j|^{\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)}}\right)\ i=1,\ldots,n\ , \tag{7}$$

where $R(z,t) = (R_1(z,t), \ldots, R_n(z,t))$, the origin of the system :

$$\dot{z} = \mathcal{S}z + K(z_1) + R(z,t) \tag{8}$$

is globally and asymptotically stable.

Note that since $\mathfrak{d}_0 < 0 < \mathfrak{d}_\infty$, it follows from Young's inequality that, given a continuous function R satisfying

$$R_i(z,t) \leq c_G \sum_{j=1}^{i} |z_j| \ .$$

then the R_i's satisfy also the bound (7) and in this case, the origin of system (8) is globally and asymptotically stable.

We introduce now the scaled coordinates defined as :

$$e_i \ = \ L^{1-i}\,\tilde{x}_i \quad , \qquad i = 1, \ldots, n \ , \tag{9}$$

where L, the high-gain parameter, is a positive real number which will be selected later. We can rewrite this change of coordinates in compact form as:

$$E = \mathfrak{L}\tilde{x} \quad , \qquad \mathfrak{L} = \mathtt{diag}\left(1, L^{-1}, L^{-2}, \ldots, L^{1-n}\right) \ .$$

We get along the trajectory of the error system (3) :

$$\dot{E} = L\Big[\mathcal{S}E + \Delta(L,\hat{x},\tilde{x},t) \ + \ K(e_1)\Big] \ ,$$

where

$$\Delta(L,\hat{x},\tilde{x},t) = L^{-1}\mathcal{L}[\delta(\hat{x},t) - \delta(\hat{x}-\tilde{x},t)] \ .$$

Moreover, due to (4), with $L \geq 1$, we get :

$$|\Delta_i(L,\hat{x},\tilde{x},t)| \leq L^{-i}c\sum_{j=1}^{i} |\tilde{x}_j| \leq L^{-1}c\sum_{j=1}^{i} |e_j|$$

Consequently with c_G defined in (7) and taking $L^* > \frac{c}{c_G}$, we get that, for all L in $[L^*, +\infty)$, the origin of the system:

$$\dot{E} = L\Big[\mathcal{S}E + \Delta(L,\tilde{x},t) \ + \ K(e_1)\Big] \ ,$$

is globally and asymptotically stable. Hence, the estimate $\hat{x}$ converges toward the state x.

Moreover, the origin is also globally and asymptotically stable for the homogeneous approximations:

$$\begin{aligned}
\dot{E} \ &= \ L\Big(\mathcal{S}E + K_0(e_1)\Big) \ , \\
\dot{E} \ &= \ L\Big(\mathcal{S}E + K_\infty(e_1)\Big) \ ,
\end{aligned}$$

and with (5), we can apply [1, Corollary 2.24] to obtain the result. □

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