# High-gain observer with uniform in the initial condition finite time convergence

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## 1 Introduction

In this short note, we show how the framework introduced in [1] allows to obtain an observer with finite time convergence for globally Lipchitz upper triangular systems.

## 2 Finite time observer

we introduce an observer for systems of the form :

$$\dot{x} \;=\; \mathcal{S}\, x \;+\; B\, u \;+\; \delta(x,t) \quad , \qquad y \;=\; x_1 \;, \tag{1}$$

where $x = (x_1, \ldots, x_n)$ is in $\mathbb{R}^n$ and $\delta \; : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is a continuous function globally Lipchitz in its first argument (uniformly in $t$).

The domination approach has been used to design observer for systems of the form (1). This approach has been popularized by high-gain observer [2]. These observers are given as:

$$\dot{\hat{x}} = \mathcal{S}\, \hat{x} + Bu + \delta(\hat{x}, t) \;+\; L\mathfrak{L}^{-1}K(\hat{x}_1 - y) \tag{2}$$

where $L$ is the extra high-gain parameter, $\mathfrak{L} = \mathtt{diag}\,(1, L^{-1}, L^{-2}, \ldots, L^{1-n})$ and $K$ is the output injection which have to be designed to ensure that the state of the error system:

$$\dot{\tilde{x}} = \mathcal{S}\, \tilde{x} + \delta(\tilde{x}, t) - \delta(\hat{x} - \tilde{x}, t) \;+\; L\mathfrak{L}^{-1}K(\tilde{x}_1) \tag{3}$$

converges to the origin.

The error system (3) has the structure of a chain of integrators disturbed by nonlinear terms which, assuming a global Lipschitz condition (as in [2]), is linearly bounded. In [2], the

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domination approach has been employed and a linear vector field $K$ in the observer (2) was introduced to ensure global and asymptotic convergence of the error $\tilde{x}$ toward the origin.

Recently, this approach has been extended in [4] (see also [3]) to a homogeneous vector field $K$ with negative degree to allow semi-global and finite time estimation.

The homogeneous in the bi-limit vector field $K$ obtained from [1, Section 3] allows us to get a global observer with finite-time estimation and with an estimation time uniform in the initial condition:

**Corollary 1 (Finite time observer)** *If for* $(x, \tilde{x})$ *in* $\mathbb{R}^{2n}$,

$$|\delta_i(x + \tilde{x}, t) - \delta_i(x, t)| \ \leq c \sum_{j=1}^{i} |\tilde{x}_j| \tag{4}$$

*where* $c$ *is a positive real numbers, then there exist a continuous vector field* $K : \mathbb{R} \to \mathbb{R}^n$ *and a real number* $L^* > 0$ *such that for every* $L$ *in* $[L^*, +\infty)$*, the estimate given by the system (2) converges to the state of system (1) in finite time uniformly in the initial condition, i.e., there exists a positive real number* $T$ *such that for all initial state* $x_0$ *in* $\mathbb{R}^n$*, initial estimate* $\hat{x}_0$ *in* $\mathbb{R}^n$ *and all locally bounded continuous function* $u : [0, T] \to \mathbb{R}$*, we get:*

$$x(T) = \hat{x}(T)$$

*where* $(x, \hat{x}) : \mathbb{R} \to \mathbb{R}^{2n}$ *is a* $C^1$ *functions solution of systems (1) and (2) such that* $x(0) = x_0$ *and* $\hat{x}(0) = \hat{x}_0$.

**Proof :** To construct the vector field $K$ we employ the homogeneous in the bi-limit framework and the procedure introduced in [1]. We introduce two real numbers $\mathfrak{d}_0$ and $\mathfrak{d}_\infty$ (the degree of the homogeneous in the bi-limit vector field $K$) such that

$$-1 < \mathfrak{d}_0 < 0 < \mathfrak{d}_\infty < \frac{1}{n-1} \ . \tag{5}$$

As in [1], we introduce the associated weights vector $r_0$ and $r_\infty$ both in $\mathbb{R}^n_+$ defined as

$$r_{b,n} = 1 \ , \quad r_{b,i} = r_{b,i+1} \ - \ \mathfrak{d}_b = 1 \ - \ \mathfrak{d}_b \, (n - i) \ , \tag{6}$$

where the letter "$b$" stand for "0" or "$\infty$". Following the procedure [1, Section 3], we obtain a homogeneous in the bi-limit vector field $K : \mathbb{R} \to \mathbb{R}^n$ with associated triples $(r_0, \mathfrak{d}_0, K_0)$ and $(r_\infty, \mathfrak{d}_\infty, K_\infty)$ such that the origin of the systems with state $z = (z_1, \ldots, z_n)$ in $\mathbb{R}^n$ :

$$\begin{aligned} \dot{z} &= \mathcal{S}z + K(z_1) \ , \\ \dot{z} &= \mathcal{S}z + K_0(z_1) \ , \\ \dot{z} &= \mathcal{S}z + K_\infty(z_1) \ , \end{aligned}$$

is globally and asymptotically stable. Hence, we can employ [1, Corollary 2.22] to get a positive real number $c_G$ such that for all continuous function $R : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, satisfying

$$R_i(z, t) \leq c_G \left( \sum_{j=1}^{i} |z_j|^{\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)}} + \sum_{j=1}^{i} |z_j|^{\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)}} \right) \ i = 1, \ldots, n \ , \tag{7}$$

where $R(z,t) = (R_1(z,t), \dots, R_n(z,t))$, the origin of the system :

$$\dot{z} = \mathcal{S}z + K(z_1) + R(z,t) \tag{8}$$

is globally and asymptotically stable.

Note that since $\mathfrak{d}_0 < 0 < \mathfrak{d}_\infty$, it follows from Young's inequality that, given a continuous function $R$ satisfying

$$R_i(z,t) \leq c_G \sum_{j=1}^{i} |z_j| \ .$$

then the $R_i$'s satisfy also the bound (7) and in this case, the origin of system (8) is globally and asymptotically stable.

We introduce now the scaled coordinates defined as :

$$e_i \ = \ L^{1-i}\,\tilde{x}_i \quad , \qquad i = 1, \dots, n \ , \tag{9}$$

where $L$, the high-gain parameter, is a positive real number which will be selected later. We can rewrite this change of coordinates in compact form as:

$$E = \mathfrak{L}\tilde{x} \quad , \qquad \mathfrak{L} = \mathtt{diag}\left(1, L^{-1}, L^{-2}, \dots, L^{1-n}\right) \ .$$

We get along the trajectory of the error system (3) :

$$\dot{E} = L\Big[\mathcal{S}E + \Delta(L, \hat{x}, \tilde{x}, t) \ + \ K(e_1)\Big] \ ,$$

where

$$\Delta(L, \hat{x}, \tilde{x}, t) = L^{-1}\mathcal{L}[\delta(\hat{x}, t) - \delta(\hat{x} - \tilde{x}, t)] \ .$$

Moreover, due to (4), with $L \geq 1$, we get :

$$|\Delta_i(L, \hat{x}, \tilde{x}, t)| \leq L^{-i} c \sum_{j=1}^{i} |\tilde{x}_j| \leq L^{-1} c \sum_{j=1}^{i} |e_j|$$

Consequently with $c_G$ defined in (7) and taking $L^* > \frac{c}{c_G}$, we get that, for all $L$ in $[L^*, +\infty)$, the origin of the system:

$$\dot{E} = L\Big[\mathcal{S}E + \Delta(L, \tilde{x}, t) \ + \ K(e_1)\Big] \ ,$$

is globally and asymptotically stable. Hence, the estimate $\hat{x}$ converges toward the state $x$.

Moreover, the origin is also globally and asymptotically stable for the homogeneous approximations:

$$\begin{aligned} \dot{E} \ &= \ L\Big(\mathcal{S}E + K_0(e_1)\Big) \ , \\ \dot{E} \ &= \ L\Big(\mathcal{S}E + K_\infty(e_1)\Big) \ , \end{aligned}$$

and with (5), we can apply [1, Corollary 2.24] to obtain the result. □

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