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SMASH-NILPOTENT CYCLES ON ABELIAN 3-FOLDS

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ABSTRACT. We show that homologically trivial algebraic cycles on a 3-dimensional abelian variety are smash-nilpotent.

INTRODUCTION

Let $X$ be a smooth projective variety over a field $k$. An algebraic cycle $Z$ on $X$ (with rational coefficients) is smash-nilpotent if there exists $n > 0$ such that $Z^n$ is rationally equivalent to 0 on $X^n$. Smash-nilpotent cycles have the following properties:

1. The sum of two smash-nilpotent cycles is smash-nilpotent.
2. The subgroup of smash-nilpotent cycles forms an ideal under the intersection product as $(x \cdot y) \times (x \cdot y) \cdots \times (x \cdot y) = (x \times x \times \cdots \times x) \cdot (y \times y \times \cdots \times y)$.
3. On an abelian variety, the subgroup of smash-nilpotent cycles forms an ideal under the Pontryagin product as $(x \ast y) \times (x \ast y) \cdots \times (x \ast y) = (x \times x \times \cdots \times x) \ast (y \times y \times \cdots \times y)$ where $\ast$ denotes the Pontryagin product.

Voevodsky [11, Cor. 3.3] and Voisin [12, Lemma 2.3] proved that any cycle algebraically equivalent to 0 is smash-nilpotent. On the other hand, because of cohomology, any smash-nilpotent cycle is numerically equivalent to 0; Voevodsky conjectured that the converse is true [11, Conj. 4.2].

This conjecture is open in general. The main result of this note is:

Theorem 1. Let $A$ be an abelian variety of dimension $\leq 3$. Any homologically trivial cycle on $A$ is smash-nilpotent.

In characteristic 0 we can improve “homologically trivial” to “numerically trivial”, thanks to Lieberman’s theorem [7].

Nori’s results in [8] give an example of a group of smash-nilpotent cycles which is not finitely generated modulo algebraic equivalence. The proof of Theorem 1 actually gives the uniform bound 21 for the degree.

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of smash-nilpotence on this group, see Remark 2. See Proposition 2 for partial results in dimension 4.

1. Beauville’s decomposition, motivically

For any smooth projective variety $X$ and any integer $n \geq 0$, we write as in [1] $CH^n_{\mathbb{Q}}(X) = CH^n(X) \otimes \mathbb{Q}$, where $CH^n(X)$ is the Chow group of cycles of codimension $n$ on $X$ modulo rational equivalence.

Let $A$ be an abelian variety of dimension $g$. For $m \in \mathbb{Z}$, we denote by $\langle m \rangle$ the endomorphism of multiplication by $m$ on $A$, viewed as an algebraic correspondence. In [1], Beauville introduces an eigenspace decomposition of the rational Chow groups of $A$ for the actions of the operators $\langle m \rangle$, using the Fourier transform. Here is an equivalent definition: in the category of Chow motives with rational coefficients, the endomorphism $1_A \in \text{End}(h(A)) = CH^g_{\mathbb{Q}}(A \times A)$ is given by the class of the diagonal $\Delta_A$. We have the canonical Chow-Künneth decomposition

$$1_A = \sum_{i=0}^{2g} \pi_i$$

[4, Th. 3.1], where the $\pi_i$ are orthogonal idempotents and $\pi_i$ is characterised by $\pi_i \langle m \rangle^* = m^i \pi_i$ for any $m \in \mathbb{Z}$. This yields a canonical Chow-Künneth decomposition of the Chow motive $h(A)$ of $A$:

$$h(A) = \bigoplus_{i=0}^{2g} h^i(A), \quad h^i(A) = (A, \pi_i)$$

(see [10, Th. 5.2]). Then, under the isomorphism

$$CH^n_{\mathbb{Q}}(A) = \text{Hom}(\mathbb{L}^n, h(A))$$

(where $\mathbb{L}$ is the Lefschetz motive) we have

$$CH^n(A)_{[r]} := \{ x \in CH^n_{\mathbb{Q}}(A) \mid \langle m \rangle^* x = m^r x \forall m \in \mathbb{Z} \} = \text{Hom}(\mathbb{L}^n, h^r(A)).$$

**Remark 1.** In Beauville’s notation, we have

$$CH^n(A)_{[r]} = CH^n_{2m-r}(A).$$

We shall use his notation in §3.

2. Skew cycles on abelian varieties

Let $\beta \in CH^*_Q(A)$. Assume that $(-1)^* \beta = -\beta$: we say that $\beta$ is skew. This implies that $\beta$ is homologically equivalent to 0.

For $g \leq 2$, the Griffiths group of $A$ is 0 and there is nothing to prove. For $g = 3$, the Griffiths group of $A$ is a quotient of $CH^2(A)[3]$ [1, Prop. 6]; thus Theorem 1 follows from the more general
Proposition 1. Any skew cycle on an abelian variety is smash-nilpotent.

This applies notably to the Ceresa cycle [3], for any genus.

Proof. We may assume $\beta$ homogeneous, say, $\beta \in CH^n(Q)(A)$. View $\beta$ as a morphism $L^n \to h(A)$ in the category of Chow motives. Thus, for all $i$:

$$-\pi_i \beta = \pi_i (-1)^i \beta = (-1)^i \pi_i \beta$$

hence $\pi_i \beta = 0$ for $i$ even.

This shows that $\beta$ factors through a morphism

$$\tilde{\beta} : L^n \to h^{\text{odd}}(A)$$

with $h^{\text{odd}}(A) = \bigoplus_{i \text{ odd}} h^i(A)$.

But $L^n$ is evenly finite-dimensional and $h^{\text{odd}}(A)$ is oddly finite-dimensional in the sense of S.-I. Kimura. (Indeed, $S^{2g+1}(h^1(A)) = h^{2g+1}(A) = 0$ by [9, Theorem], and a direct summand of an odd tensor power of an oddly finite-dimensional motive is oddly finite dimensional by [6, Prop. 5.10 p. 186].) Hence the conclusion follows from [6, prop. 6.1 p. 188].

Remark 2. Kimura’s proposition 6.1 shows in fact that all $z \in \text{Hom}(L^n, h^{\text{odd}}(A))$ verify $z^{\otimes N + 1} = 0$ for a fixed $N$, namely, the sum of the odd Betti numbers of $A$. If $z \in \text{Hom}(L^n, h^i(A))$ for some odd $i$, then we may take for $N$ the $i$-th Betti number of $A$. Thus, for $i = 3$ and if $A$ is a 3-fold, we find that all $z \in \text{Hom}(L, h^3(A))$ verify $z^{\otimes 21} = 0$.

3. The 4-Dimensional Case

Proposition 2. If $g = 4$, homologically trivial cycles on $A$, except perhaps those which occur in parts $CH^2_0(A)$ or $CH^3_2(A)$ of the Beauville decomposition, are smash-nilpotent.

Proof. Let $A$ be an abelian variety and let $\hat{A}$ denote its dual abelian variety. We know, from [1], the following:

1. $CH^p(A) = 0$ for $p \in \{0, 1, g - 2, g - 1, g\}$ and $s < 0$. [1, Prop. 3a].

2. $CH^p(A)$ and $CH^g(A)$ consist of cycles algebraically equivalent to 0 for all values of $p$ and all values of $s > 0$. [1, Prop. 4].

For $g = 4$, using these results and Proposition 1 one can conclude smash nilpotence for homologically trivial cycles which are not in $CH^0_0(A)$ or $CH^2_2(A)$. Note that, with the notation of §1,

$$CH^3_2(A) = \text{Hom}(\mathbb{L}^3, h^4(A)), \quad CH^2_0(A) = \text{Hom}(\mathbb{L}^2, h^4(A)).$$
In the case of $CH^2(A)$, the problem is whether there are any homologically trivial cycles: in view of the above expression, this is conjecturally not the case, cf. [5, Prop. 5.8]. □

Remark 3. Some of these arguments also follow from a paper of Bloch and Srinivas [2].

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