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# Branchwidth of graphic matroids.

Frédéric Mazoit\* and Stéphan Thomassé†

## Abstract

Answering a question of Geelen, Gerards, Robertson and Whittle [2], we prove that the branchwidth of a bridgeless graph is equal to the branchwidth of its cycle matroid. Our proof is based on branch-decompositions of hypergraphs.

## 1 Introduction.

Let  $H = (V, E)$  be a hypergraph and  $(E_1, E_2)$  be a partition of  $E$ . The *border* of  $(E_1, E_2)$  is the set of vertices  $\delta(E_1, E_2)$  which belong to both an edge of  $E_1$  and an edge of  $E_2$ . We often write it  $\delta(E_1, E_2)$ , or often simply  $\delta(E_1)$ . A *component* of  $E$  is a minimum nonempty subset  $C \subseteq E$  such that  $\delta(C) = \emptyset$ . Let  $F$  be a subset of  $E$ . We denote by  $c(F)$  the number of components of the subhypergraph of  $H$  spanned by  $F$ , i.e. the hypergraph  $(V(F), F)$ . A hypergraph  $H$  is *2-edge connected* if every vertex belongs to at least two edges and  $c(E \setminus e) = 1$  for every  $e \in E$ .

A *branch-decomposition*  $\mathcal{T}$  of  $H$  is a ternary tree  $\mathcal{T}$  and a bijection from the set of leaves of  $\mathcal{T}$  into the set of edges of  $H$ . Practically, we simply identify the leaves of  $\mathcal{T}$  to the edges of  $H$ . Observe that every edge  $e$  of  $\mathcal{T}$  partitions  $\mathcal{T} \setminus e$  into two subtrees, and thus correspond to a bipartition of  $E$ , called *e-separation*. More generally, a  $\mathcal{T}$ -separation is an *e-separation* for some edge  $e$  of  $\mathcal{T}$ . We will often identify the edge  $e$  of  $\mathcal{T}$  with the *e-separation*, allowing us to write, for instance,  $\delta(e)$  instead of  $\delta(E_1, E_2)$ , where  $(E_1, E_2)$  is the *e-separation*.

Let  $f$  be a real function defined on the set of bipartitions of  $E$ . For sake of simplicity we often write  $f(E_1)$  instead of  $f(E_1, E \setminus E_1)$ . Let  $\mathcal{T}$  be a branch-decomposition of  $H$ . The *f-width* of  $\mathcal{T}$ , denoted by  $w_f(\mathcal{T})$ , is the maximum value of  $f(e)$ , for all edges  $e$  of  $\mathcal{T}$ . The *f-branchwidth* of  $H$ , denoted by  $\text{bw}_f(H)$ , is the minimum *f-width* of a branch-decomposition of  $H$ . A branch-decomposition achieving  $\text{bw}_f(H)$  is *f-optimal*.

The  $|\delta|$ -branchwidth (i.e. when  $f(E_1, E_2) = |\delta(E_1, E_2)|$ ) of a graph  $G$  is the usual branchwidth introduced by Robertson and Seymour in [5]. In this paper, we study the branchwidth associated to the function  $\rho(E_1, E_2) = |\delta(E_1, E_2)| +$

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$2 - c(E_1) - c(E_2)$ . Our goal is to prove that in the class of 2-edge connected hypergraphs, the  $|\delta|$ -branchwidth is equal to the  $\rho$ -branchwidth. The proof simply consists to show that every 2-edge connected hypergraph admits a  $\rho$ -optimal decomposition such that  $c(E_1) = c(E_2) = 1$  for every  $\mathcal{T}$ -separation  $(E_1, E_2)$ .

Our motivation comes from the following: Let  $M$  be a matroid on base set  $E$  with rank function  $r$ . The *weight* of every non-trivial partition  $(E_1, E_2)$  of  $E$  is  $w(E_1, E_2) := r(E_1) + r(E_2) - r(E) + 1$ . When  $\mathcal{T}$  is a branch-decomposition of  $M$ , i.e. a ternary tree whose leaves are labelled by  $E$ , the *width* of  $\mathcal{T}$  is the maximum weight of a  $\mathcal{T}$ -separation. Again, the *branchwidth* of  $M$  is the minimum width of a branch-decomposition of  $M$ . Let  $M$  be the cycle matroid of a 2-edge connected graph  $G$ , i.e. the matroid which base set is the set of edges of a graph and which independent sets are the acyclic subsets of edges. Branch-decompositions of  $G$  are exactly branch-decompositions of  $M$ . Moreover,  $r(E_1) + r(E_2) - r(E) + 1$  is exactly  $n_1 - c(E_1) + n_2 - c(E_2) - n + c(E) + 1$  where  $n_1, n_2, n$  are the number of vertices respectively spanned by  $E_1, E_2, E$ . Thus we have  $w(E_1, E_2) = \delta(E_1) + 2 - c(E_1) - c(E_2) = \rho(E_1, E_2)$ . In particular, the branchwidth of  $M$  is exactly the  $\rho$ -branchwidth, and thus is equal to the  $|\delta|$ -branchwidth.

Unless stated otherwise, we always assume that  $H$  is a 2-edge connected hypergraph and  $\mathcal{T}$  is a branch-decomposition of  $H$ . Also, when speaking about width, branchwidth, etc, we implicitly mean  $\rho$ -width,  $\rho$ -branchwidth, etc.

## 2 Faithful branch-decompositions.

Let  $(E_1, E_2)$  be a  $\mathcal{T}$ -separation. The decomposition  $\mathcal{T}$  is *faithful* to  $E_1$  if for every component  $C$  of  $E_1$ , the partition  $(C, E \setminus C)$  is a  $\mathcal{T}$ -separation. The *border graph*  $G_{\mathcal{T}}$  has vertex set  $V$  and contains all edges  $xy$  for which there exists an edge  $e$  of  $\mathcal{T}$  such that  $\{x, y\} \subseteq \delta(e)$ . A branch-decomposition  $\mathcal{T}'$  is *tighter* than  $\mathcal{T}$  if  $w_{\rho}(\mathcal{T}') < w_{\rho}(\mathcal{T})$  or if  $w_{\rho}(\mathcal{T}) = w_{\rho}(\mathcal{T}')$  and  $G_{\mathcal{T}'}$  is a subgraph of  $G_{\mathcal{T}}$ . Moreover,  $\mathcal{T}'$  is *strictly tighter* than  $\mathcal{T}$  if  $\mathcal{T}'$  is tighter than  $\mathcal{T}$ , and  $\mathcal{T}$  is not tighter than  $\mathcal{T}'$ . Finally,  $\mathcal{T}$  is *tight* if no  $\mathcal{T}'$  is strictly tighter than  $\mathcal{T}$ .

**Lemma 1** *Let  $(E_1, E_2)$  be a partition of  $E$ . For any union  $E'_1$  of connected components of  $E_1$  and  $E_2$ , we have both  $\delta(E'_1) \subseteq \delta(E_1)$  and  $\rho(E'_1) \leq \rho(E_1)$ .*

□ Clearly,  $\delta(E'_1) \subseteq \delta(E_1)$ . Moreover, every vertex of  $\delta(E_1)$  belongs to one component of  $E_1$  and one component of  $E_2$ . Therefore, if  $C$  is a component of  $E'_1$  which is the union of  $k$  components of  $E_1$  and  $E_2$ , there are at least  $k - 1$  vertices of  $C \setminus \delta(C)$  which belong to  $\delta(E_1)$ . In all, the weight of the separation increased by  $k - 1$  since we merge  $k$  components into one, but it also decreased by at least  $k - 1$  since we lose at least that many vertices on the border. Since this is the case for every component of  $E'_1$  or of  $E \setminus E'_1$ , we have  $\rho(E'_1) \leq \rho(E_1)$ . ■

**Lemma 2** *Let  $(E_1, E_2)$  be an  $e$ -separation of  $\mathcal{T}$ . Let  $\mathcal{T}_1$  be the subtree of  $\mathcal{T} \setminus e$  with set of leaves  $E_1$ . If  $\mathcal{T}$  is not faithful to  $E_1$ , one can modify  $\mathcal{T}_1$  in  $\mathcal{T}$  to form a tighter branch-decomposition  $\mathcal{T}'$  of  $H$ .*

□ Fix the vertex  $e \cap \mathcal{T}_1$  as a root of  $\mathcal{T}_1$ . Our goal is to change the binary rooted tree  $\mathcal{T}_1$  into another binary rooted tree  $\mathcal{T}'_1$ . For every connected component  $C$  of  $E_1$ , consider the subtree  $\mathcal{T}_C$  of  $\mathcal{T}_1$  which contains the root of  $\mathcal{T}_1$  and has set of leaves  $C$ . Observe that  $\mathcal{T}_C$  is not necessarily binary since  $\mathcal{T}_C$  may contain paths having internal vertices with only one descendant. We simply replace these paths by edges to obtain our rooted tree  $\mathcal{T}'_C$ . Now, consider any rooted binary tree  $BT$  with  $c(E_1)$  leaves and identify these leaves to the roots of  $\mathcal{T}'_C$ , for all components  $C$  of  $E_1$ . This rooted binary tree is our  $\mathcal{T}'_1$ . We denote by  $\mathcal{T}'$  the branch-decomposition we obtain from  $\mathcal{T}$  by replacing  $\mathcal{T}_1$  by  $\mathcal{T}'_1$ . Roughly speaking, we merged all subtrees of  $\mathcal{T}_1$  induced by the components of  $E_1$  together with  $\mathcal{T} \setminus \mathcal{T}_1$  to form  $\mathcal{T}'$ . Let us prove that  $\mathcal{T}'$  is tighter than  $\mathcal{T}$ . For this, consider an edge  $f'$  of  $\mathcal{T}'$ . If  $f' \notin \mathcal{T}'_1$ , the  $f'$ -separations of  $\mathcal{T}$  and  $\mathcal{T}'$  are the same. If  $f' \in BT$ , by Lemma 1, we have  $\rho(f') \leq \rho(e)$  and  $\delta(f') \subseteq \delta(e)$ . So the only case we have to care of is when  $f'$  is an edge of some tree  $\mathcal{T}'_C$ , where  $C$  is a component of  $E_1$ . Recall that  $f'$  corresponds to a path  $P$  of  $\mathcal{T}_C$ . Let  $f$  be any edge of  $P$ . Let  $(F, E \setminus F)$  be the  $f$ -separation of  $\mathcal{T}$ , where  $F \subseteq E_1$ . Therefore, the  $f'$ -separation of  $\mathcal{T}'$  is  $(F \cap C, E \setminus (F \cap C))$ . Since  $F$  is a subset of  $E_1$ , the connected components of  $F$  are subsets of the connected components of  $E_1$ . Thus  $F \cap C$  is a union of connected components of  $F$ . This implies that  $\delta(f') \subseteq \delta(f)$ . Also, by Lemma 1,  $\rho(f') \leq \rho(f)$ .

We have proved that  $w(\mathcal{T}') \leq w(\mathcal{T})$  and that  $G_{\mathcal{T}'}$  is a subgraph of  $G_{\mathcal{T}}$ , thus  $\mathcal{T}'$  is tighter than  $\mathcal{T}$ . ■

### 3 Connected branch-decompositions.

Let  $F \subseteq E$  be a component. The hypergraph  $H * F$  on vertex set  $V$  and edge set  $(E \setminus F) \cup \{V(F)\}$  is denoted by  $H * F$ . In other words,  $H * F$  is obtained by merging the edges of  $F$  into one edge. A partition  $(E_1, E_2)$  of  $E$  is *connected* if  $c(E_1) = c(E_2) = 1$ . A branch-decomposition  $\mathcal{T}$  is *connected* if all its  $\mathcal{T}$ -separations are connected.

**Lemma 3** *If  $\mathcal{T}$  is tight, every  $\mathcal{T}$ -separation  $(E_1, E_2)$  is such that  $E_1$  or  $E_2$  is connected.*

□ Suppose for contradiction that there exists a  $\mathcal{T}$ -separation  $(E_1, E_2)$  such that neither  $E_1$  nor  $E_2$  is connected. By Lemma 2, we can assume that  $\mathcal{T}$  is faithful to  $E_1$  and to  $E_2$ . Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be respectively the sets of components of  $E_1$  and  $E_2$ . Consider the graph on set of vertices  $\mathcal{C}_1 \cup \mathcal{C}_2$  where  $C_1 C_2$  is an edge whenever  $C_1 \in \mathcal{C}_1$  and  $C_2 \in \mathcal{C}_2$  have nonempty intersection. This graph is connected and is not a star. Thus, it has a vertex-partition into two connected subgraphs, each having at least two vertices. This vertex-partition corresponds to a partition  $(\mathcal{C}'_1, \mathcal{C}'_2)$  of  $\mathcal{C}_1 \cup \mathcal{C}_2$ .

Consider any rooted binary tree  $BT$  with  $|\mathcal{C}'_1|$  leaves. Since every  $C \in \mathcal{C}'_1$  is an element of  $\mathcal{C}_1 \cup \mathcal{C}_2$ ,  $(C, E \setminus C)$  is an  $e$ -separation of  $\mathcal{T}$ . We denote by  $\mathcal{T}_C$  the tree of  $\mathcal{T} \setminus e$  with set of leaves  $C$ . Root  $\mathcal{T}_C$  with the vertex  $e \cap \mathcal{T}_C$  in order to get a binary rooted tree. Now identify the leaves of  $BT$  with the roots of  $\mathcal{T}_C$ , for  $C \in \mathcal{C}'_1$ . This rooted tree is our  $\mathcal{T}'_1$ . We construct similarly  $\mathcal{T}'_2$ . Adding an edge between the roots of  $\mathcal{T}'_1$  and  $\mathcal{T}'_2$  gives the branch-decomposition  $\mathcal{T}'$  of  $H$ . By Lemma 1,  $w(\mathcal{T}') \leq w(\mathcal{T})$ . Moreover,  $G_{\mathcal{T}'}$  is a subgraph of  $G_{\mathcal{T}}$ . Let us now show that  $G_{\mathcal{T}'}$  is a strict subgraph of  $G_{\mathcal{T}}$ . Indeed, since  $\mathcal{C}'_1$  is connected and has at least two elements, it contains  $C_1 \in \mathcal{C}_1$  and  $C_2 \in \mathcal{C}_2$  such that  $C_1 \cap C_2$  is nonempty. By construction, every vertex  $x$  of  $C_1 \cap C_2$  is such that  $x \notin \delta(\mathcal{C}'_1)$  and  $x \in \delta(\mathcal{C}_1)$ . Similarly, there is a vertex  $y$  spanned by  $\mathcal{C}'_2$  such that  $y \notin \delta(\mathcal{C}'_2)$  and  $y \in \delta(\mathcal{C}_2)$ . Thus  $xy$  is an edge of  $G_{\mathcal{T}}$  but not of  $G_{\mathcal{T}'}$ , contradicting the fact that  $\mathcal{T}$  is tight. ■

**Theorem 1** *For every branch-decomposition  $\mathcal{T}$  of a hypergraph  $H$ , there exists a tighter branch-decomposition  $\mathcal{T}'$  such that for every  $\mathcal{T}'$ -separation  $(E_1, E_2)$  with  $c(E_1) > 1$ ,  $E_1$  consists of components of  $H \setminus e$ , for some  $e \in E$ . In particular, if  $H$  is 2-edge connected, it has an optimal connected branch-decomposition.*

□ Let us prove the theorem by induction on  $|V| + |E|$ . The statement is obvious if  $|E| \leq 3$ , so we assume now that  $H$  has at least four edges. Call *achieved* a branch-decomposition satisfying the conclusion of Theorem 1. If  $\mathcal{T}$  is not tight, we can replace it by a tight branch-decomposition tighter than  $\mathcal{T}$ . So we may assume that  $\mathcal{T}$  is tight.

If  $H$  is not connected, apply induction on every components of  $H$  in order to find an achieved branch-decomposition. Then merge these branch-decompositions into one branch-decomposition of  $H$ .

If there is an edge  $e \in E$  such that  $H \setminus e$  is not connected, we can assume by Lemma 2 that  $\mathcal{T}$  is faithful to  $E \setminus e$ . Let  $E_1$  be a connected component of  $E \setminus e$ . Let  $\mathcal{T}_1$  be the branch-decomposition induced by  $\mathcal{T}$  on  $E_1 \cup e$ . Let also  $\mathcal{T}_2$  be the branch-decomposition induced by  $\mathcal{T}$  on  $(E \setminus E_1) \cup e$ . By the induction hypothesis, there exists two achieved branch-decompositions  $\mathcal{T}'_1$  and  $\mathcal{T}'_2$ , respectively tighter than  $\mathcal{T}_1$  and  $\mathcal{T}_2$ . Identify the leaf  $e$  of the trees  $\mathcal{T}'_1$  and  $\mathcal{T}'_2$ , and attach a leaf labelled by  $e$  to the identified vertex. Call  $\mathcal{T}'$  this branch-decomposition of  $H$ . Observe that it is tighter than  $\mathcal{T}$  and achieved.

So we assume now that  $H$  is 2-edge connected. The key-observation is that if there is a connected  $\mathcal{T}$ -separation  $(E_1, E_2)$  with  $|E_1| \geq 2$  and  $|E_2| \geq 2$ , we can apply the induction hypothesis on  $H * E_1$  and  $H * E_2$  and merge the two branch-decompositions to obtain an optimal connected branch-decomposition of  $H$ . Therefore, we assume that every  $\mathcal{T}$ -separation  $(E_1, E_2)$  with  $|E_1| \geq 2$  and  $|E_2| \geq 2$  is such that  $E_1$  or  $E_2$  is connected.

We now orient the edges of  $\mathcal{T}$ . If  $(E_1, E_2)$  is an  $e$ -separation such that  $E_2$  is connected but not an edge of  $H$ , we orient  $e$  from  $E_1$  to  $E_2$ . Since  $H$  is 2-edge-connected, every edge of  $\mathcal{T}$  incident to a leaf is oriented from the leaf. By Lemma 3, every edge get at least one orientation. And by the key-observation, every edge of  $\mathcal{T}$  has exactly one orientation.

This orientation of  $\mathcal{T}$  has no circuit, thus there is a vertex  $t \in T$  with outdegree zero. Since every leaf has outdegree one,  $t$  has indegree three. Let us denote by  $A, B, C$  the set of leaves of the three trees of  $\mathcal{T} \setminus t$ . Observe that by construction,  $A \cup B$ ,  $A \cup C$  and  $B \cup C$  are connected. By Lemma 2, we can assume moreover that  $\mathcal{T}$  is faithful to  $A, B$  and  $C$ . We claim that  $A$  is a disjoint union of edges, i.e. the connected components of  $A$  are edges of  $H$ . To see this, pick any component  $C_A$  of  $A$ . Since  $\mathcal{T}$  is faithful to  $A$ ,  $(C_A, E \setminus C_A)$  is a  $\mathcal{T}$ -separation. But this is simply impossible since  $B \cup C$  being connected,  $E \setminus C_A$  is also connected, against the fact that every edge has a single orientation. So the hypergraph  $H$  consists of three sets of disjoint edges  $A, B, C$ . Call this partition the *canonical partition* of  $\mathcal{T}$ . Call  $(A, E \setminus A)$ ,  $(B, E \setminus B)$  and  $(C, E \setminus C)$  the *main*  $\mathcal{T}$ -separations. The width of every other  $\mathcal{T}$ -separation is strictly less than  $\text{bw}(H)$ . Since every vertex of  $H$  belongs to two or three edges, it is spanned by at least two of the sets  $\delta(A), \delta(B), \delta(C)$ . In particular  $G_{\mathcal{T}}$  is a complete graph, and thus every optimal branch-decomposition of  $H$  is tighter than  $\mathcal{T}$ . Set  $\delta_{AB} := |\delta(A) \cap \delta(B)|$ ,  $\delta_{AC} := |\delta(A) \cap \delta(C)|$ ,  $\delta_{BC} := |\delta(B) \cap \delta(C)|$  and  $\delta_{ABC} := |\delta(A) \cap \delta(B) \cap \delta(C)|$ . We now prove some properties of  $H$ .

1. Two of the sets  $A, B, C$  have at least two edges. Indeed, assume for contradiction that  $A = \{a\}$  and  $B = \{b\}$ . Since  $|E| \geq 4$ , there are at least two edges in  $C$ . Let  $c \in C$ . Assume without loss of generality that  $|a \cap c| \geq |b \cap c|$ . Now form a new branch-decomposition  $\mathcal{T}'$  by *moving*  $c$  to  $A$ , i.e.  $\mathcal{T}'$  has a separation  $(A \cup c, B \cup (C \setminus c))$  and then four branches  $A, c, B, (C \setminus c)$ . We have

$$\rho(A \cup c, B \cup (C \setminus c)) = \rho(A) + |b \cap c| - |a \cap c|$$

since both parts are connected. In particular  $\mathcal{T}'$  is tighter than  $\mathcal{T}$ , and since the  $\mathcal{T}'$ -separation  $(A \cup c, B \cup (C \setminus c))$  is connected and both of its branches have at least two vertices, we can apply induction to conclude.

2. Every set  $A, B, C$  have at least two edges. Indeed, assume for contradiction that  $A$  consists of a single edge  $a$ . Let  $b$  be an edge of  $B$ . If  $|b \cap \delta(C)| \leq |b \cap a|$ , we can as previously move  $b$  to  $A$  in order to conclude. Call  $|b \cap \delta(C)| - |b \cap a|$  the *excess* of  $b$ . Similarly, call  $|c \cap \delta(B)| - |c \cap a|$  the *excess* of an edge  $c \in C$ . Let  $s$  be the minimum excess of an edge  $e_s$  of  $B \cup C$ . Observe that  $s \geq 1$  and that every  $b \in B$  satisfies  $|b \cap \delta(C)| \geq |b \cap \delta(A)| + s$ . Thus, summing for all edges of  $B$ , we obtain  $\delta_{BC} \geq \delta_{AB} + s|B|$ . Similarly,  $\delta_{BC} \geq \delta_{AC} + s|C|$ . Note also that  $\text{bw}(H) \geq \rho(C) = \delta_{BC} + \delta_{AC} - \delta_{ABC} - |C| + 1$  and  $\text{bw}(H) \geq \rho(B) = \delta_{BC} + \delta_{AB} - \delta_{ABC} - |B| + 1$ . In all

$$2 \text{bw}(H) \geq 2\delta_{BC} - 2\delta_{ABC} + \delta_{AC} - |C| + \delta_{AB} - |B| + 2.$$

Then  $2 \text{bw}(H) \geq \delta_{AB} + s|B| + \delta_{AC} + s|C| - 2\delta_{ABC} + \delta_{AC} - |C| + \delta_{AB} - |B| + 2$ . Finally,  $\text{bw}(H) \geq \delta_{AC} + \delta_{AB} - \delta_{ABC} + 1 + ((s-1)|C| + (s-1)|B|)/2$ . Since  $\rho(A) = \delta_{AC} + \delta_{AB} - \delta_{ABC}$ , we have  $\text{bw}(H) \geq \rho(A) + s$ . But then we can move  $e_s$  to  $A$  to conclude.

3. We have  $\text{bw}(H) = \rho(A)$ . If not, pick two edges  $a, a'$  of  $A$  and merge them together. The hypergraph we obtain is still 2-edge-connected, and the branch-decomposition still has the same width. Apply induction to get an achieved branch-decomposition. Then replace the merged edge by the two original edges. This branch-decomposition  $\mathcal{T}'$  is optimal, hence tighter than  $\mathcal{T}$ . So either we can apply induction on  $\mathcal{T}'$ , or  $\mathcal{T}'$  has a canonical partition. But in this last case, the canonical partition of  $\mathcal{T}'$  is exactly  $\{a\}, \{a'\}, E \setminus \{a, a'\}$ . Thus by 1, we can apply induction. Similarly,  $\text{bw}(H) = \rho(B) = \rho(C)$ .
4. We have  $\text{bw}(H) \geq \beta + 1$ , where  $\beta$  is the size of a maximum edge  $e$  of  $H$ . Since edges of  $H$  with only one vertex play no role here, we can ignore them. So the size of an edge of  $H$  is at least two. Assume for instance that  $e \in A$ . Since  $A$  has at least two components, we have  $\text{bw}(H) = \rho(A) \geq |\delta(A)| - c(B \cup C) - c(A) + 2 \geq 2 + \beta - 1$ .
5. We have  $\delta_{ABC} = 0$ . Indeed, suppose for contradiction that there exists a vertex  $z$  in  $\delta(A) \cap \delta(B) \cap \delta(C)$ . Consider the hypergraph  $H_z$  obtained from  $H$  by removing the vertex  $z$  from all its edges. The branch-decomposition  $\mathcal{T}$  induces a branch-decomposition  $\mathcal{T}_z$  of  $H_z$  having width at most  $\text{bw}(\mathcal{T}) - 1$ . Observe also that  $H_z$  is connected since  $z$  is incident to three edges and  $H$  is 2-edge connected. We apply induction on  $\mathcal{T}_z$  to obtain an achieved branch-decomposition  $\mathcal{T}'_z$  of  $H_z$ . Now add back the vertex  $z$  to the edges of  $H_z$  and call  $\mathcal{T}'$  the branch-decomposition obtained from  $\mathcal{T}'_z$ . Observe that if a  $\mathcal{T}'_z$ -separation  $(E_1, E_2)$  is connected, adding  $z$  will raise by at most one its width in  $\mathcal{T}'$ . Moreover if a  $\mathcal{T}'_z$ -separation  $(E_1, E_2)$  is not connected, say  $c(E_2) > 1$ , adding  $z$  can raise by at most two its width in  $\mathcal{T}'$  (either by merging three components of  $E_2$  into one, or by merging two and increasing the border by one). Since  $\mathcal{T}'_z$  is achieved,  $E_2$  is a set of components of  $E \setminus e$  for some edge  $e$  of  $E$ . But then,  $\rho_{\mathcal{T}'_z}(E_1, E_2) \leq |e| - 3 + 2 \leq \beta - 1 \leq \text{bw}(H) - 2$ , and thus  $\rho_{\mathcal{T}'}(E_1, E_2) \leq \text{bw}(H)$ . Finally  $\mathcal{T}'$  is optimal. Moreover every  $\mathcal{T}'$ -separation  $(E_1, E_2)$  is connected. Indeed, if  $E_1$  is connected in  $\mathcal{T}'_z$ , we are done. If  $E_1$  is not connected in  $\mathcal{T}'_z$ ,  $E_1$  consists of components of  $H_z \setminus e$ , for some edge  $e$  of  $H_z$ . But since  $H$  is 2-edge connected, every component of  $E_1$  in  $H$  contains  $z$ , otherwise they would be components of  $H \setminus e$ . Consequently  $E_1$  is connected.
6. Every edge of  $H$  is incident to at least four other edges. Indeed, assume for contradiction that an edge  $a$  of  $A$  is incident to only one edge  $b$  of  $B$  and at most two edges of  $C$ . Moving  $a$  to  $B$  increases  $\rho(B)$  by  $|a \cap \delta(C)| - |a \cap b|$  and does not increase  $\rho(A)$  and  $\rho(C)$ . Therefore, if  $|a \cap \delta(C)| \leq |a \cap b|$ , we can move  $a$  to  $B$ , and this new branch-decomposition  $\mathcal{T}'$  is strictly tighter than  $\mathcal{T}$  since the vertices of  $a \cap b$  are no more joined to  $(\delta(A) \setminus a) \cap \delta(C)$  in the graph  $G_{\mathcal{T}'}$ . Thus  $|a \cap \delta(C)| \geq |a \cap b| + 1$ . Moreover, moving  $a$  to  $C$ , increases  $\rho(C)$  by at most  $|a \cap b| - |a \cap \delta(C)| + 1$ , since at most two components of  $C$  can merge. So  $|a \cap b| + 1 > |a \cap \delta(C)|$ , a contradiction.

This implies in particular that  $\rho(e) \leq \text{bw}(H) - 3$  whenever  $e$  is not one of the main  $\mathcal{T}$ -separations. In particular  $\beta \leq \text{bw}(H) - 3$ .

7. The hypergraph  $H$  is triangle-free. Indeed, suppose that there exists three edges  $a \in A$ ,  $b \in B$  and  $c \in C$  and three vertices  $x \in a \cap b$ ,  $y \in b \cap c$  and  $z \in c \cap a$ . Let  $H_{xyz}$  be the hypergraph obtained by removing  $x, y, z$  in the vertex set of  $H$  and in every edge of  $H$ . The branchwidth of  $H_{xyz}$  is at most  $\text{bw}(H) - 2$ , since we removed two vertices in the border of every main separation. As in 5,  $\mathcal{T}$  induces a branch-decomposition  $\mathcal{J}_{xyz}$  of  $H_{xyz}$  which can be improved by induction to an achieved branch-decomposition  $\mathcal{J}'_{xyz}$  of  $H'_{xyz}$ . Adding back the vertices  $x, y, z$ , we obtain a branch-decomposition  $\mathcal{J}'$  of  $H$ . We claim that  $w(\mathcal{J}') \leq \text{bw}(H)$ . Let  $(E_1, E_2)$  be a  $\mathcal{J}'$ -separation, without loss of generality, we assume that  $E_1$  contains at least two edges of  $a, b, c$ , say  $a$  and  $b$ . Assume first that  $(E_1, E_2)$  is connected in  $\mathcal{J}'_{xyz}$ . If  $c \in E_1$ , we have  $\rho_{\mathcal{J}'_{xyz}}(E_1) = \rho_{\mathcal{J}'}(E_1)$ . If  $c \notin E_1$ ,  $\rho_{\mathcal{J}'_{xyz}}(E_1) + 2 = \rho_{\mathcal{J}'}(E_1)$  since we add the endvertices of  $c$  to the border of  $E_1$ . Now, if  $(E_1, E_2)$  is not connected in  $\mathcal{J}'_{xyz}$ , then  $\rho_{\mathcal{J}'_{xyz}}(E_1)$  is at most the size of an edge of  $H_{xyz}$ , hence at most  $\beta$ . Since  $x, y, z$  have degree two in  $H$ , each can either increase the border of a separation by one, or merge two components. In all,  $\rho_{\mathcal{J}'_{xyz}}(E_1)$  increases by at most three. Since  $\beta \leq \text{bw}(H) - 3$ , we have that  $w(\mathcal{J}') \leq \text{bw}(H)$ . To conclude, we prove that  $\mathcal{J}'$  is connected. Indeed, if  $c(E_2) > 1$  for some  $\mathcal{J}'$ -separation  $(E_1, E_2)$ ,  $E_2$  consists of components of  $H_{xyz} \setminus e$  for some edge  $e$  of  $H_{xyz}$ . Since  $H$  is 2-edge connected, these components are not components of  $H \setminus e$ , so each of them must contain one of the edges  $a, b$  or  $c$ , and therefore  $E_2$  is connected in  $\mathcal{J}'$ .

Now we are ready to finish the proof. Note that  $\text{bw}(H) = (\rho(A) + \rho(B) + \rho(C))/3 = (2|V| - |E|)/3 + 1$ . Consider the line multigraph  $L(H)$  of  $H$ , i.e. the multigraph on vertex set  $A \cup B \cup C$  and edge set  $V$  such that  $v \in V$  is the edge which joins the two edges  $e, f$  of  $H$  such that  $v \in e$  and  $v \in f$ . The multigraph  $L(H)$  satisfies the hypothesis of Lemma 4 (proved in the next section), thus it admits a vertex-partition of its vertices as in the conclusion of Lemma 4. This corresponds to a partition of  $A \cup B \cup C$  into two subsets  $E_1 := A_1 \cup B_1 \cup C_1$  and  $E_2 := A_2 \cup B_2 \cup C_2$  such that  $|\delta(E_1, E_2)| \leq (2|V| - |E|)/3 + 1$  and both  $E_1$  and  $E_2$  have at least  $\lfloor |E|/2 \rfloor - 1$  internal vertices. In particular, the separation  $(E_1, E_2)$  has width at most  $\text{bw}(H)$ . Let us show that one of  $\rho(A_1 \cup B_1)$ ,  $\rho(B_1 \cup C_1)$ , and  $\rho(C_1 \cup A_1)$  is also at most  $\text{bw}(H)$ . For this, observe that

$$\delta(A_1 \cup B_1) + \delta(B_1 \cup C_1) + \delta(C_1 \cup A_1) \leq 2(|V| - (|E_2| - |\delta(E_2)|)).$$

Thus  $\delta(A_1 \cup B_1) + \delta(B_1 \cup C_1) + \delta(C_1 \cup A_1) \leq 2|V| - 2\lfloor |E|/2 \rfloor + 2 \leq 2|V| - |E| + 3$ . Without loss of generality, we can assume that  $\delta(A_1 \cup B_1) \leq (2|V| - |E|)/3 + 1 = \text{bw}(H)$ , and thus we split  $E_1$  into two branches  $A_1 \cup B_1$  and  $C_1$ . We similarly split  $E_2$  to obtain an optimal branch-decomposition  $\mathcal{J}'$  of  $H$ . Observe that since both  $E_1 \setminus \delta(E_1)$  and  $E_2 \setminus \delta(E_2)$  are not empty, the graph  $G_{\mathcal{J}'}$  is not complete, against the fact that  $\mathcal{T}$  is tight, a contradiction. ■



## 4 The technical Lemma.

Let  $G$  be a multigraph and  $X, Y$  two subsets of its vertices. We denote by  $e(X, Y)$  the number of edges of  $G$  between  $X$  and  $Y$ . We also denote by  $e(X)$  the number of edges in  $X$ .

**Lemma 4** *Let  $G$  be a 2-connected triangle-free multigraph on  $n \geq 5$  vertices and  $m$  edges. Assume that its minimum underlying degree (forgetting the multiplicity of edges) is four. Assume moreover that its maximum degree is at most  $(2m - n)/3 + 1$ . There exists a partition  $(X, Y)$  of the vertex set of  $G$  such that  $e(X) \geq \lfloor n/2 \rfloor - 1$ ,  $e(Y) \geq \lfloor n/2 \rfloor - 1$  and  $e(X, Y) \leq (2m - n)/3 + 1$ .*

□ Call *good* a partition which satisfies the conclusion of Lemma 4. Assume first that there are vertices  $x, y$  such that  $e(x, y) \geq \lfloor n/2 \rfloor - 1$ . The minimum degree in  $V \setminus \{x, y\}$  is at least two, so  $e(V \setminus \{x, y\})$  is at least  $\lfloor n/2 \rfloor - 1$ . Thus, if the partition  $(V \setminus \{x, y\}, \{x, y\})$  is not good, we necessarily have  $d(x) + d(y) - 2e(x, y) > (2m - n)/3 + 1$ . By the maximum degree hypothesis, both  $d(x)$  and  $d(y)$  are greater than  $2e(x, y)$ . Since  $G$  is triangle-free, there exists a partition  $(X, Y)$  where  $(N(x) \cup x) \setminus y \subseteq X$  and  $(N(y) \cup y) \setminus x \subseteq Y$ . Observe that  $e(X) \geq d(x) - e(x, y) > e(x, y) \geq \lfloor n/2 \rfloor - 1$ . Similarly  $e(Y) \geq \lfloor n/2 \rfloor - 1$ . Moreover, since  $m \geq 2n$  by the minimum degree four hypothesis, we have

$$\begin{aligned} e(X, Y) &\leq m - (d(x) + d(y) - 2e(x, y)) \\ &< m - (2m - n)/3 - 1 \\ &\leq (m + n)/3 - 1 \\ &\leq (2m - n)/3 + 1. \end{aligned}$$

So  $(X, Y)$  is a good partition. We assume from now on that the multiplicity of an edge is less than  $\lfloor n/2 \rfloor - 1$ .

Let  $a + b = n$ , where  $a \leq b$ . A partition  $(X, Y)$  of  $V$  is an *a-partition* if  $|X| \leq a$ ,  $e(X) \geq a - 1$ ,  $e(Y) \geq b - 1$ ,  $e(X, Y) \leq (2m - n)/3 + 1$ , and the additional requirement that  $X$  contains a vertex of  $G$  with maximum degree.

Note that there exists a 1-partition, just consider for this  $X := \{x\}$ , where  $x$  has maximum degree in  $G$  (the minimum degree in  $Y$  is at least three, insuring that  $e(Y) \geq n - 2$ ). We consider now an *a-partition*  $(X, Y)$  with maximum  $a$ . If  $a \geq b - 1$ , this partition is good and we are done. So we assume that  $a < b - 1$ . In particular  $e(X) = a - 1$ . The *excess* of a vertex  $y \in Y$  is  $exc(y) := d_Y(y) - d_X(y)$ .

The key-observation is that there exists at most one vertex  $y \in Y$  such that  $e(Y \setminus y) < b - 2$ . Indeed, if there is a vertex of  $Y$  with degree one in  $Y$ , we simply move it to  $X$ , and we obtain an  $(a + 1)$ -partition ( $e(X)$  increases,  $e(Y)$  decreases by one, and  $e(X, Y)$  decreases). Thus the minimum degree in  $Y$  is at least two. Moreover, if there is a vertex of  $Y$  with degree two, we can still move it to  $X$  ( $e(X)$  increases,  $e(Y) \geq |Y| - 2$  and  $e(X, Y)$  does not increase). So the minimum degree in  $Y$  is at least three (but the underlying minimum degree may be one). This implies that  $e(Y) \geq 3|Y|/2$ . Let  $Y := \{y_1, \dots, y_{|Y|}\}$  where the

vertices are indexed in the increasing order according to their degree in  $Y$ . For every  $i \neq |Y|$ , we have  $e(Y) \geq (3(|Y| - 2) + d_Y(y_i) + d_Y(y_{|Y|}))/2$ . Furthermore,

$$\begin{aligned} e(Y \setminus y_i) &\geq (3(|Y| - 2) + d_Y(y_i) + d_Y(y_{|Y|}))/2 - d_Y(y_i) \\ &\geq 3(|Y| - 2)/2 \\ &\geq |Y| - 2 \\ &\geq b - 2. \end{aligned}$$

We now discuss the two different cases.

- Assume that  $e(Y \setminus y) \geq b - 2$  for every  $y \in Y$ . We denote by  $Y'$  the (nonempty) set of vertices of  $Y$  with at least one neighbour in  $X$ . Set  $Y'' := Y \setminus Y'$ . Denote by  $c$  the minimum excess of a vertex of  $Y'$ . Observe that every vertex of  $Y''$  has degree at least four in  $Y$ . Thus summing the degrees of the vertices of  $Y$  gives

$$2e(Y) \geq e(X, Y) + 4|Y''| + c|Y'| \quad (1)$$

Let  $y \in Y'$  such that  $exc(y) = c$ . Since the partition  $(X \cup y, Y \setminus y)$  is not an  $(a + 1)$ -partition, we have  $e(X, Y) + c > (2m - n)/3 + 1$ . Since  $m = e(X, Y) + e(X) + e(Y)$ , this implies

$$e(X, Y) + 3c > 2e(X) + 2e(Y) - n + 3 \quad (2)$$

Equations (1) and (2) give:

$$e(X, Y) + 3c > 2e(X) + e(X, Y) + 4|Y''| + c|Y'| - n + 3 \quad (3)$$

Since  $e(X) \geq n - |Y| - 1$ , we get  $3c > n - 2|Y| + 4|Y''| + c|Y'| + 1$ . So  $3c > n + 2|Y| + (c - 4)|Y'| + 1$ , and finally  $n + 2|Y| < (c - 4)(3 - |Y'|) + 11$ . If  $c = 4$ , we get  $n + 2|Y| \leq 10$ , impossible. If  $c = 3$ , we get  $n + 2|Y| - |Y'| \leq 7$ , impossible. If  $c = 2$ , we get  $n + 2|Y| - 2|Y'| \leq 4$ , impossible. If  $c = 1$ , we get  $n + 2|Y| - 3|Y'| \leq 1$ , which can only hold if  $|Y| = |Y'| = n - 1$ . Thus,  $X$  consists of a single vertex, completely joined to  $Y$ , against the fact that  $G$  is triangle-free. Finally  $c > 4$ , and consequently  $|Y'| < 3$ . Observe that  $|Y'| > 1$  since  $G$  is 2-connected. Thus  $|Y'| = 2$ . Let  $y_1, y_2$  be the vertices of  $Y'$ , indexed in such a way that  $e(y_1, X) + e(y_2, Y'') \geq e(y_2, X) + e(y_1, Y'')$ . Let  $X_1 := X \cup y_1$  and  $Y_1 := Y \setminus y_1$ . Since  $y_1 \in Y'$ , we have that  $e(X_1) \geq a$ . Moreover  $e(Y_1) \geq b - 2$ . Observe also that  $e(y_1, y_2) \leq e(Y \setminus \{y_1, y_2\})$ : this is obvious if  $e(y_1, y_2) = 0$ , and if there is an edge between  $y_1$  and  $y_2$ , since  $G$  is triangle-free with minimum degree four, the minimum degree in  $Y \setminus \{y_1, y_2\}$  is at least two. So

$$e(Y \setminus \{y_1, y_2\}) \geq |Y| - 2 \geq \lfloor n/2 \rfloor - 1 \geq e(y_1, y_2).$$

In all  $e(X_1, Y_1) \leq e(X_1) + e(Y_1)$ . And since  $n - 2 \leq e(X_1) + e(Y_1)$ , we have  $e(X_1, Y_1) \leq 2e(X_1) + 2e(Y_1) - n + 2$ , which implies  $e(X_1, Y_1) \leq (3m - n)/2 + 1$ . So the partition  $(X_1, Y_1)$  is good.

- Now assume that there exists a vertex  $y \in Y$  such that  $e(Y \setminus y) \leq b - 3$ . We denote by  $Y'$  the set of vertices of  $Y \setminus y$  with at least one neighbour in  $X$ . Set  $Y'' := Y \setminus (Y' \cup y)$ . Observe that since every vertex of  $Y''$  has underlying degree four in  $Y$ , we have  $e(Y \setminus y) \geq 3|Y''|/2$ . Thus,  $|Y''| \leq (2|Y| - 6)/3$ . Since  $|Y| \geq 3$ , we have  $|Y''| \leq |Y| - 3$ , and finally  $|Y'| \geq 3$ . Denote by  $c$  the minimum excess of a vertex of  $Y'$ . Summing the degrees of the vertices of  $Y$  gives  $2e(Y) \geq e(X, Y) + 4|Y''| + c|Y'| + exc(y)$ . Equation (2) still holds, so

$$exc(y) < 3c + n - 3 - 2e(X) - 4|Y''| - c|Y'| \leq 3c - 1 - e(X) - 3|Y''| - (c-1)|Y'|$$

since  $e(X) + |Y''| + |Y'| \geq n - 2$ . Therefore  $exc(y) < -e(X) - 3|Y''| - (c-1)(|Y'| - 3) + 2$ . Since  $|Y'| \geq 3$  and  $c \geq 1$ , we have  $exc(y) \leq 1 - e(X)$ . Moreover, since the minimum degree in  $Y$  is at least three, summing the degrees in  $Y$  of the vertices of  $Y \setminus y$  gives that  $3(|Y| - 1) \leq 2e(Y \setminus y) + d_Y(y) \leq 2b - 6 + d_Y(y)$ . Finally,  $d_Y(y) \geq |Y| + 3$  and by the fact that  $exc(y) \leq 1 - e(X)$ , we have  $d_X(y) \geq |Y| + e(X) + 2$ . Recall that  $X$  contains a vertex  $x$  with maximum degree in  $G$ . In particular both  $x$  and  $y$  have degree at least  $2|Y| + e(X) + 5$ . Observe that  $d_Y(x)$  is at least  $2|Y| + 5$ . Now the end of the proof is straightforward, it suffices to switch  $x$  and  $y$  to obtain the good partition  $(X_1, Y_1) := ((X \cup y) \setminus x, (Y \cup x) \setminus y)$ . The only fact to care of is  $e(x, y)$ . Indeed if  $e(x, y)$  is at most  $e(X)$ , we have:

1.  $e(Y_1) \geq d_{Y_1}(x) \geq 2|Y| + 5 - e(x, y) \geq 2|Y| - e(X) \geq |Y| \geq n/2$ .
2.  $e(X_1) \geq d_{X_1}(y) \geq |Y| + e(X) + 2 - e(x, y) \geq n/2$ .
3. Finally, since the excess of  $y$  is at most  $1 - e(X)$ , we have  $d_X(y) \geq d_Y(y) + e(X) - 1$ , hence  $d_{X_1}(y) \geq d_Y(y) - 1$ . Moreover  $d_{Y_1}(x) \geq 2|Y| + 5 - e(X) \geq e(X) + 5 \geq d_X(x) + 5$ . In all, we have  $e(X_1, Y_1) \leq e(X, Y) - 4 \leq (2m - n)/3 + 1$ , since  $(X, Y)$  is an  $a$ -partition.

To conclude, we just have to show that  $e(x, y)$  is at most  $e(X)$ . Assume for contradiction that  $e(x, y) \geq a$ . We consider the partition into  $X_2 := \{x, y\}$  and  $Y_2 := V \setminus \{x, y\}$ . Observe that the minimum underlying degree in  $Y_2$  is at least two, since  $G$  being triangle-free, a vertex of  $Y_2$  can only be joined to at most one vertex of  $X_2$ . Thus  $e(Y_2) \geq b - 2$ . By maximality of  $a$ ,  $(X_2, Y_2)$  is not an  $(a+1)$ -partition, thus  $e(X_2, Y_2) > (2m - n)/3 + 1$ , that is  $d(x) + d(y) - 2e(x, y) > (2m - n)/3 + 1$ .

Consider now any partition  $(X_3, Y_3)$  such that  $(x \cup N(x)) \setminus y \subseteq X_3$  and  $(y \cup N(y)) \setminus x \subseteq Y_3$ . We have  $e(X_3) \geq d(x) - e(x, y) \geq n/2$ . Similarly  $e(Y_3) \geq n/2$ . So, if this partition is not good, we must have  $e(X_3, Y_3) > (2m - n)/3 + 1$ . Thus  $m - (d(x) + d(y) - 2e(x, y)) > (2m - n)/3 + 1$ . This gives  $m > 2(2m - n)/3 + 2$ , and finally  $m < 2n - 6$  which is impossible since the minimum degree in  $G$  is at least four. ■

## 5 Conclusion

Note that the only case where  $\text{bw}(G) > \text{bw}(M_G)$  is when  $G$  has a bridge  $e$  and that  $\text{bw}(M_G) = 1$ , that is when  $G$  is a tree that is not a star.

A consequence of our result is a new proof of the fact that the branch-width of a connected planar graph that is not a tree and the branch-width of its dual are the same, for previous proofs see [4] and [1]. Indeed, if  $r_M$  is the rank function of a matroid, the rank function  $r_{M^*}$  of the dual matroid is such that  $r_{M^*}(U) = |U| + r_M(E \setminus U) - r_M(E)$  which implies that  $w_M(E_1, E_2) = w_{M^*}(E_1, E_2)$  and that  $\text{bw}(M) = \text{bw}(M^*)$ . The result follows from the fact that if  $G$  is a planar graph and  $M_G$  its graphic matroid, the dual matroid of  $M_G$  is  $M_{G^*}$ .

Note that this dual property also holds for stars and thus the only planar graphs  $G$  such that  $\text{bw}(G) = \text{bw}(G^*)$  are exactly the planar graphs such that  $\text{bw}(G) = \text{bw}(M_G)$ . We feel that the natural definition for the branch-width of graphs is the matroidal one.

An independent proof of the equality of branchwidth of cycle matroids and graphs was also given by Hicks and McMurray [3]. Their method is based on matroid tangles and is slightly more involved than ours.

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