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Branchwidth of graphic matroids.

Frédéric Mazoit* and Stéphan Thomassé†

Abstract

Answering a question of Geelen, Gerards, Robertson and Whittle [2], we prove that the branchwidth of a bridgeless graph is equal to the branchwidth of its cycle matroid. Our proof is based on branch-decompositions of hypergraphs.

1 Introduction.

Let $H = (V, E)$ be a hypergraph and $(E_1, E_2)$ be a partition of $E$. The border of $(E_1, E_2)$ is the set of vertices $\delta(E_1, E_2)$ which belong to both an edge of $E_1$ and an edge of $E_2$. We often write it $\delta(E_1, E_2)$, or often simply $\delta$. A component of $E$ is a minimum nonempty subset $C \subseteq E$ such that $\delta(C) = \emptyset$. Let $F$ be a subset of $E$. We denote by $c(F)$ the number of components of the subhypergraph of $H$ spanned by $F$, i.e. the hypergraph $(V(F), F)$. A hypergraph $H$ is 2-edge connected if every vertex belongs to at least two edges and $c(E \setminus e) = 1$ for every $e \in E$.

A branch-decomposition $\mathcal{T}$ of $H$ is a ternary tree $\mathcal{T}$ and a bijection from the set of leaves of $\mathcal{T}$ into the set of edges of $H$. Practically, we simply identify the leaves of $\mathcal{T}$ to the edges of $H$. Observe that every edge $e$ of $\mathcal{T}$ partitions $\mathcal{T} \setminus e$ into two subtrees, and thus correspond to a bipartition of $E$, called $e$-separation. More generally, a $\mathcal{T}$-separation is an $e$-separation for some edge $e$ of $\mathcal{T}$. We will often identify the edge $e$ of $\mathcal{T}$ with the $e$-separation, allowing us to write, for instance, $\delta(e)$ instead of $\delta(E_1, E_2)$, where $(E_1, E_2)$ is the $e$-separation.

Let $f$ be a real function defined on the set of bipartitions of $E$. For sake of simplicity we often write $f(E_1)$ instead of $f(E_1, E \setminus E_1)$. Let $\mathcal{T}$ be a branch-decomposition of $H$. The $f$-width of $\mathcal{T}$, denoted by $w_f(\mathcal{T})$, is the maximum value of $f(e)$, for all edges $e$ of $\mathcal{T}$. The $f$-branchwidth of $H$, denoted by $bw_f(H)$, is the minimum $f$-width of a branch-decomposition of $H$. A branch-decomposition achieving $bw_f(H)$ is $f$-optimal.

The $|\delta|$-branchwidth (i.e. when $f(E_1, E_2) = |\delta(E_1, E_2)|$) of a graph $G$ is the usual branchwidth introduced by Robertson and Seymour in [5]. In this paper, we study the branchwidth associated to the function $\rho(E_1, E_2) = |\delta(E_1, E_2)| + $
Let \(2 - c(E_1) - c(E_2)\). Our goal is to prove that in the class of 2-edge connected hypergraphs, the \(|\delta|\)-branchwidth is equal to the \(\rho\)-branchwidth. The proof simply consists to show that every 2-edge connected hypergraph admits a \(\rho\)-optimal decomposition such that \(c(E_1) = c(E_2) = 1\) for every \(\mathcal{T}\)-separation \((E_1, E_2)\).

Our motivation comes from the following: Let \(M\) be a matroid on base set \(E\) with rank function \(r\). The weight of every non-trivial partition \((E_1, E_2)\) of \(E\) is \(w(E_1, E_2) := r(E_1) + r(E_2) - r(E) + 1\). When \(\mathcal{T}\) is a branch-decomposition of \(M\), i.e. a ternary tree whose leaves are labelled by \(E\), the width of \(\mathcal{T}\) is the maximum weight of a \(\mathcal{T}\)-separation. Again, the branchwidth of \(M\) is the minimum width of a branch-decomposition of \(M\). Let \(M\) be the cycle matroid of a 2-edge connected graph \(G\), i.e. the matroid which base set is the set of edges of a graph and which independent sets are the acyclic subsets of edges. Branch-decompositions of \(G\) are exactly branch-decompositions of \(M\). Moreover, \(r(E_1) + r(E_2) - r(E) + 1\) is exactly \(n_1 - c(E_1) + n_2 - c(E_2) - n + c(E) + 1\) where \(n_1, n_2, n\) are the number of vertices respectively spanned by \(E_1, E_2, E\). Thus we have \(w(E_1, E_2) = \delta(E_1) + 2 - c(E_1) - c(E_2) = \rho(E_1, E_2)\). In particular, the branchwidth of \(M\) is exactly the \(\rho\)-branchwidth, and thus is equal to the \(|\delta|\)-branchwidth.

Unless stated otherwise, we always assume that \(H\) is a 2-edge connected hypergraph and \(\mathcal{T}\) is a branch-decomposition of \(H\). Also, when speaking about width, branchwidth, etc, we implicitly mean \(\rho\)-width, \(\rho\)-branchwidth, etc.

\[\text{2 Faithful branch-decompositions.}\]

Let \((E_1, E_2)\) be a \(\mathcal{T}\)-separation. The decomposition \(\mathcal{T}\) is faithful to \(E_1\) if for every component \(C\) of \(E_1\), the partition \((C, E \setminus C)\) is a \(\mathcal{T}\)-separation. The border graph \(G_{\mathcal{T}}\) has vertex set \(V\) and contains all edges \(xy\) for which there exists an edge \(e\) of \(\mathcal{T}\) such that \(\{x, y\} \subseteq \delta(e)\). A branch-decomposition \(\mathcal{T}'\) is tighter than \(\mathcal{T}\) if \(w_{\rho}(\mathcal{T}') < w_{\rho}(\mathcal{T})\) or if \(w_{\rho}(\mathcal{T}) = w_{\rho}(\mathcal{T}')\) and \(G_{\mathcal{T}'}\) is a subgraph of \(G_{\mathcal{T}}\). Moreover, \(\mathcal{T}'\) is strictly tighter than \(\mathcal{T}\) if \(\mathcal{T}'\) is tighter than \(\mathcal{T}\) and \(\mathcal{T}\) is not tighter than \(\mathcal{T}'\). Finally, \(\mathcal{T}\) is tight if no \(\mathcal{T}'\) is strictly tighter than \(\mathcal{T}\).

**Lemma 1** Let \((E_1, E_2)\) be a partition of \(E\). For any union \(E'_1\) of connected components of \(E_1\) and \(E_2\), we have both \(\delta(E'_1) \leq \delta(E_1)\) and \(\rho(E'_1) \leq \rho(E_1)\).

\(\Box\) Clearly, \(\delta(E'_1) \leq \delta(E_1)\). Moreover, every vertex of \(\delta(E_1)\) belongs to one component of \(E_1\) and one component of \(E_2\). Therefore, if \(C\) is a component of \(E'_1\) which is the union of \(k\) components of \(E_1\) and \(E_2\), there are at least \(k - 1\) vertices of \(C \setminus \delta(C)\) which belong to \(\delta(E_1)\). In all, the weight of the separation increased by \(k - 1\) since we merge \(k\) components into one, but it also decreased by at least \(k - 1\) since we lose at least that many vertices on the border. Since this is the case for every component of \(E'_1\) or of \(E \setminus E'_1\), we have \(\rho(E'_1) \leq \rho(E_1)\).
Lemma 2  Let \((E_1, E_2)\) be an e-separation of \(T\). Let \(T_1\) be the subtree of \(T \setminus e\) with set of leaves \(E_1\). If \(T\) is not faithful to \(E_1\), one can modify \(T_1\) in \(T\) to form a tighter branch-decomposition \(T'\) of \(H\).

\(\square\) Fix the vertex \(e \cap T_1\) as a root of \(T_1\). Our goal is to change the binary rooted tree \(T_1\) into another binary rooted tree \(T'_1\). For every connected component \(C\) of \(E_1\), consider the subtree \(T_C\) of \(T_1\) which contains the root of \(T_1\) and has set of leaves \(C\). Observe that \(T_C\) is not necessarily binary since \(T_C\) may contain paths having internal vertices with only one descendant. We simply replace these paths by edges to obtain our rooted tree \(T'_C\). Now, consider any rooted binary tree \(BT\) with \(c(E_1)\) leaves and identify these leaves to the roots of \(T'_C\), for all components \(C\) of \(E_1\). This rooted binary tree is our \(T'_1\). We denote by \(T'\) the branch-decomposition we obtain from \(T\) by replacing \(T_1\) by \(T'_1\). Roughly speaking, we merged all subtrees of \(T_1\) induced by the components of \(E_1\) together with \(T \setminus T_1\) to form \(T'\). Let us prove that \(T'\) is tighter than \(T\). For this, consider an edge \(f'\) of \(T'\). If \(f' \notin T'_1\), the \(f'\)-separations of \(T\) and \(T'\) are the same. If \(f' \in BT\), by Lemma 1, we have \(\rho(f') \leq \rho(e)\) and \(\delta(f') \subseteq \delta(e)\). So the only case we have to care of is when \(f'\) is an edge of some tree \(T'_{C'}\), where \(C\) is a component of \(E_1\). Recall that \(f'\) corresponds to a path \(P\) of \(T_{C'}\). Let \(f\) be any edge of \(P\). Let \((F, E \setminus F)\) be the \(f\)-separation of \(T\), where \(F \subseteq E_1\). Therefore, the \(f'\)-separation of \(T'\) is \((F \cap C, E \setminus (F \cap C))\). Since \(F\) is a subset of \(E_1\), the connected components of \(F\) are subsets of the connected components of \(E_1\). Thus \(F \cap C\) is a union of connected components of \(F\). This implies that \(\delta(f') \subseteq \delta(f)\). Also, by Lemma 1, \(\rho(f') \leq \rho(f)\).

We have proved that \(w(T') \leq w(T)\) and that \(G_{T'}\) is a subgraph of \(G_T\), thus \(T'\) is tighter than \(T\).

\(\blacksquare\)

3  Connected branch-decompositions.

Let \(F \subseteq E\) be a component. The hypergraph \(H \ast F\) on vertex set \(V\) and edge set \((E \setminus F) \cup \{V(F)\}\) is denoted by \(H \ast F\). In other words, \(H \ast F\) is obtained by merging the edges of \(F\) into one edge. A partition \((E_1, E_2)\) of \(E\) is connected if \(c(E_1) = c(E_2) = 1\). A branch-decomposition \(T\) is connected if all its \(T\)-separations are connected.

Lemma 3  If \(T\) is tight, every \(T\)-separation \((E_1, E_2)\) is such that \(E_1\) or \(E_2\) is connected.

\(\square\) Suppose for contradiction that there exists a \(T\)-separation \((E_1, E_2)\) such that neither \(E_1\) nor \(E_2\) is connected. By Lemma 2, we can assume that \(T\) is faithful to \(E_1\) and to \(E_2\). Let \(E_1\) and \(E_2\) be respectively the sets of components of \(E_1\) and \(E_2\). Consider the graph on set of vertices \(E_1 \cup E_2\) where \(C_1C_2\) is an edge whenever \(C_1 \in E_1\) and \(C_2 \in E_2\) have nonempty intersection. This graph is connected and is not a star. Thus, it has a vertex-partition into two connected subgraphs, each having at least two vertices. This vertex-partition corresponds to a partition \((E'_1, E'_2)\) of \(E_1 \cup E_2\).
Consider any rooted binary tree $BT$ with $|C_1'|$ leaves. Since every $C \in C_1'$ is an element of $C_1' \cup C_2'$, $(C, E \setminus C)$ is an $e$-separation of $H$. We denote by $T_C$ the tree of $C \setminus e$ with set of leaves $C$. Root $T_C$ with the vertex $e \cap T_C$, in order to get a binary rooted tree. Now identify the leaves of $BT$ with the roots of $T_C$, for $C \in C_1'$. This rooted tree is our $T'$. We construct similarly $T'_2$. Adding an edge between the roots of $T'_1$ and $T'_2$ gives the branch-decomposition $T'$ of $H$. By Lemma 1, $w(T') \leq w(T)$. Moreover, $G_{T'}$ is a subgraph of $G_T$. Let us now show that $G_{T'}$ is a strict subgraph of $G_T$. Indeed, since $C_1'$ is connected and has at least two elements, it contains $C_1 \in C_1'$ and $C_2 \in C_2'$ such that $C_1 \cap C_2$ is nonempty. By construction, every vertex $x$ of $C_1 \cap C_2$ is such that $x \notin \delta(C_1')$ and $x \in \delta(C_1)$. Similarly, there is a vertex $y$ spanned by $C_2'$ such that $y \notin \delta(C_2')$ and $y \in \delta(C_2)$. Thus $xy$ is an edge of $G_{T'}$ but not of $G_T$, contradicting the fact that $T$ is tight.

**Theorem 1** For every branch-decomposition $T$ of a hypergraph $H$, there exists a tighter branch-decomposition $T'$ such that for every $T'$-separation $(E_1, E_2)$ with $c(E_1) > 1$, $E_1$ consists of components of $H \setminus e$, for some $e \in E$. In particular, if $H$ is 2-edge connected, it has an optimal connected branch-decomposition.

Let us prove the theorem by induction on $|V|+|E|$. The statement is obvious if $|E| \leq 3$, so we assume now that $H$ has at least four edges. Call achieved a branch-decomposition satisfying the conclusion of Theorem 1. If $T$ is not tight, we can replace it by a tight branch-decomposition tighter than $T$. So we may assume that $T$ is tight.

If $H$ is not connected, apply induction on every components of $H$ in order to find an achieved branch-decomposition. Then merge these branch-decompositions into one branch-decomposition of $H$.

If there is an edge $e \in E$ such that $H \setminus e$ is not connected, we can assume by Lemma 2 that $T$ is faithful to $E \setminus e$. Let $E_1$ be a connected component of $E \setminus e$. Let $T_1$ be the branch-decomposition induced by $T$ on $E_1 \cup e$. Let also $T_2$ be the branch-decomposition induced by $T$ on $(E \setminus E_1) \cup e$. By the induction hypothesis, there exists two achieved branch-decompositions $T'_1$ and $T'_2$, respectively tighter than $T_1$ and $T_2$. Identify the leaf $e$ of the trees $T'_1$ and $T'_2$, and attach a leaf labelled by $e$ to the identified vertex. Call $T'$ this branch-decomposition of $H$.

Observe that it is tighter than $T$ and achieved.

So we assume now that $H$ is 2-edge connected. The key-observation is that if there is a connected $T$-separation $(E_1, E_2)$ with $|E_1| \geq 2$ and $|E_2| \geq 2$, we can apply the induction hypothesis on $H + E_1$ and $H + E_2$ and merge the two branch-decompositions to obtain an optimal connected branch-decomposition of $H$. Therefore, we assume that every $T$-separation $(E_1, E_2)$ with $|E_1| \geq 2$ and $|E_2| \geq 2$ is such that $E_1$ or $E_2$ is connected.

We now orient the edges of $T$. If $(E_1, E_2)$ is an $e$-separation such that $E_2$ is connected but not an edge of $H$, we orient $e$ from $E_1$ to $E_2$. Since $H$ is 2-edge-connected, every edge of $T$ incident to a leaf is oriented from the leaf. By Lemma 3, every edge get at least one orientation. And by the key-observation, every edge of $T$ has exactly one orientation.
This orientation of $\mathcal{T}$ has no circuit, thus there is a vertex $t \in T$ with outdegree zero. Since every leaf has outdegree one, $t$ has indegree three. Let us denote by $A, B, C$ the set of leaves of the three trees of $\mathcal{T} \setminus t$. Observe that by construction, $A \cup B, A \cup C$ and $B \cup C$ are connected. By Lemma 2, we can assume moreover that $\mathcal{T}$ is faithful to $A, B$ and $C$. We claim that $A$ is a disjoint union of edges, i.e. the connected components of $A$ are edges of $H$. To see this, pick any component $C_A$ of $A$. Since $\mathcal{T}$ is faithful to $A, (C_A, E \setminus C_A)$ is a $T$-separation. But this is simply impossible since $B \cup C$ being connected, $E \setminus C_A$ is also connected, against the fact that every edge has a single orientation. So the separation. But this is simply impossible since $B \cup C$ being connected, $E \setminus C_A$ is also connected, against the fact that every edge has a single orientation. So the connected components of $A$ are edges of $H$. By this, we can assume moreover that $\mathcal{T}$ is faithful to $A, B, C$. Let $\delta = \delta(\mathcal{T})$ be the connected components of $A$ by construction, and thus every optimal branch-decomposition of $H$ is tighter than $\mathcal{T}$. Set $\delta_{AB} := |\delta(A) \cap \delta(B)|, \delta_{AC} := |\delta(A) \cap \delta(C)|, \delta_{BC} := |\delta(B) \cap \delta(C)|$ and $\delta_{ABC} := |\delta(A) \cap \delta(B) \cap \delta(C)|$. We now prove some properties of $H$.

1. Two of the sets $A, B, C$ have at least two edges. Indeed, assume for contradiction that $A = \{a\}$ and $B = \{b\}$. Since $|E| \geq 4$, there are at least two edges in $C$. Let $c \in C$. Assume without loss of generality that $|a \cap c| \geq |b \cap c|$. Now form a new branch-decomposition $\mathcal{T}'$ by moving $c$ to $A$, i.e. $\mathcal{T}'$ has a separation $(A \cup c, B \cup (C \setminus c))$ and then four branches $A, c, B, (C \setminus c)$. We have

$$\rho(A \cup c, B \cup (C \setminus c)) = \rho(A) + |b \cap c| - |a \cap c|$$

since both parts are connected. In particular $\mathcal{T}'$ is tighter than $\mathcal{T}$, and since the $\mathcal{T}'$-separation $(A \cup c, B \cup (C \setminus c))$ is connected and both of its branches have at least two vertices, we can apply induction to conclude.

2. Every set $A, B, C$ have at least two edges. Indeed, assume for contradiction that $A$ consists of a single edge $a$. Let $b$ be an edge of $B$. If $|b \cap \delta(C)| \leq |b \cap a|$, we can as previously move $b$ to $A$ in order to conclude. Call $|b \cap \delta(C)| - |b \cap a|$ the excess of $b$. Similarly, call $|c \cap \delta(B)| - |c \cap a|$ the excess of an edge $c \in C$. Let $s$ be the minimum excess of an edge $e_s$ of $B \cup C$. Observe that $s \geq 1$ and that every $b \in B$ satisfies $|b \cap \delta(C)| \geq |b \cap \delta(A)| + s$. Thus, summing for all edges of $B$, we obtain $\delta_{BC} \geq \delta_{AB} + s|B|$. Similarly, $\delta_{BC} \geq \delta_{AC} + s|C|$. Note also that $bw(H) \geq \rho(C) = \delta_{BC} + \delta_{AC} - \delta_{ABC} - |C| + 1$ and $bw(H) \geq \rho(B) = \delta_{BC} + \delta_{AB} - \delta_{ABC} - |B| + 1$. In all

$$2bw(H) \geq 2\delta_{BC} - 2\delta_{ABC} + \delta_{AC} - |C| + \delta_{AB} - |B| + 2.$$

Then $2bw(H) \geq \delta_{AB} + s|B| + \delta_{AC} + s|C| - 2\delta_{ABC} + \delta_{AC} - |C| + \delta_{AB} - |B| + 2$. Finally, $bw(H) \geq \delta_{AC} + \delta_{AB} - \delta_{ABC} + 1 + ((s - 1)|C| + (s - 1)|B|)/2$. Since $\rho(A) = \delta_{AC} + \delta_{AB} - \delta_{ABC}$, we have $bw(H) \geq \rho(A) + s$. But then we can move $e_s$ to $A$ to conclude.
3. We have $bw(H) = \rho(A)$. If not, pick two edges $a, a'$ of $A$ and merge them together. The hypergraph we obtain is still 2-edge-connected, and the branch-decomposition still has the same width. Apply induction to get an achieved branch-decomposition. Then replace the merged edge by the two original edges. This branch-decomposition $T'$ is optimal, hence tighter than $T$. So either we can apply induction on $T'$, or $T'$ has a canonical partition. But in this last case, the canonical partition of $T'$ is exactly $\{a\}, \{a'\}, E \setminus \{a, a'\}$. Thus by 1, we can apply induction. Similarly, $bw(H) = \rho(\emptyset) = \rho(C)$.

4. We have $bw(H) \geq \beta + 1$, where $\beta$ is the size of a maximum edge $e$ of $H$. Since edges of $H$ with only one vertex play no role here, we can ignore them. So the size of an edge of $H$ is at least two. Assume for instance that $e \in A$. Since $A$ has at least two components, we have $bw(H) = \rho(A) \geq |\delta(A)| - c(B \cup C) - c(A) + 2 \geq 2 + \beta - 1$.

5. We have $\delta_{ABC} = 0$. Indeed, suppose for contradiction that there exists a vertex $z$ in $\delta(A) \cap \delta(B) \cap \delta(C)$. Consider the hypergraph $H_z$ obtained from $H$ by removing the vertex $z$ from all its edges. The branch-decomposition $T$ induces a branch-decomposition $T_z$ of $H_z$ having at most $bw(T) - 1$. Observe also that $H_z$ is connected since $z$ is incident to three edges and $H$ is 2-edge connected. We apply induction on $T_z$ to obtain an achieved branch-decomposition $T'_z$ of $H_z$. Now add back the vertex $z$ to the edges of $H_z$ and call $T'$ the branch-decomposition obtained from $T'_z$. Observe that if a $T'_z$-separation $(E_1, E_2)$ is connected, adding $z$ will raise by at most one its width in $T'$. Moreover if a $T'_z$-separation $(E_1, E_2)$ is not connected, say $c(E_2) > 1$, adding $z$ can raise by at most two its width in $T'$ (either by merging three components of $E_2$ into one, or by merging two and increasing the border by one). Since $T'_z$ is achieved, $E_2$ is a set of components of $E \setminus e$ for some edge $e$ of $E$. But then, $\rho_{T'}(E_1, E_2) \leq |e| - 3 + 2 \leq \beta - 1 \leq bw(H) - 2$, and thus $\rho_{T'}(E_1, E_2) \leq bw(H)$. Finally $T'$ is optimal. Moreover every $T'$-separation $(E_1, E_2)$ is connected. Indeed, if $E_1$ is connected in $T'_z$, we are done. If $E_1$ is not connected in $T'_z$, $E_1$ consists of components of $H_z \setminus e$, for some edge $e$ of $H_z$. But since $H$ is 2-edge connected, every component of $E_1$ in $H$ contains $z$, otherwise they would be components of $H \setminus e$. Consequently $E_1$ is connected.

6. Every edge of $H$ is incident to at least four other edges. Indeed, assume for contradiction that an edge $a$ of $A$ is incident to only one edge $b$ of $B$ and at most two edges of $C$. Moving $a$ to $B$ increases $\rho(B)$ by $|a \cap \delta(C)| - |a \cap b|$ and does not increase $\rho(A)$ and $\rho(C)$. Therefore, if $|a \cap \delta(C)| \leq |a \cap b|$, we can move $a$ to $B$, and this new branch-decomposition $T'$ is strictly tighter than $T$ since the vertices of $a \cap b$ are no more joined to $(\delta(A) \setminus a) \cap \delta(C)$ in the graph $G_T$. Thus $|a \cap \delta(C)| > |a \cap b| + 1$. Moreover, moving $a$ to $C$, increases $\rho(C)$ by at most $|a \cap b| - |a \cap \delta(C)| + 1$, since at most two components of $C$ can merge. So $|a \cap b| + 1 > |a \cap \delta(C)|$, a contradiction.
This implies in particular that \( \rho(e) \leq \text{bw}(H) - 3 \) whenever \( e \) is not one of the main \( \mathcal{T} \)-separations. In particular \( \beta \leq \text{bw}(H) - 3 \).

7. The hypergraph \( H \) is triangle-free. Indeed, suppose that there exists three edges \( a \in A, b \in B \) and \( c \in C \) and three vertices \( x \in a \cap b, y \in b \cap c \) and \( z \in c \cap a \). Let \( H \) be the hypergraph obtained by removing \( x, y, z \) in the vertex set of \( H \) and in every edge of \( H \). The branchwidth of \( H \) is at most \( \text{bw}(H) - 2 \), since we removed two vertices in the border of every main separation. As in 5, \( \mathcal{T} \) induces a branch-decomposition \( \mathcal{T}_x \) of \( H \), which can be improved by induction to an achieved branch-decomposition \( \mathcal{T}'_x \) of \( H \). Adding back the vertices \( x, y, z \), we obtain a branch-decomposition \( \mathcal{T}' \) of \( H \). We claim that \( w(\mathcal{T}') \leq \text{bw}(H) \). Let \( (E_1, E_2) \) be a \( \mathcal{T}' \)-separation, without loss of generality, we assume that \( E_1 \) contains at least two edges of \( a, b, c \), say \( a \) and \( b \). Assume first that \( (E_1, E_2) \) is connected in \( \mathcal{T}_x \). If \( c \in E_1 \), we have \( \rho_{\mathcal{T}_x}(E_1) = \rho_{\mathcal{T}_x}(E_1) \). If \( c \notin E_1 \), \( \rho_{\mathcal{T}_x}(E_1) + 2 = \rho_{\mathcal{T}_x}(E_1) \) since we add the endvertices of \( c \) to the border of \( E_1 \). Now, if \( (E_1, E_2) \) is not connected in \( \mathcal{T}_x \), then \( \rho_{\mathcal{T}_x}(E_1) \) is at most the size of an edge of \( H \), hence at most \( \beta \). Since \( x, y, z \) have degree two in \( H \), each can either increase the border of a separation by one, or merge two components. In all, \( \rho_{\mathcal{T}_x}(E_1) \) increases by at most three. Since \( \beta \leq \text{bw}(H) - 3 \), we have that \( w(\mathcal{T}') \leq \text{bw}(H) \). To conclude, we prove that \( \mathcal{T}' \) is connected. Indeed, if \( c(E_2) > 1 \) for some \( \mathcal{T}' \)-separation \( (E_1, E_2) \), \( E_2 \) consists of components of \( H \) \( \cap e \) for some edge \( e \) of \( H \). Since \( H \) is 2-edge connected, these components are not components of \( H \) \( \cap e \), so each of them must contain one of the edges \( a, b, c \), and therefore \( E_2 \) is connected in \( \mathcal{T}' \).

Now we are ready to finish the proof. Note that \( \text{bw}(H) = (\rho(A) + \rho(B) + \rho(C))/3 = (2|V| - |E|)/3 + 1. \) Consider the line multigraph \( L(H) \) of \( H \), i.e. the multigraph on vertex set \( A \cup B \cup C \) and edge set \( V \) such that \( v \in V \) is the edge which joins the two edges \( e, f \) of \( H \) such that \( v \in e \) and \( v \in f \). The multigraph \( L(H) \) satisfies the hypothesis of Lemma 4 (proved in the next section), thus it admits a vertex-partition of its vertices as in the conclusion of Lemma 4. This corresponds to a partition of \( A \cup B \cup C \) into two subsets \( E_1 := A_1 \cup B_1 \cup C_1 \) and \( E_2 := A_2 \cup B_2 \cup C_2 \) such that \( |E_1| \geq 2|V| - |E|/3 \) and both \( E_1 \) and \( E_2 \) have at least \(|E|/2 + 1\) internal vertices. In particular, the separation \( (E_1, E_2) \) has width at most \( \text{bw}(H) \). Let us show that one of \( \rho(A_1 \cup B_1), \rho(B_1 \cup C_1) \), and \( \rho(C_1 \cup A_1) \) is also at most \( \text{bw}(H) \). For this, observe that

\[
\delta(A_1 \cup B_1) + \delta(B_1 \cup C_1) + \delta(C_1 \cup A_1) \leq 2(|V| - (|E_2| - |\delta(E_2)|)).
\]

Thus \( \delta(A_1 \cup B_1) + \delta(B_1 \cup C_1) + \delta(C_1 \cup A_1) \leq 2|V| - 2|E|/2 + 2 \leq 2|V| - |E| + 3. \) Without loss of generality, we can assume that \( \delta(A_1 \cup B_1) \leq (2|V| - |E|)/3 + 1 = \text{bw}(H), \) and thus we split \( E_1 \) into two branches \( A_1 \cup B_1 \) and \( C_1 \). We similarly split \( E_2 \) to obtain an optimal branch-decomposition \( \mathcal{T}' \) of \( H \). Observe that since both \( E_1 \) and \( E_2 \) are not empty, the graph \( G_{\mathcal{T}'} \) is not complete, against the fact that \( \mathcal{T} \) is tight, a contradiction.  

\[\blacksquare\]
4 The technical Lemma.

Let $G$ be a multigraph and $X,Y$ two subsets of its vertices. We denote by $e(X,Y)$ the number of edges of $G$ between $X$ and $Y$. We also denote by $e(X)$ the number of edges in $X$.

**Lemma 4** Let $G$ be a 2-connected triangle-free multigraph on $n \geq 5$ vertices and $m$ edges. Assume that its minimum underlying degree (forgetting the multiplicity of edges) is four. Assume moreover that its maximum degree is at most $2+3/2$. There exists a partition $(X,Y)$ of the vertex set of $G$ such that $e(X) \geq \lfloor n/2 \rfloor - 1$, $e(Y) \geq \lfloor n/2 \rfloor - 1$ and $e(X,Y) \leq (2m-n)/3 + 1$.

Proof Call good a partition which satisfies the conclusion of Lemma 4. Assume first that there are vertices $x,y$ such that $e(x,y) \geq \lfloor n/2 \rfloor - 1$. The minimum degree in $V \setminus \{x,y\}$ is at least two, so $e(V \setminus \{x,y\})$ is at least $\lfloor n/2 \rfloor - 1$. Thus, if the partition $(V \setminus \{x,y\}, \{x,y\})$ is not good, we necessarily have $d(x) + d(y) \geq 2e(x,y) > (2m-n)/3 + 1$. By the maximum degree hypothesis, both $d(x)$ and $d(y)$ are greater than $2e(x,y)$. Since $G$ is triangle-free, there exists a partition $(X,Y)$ where $(N(x) \cup x) \setminus y \subseteq X$ and $(N(y) \cup y) \setminus x \subseteq Y$. Observe that $e(X) \geq d(x) - e(x,y) > e(x,y) \geq \lfloor n/2 \rfloor - 1$. Similarly $e(Y) \geq \lfloor n/2 \rfloor - 1$. Moreover, since $m \geq 2n$ by the minimum degree four hypothesis, we have

\[
e(X,Y) \leq m - (d(x) + d(y) - 2e(x,y)) < m - (2m-n)/3 - 1 \leq (m+n)/3 - 1 \leq (2m-n)/3 + 1.
\]

So $(X,Y)$ is a good partition. We assume from now on that the multiplicity of an edge is less than $\lfloor n/2 \rfloor - 1$.

Let $a + b = n$, where $a \leq b$. A partition $(X,Y)$ of $V$ is an $a$-partition if $|X| \leq a$, $e(X) \geq a - 1$, $e(Y) \geq b - 1$, $e(X,Y) \leq (2m-n)/3 + 1$, and the additional requirement that $X$ contains a vertex of $G$ with maximum degree.

Note that there exists a 1-partition, just consider for this $X := \{x\}$, where $x$ has maximum degree in $G$ (the minimum degree in $Y$ is at least three, insuring that $e(Y) \geq n - 2$). We consider now an $a$-partition $(X,Y)$ with maximum $a$. If $a \geq b - 1$, this partition is good and we are done. So we assume that $a < b - 1$. In particular $e(X) = a - 1$. The excess of a vertex $y \in Y$ is $\text{exc}(y) := d_Y(y) - d_X(y)$.

The key-observation is that there exists at most one vertex $y \in Y$ such that $e(Y \setminus y) < b - 2$. Indeed, if there is a vertex of $Y$ with degree one in $Y$, we simply move it to $X$, and we obtain an $(a+1)$-partition $(e(X)$ increases, $e(Y)$ decreases by one, and $e(X,Y)$ decreases). Thus the minimum degree in $Y$ is at least two. Moreover, if there is a vertex of $Y$ with degree two, we can still move it to $X$ $(e(X)$ increases, $e(Y) \geq |Y| - 2$ and $e(X,Y)$ does not increase). So the minimum degree in $Y$ is at least three (but the underlying minimum degree may be one). This implies that $e(Y) \geq 3|Y|/2$. Let $Y := \{y_1, \ldots, y_{|Y|}\}$ where the
vertices are indexed in the increasing order according to their degree in $Y$. For every $i \neq |Y|$, we have $e(Y) \geq (3(|Y| - 2) + d_Y(y_i) + d_Y(y_{|Y|})) / 2$. Furthermore,

$$
e(Y \setminus y_i) \geq (3(|Y| - 2) + d_Y(y_i) + d_Y(y_{|Y|})) / 2 - d_Y(y_i) \\
\geq 3(|Y| - 2) / 2 \\
\geq |Y| - 2 \\
\geq b - 2.
$$

We now discuss the two different cases.

- Assume that $e(Y \setminus y) \geq b - 2$ for every $y \in Y$. We denote by $Y'$ the (nonempty) set of vertices of $Y$ with at least one neighbour in $X$. Set $Y'' := Y \setminus Y'$. Denote by $c$ the minimum excess of a vertex of $Y'$. Observe that every vertex of $Y''$ has degree at least four in $Y$. Thus summing the degrees of the vertices of $Y$ gives

$$
2e(Y) \geq e(X, Y') + 4|Y''| + c|Y'|
$$

Let $y \in Y'$ such that $exc(y) = c$. Since the partition $(X \cup y, Y \setminus y)$ is not an $(a + 1)$-partition, we have $e(X, Y) + c > (2m - n)^3 + 1$. Since $m = e(X, Y') + e(X) + e(Y)$, this implies

$$
e(X, Y) + 3c > 2e(X) + 2e(Y) - n + 3
$$

Equations (1) and (2) give:

$$
e(X, Y) + 3c > 2e(X) + e(X, Y) + 4|Y''| + c|Y'| - n + 3
$$

Since $e(X) \geq n - |Y| - 1$, we get $3c > n - 2|Y| + 4|Y''| + c|Y'| + 1$. So $3c > n + 2|Y| + (c - 4)|Y''| + 1 + 1$, and finally $n + 2|Y| < (c - 4)(3 - |Y'|) + 11$. If $c = 4$, we get $n + 2|Y| \leq 10$, impossible. If $c = 3$, we get $n + 2|Y| - |Y'| \leq 7$, impossible. If $c = 2$, we get $n + 2|Y| - 2|Y'| \leq 4$, impossible. If $c = 1$, we get $n + 2|Y| - 3|Y'| \leq 1$, which can only hold if $|Y'| = |Y''| = n - 1$. Thus, $X$ consists of a single vertex, completely joined to $Y$, against the fact that $G$ is triangle-free. Finally $c > 4$, and consequently $|Y'| < 3$. Observe that $|Y'| > 1$ since $G$ is 2-connected. Thus $|Y'| = 2$. Let $y_1, y_2$ be the vertices of $Y'$, indexed in such a way that $e(y_1, X) + e(y_2, Y'') \geq e(y_2, X) + e(y_1, Y'')$. Let $X_1 := X \cup y_1$ and $Y_1 := Y \setminus y_1$. Since $y_1 \in Y'$, we have that $e(X_1) \geq a$. Moreover $e(Y_1) \geq b - 2$. Observe also that $e(y_1, y_2) \leq e(Y \setminus \{y_1, y_2\})$: this is obvious if $e(y_1, y_2) = 0$, and if there is an edge between $y_1$ and $y_2$, since $G$ is triangle-free with minimum degree four, the minimum degree in $Y \setminus \{y_1, y_2\}$ is at least two. So

$$
e(Y \setminus \{y_1, y_2\}) \geq |Y| - 2 \geq |n/2| - 1 \geq e(y_1, y_2).
$$

In all $e(X_1, Y_1) \leq e(X_1) + e(Y_1)$. And since $n - 2 \leq e(X_1) + e(Y_1)$, we have $e(X_1, Y_1) \leq 2e(X_1) + 2e(Y_1) - n + 2$, which implies $e(X_1, Y_1) \leq (3m - n)/2 + 1$. So the partition $(X_1, Y_1)$ is good.
Now assume that there exists a vertex \( y \in Y \) such that \( e(Y \setminus y) \leq b - 3 \). We denote by \( Y' \) the set of vertices of \( Y \setminus y \) with at least one neighbour in \( X \). Set \( Y'' := Y \setminus (Y' \cup y) \). Observe that since every vertex of \( Y'' \) has underlying degree four in \( Y \), we have \( e(Y \setminus y) \geq 3|Y''|/2 \). Thus, \( |Y''| \leq (2|Y| - 6)/3 \). Since \( |Y| \geq 3 \), we have \( |Y''| \leq |Y| - 3 \), and finally \( |Y'| \geq 3 \). Denote by \( c \) the minimum excess of a vertex of \( Y' \). Summing the degrees of the vertices of \( Y \) gives \( 2e(Y) \geq e(X, Y) + 4|Y''| + c|Y'| + \text{exc}(y) \). Equation (2) still holds, so
\[ \text{exc}(y) \leq 3c + n - 3 - 2e(X) - 4|Y''| - c|Y'| \leq 3c - 1 - e(X) - 3|Y''| - (c - 1)|Y'| \]
since \( e(X) + |Y''| + |Y'| \geq n - 2 \). Therefore \( \text{exc}(y) < -e(X) - 3|Y''| - (c - 1)|Y'| - 3 + 2. \) Since \( |Y'| = 3 \) and \( c \geq 1 \), we have \( \text{exc}(y) \leq 1 - e(X) \). Moreover, since the minimum degree in \( Y \) is at least three, summing the degrees in \( Y \) of the vertices of \( Y \) gives that \( 3(|Y| - 1) \leq 2e(Y \setminus y) + d_Y(y) \leq 2b - 6 + d_Y(y) \). Finally, \( d_Y(y) \geq |Y| + 3 \) and by the fact that \( \text{exc}(y) \leq 1 - e(X) \), we have \( d_X(y) \geq |Y| + e(X) + 2. \) Recall that \( X \) contains a vertex \( x \) with maximum degree in \( G \). In particular both \( x \) and \( y \) have degree at least \( 2|Y| + e(X) + 5 \). Observe that \( d_Y(x) \) is at least \( 2|Y| + 5 \). Now the end of the proof is straightforward, it suffices to switch \( x \) and \( y \) to obtain the good partition \( (X_1, Y_1) := ((X \cup y) \setminus x, (Y \cup x) \setminus y) \). The only fact to care of is \( e(x, y) \). Indeed if \( e(x, y) \) is at most \( e(X) \), we have:

1. \( e(Y_1) \geq d_{Y_1}(x) \geq 2|Y| + 5 - e(x, y) \geq 2|Y| - e(X) \geq |Y| \geq n/2 \).
2. \( e(X_1) \geq d_{X_1}(y) \geq |Y| + e(X) + 2 - e(x, y) \geq n/2 \).
3. Finally, since the excess of \( y \) is at most \( 1 - e(X) \), we have \( d_X(y) \geq d_Y(y) + e(X) - 1 \), hence \( d_X(y) \geq d_Y(y) - 1 \). Moreover \( d_Y(x) \geq 2|Y| + 5 - e(X) \geq e(X) + 5 \geq d_X(x) + 5 \). In all, we have \( e(X_1, Y_1) \leq e(X, Y) - 4 \leq (2m - n)/3 + 1 \), since \( (X, Y) \) is an \( n \)-partition.

To conclude, we just have to show that \( e(x, y) \) is at most \( e(X) \). Assume for contradiction that \( e(x, y) \geq a \). We consider the partition into \( X_2 := \{ x, y \} \) and \( Y_2 := V \setminus \{ x, y \} \). Observe that the minimum underlying degree in \( Y_2 \) is at least two, since \( G \) being triangle-free, a vertex of \( V_2 \) can only be joined to at most one vertex of \( X_2 \). Thus \( e(Y_2) \geq b - 2 \). By maximality of \( a \), \( (X_2, Y_2) \) is not an \( (a + 1) \)-partition, thus \( e(X_2, Y_2) > (2m - n)/3 + 1 \), that is \( d(x) + d(y) - 2e(x, y) > (2m - n)/3 + 1 \).

Consider now any partition \( (X_3, Y_3) \) such that \( (x \cup N(x)) \setminus y \subseteq X_3 \) and \( (y \cup N(y)) \setminus x \subseteq Y_3 \). We have \( e(X_3) \geq d(x) - e(x, y) \geq n/2 \). Similarly \( e(Y_3) \geq n/2 \). So, if this partition is not good, we must have \( e(X_3, Y_3) > (2m - n)/3 + 1 \). Thus \( m - (d(x) + d(y) - 2e(x, y)) > (2m - n)/3 + 1 \). This gives \( m > 2(2m - n)/3 + 2 \), and finally \( m < 2n - 6 \) which is impossible since the minimum degree in \( G \) is at least four.
5 Conclusion

Note that the only case where $bw(G) > bw(M_G)$ is when $G$ has a bridge $e$ and that $bw(M_G) = 1$, that is when $G$ is a tree that is not a star.

A consequence of our result is a new proof of the fact that the branch-width of a connected planar graph that is not a tree and the branch-width of its dual are the same, for previous proofs see [4] and [1]. Indeed, if $r_M$ is the rank function of a matroid, the rank function $r_{M^*}$ of the dual matroid is such that $r_{M^*}(U) = |U| + r_M(E \setminus U) - r_M(E)$ which implies that $w_M(E_1, E_2) = w_{M^*}(E_1, E_2)$ and that $bw(M) = bw(M^*)$. The result follows from the fact that if $G$ is a planar graph and $M_G$ its graphic matroid, the dual matroid of $M_G$ is $M_G^*$.

Note that this dual property also holds for stars and thus the only planar graphs $G$ such that $bw(G) = bw(G^*)$ are exactly the planar graphs such that $bw(G) = bw(M_G)$. We feel that the natural definition for the branch-width of graphs is the matroidal one.

An independent proof of the equality of branchwidth of cycle matroids and graphs was also given by Hicks and McMurray [3]. Their method is based on matroid tangles and is slightly more involved than ours.

References


