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Self-interacting diffusions IV: Rate of convergence

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Abstract

Self-interacting diffusions are processes living on a compact Riemannian manifold defined by a stochastic differential equation with a drift term depending on the past empirical measure \( \mu_t \) of the process. The asymptotics of \( \mu_t \) is governed by a deterministic dynamical system and under certain conditions \( (\mu_t) \) converges almost surely towards a deterministic measure \( \mu^* \) (see Benaïm, Ledoux, Raimond (2002) and Benaïm, Raimond (2005)). We are interested here in the rate of convergence of \( \mu_t \) towards \( \mu^* \). A central limit theorem is proved. In particular, this shows that greater is the interaction repelling faster is the convergence.

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1 Introduction

Self-interacting diffusions

Let $M$ be a smooth compact Riemannian manifold and $V : M \times M \to \mathbb{R}$ a sufficiently smooth mapping. For all finite Borel measure $\mu$, let $V_{\mu} : M \to \mathbb{R}$ be the smooth function defined by

$$V_{\mu}(x) = \int_M V(x, y)\mu(dy).$$

Let $(e_{\alpha})$ be a finite family of vector fields on $M$ such that

$$\sum_{\alpha} e_{\alpha}(e_{\alpha}f)(x) = \Delta f(x),$$

where $\Delta$ is the Laplace operator on $M$ and $e_{\alpha}(f)$ stands for the Lie derivative of $f$ along $e_{\alpha}$. Let $(B^\alpha)$ be a family of independent Brownian motions.

A self-interacting diffusion on $M$ associated to $V$ can be defined as the solution to the stochastic differential equation (SDE)

$$dX_t = \sum_{\alpha} e_{\alpha}(X_t) \circ dB^\alpha_t - \nabla(V_{\mu_t})(X_t)dt.$$ 

where

$$\mu_t = \frac{1}{t} \int_0^t \delta_{X_s} ds$$

is the empirical occupation measure of $(X_t)$.

In absence of drift (i.e $V = 0$), $(X_t)$ is just a Brownian motion on $M$ but in general it defines a non Markovian process whose behavior at time $t$ depends on its past trajectories through $\mu_t$. This type of process was introduced in Benaim, Ledoux and Raimond (2002) (hence after referred as [3]) and further analyzed in a series of papers by Benaim and Raimond (2003, 2005, 2007) (hence after referred as [4], [5] and [6]). We refer the reader to these papers for more details and especially to [3] for a detailed construction of the process and its elementary properties. For a general overview of processes with reinforcement we refer the reader to the recent survey paper by Pemantle (2007) ([15]).

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The mapping $V_x : M \to \mathbb{R}$ defined by $V_x(y) = V(x, y)$ is $C^2$ and its derivatives are continuous in $(x, y)$.
Notation and Background

Standing Notation We let $\mathcal{M}(M)$ denote the space of finite Borel measures on $M$, $\mathcal{P}(M) \subset \mathcal{M}(M)$ the space of probability measures. If $I$ is a metric space (typically, $I = M, \mathbb{R}^+ \times M$ or $[0, T] \times M$) we let $C(I)$ denote the space of real valued continuous functions on $I$ equipped with the topology of uniform convergence on compact sets. When $I$ is compact and $f \in C(I)$ we let $\|f\| = \sup_{x \in I} |f(x)|$. The normalized Riemann measure on $M$ will be denoted by $\lambda$.

Let $\mu \in \mathcal{P}(M)$ and $f : M \to \mathbb{R}$ a nonnegative or $\mu-$integrable Borel function. We write $\mu f$ for $\int f d\mu$, and $f \mu$ for the measure defined as $f \mu(A) = \int_A f d\mu$. We let $L^2(\mu)$ denote the space of such functions for which $\mu|f|^2 < \infty$, equipped with the inner product

$$\langle f, g \rangle_\mu = \mu(fg)$$

and the norm

$$\|f\|_\mu = \sqrt{\mu f^2}.$$ 

We simply write $L^2$ for $L^2(\lambda)$.

Of fundamental importance in the analysis of the asymptotics of $(\mu_t)$ is the mapping $\Pi : \mathcal{M}(M) \to \mathcal{P}(M)$ defined by

$$\Pi(\mu) = \xi(V \mu) \lambda$$

where $\xi : C(M) \to C(M)$ is the function defined by

$$\xi(f)(x) = \frac{e^{-f(x)}}{\int_M e^{-f(y)} \lambda(dy)}.$$ 

In [3], it is shown that the asymptotics of $\mu_t$ can be precisely related to the long term behavior of a certain semiflow on $\mathcal{P}(M)$ induced by the ordinary differential equation (ODE) on $\mathcal{M}(M)$:

$$\dot{\mu} = -\mu + \Pi(\mu).$$ 

Depending on the nature of $V$, the dynamics of (3) can either be convergent or nonconvergent leading to similar behaviors for $\{\mu_t\}$ (see [3]). When $V$ is symmetric, (3) happens to be a quasigradient and the following convergence result hold.
Theorem 1.1 ([5]) Assume that $V$ is symmetric, i.e. $V(x, y) = V(y, x)$. Then the limit set of $\{\mu_t\}$ (for the topology of weak* convergence) is almost surely a compact connected subset of

$$\text{Fix}(\Pi) = \{\mu \in \mathcal{P}(M) : \mu = \Pi(\mu)\}.$$ 

In particular, if $\text{Fix}(\Pi)$ is finite then $(\mu_t)$ converges almost surely toward a fixed point of $\Pi$. This holds for a generic function $V$ (see [5]).

Sufficient conditions ensuring that $\text{Fix}(\Pi)$ has cardinal one are as follows:

Theorem 1.2 ([5], [6]) Assume that $V$ is symmetric and that one of the two following conditions hold

(i) Up to an additive constant $V$ is a Mercer kernel, That is

$$V(x, y) = K(x, y) + C$$

and

$$\int K(x, y)f(x)f(y)\lambda(dx)\lambda(dy) \geq 0$$

for all $f \in L^2$.

(ii) For all $x \in M, y \in M, u \in T_x M, v \in T_y M$

$$\text{Ric}_x(u, u) + \text{Ric}_y(v, v) + \text{Hess}_{x,y}V((u, v), (u, v)) \geq K(\|u\|^2 + \|v\|^2)$$

where $K$ is some positive constant. Here $\text{Ric}_x$ stands for the Ricci tensor at $x$ and $\text{Hess}_{x,y}$ is the Hessian of $V$ at $(x, y)$.

Then $\text{Fix}(\Pi)$ reduces to a singleton $\{\mu^*\}$ and $\mu_t \to \mu^*$ with probability one.

As observed in [6] the condition $(i)$ in Theorem 1.2 seems well suited to describe self-repelling diffusions. On the other hand, it is not clearly related to the geometry of $M$. Condition $(ii)$ has a more geometrical flavor and is robust to smooth perturbations (of $M$ and $V$). It can be seen as a Bakry-Emery type condition for self interacting diffusions.

In [5], it is also proved that every stable (for the ODE (3)) fixed point of $\Pi$ has a positive probability to be a limit point for $\mu_t$; and any unstable fixed point cannot be a limit point for $\mu_t$. 

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Organisation of the paper

Let $\mu^* \in \text{Fix}(\Pi)$. We will assume that

**Hypothesis 1.3** $\mu_t$ converges a.s. towards $\mu^*$.

Sufficient conditions are given by Theorem 1.2

In this paper we intend to study the rate of this convergence. Let

$$\Delta_t = e^{t/2}(\mu_t - \mu^*).$$

It will be shown that, under some conditions to be specified later, for all $g = (g_1, \ldots, g_n) \in C(M)^n$ the process

$$[\Delta_t g_1, \ldots, \Delta_t g_n, V\Delta_t]_{s \geq t}$$

converges in law, as $t \to \infty$, toward a certain stationary Ornstein-Uhlenbeck process $(Z^g, Z)$ on $\mathbb{R}^n \times C(M)$. This process is defined in Section 2. The main result is stated in section 3 and some examples are developed. It is in particular observed that a strong repelling interaction gives a faster convergence. The section 4 is a proof section. The appendix, section 5, contains general material on random variables and Ornstein-Uhlenbeck processes on $C(M)$.

In the following $K$ (respectively $C$) denotes a positive constant (respectively a positive random constant). These constants may change from line to line.

2 The Ornstein-Uhlenbeck process $(Z^g, Z)$.

Throughout all this section we let $\mu \in \mathcal{P}(M)$. For $x \in M$ we set $V_x : M \to \mathbb{R}$ defined by $V_x(y) = V(x, y)$.

2.1 The operator $G_{\mu}$

Let $g \in C(M)$ and let $G_{\mu, g} : \mathbb{R} \times C(M) \to \mathbb{R}$ be the linear operator defined by

$$G_{\mu, g}(u, f) = u/2 + \text{Cov}_\mu(g, f),$$

(4)
where Cov$_\mu$ is the covariance on $L^2(\mu)$, that is the bilinear form acting on $L^2 \times L^2$ defined by

$$Cov_\mu(f, g) = \mu(fg) - (\mu f)(\mu g).$$

We define the linear operator $G_\mu : C(M) \to C(M)$ by

$$G_\mu f(x) = G_{\mu, V_x}(f(x), f) = f(x)/2 + Cov_\mu(V_x, f).$$

It is easily seen that $\|G_\mu f\| \leq (2\|V\| + 1/2)\|f\|$. In particular, $G_\mu$ is a bounded operator. Let $\{e^{-tG_\mu}\}$ denotes the semigroup acting on $C(M)$ with generator $-G_\mu$. From now on we will assume the following:

**Hypothesis 2.1** There exists $\kappa > 0$ and $\hat{\lambda} \in P(M)$ such that $\mu << \hat{\lambda}$ with $\|d\mu/d\hat{\lambda}\|_\infty < \infty$, $\lambda$ and $\hat{\lambda}$ are equivalent measures with $\|d\lambda/d\hat{\lambda}\|_\infty < \infty$ and $\|\frac{d\hat{\lambda}}{d\lambda}\|_\infty < \infty$, and such that for all $f \in L^2(\hat{\lambda})$,

$$\langle G_\mu f, f \rangle_{\hat{\lambda}} \geq \kappa \|f\|_{\hat{\lambda}}^2.$$

Let

$$\lambda(-G_\mu) = \lim_{t \to \infty} \frac{\log(\|e^{-tG_\mu}\|)}{t}.$$

This limit exists by subadditivity. Then

**Lemma 2.2** Hypothesis 2.1 implies that $\lambda(-G_\mu) \leq -\kappa < 0$.

**Proof:** For all $f \in L^2(\hat{\lambda})$,

$$\frac{d}{dt}\|e^{-tG_\mu} f\|_{\hat{\lambda}}^2 = -2\langle G_\mu e^{-tG_\mu} f, e^{-tG_\mu} f \rangle_{\hat{\lambda}} \leq -2\kappa \|e^{-tG_\mu} f\|_{\hat{\lambda}}.$$

This implies that $\|e^{-tG_\mu} f\|_{\hat{\lambda}} \leq e^{-\kappa t}\|f\|_{\hat{\lambda}}$.

Denote by $g_t$ the solution of the differential equation

$$\frac{dg_t}{dt} = Cov_\mu(V_x, g_t)$$

with $g_0 = f$, where $f \in C(M)$. Note that $e^{-tG_\mu} f = e^{-t/2} g_t$. It is straightforward to check that (using the fact that $\|d\mu/d\lambda\|_\infty < \infty$)

$$\frac{d}{dt}\|g_t\|_{\hat{\lambda}} \leq K\|g_t\|_{\hat{\lambda}}$$
with $K$ a constant depending only on $V$ and $\mu$. Thus

$$\sup_{t \in [0,1]} \| g_t \|_\lambda \leq K \| f \|_\lambda.$$ 

Now, since for all $x \in M$ and $t \in [0,1]$

$$\left| \frac{d}{dt} g_t(x) \right| \leq K \| g_t \|_\lambda \leq K \| f \|_\lambda,$$

we have $\| g_1 \| \leq K \| f \|_\lambda$. This implies that

$$\| e^{-G_\mu} f \| \leq K \| f \|_\lambda.$$

Now for all $t > 1$, and $f \in C(M)$,

$$\| e^{-tG_\mu} f \| = \| e^{-G_\mu} e^{-(t-1)G_\mu} f \| \leq K \| e^{-(t-1)G_\mu} f \|_\lambda \leq K e^{-\kappa(t-1)} \| f \|_\lambda \leq K e^{-\kappa t} \| f \|_\infty.$$ 

This implies that $\| e^{-tG_\mu} \| \leq Ke^{-\kappa t}$, which proves the lemma. QED

The adjoint of $G_\mu$ is the operator on $\mathcal{M}(M)$ defined by the relation

$$m(G_\mu f) = (G_\mu^* m) f$$

for all $m \in \mathcal{M}(M)$ and $f \in C(M)$. It is not hard to verify that

$$G_\mu^* m = \frac{1}{2} m + (Vm)\mu - (\mu(Vm))\mu. \quad (6)$$

2.2 The generator $A_\mu$ and its inverse $Q_\mu$

Let $H^2$ be the Sobolev space of real valued functions on $M$, associated with the norm $\| f \|_2^2 = \| f \|_\lambda^2 + \| \nabla f \|_\lambda^2$. Since $\Pi(\mu)$ and $\lambda$ are equivalent measures with continuous Radon-Nykodim derivative, $L^2(\Pi(\mu)) = L^2(\lambda) := L^2$. We denote by $K_\mu$ the projection operator, acting on $L^2(\Pi(\mu))$, defined by

$$K_\mu f = f - \Pi(\mu)f.$$
We denote by $A_\mu$ the operator acting on $H^2$ defined by

\[ A_\mu f = \frac{1}{2} \Delta f - \langle \nabla V_\mu, \nabla f \rangle. \]

Note that for $f$ and $g$ in $L^2$,

\[ \langle A_\mu f, g \rangle_{\Pi(\mu)} = -\frac{1}{2} \int \langle \nabla f, \nabla g \rangle(x) \Pi(\mu)(dx) \]

where $\langle \cdot, \cdot \rangle$ denotes the Riemannian inner product on $M$.

For all $f \in C(M)$ there exists $Q_\mu f \in H^2$ such that $\Pi(\mu)(Q_\mu f) = 0$ and

\[ f - \Pi(\mu)f = K_\mu f = -A_\mu Q_\mu f. \tag{7} \]

Note that if $P^\mu_t$ denotes the semigroup with generator $A_\mu$, then

\[ Q_\mu f = \int_0^\infty P^\mu_t K_\mu f \, dt. \]

Since there exists $p^\mu_t (\cdot, \cdot)$ such that

\[ P^\mu_t f(x) = \int_M p^\mu_t(x, y) f(y) \Pi(\mu)(dy), \]

we have

\[ Q_\mu f(x) = \int_M q_\mu(x, y) f(y) \Pi(\mu)(dy) \]

where

\[ q_\mu(x, y) = \int_0^\infty (p^\mu_t(x, y) - 1) \, dt. \]

Then, as shown in [3], $Q_\mu f$ is $C^1$ and there exists a constant $K$ such that for all $f \in C(M)$ and $\mu \in \mathcal{P}(M)$,

\[ \|Q_\mu f\|_{\infty} \leq K \|f\|_{\infty} \tag{8} \]

\[ \|\nabla Q_\mu f\|_{\infty} \leq K \|f\|_{\infty}. \tag{9} \]

Finally, note that for $f$ and $g$ in $L^2$,

\[ \int \langle \nabla Q_\mu f, \nabla Q_\mu g \rangle(x) \Pi(\mu)(dx) = -2\langle A_\mu Q_\mu f, Q_\mu g \rangle_{\Pi(\mu)} = 2\langle f, Q_\mu g \rangle_{\Pi(\mu)}. \tag{10} \]
2.3 The covariance $C_\mu$

We let $\hat{C}_\mu$ denote the bilinear continuous form $\hat{C}_\mu : C(M) \times C(M) \to \mathbb{R}$ defined by

$$\hat{C}_\mu(f, g) = 2\langle f, Q_\mu g \rangle \Pi(\mu).$$

This form is symmetric (see its expression given by (10)). Note also that for some constant depending on $\mu$,

$$|\hat{C}_\mu(f, g)| \leq K \|f\| \times \|g\|.$$

We let $C_\mu$ denote the mapping $C_\mu : M \times M \to \mathbb{R}$ defined by

$$C_\mu(x, y) = \hat{C}_\mu(V_x, V_y).$$

Then $C_\mu$ is a covariance function (or a Mercer kernel), i.e. it is continuous, symmetric and $\sum_{i,j} \lambda_i \lambda_j C_\mu(x_i, x_j) \geq 0$.

2.4 The process $Z$

We now define an Ornstein-Uhlenbeck process on $C(M)$ of covariance $C_\mu$ and drift $-G_\mu$. This heavily relies on the general construction given in the appendix.

A Brownian motion on $C(M)$ with covariance $C_\mu$ is a $C(M)$-valued stochastic process $W = \{W_t\}_{t \geq 0}$ such that

(i) $W_0 = 0$;

(ii) $t \mapsto W_t$ is continuous;

(iii) For every finite subset $S \subset \mathbb{R} \times M$, $\{W_t(x)\}_{(t,x) \in S}$ is a centered Gaussian random vector;

(iv) $E[W_s(x)W_t(y)] = (s \wedge t)C_\mu(x, y)$.

Lemma 2.3 There exists a Brownian motion on $C(M)$ with covariance $C_\mu$.

Proof: Let

$$d_{C_\mu}(x, y) := \sqrt{C_\mu(x, x) - 2C_\mu(x, y) + C_\mu(y, y)}$$

$$= \|\nabla Q_\mu(V_x - V_y)\| \Pi(\mu)$$

$$\leq K \|V_x - V_y\|$$
where the last inequality follows from (9). Then
\[ d_{C_{\mu}}(x, y) \leq K d(x, y) \]
and the result follows from Proposition 5.8 and Remark 5.7 in the appendix.

QED

We say that a \( C(M) \)-valued process \( Z \) is an Ornstein-Uhlenbeck process of covariance \( C_{\mu} \) and drift \( -G_{\mu} \) if
\[
Z_t = Z_0 - \int_0^t G_{\mu} Z_s ds + W_t
\]  
(11)

where

(i) \( W \) is a \( C(M) \)-valued Brownian motion of covariance \( C_{\mu} \);

(ii) \( Z_0 \) is a \( C(M) \)-valued random variable;

(iii) \( W \) and \( Z_0 \) are independent.

Note that we can think of \( Z \) as a solution to the linear SDE
\[
dZ_t = dW_t - G_{\mu} Z_t dt.
\]

It follows from section 5.3 in the appendix that such a process exists and defines a Markov process. Furthermore

**Proposition 2.4** Under hypothesis 2.1,

(i) \((Z_t)\) converges in law toward a \( C(M) \)-valued random variable \( Z_{\infty} \);

(ii) \( Z_{\infty} \) is Gaussian, in the sense that for every finite set \( S \subset M, \{Z_{\infty}(x)\}_{x \in S} \) is a centered Gaussian random vector;

(iii) Let \( \pi^{\mu} \) denotes the law of \( Z_{\infty} \). Then \( \pi^g \) is characterized by its variance
\[
\text{Var}(\pi^{\mu}) : \mathcal{M}(M) \to \mathbb{R},
\]
\[
m \mapsto \mathbb{E}((m Z_{\infty})^2),
\]

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and for all \( m \in \mathcal{M} \),

\[
\text{Var}(\pi^n)(m) = \int_0^\infty \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathcal{C}_\mu(x, y) m_t(dx)m_t(dy)dt
\]

\[
= \int_0^\infty \mathring{\mathcal{C}}_\mu(Vm_t, Vm_t)dt
\]

where

\[m_t = e^{-tG^*\nu}m.\]

**Proof:** This follows from Proposition 5.16 in the appendix. Example 5.18 shows that assertion (iii) of this proposition is satisfied. \( \text{QED} \)

### 2.5 The process \( Z^g \).

For \( g = (g_1, \ldots, g_n) \in C(M)^n \), let \( \tilde{M} = \{1, \ldots, n\} \cup M \) be the disjoint union of \( \{1, \ldots, n\} \) and \( M \), and \( \mathcal{C}_\mu^g : \tilde{M} \times \tilde{M} \to \mathbb{R} \) be the function defined by

\[
\mathcal{C}_\mu^g(x, y) = \begin{cases} 
\mathring{\mathcal{C}}_\mu(g_x, g_y) & \text{for } x, y \in \{1, \ldots, n\}, \\
\mathcal{C}_\mu(x, y) & \text{for } x, y \in M, \\
\mathring{\mathcal{C}}_\mu(Vx, g_y) & \text{for } x \in M, y \in \{1, \ldots, n\}.
\end{cases}
\]

Then \( \mathcal{C}_\mu^g \) is a Mercer kernel (see section 5.2).

A Brownian motion on \( \mathbb{R}^n \times C(M) \) with covariance \( \mathcal{C}_\mu^g \) is a \( \mathbb{R}^n \times C(M) \)-valued stochastic process \( (W^g, W) = \{(W^g_1, \ldots, W^g_n, W_t)\}_{t \geq 0} \) such that:

(i) \( W = \{W_t\}_{t \geq 0} \) is a \( C(M) \)-valued Brownian motion with covariance \( \mathcal{C}_\mu^g \);

(ii) For every finite subset \( S \subset \mathbb{R} \times M, \{W^g_t, W_t(x)\}_{(t, x) \in S} \) is a centered Gaussian random vector;

(iii) \( \mathbb{E}(W^g_s W^g_t) = (s \wedge t)\mathring{\mathcal{C}}_\mu(g_i, g_j) \) and \( \mathbb{E}(W_s(x)W^g_t) = (s \wedge t)\mathring{\mathcal{C}}_\mu(Vx, g_i) \).

**Lemma 2.5** There exists a Brownian motion on \( \mathbb{R}^n \times C(M) \) with covariance \( \mathcal{C}_\mu^g \).

**Proof:** Let \( \tilde{d} \) be the distance on \( \tilde{M} \) defined by

\[
\tilde{d}(x, y) = \begin{cases} 
1_{x \neq y} & \text{for } x, y \in \{1, \ldots, n\}, \\
d(x, y) & \text{for } x, y \in M, \\
d(x, x_0) + 1 & \text{for } x \in M, y \in \{1, \ldots, n\}
\end{cases}
\]
where $x_0$ is some arbitrary point in $M$. This makes $\tilde{M}$ a compact metric space, and it is easy to show that the function $C^g_\mu$ verifies hypothesis 5.4 (use the proof of Lemma 2.3). The result follows by application of Proposition 5.8. QED

Let now be $Z^g_t = (Z^g_{1t}, \ldots, Z^g_{nt}) \in \mathbb{R}^n$ denote the solution to the SDE

$$dZ^g_t = dW^g_t - (Z^g_t/2 + \text{Cov}_\mu(Z_t, g_t)) \, dt, \quad i = 1, \ldots, n \quad (12)$$

where $(W^g, W)$ is as above and $Z = (Z_t)$ is given by (11).

The following result generalizes Proposition 2.4.

**Proposition 2.6** Under hypothesis 2.1,

(i) The process $(Z^g_t, Z_t)$ converges in law toward a centered $\mathbb{R}^n \times C(M)$ valued Gaussian random variable $(Z^g_\infty, Z_\infty)$.

(ii) Let $\pi^{g,\mu}$ denotes the law of $(Z^g_\infty, Z_\infty)$. Then $\pi^{g,\mu}$ is characterized by its variance

$$\text{Var}(\pi^{g,\mu}) : \mathbb{R}^n \times C(M) \to \mathbb{R},$$

$$(u, m) \mapsto \mathbb{E}((mZ^g_\infty + \langle u, Z^g_\infty \rangle)^2);$$

and for all $u \in \mathbb{R}^n, m \in C(M)$,

$$\text{Var}(\pi^{g,\mu})(u, m) = \int_0^\infty \hat{C}_\mu(f_t, f_t) \, dt$$

with

$$f_t = e^{-t/2} \sum_{i=1}^n u_i g_i + V m_t,$$

and where $m_t$ is defined by

$$m_t f = m_0(e^{-tG_\mu} f) + \sum_{i=1}^n u_i \int_0^t e^{-(t-s)/2} \text{Cov}_\mu(g_i, e^{-(t-s)G_\mu} f) \, ds. \quad (13)$$

**Proof:** Let $G^g_\mu : \mathbb{R}^n \times C(M) \to \mathbb{R}^n \times C(M)$ be the operator defined by

$$G^g_\mu = \begin{pmatrix} I/2 & A^g_\mu \\ 0 & G_\mu \end{pmatrix} \quad (14)$$
where $A^g_\mu : C(M) \to \mathbb{R}^n$ is the linear map defined by

$$A^g_\mu(f) = \left( \text{Cov}_\mu(f, g_1), \ldots, \text{Cov}_\mu(f, g_n) \right).$$

Then $(Z^g, Z)$ is a $C(\hat{M})$-valued Ornstein-Uhlenbeck process of covariance $C^g_\mu$ and drift $-G^g_\mu$. It is not hard to verify that under hypothesis 2.1, the assumptions of Proposition 5.16 hold, so that $(Z^g_t, Z_t)$ converges in law toward a centered $\mathbb{R}^n \times C(M)$ valued Gaussian random variable $(Z^g_\infty, Z_\infty)$ with variance

$$\text{Var}(\pi^{g,\mu})(u, m) = \int_0^\infty \hat{C}_\mu(f_t, f_t) dt$$

with $f_t = \sum_i u_t(i)g_i + Vm_t$ and where $(u_t, m_t) = e^{-t(G^\mu_\mu)^*}(u, m)$. Now

$$(G^g_\mu)^* = \begin{pmatrix} I/2 & 0 \\ (A^g_\mu)^* & (G_\mu)^* \end{pmatrix}$$

and $(A^g_\mu)^*u = \sum_i u_i(g_i - \mu g_i)$. Thus $u_t = e^{-t/2}u$ and

$$\frac{d m_t}{dt} = -(A^g_\mu)^*u_t - (G_\mu)^*m_t$$

Thus $m_t$ is the solution with $m_0 = m$ of

$$\frac{d m_t}{dt} = -e^{-t/2} \left( \sum_i u_i(g_i - \mu g_i) \right) \mu - G^*_\mu m_t$$

(15)

Note that (15) is equivalent to

$$\frac{d}{dt}(m_t f) = -e^{-t/2} \text{Cov}_\mu \left( \sum_i u_i g_i, f \right) - m_t (G_\mu f)$$

for all $f \in C(M)$, and $m_0 = m$. From which we deduce that

$$m_t = e^{-tG^*_\mu}m_0 - \int_0^t e^{-s/2} e^{-(t-s)G^*_\mu} \left( \sum_i u_i(g_i - \mu g_i) \mu \right) ds$$

which implies the formula for $m_t$ given by (13). QED

For further reference we call $(Z^g, Z)$ an Ornstein-Uhlenbeck process of covariance $C^g_\mu$ and drift $-G^g_\mu$. It is called stationary when its initial distribution is $\pi^{g,\mu}$.
3 A central limit theorem for $\mu_t$

We state here the main results of this article. We assume $\mu^* \in \text{Fix}(\Pi)$ satisfies hypotheses $1.3$ and $2.1$. Set $\Delta_t = e^{t/2}(\mu_{t} - \mu^*)$, $D_t = V\Delta_t$ and $D_{t+} = \{D_{t+s} : s \geq 0\}$. Then

**Theorem 3.1** $D_{t+}$ converges in law, as $t \to \infty$, towards a stationary Ornstein-Uhlenbeck process of covariance $C_{\mu^*}$ and drift $-G_{\mu^*}$.

For $g = (g_1, \ldots, g_n) \in C(M)^n$, we set $D^g_t = (\Delta_t g, D_t)$ and $D^g_{t+} = \{D^g_{t+s} : s \geq 0\}$. Then

**Theorem 3.2** $(D^g_{t+s})_{s \geq 0}$ converges in law towards a stationary Ornstein-Uhlenbeck process of covariance $C^g_{\mu^*}$ and drift $-G^g_{\mu^*}$.

Define $\hat{C} : C(M) \times C(M) \to \mathbb{R}$ the symmetric bilinear form defined by

$$
\hat{C}(f, g) = \int_0^\infty \hat{C}_{\mu^*}(f_t, g_t)dt,
$$

with $(g_t$ is defined by the same formula, with $g$ in place of $f$)

$$
f_t(x) = e^{-t/2}f(x) - \int_0^t e^{-s/2}\text{Cov}_{\mu^*}(f, e^{-(t-s)G_{\mu^*}V})ds.
$$

**Corollary 3.3** $\Delta g$ converges in law towards a centered Gaussian variable $Z^g_\infty$ of covariance

$$
\mathbb{E}[Z^g_\infty Z^{g_j}_\infty] = \hat{C}(f, g).
$$

**Proof**: Follows from theorem 3.2 and the calculus of $\text{Var}(\pi^{g,\mu})(u, 0)$. QED

3.1 Examples

3.1.1 Diffusions

Suppose $V(x, y) = V(x)$, so that $(X_t)$ is just a standard diffusion on $M$ with invariant measure $\mu^* = \frac{\exp(-V)}{\lambda \exp(-V)}$.

Let $f \in C(M)$. Then $f_t$ defined by (17) is equal to (using $e^{-tG_{\mu^*}1} = e^{-t/2}1$) $e^{-t/2}f$. Thus

$$
\hat{C}(f, g) = 2\mu^*(fQ_{\mu^*}g).
$$

Corollary 3.3 says that
Theorem 3.4 For all \( g \in C(M)^n \), \( \Delta_t \) converges in law toward a centered Gaussian variable \((Z_{\infty}^g, \ldots, Z_{\infty}^g)\), with covariance given by
\[
E(Z_{\infty}^g Z_{\infty}^g) = 2\mu^*(g, Q_{\mu^*} g_j).
\]

Remark 3.5 This central limit theorem for Brownian motions on compact manifolds has already been considered by Baxter and Brosamler in [4] and [5]; and by Bhattacharya in [7] for ergodic diffusions.

3.1.2 The case \( \mu^* = \lambda \) and \( V \) symmetric.

Suppose here that \( \mu^* = \lambda \) and that \( V \) is symmetric. We assume (without loss of generality since \( \Pi(\lambda) = \lambda \) implies that \( V\lambda \) is a constant function) that \( V\lambda = 0 \).

Since \( V \) is compact and symmetric, there exists an orthonormal basis \((e_\alpha)_{\alpha \geq 0}\) in \( L^2(\lambda) \) and a sequence of reals \((\lambda_\alpha)_{\alpha \geq 0}\) such that \( e_0 \) is a constant function and
\[
V = \sum_{\alpha \geq 1} \lambda_\alpha e_\alpha \otimes e_\alpha.
\]

Assume that for all \( \alpha \), \( 1/2 + \lambda_\alpha > 0 \). Then hypothesis [3] holds with \( \hat{\lambda} = \lambda \), and the convergence of \( \mu_t \) towards \( \lambda \) holds with positive probability (see [8]).

Let \( f \in C(M) \) and \( f_t \) defined by (17), denoting \( f^\alpha = \langle f, e_\alpha \rangle_\lambda \) and \( f_\alpha^t = \langle f_t, e_\alpha \rangle_\lambda \), we have \( f^0_t = e^{-t/2} f^0 \) and for \( \alpha \geq 1 \),
\[
f_\alpha^t = e^{-t/2} f^\alpha - \lambda_\alpha e^{-(1/2 + \lambda_\alpha)t} \left( \frac{e^{\lambda_\alpha t} - 1}{\lambda_\alpha} \right) f^\alpha
\]
\[
= e^{-(1/2 + \lambda_\alpha)t} f^\alpha.
\]

Using the fact that
\[
\hat{C}_\lambda(f, g) = 2\lambda(f Q_\lambda g),
\]
this implies that
\[
\hat{C}(f, g) = 2 \sum_{\alpha \geq 1} \sum_{\beta \geq 1} \frac{1}{1 + \lambda_\alpha + \lambda_\beta} \langle f, e_\alpha \rangle_\lambda \langle g, e_\beta \rangle_\lambda \lambda(e_\alpha Q_\lambda e_\beta).
\]

This, with corollary [3.3], proves
Theorem 3.6 Assume hypothesis \([1.3]\) and that \(1/2 + \lambda_\alpha > 0\) for all \(\alpha\). Then for all \(g \in C(M)^n\), \(\Delta^g_t\) converges in law toward a centered Gaussian variable \((Z_{\infty}^{g_1}, \ldots, Z_{\infty}^{g_n})\), with covariance given by
\[
E(Z_{\infty}^{g_i}Z_{\infty}^{g_j}) = \hat{C}(g_i, g_j).
\]
In particular,
\[
E(Z_{\infty}^{e_\alpha}Z_{\infty}^{e_\beta}) = \frac{2}{1 + \lambda_\alpha + \lambda_\beta} \lambda(e_\alpha Q e_\beta).
\]
Note that when all \(\lambda_\alpha\) are positive, which corresponds to what is named a self-repelling interaction in [6], the rate of convergence of \(\mu_t\) towards \(\lambda\) is bigger than when there is no interaction, and the bigger is the interaction (that is larger \(\lambda_\alpha\)'s) faster is the convergence.

4 Proof of the main results

We assume hypothesis [1,3] and \(\mu^*\) satisfies hypothesis [2.1]. It is possible to choose \(\kappa\) in hypothesis [2.1] such that \(\kappa < 1/2\). In the following \(\kappa\) will denote such constant. Note that we have \(\lambda(-G_{\mu^*}) < -\kappa\). Such \(\kappa\) exists when hypothesis [2.1] holds.

4.1 A lemma satisfied by \(Q_\mu\)

We denote by \(\mathcal{X}(M)\) the space of continuous vector fields on \(M\), and equip the spaces \(\mathcal{P}(M)\) and \(\mathcal{X}(M)\) respectively with the weak convergence topology and with the uniform convergence topology.

Lemma 4.1 For all \(f \in C(M)\), the mapping \(\mu \mapsto \nabla Q_\mu f\) is a continuous mapping from \(\mathcal{P}(M)\) in \(\mathcal{X}(M)\).

Proof : Let \(\mu\) and \(\nu\) be in \(\mathcal{M}(M)\), and \(f \in C(M)\). Set \(g = Q_\mu f\). Then \(f = -A_\mu g + \Pi(\mu)f\) and
\[
\|\nabla Q_\mu f - \nabla Q_\nu f\|_\infty = \| - \nabla Q_\mu A_\mu g + \nabla Q_\nu A_\mu g\|_\infty \\
= \|\nabla g + \nabla Q_\nu A_\mu g\|_\infty \\
\leq \|\nabla (g + Q_\nu A_\mu g)\|_\infty + \|\nabla Q_\nu (A_\mu - A_\nu)g\|_\infty.
\]
since $\nabla (g + Q_{\nu} A_{\nu} g) = 0$ and $(A_{\mu} - A_{\nu}) g = \langle \nabla V_{\mu-\nu}, \nabla g \rangle$, we get
\begin{equation}
\| \nabla Q_{\mu} f - \nabla Q_{\nu} f \|_{\infty} \leq K \| \langle \nabla V_{\mu-\nu}, \nabla g \rangle \|_{\infty}.
\end{equation}

Using the fact that $(x, y) \mapsto \nabla V_{x}(y)$ is uniformly continuous, the right hand term of (19) converges towards 0, when $d(\mu, \nu)$ converges towards 0, $d$ being a distance compatible with the weak convergence. \[ \text{QED} \]

\section{4.2 The process $\Delta$}

Set $h_{t} = V_{\mu_{t}}$ and $h^{*} = V_{\mu^{*}}$. Recall $\Delta_{t} = e^{t/2}(\mu_{t\cdot} - \mu^{*})$ and $D_{t} = V \Delta_{t}$. Note that $D_{t}(x) = \Delta_{t} V_{x}$.

To simplify the notation, we set $K_{s} = K_{\mu_{s}}$, $Q_{s} = Q_{\mu_{s}}$, and $A_{s} = A_{\mu_{s}}$. Let $(M^{f}_{t})_{t \geq 1}$ be the martingale defined by
\begin{equation}
M^{f}_{t} = \sum_{\alpha} \int_{1}^{t} e_{\alpha}(Q_{s} f)(X_{s}) dB_{s}^{\alpha}.
\end{equation}

The quadratic covariation of $M^{f}$ and $M^{g}$ (with $f$ and $g$ in $C(M)$) is given by
\begin{equation}
\langle M^{f}, M^{g} \rangle_{t} = \int_{1}^{t} \langle \nabla Q_{s} f, \nabla Q_{s} g \rangle(X_{s}) ds.
\end{equation}

Then for all $t \geq 1$ (with $Q^{t}_{\mu_{t}} = \frac{d}{dt} Q_{\mu_{t}}$),
\begin{equation}
Q_{t} f(X_{t}) - Q_{1} f(X_{1}) = M^{f}_{t} + \int_{1}^{t} Q_{s} f(X_{s}) ds - \int_{1}^{t} K_{s} f(X_{s}) ds.
\end{equation}

Thus
\begin{align*}
\mu_{t} f &= \frac{1}{t} \int_{1}^{t} K_{s} f(X_{s}) ds + \frac{1}{t} \int_{1}^{t} \Pi(\mu_{s}) f ds + \frac{1}{t} \int_{0}^{1} f(X_{s}) ds \\
&= -\frac{1}{t} \left( Q_{t} f(X_{t}) - Q_{1} f(X_{1}) - \int_{1}^{t} Q_{s} f(X_{s}) ds \right) \\
&\quad + \frac{M^{f}_{t}}{t} + \frac{1}{t} \int_{1}^{t} \langle \xi(h_{s}), f \rangle ds + \frac{1}{t} \int_{0}^{1} f(X_{s}) ds.
\end{align*}

Note that $(D_{t})$ is a continuous process taking its values in $C(M)$ and that $D_{t} = e^{t/2}(h_{t\cdot} - h^{*})$. For $f \in C(M)$ (using the fact that $\mu^{*} f = \langle \xi(h^{*}), f \rangle$),
\begin{equation}
\Delta_{t} f = \sum_{i=1}^{5} \Delta_{i} f
\end{equation}
with
\[
\Delta^1_t f = e^{-t/2} \left( -Q^1_t f(X^-) + Q^1_t f(X^+) + \int_1^t \dot{Q}_s f(X_s) ds \right)
\]
\[
\Delta^2_t f = e^{-t/2} M^f_{t} 
\]
\[
\Delta^3_t f = e^{-t/2} \int_1^t \langle \xi(h_s) - \xi(h^*) - D\xi(h^*)(h_s - h^*), f \rangle ds 
\]
\[
\Delta^4_t f = e^{-t/2} \int_1^t \langle D\xi(h^*)(h_s - h^*), f \rangle ds 
\]
\[
\Delta^5_t f = e^{-t/2} \left( \int_0^1 f(X_s) ds - \mu^* f \right).
\]

Then \( D_t = \sum_{i=1}^5 D^i_t \), where \( D^i_t = V \Delta^i_t \). Finally, note that
\[
\langle D\xi(h^*)(h - h^*), f \rangle = -\text{Cov}_{\mu^*}(h - h^*, f). \tag{21}
\]

### 4.3 First estimates

We recall some estimates from [3]: There exists a constant \( K \) such that for all \( f \in C(M) \) and \( t > 0 \),
\[
\|Q_t f\|_\infty \leq K \|f\|_\infty \\
\|\nabla Q_t f\|_\infty \leq K \|f\|_\infty \\
\|
\dot{Q}_t f\|_\infty \leq \frac{K}{t} \|f\|_\infty.
\]

These estimates imply in particular that
\[
\langle M^f - M^g \rangle_t \leq K \|f - g\|_\infty \times t
\]

and that

**Lemma 4.2** There exists a constant \( K \) depending on \( \|V\|_\infty \) such that for all \( t \geq 1 \), and all \( f \in C(M) \)
\[
\|\Delta^1_t f\|_\infty + \|\Delta^5_t f\|_\infty \leq K \times (1 + t)e^{-t/2}\|f\|_\infty, \tag{22}
\]

which implies that \( ((\Delta^1 + \Delta^5)_{t+s})_{s \geq 0} \) and \( ((D^1 + D^5)_{t+s})_{s \geq 0} \) both converge towards 0 (respectively in \( \mathcal{M}(M) \) and in \( C(\mathbb{R}^+ \times M) \)).
We also have

**Lemma 4.3** There exists a constant $K$ such that for all $t \geq 0$ and all $f \in C(M)$,

\[
E[(\Delta^2_t f)^2] \leq K\|f\|_{\infty}^2,
\]

\[
|\Delta^3_t f| \leq K\|f\|_{\lambda} \times e^{-t/2} \int_0^t \|D_s\|_{\lambda}^2 ds,
\]

\[
|\Delta^4_t f| \leq K\|f\|_{\lambda} \times e^{-t/2} \int_0^t e^{s/2}\|D_s\|_{\lambda} ds.
\]

**Proof:** The first estimate follows from

\[
E[(\Delta^2_t f)^2] = e^{-t}E[(M^f_{e^t})^2] = e^{-t}E[(M^f)_{e^t}^2] \leq e^{-t} \int_1^{e^t} \|\nabla Q_s f\|_{\infty}^2 ds \leq K\|f\|_{\infty}^2.
\]

The second estimate follows from the fact that

\[
\|\xi(h) - \xi(h^*) - D\xi(h^*)(h-h^*)\|_{\lambda} = O(\|h - h^*\|_{\lambda}^2).
\]

The last estimate follows easily after having remarked that

\[
|\langle D\xi(h^*)(h_s - h^*), f \rangle| = |\text{Cov}_{\mu^*} (h_s - h^*, f)| \leq K\|f\|_{\lambda} \times \|h_s - h^*\|_{\lambda} \leq K\|f\|_{\lambda} \times s^{-1/2}\|D_{\log(s)}\|_{\lambda}.
\]

This proves this lemma. QED

**4.4 The processes $\Delta'$ and $D'$**

Set $\Delta' = \Delta^2 + \Delta^3 + \Delta^4$ and $D' = D^2 + D^3 + D^4$. For $g \in C(M)$, set

\[
e_i^g = e^{t/2}\langle \xi(h_{e_t}) - \xi(h^*) - D\xi(h^*)(h_{e_t} - h^*), g \rangle_{\lambda}.
\]

Then

\[
d\Delta'_t g = -\frac{\Delta'_t g}{2} dt + dN^g_t + e^g_t dt + \langle D\xi(h^*) (D_t), g \rangle_{\lambda} dt.
\]
where for all $g \in C(M)$, $N^g$ is a martingale. Moreover, for $f$ and $g$ in $C(M)$,

$$\langle N^f, N^g \rangle_t = \int_0^t \langle \nabla Q_e^s f(X_e^s), \nabla Q_e^s g(X_e^s) \rangle ds.$$ 

Then, for all $x$,

$$dD'_t(x) = -\frac{D'_t(x)}{2}dt + dM_t(x) + \epsilon_t(x)dt + \langle D\xi(h^*)(D'_t), V_x \rangle \lambda dt$$

where $M$ is the martingale in $C(M)$ defined by $M(x) = N^V_x$ and $\epsilon_t(x) = \epsilon^V_t$. We also have

$$G_{\mu^*}(D'_t)(x) = \frac{D'_t(x)}{2} - \langle D\xi(h^*)(D'_t), V_x \rangle \lambda.$$ 

Denoting $L_{\mu^*} = L - G_{\mu^*}$ (defined by equation (32) in the appendix), this implies that

$$dL_{\mu^*}(D'_t)(x) = dD'_t(x) + G_{\mu^*}(D'_t)(x)dt$$

$$= dM_t(x) + \langle D\xi(h^*)((D^1 + D^5)_t), V_x \rangle \lambda dt + \epsilon_t(x)dt$$

Thus

$$L_{\mu^*}(D'_t)(x) = M_t(x) + \int_0^t \epsilon'_s(x)ds$$

with $\epsilon'_s(x) = \epsilon'_sV_x$ where for all $f \in C(M)$,

$$\epsilon'_sf = \epsilon'_s + \langle D\xi(h^*)((D^1 + D^5)_s), f \rangle \lambda.$$ 

Using lemma 5.10,

$$D'_t = L^{-1}_{\mu^*}(M)_t + \int_0^t e^{-(t-s)G_{\mu^*}} \epsilon'_s ds. \quad (23)$$

For $g = (g_1, \ldots, g_n) \in C(M)^n$, we denote $\Delta'_tg = (\Delta'_tg_1, \ldots, \Delta'_tg_n)$, $N^g = (N^{g_1}, \ldots, N^{g_n})$ and $\epsilon'_tg = (\epsilon'_tg_1, \ldots, \epsilon'_tg_n)$. Then, denoting $L_{\mu^*}^g = L - G_{\mu^*}^g$ (with $G_{\mu^*}^g$ defined by (14)) we have

$$L_{\mu^*}^g(\Delta'_tg, D'_t) = (N^g_t, M_t) + \int_0^t (\epsilon'_tg, \epsilon'_s)ds$$

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so that (using lemma 5.11 and integrating by parts)

\[(\Delta_t g, D_t') = (L_{\mu^*}^{-1}(N^g, M)_t + \int_0^t e^{-(t-s)G_{\mu^*}^a} (\epsilon'_s g, \epsilon'_s) ds). \quad (24)\]

Moreover

\[(L_{\mu^*}^{-1}(N^g, M)_t = \left(\hat{N}_{t}^{g_1}, \ldots, \hat{N}_{t}^{g_n}, L_{\mu^*}^{-1}(M)_t \right),\]

where

\[\hat{N}_{t}^{g_i} = N_{t}^{g_i} - \int_0^t \left(\frac{N_{s}^{g_i}}{2} + \hat{C}_{\mu^*}(L_{\mu^*}^{-1}(M)_s, g_i)\right) ds.\]

4.5 Estimation of \(\epsilon'_t\)

4.5.1 Estimation of \(\|L_{\mu^*}^{-1}(M)_t\|_\Lambda\)

Lemma 4.4 (i) For all \(\alpha \geq 2\), there exists a constant \(K_\alpha\) such that for all \(t \geq 0\),

\[E[\|L_{\mu^*}^{-1}(M)_t\|_\Lambda^{\alpha/2}] \leq K_\alpha.\]

(ii) a.s. there exists \(C\) with \(E[C] < \infty\) such that for all \(t \geq 0\),

\[\|L_{\mu^*}^{-1}(M)_t\|_\Lambda \leq C(1 + t).\]

Proof: Since \(\|L_{\mu^*}^{-1}(M)_t\|_\Lambda \leq K\|L_{\mu}^{-1}(M)_t\|_\Lambda\), we estimate \(\|L_{\mu^*}^{-1}(M)_t\|_\Lambda\). We have

\[dL_{\mu^*}^{-1}(M)_t = dM_t - G_{\mu^*}L_{\mu^*}^{-1}(M)_t dt.\]

Let \(N\) be the martingale defined by

\[N_t = \int_0^t \left\langle \frac{L_{\mu^*}^{-1}(M)_s}{\|L_{\mu^*}^{-1}(M)_s\|_\Lambda}, dM_s \right\rangle_{\hat{\Lambda}}.\]

We have \(\langle N \rangle_t \leq Kt\) for some constant \(K\). Then

\[d\|L_{\mu^*}^{-1}(M)_t\|_\Lambda^2 = 2\|L_{\mu^*}^{-1}(M)_t\|_{\hat{\Lambda}} dN_t - 2\langle L_{\mu^*}^{-1}(M)_t, G_{\mu^*}L_{\mu^*}^{-1}(M)_t \rangle_{\hat{\Lambda}} dt + d \left(\int \langle M(x) \rangle_t \hat{\lambda}(dx) \right).\]

Note that there exists a constant \(K\) such that

\[\frac{d}{dt} \left(\int \langle M(x) \rangle_t \hat{\lambda}(dx) \right) \leq K\]
and that (see hypothesis [2.1])

\[ \langle L^{-1}_\mu(M) t, G_\mu, L^{-1}_\mu(M) t \rangle \geq \kappa \| L^{-1}_\mu(M) t \|_\lambda^2. \]

This implies that

\[ \frac{d}{dt} E[\| L^{-1}_\mu(M) t \|_\lambda^2] \leq -2\kappa E[\| L^{-1}_\mu(M) t \|_\lambda^2] + K \]

which implies (i) for \( \alpha = 2 \). For \( \alpha > 2 \), we find that

\[
\frac{d}{dt} E[\| L^{-1}_\mu(M) t \|_\lambda^{\alpha}] \leq -\alpha \kappa E[\| L^{-1}_\mu(M) t \|_\lambda^{\alpha-2}] + K E[\| L^{-1}_\mu(M) t \|_\lambda^{\alpha-2}] \\
\leq -\alpha \kappa E[\| L^{-1}_\mu(M) t \|_\lambda^{\alpha}] + K E[\| L^{-1}_\mu(M) t \|_\lambda^{\alpha-2}] \\
\]

which implies that \( E[\| L^{-1}_\mu(M) t \|_\lambda^{\alpha}] \) is bounded.

We now prove (ii). Fix \( \alpha > 1 \). Then there exists a constant \( K \) such that

\[
\frac{\| L^{-1}_\mu(M) t \|_\lambda^2}{(1+t)^\alpha} \leq \| L^{-1}_\mu(M) t \|_\lambda^2 + 2 \int_0^t \frac{\| L^{-1}_\mu(M) s \|_\lambda^2}{(1+s)^\alpha} dN_s + K. 
\]

Then BDG inequality implies that

\[
E \left[ \sup_{t \geq 0} \frac{\| L^{-1}_\mu(M) t \|_\lambda^2}{(1+t)^\alpha} \right] \leq K + 2 \sup_{t \geq 0} \left( \int_0^t \frac{K ds}{(1+s)^{2\alpha}} \right)^{1/2}
\]

which is finite. This implies the lemma by taking \( \alpha = 2 \). QED

4.5.2 Estimation of \( \| D_t \|_\lambda \)

Note that \( |e^g_t| \leq Ke^{-t/2}\| D_t \|_\lambda^2 \times \| g \| \). Thus

\[ |e_t^' g| \leq Ke^{-t/2}(1 + t + \| D_t \|_\lambda^2) \times \| g \| . \]

This implies (using lemma [2.2] and the fact that \( 0 < \kappa < 1/2 \))

**Lemma 4.5** There exists \( K \) such that

\[ \left\| \int_0^t e^{-(t-s)G_\mu} e'_s ds \right\| \leq Ke^{-\kappa t} \left( 1 + \int_0^t e^{-\frac{1}{2}(1/2-\kappa)t} \| D_s \|_\lambda^2 ds \right) . \] (25)
This lemma with lemma 4.4-(ii) imply the following

**Lemma 4.6** a.s. there exists $C$ with $E[C] < \infty$ such that

$$\|D_t\|_\lambda \leq C \times \left[ 1 + t + \int_0^t e^{-s/2} \|D_s\|_\lambda^2 ds \right]. \tag{26}$$

**Proof:** First note that

$$\|D_t\|_\lambda \leq \|D'_t\|_\lambda + K(1 + t)e^{-t/2}.$$  

Using the expression of $D'_t$ given by (23), we get

$$\|D'_t\|_\lambda \leq \|L^{-1}_\mu(M)_t\|_\lambda + \left\| \int_0^t e^{-(t-s)}G_{\mu^*} \epsilon'_s ds \right\|$$

$$\leq C(1 + t) + Ke^{-\kappa t} \left( 1 + \int_0^t e^{-(1/2 - \kappa)s} \|D_s\|_\lambda^2 ds \right)$$

which implies the lemma.  QED

**Lemma 4.7** Let $x$ and $\epsilon$ be real functions. If for all $t \geq 0$,

$$x_t \leq \alpha + \int_0^t \epsilon x_s ds,$$

where $\alpha$ is a real constant, then

$$x_t \leq \alpha \exp \left( \int_0^t \epsilon_s ds \right).$$

**Proof:** Similarly to the proof of Gronwall’s lemma, we set $y_t = \int_0^t \epsilon x_s ds$. Then,

$$\dot{y}_t \leq \alpha \epsilon_t + \epsilon_t y_t.$$  

Take $\lambda_t = y_t \exp \left( -\int_0^t \epsilon_s ds \right)$, then

$$\dot{\lambda}_t \leq \alpha \epsilon_t \exp \left( -\int_0^t \epsilon_s ds \right)$$

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and

\[ y_t \leq \alpha \int_0^t \epsilon_s \exp \left( \int_s^t \epsilon_u du \right) ds \leq \alpha \exp \left( \int_0^t \epsilon_u du \right) - \alpha. \]

This implies the lemma. QED

This lemma implies that

\[ \| D_t \|_\lambda \leq C(1 + t) \times \exp \left( C \int_0^t e^{-s/2} \| D_s \|_\lambda ds \right). \]

Since hypothesis [4.3] implies that \( \lim_{s \to \infty} e^{-s/2} \| D_s \|_\lambda = 0 \), this proves that a.s. for all \( \epsilon > 0 \), there exists \( C_\epsilon \) such that

\[ \| D_t \|_\lambda \leq C_\epsilon e^{\epsilon t}. \]

Take \( \epsilon < 1/4 \). Then

\[ \int_0^\infty e^{-s/2} \| D_s \|_\lambda^2 ds \leq C_\epsilon. \]

This implies

**Lemma 4.8** a.s., there exists \( C \) such that for all \( t \),

\[ \| D_t \|_\lambda \leq C(1 + t). \]

### 4.5.3 Estimation of \( \epsilon'_t \)

**Lemma 4.9** a.s. there exists \( C \) such that for all \( f \in C(M) \),

\[ |\epsilon'_t f| \leq C(1 + t)^2 e^{-t/2} \| f \| \]

**Proof:** We have \( |\epsilon'_t f| \leq |\epsilon'_t| + K(1 + t)e^{-t/2} \| f \| \) and

\[ |\epsilon'_t| \leq K \| f \|_\lambda \times e^{-t/2} \| D_t \|_\lambda^2 \leq C \| f \| \times (1 + t)^2 e^{-t/2} \]

by lemma [4.8] QED
4.6 Estimation of $\|D_t - L_{\mu^*}^{-1}(M)_t\|

Lemma 4.10 $\|D_t - L_{\mu^*}^{-1}(M)_t\| \leq Ce^{-\kappa t}$.

Proof: We have $\|D_t - D'_t\| \leq K(1+t)e^{-t/2}$. So to prove this lemma, it suffices to prove that (see the expression of $D'_t$ given by (23))

$$\left\| \int_0^t e^{-(t-s)G_{\mu^*}} \epsilon'_s ds \right\| \leq Ce^{-\kappa t}.$$

This term is dominated by

$$K \int_0^t e^{-\kappa (t-s)} \|\epsilon'_s\| ds.$$

Using the previous lemma, it is also dominated by

$$Ce^{-\kappa t} \int_0^t e^{\kappa s}(1+s)^2 e^{-s/2} ds \leq Ce^{-\kappa t}$$

because $\kappa \in ]0, 1/2]$. The lemma is proved. QED

In addition, for $g = (g_1, \ldots, g_n) \in C(M)^n$, setting

$$\Delta_t g = (\Delta_t g_1, \ldots, \Delta_t g_n),$$

Lemma 4.11 $\|(\Delta_t g, D_t) - (L_{\mu^*}^{-1})^{-1}(N^g, M)_t\| \leq C(1 + \|g\|)e^{-\kappa t}$.

Proof: We have $\|(\Delta_t g, D_t) - (\Delta'_t g, D'_t)\| \leq K(1 + \|g\|)(1+t)e^{-\kappa t}$. So to prove this lemma, using (24), it suffices to prove that

$$\left\| \int_0^t e^{-(t-s)G_{\mu^*}} (\epsilon'_s g, \epsilon'_s) ds \right\| \leq K(1 + \|g\|)e^{-\kappa t}.$$  (27)

Using hypothesis 2.1 and the definition of $G_{\mu^*}^g$, we have that for all positive $t$, $\|e^{-tG_{\mu^*}^g}\| \leq Ke^{-\kappa t}$.

This implies

$$\|e^{-(t-s)G_{\mu^*}^g} (\epsilon'_s g, \epsilon'_s)\| \leq Ke^{\kappa(t-s)} \|\epsilon'_s\| \times (1 + \|g\|).$$

Thus the term (27) is dominated by

$$K(1 + \|g\|) \int_0^t e^{-\kappa (t-s)} \|\epsilon'_s\| ds,$$

from which we prove (27) like in the previous lemma. QED
4.7 Tightness results

We refer the reader to section 5.1.2 in the appendix, where tightness criteria for families of $C(M)$-valued random variables are given. They will be used in this section.

4.7.1 Tightness of $(L_{\mu^*}^{-1}(M))_{t \geq 0}$

In this section we prove the following lemma which in particular implies the tightness of $(D_t)_{t \geq 0}$ and of $(D'_t)_{t \geq 0}$.

**Lemma 4.12** $(L_{\mu^*}^{-1}(M))_{t \geq 0}$ is tight.

**Proof:** We have the relation (that defines $L_{\mu^*}^{-1}(M)$)

$$dL_{\mu^*}^{-1}(M)(x) = -G_{\mu^*}L_{\mu^*}^{-1}(M)(x)dt + dM_t(x).$$

Thus, using the expression of $G_{\mu^*}$

$$dL_{\mu^*}^{-1}(M)(x) = -\frac{1}{2}L_{\mu^*}^{-1}(M)(x)dt + A_t(x)dt + dM_t(x),$$

with

$$A_t(x) = \hat{C}_{\mu^*}(V_x, L_{\mu^*}^{-1}(M)).$$

Since $\mu^*$ is absolutely continuous with respect to $\lambda$, we have that

$$\|A_t\| \leq K\|L_{\mu^*}^{-1}(M)\|_{\lambda}$$

and therefore (using lemma 4.4 (i) for $\alpha = 2$)

$$\sup_{t} E[\|A_t\|^2] < \infty.$$ 

We also have

$$\text{Lip}(A_t) \leq K\|L_{\mu^*}^{-1}(M)\|_{\lambda},$$

where $\text{Lip}(A_t)$ is the Lipschitz constant of $A_t$ (see (38)).

In order to prove this tightness result, we first prove that for all $x$, $(L_{\mu^*}^{-1}(M)(x))_t$ is tight. Setting $Z_t^x = L_{\mu^*}^{-1}(M)(x)$ we have

$$\frac{d}{dt}E[\nu(Z_t^x)^2] \leq -E[(Z_t^x)^2] + 2E[|Z_t^x| \times |A_t(x)|] + \frac{d}{dt}E[(M(x))_t]$$

$$\leq -E[(Z_t^x)^2] + K E[(Z_t^x)^2]^{1/2} + K$$
which implies that \((L^{-1}_\mu(M)_t(x))_t\) is bounded in \(L^2(P)\) and thus tight.

We now estimate \(E[|Z_t^x - Z_t^y|^{1/\alpha}]\) for \(\alpha\) greater than 2 and the dimension of \(M\). Setting \(Z_t^{x,y} = Z_t^x - Z_t^y\), we have

\[
\frac{d}{dt}E[(Z_t^{x,y})^\alpha] \leq -\frac{\alpha}{2}E[(Z_t^{x,y})^\alpha] + \alpha E[|Z_t^{x,y}|^{\alpha-1}|A_t(x) - A_t(y)|] \\
+ \frac{\alpha(\alpha - 1)}{2} E\left[(Z_t^{x,y})^{\alpha-2}\frac{d}{dt}(M(x) - M(y))_t\right] \\
\leq -\frac{\alpha}{2}E[(Z_t^{x,y})^\alpha] + \alpha d(x, y)E[(Z_t^{x,y})^{\alpha-1}\text{Lip}(A_t)] \\
+ Kd(x, y)^2E[(Z_t^{x,y})^{\alpha-2}] \\
\leq -\frac{\alpha}{2}E[(Z_t^{x,y})^\alpha] + Kd(x, y)E[\|Z_t^{x,y}\|^{\alpha-1}L^{\alpha-1}(M)_t] \\
+ Kd(x, y)^2E[(Z_t^{x,y})^{\alpha-2}] \\
\leq -\frac{\alpha}{2}E[(Z_t^{x,y})^\alpha] + Kd(x, y)E[(Z_t^{x,y})^{\alpha-1}]^{\frac{\alpha-2}{\alpha}}E[\|L^{\alpha-1}(M)_t\|^\alpha]^{1/\alpha} \\
+ Kd(x, y)^2E[(Z_t^{x,y})^{\alpha-2}]^{\frac{\alpha-2}{\alpha}}.
\]

Thus, if \(x_t = E[(Z_t^{x,y})^\alpha]/d(x, y)^\alpha\),

\[
\frac{dx_t}{dt} \leq -\frac{\alpha}{2}x_t + Kx_t^{\frac{\alpha-1}{\alpha}} + Kx_t^{\frac{\alpha-2}{\alpha}}.
\]

It is now an exercise to show that \(x_t \leq K\) and so that

\[
E[(Z_t^{x,y})^\alpha]^{1/\alpha} \leq Kd(x, y).
\]

Using corollary 5.3, this completes the proof for the tightness of \((L^{-1}_\mu(M)_t)_t\).

QED

**Remark 4.13** Kolmogorov’s theorem (see theorem 1.4.1 and its proof in Kunita (1990)), with the estimates given in the proof of this lemma, implies that

\[
\sup_t E[\|L^{-1}_\mu(M)_t\|] < \infty.
\]
4.7.2 Tightness of $((L^g_{\mu^*})^{-1}(N^g, M)_t)_{t \geq 0}$

Fix $g = (g_1, \ldots, g_n) \in C(M)^n$. Let $\hat{\Delta}g$ be defined by the relation

$$(\hat{\Delta}g, L^g_{\mu^*}(M)) = (L^g_{\mu^*})^{-1}(N^g, M).$$

Set $A_t g = (A_t g_1, \ldots, A_t g_n)$ with $A_t g_i = \hat{C}_{\mu^*}(g_i, L^g_{\mu^*}(M)_t)$. Then

$$d\hat{\Delta}_t g = dN^g_t - \frac{\hat{\Delta}_t g}{2} dt + \int S^g_t ds.$$  

Thus,

$$\hat{\Delta}_t g = e^{-t/2} \int_0^t e^{-s/2} dN^g_s + e^{-t/2} \int_0^t e^{-s/2} A_s g ds.$$  

Using this expression it is easy to prove that $(\Delta_t g)_{t \geq 0}$ is bounded in $L^2(P)$. This implies, using also lemma 4.12

**Lemma 4.14** $((L^g_{\mu^*})^{-1}(N^g, M)_t)_{t \geq 0}$ is tight.

4.8 Convergence in law of $(N^g, M)_{t+} - (N^g, M)_t$

In this section, we denote by $E_t$ the conditional expectation with respect to $\mathcal{F}_{e_t}$. We also set $Q = Q_{\mu^*}$ and $C = \hat{C}_{\mu^*}$.

4.8.1 Preliminary lemmas.

For $f \in C(M)$ and $t \geq 0$, set $N^{f,t}_s = N^{f,t}_{t+s} - N^{f,t}_t$.

**Lemma 4.15** For all $f$ and $g$ in $C(M)$,

$$\lim_{t \to \infty} \langle N^{f,t}, N^{g,t} \rangle_s = s \times C(f, g).$$

**Proof:** Set

$$G(z) = \langle \nabla Q f, \nabla Q g \rangle(z) - C(f, g)$$

and

$$G_u(z) = \langle \nabla Q_u f, \nabla Q_u g \rangle(z) - C(f, g).$$
We have
\[ \langle N_{f,t}^s, N_{g,t}^s \rangle_s - s \times C(f, g) = \int_{e^t}^{e^{t+s}} G_u(X_u) \frac{du}{u} \]
\[ = \int_{e^t}^{e^{t+s}} (G_u - G)(X_u) \frac{du}{u} + \int_{e^t}^{e^{t+s}} G(X_u) \frac{du}{u}. \]

Integrating by parts, we get that
\[ \int_{e^t}^{e^{t+s}} G(X_u) \frac{du}{u} = (\mu e^{t+s}G - \mu e^tG) + \int_0^s (\mu e^{t+s}G) du. \]

Since \( \mu^*G = 0 \), this converges towards 0 on the event \( \{ \mu_t \to \mu^* \} \). The term \( \int_{e^t}^{e^{t+s}} (G_u - G)(X_u) \frac{du}{u} \) converges towards 0 because \( (\mu, z) \mapsto \nabla Q_{\mu} f(z) \) is continuous. This proves the lemma. QED

Let \( f_1, \ldots, f_n \) be in \( C(M) \). Let \( (t_k) \) be an increasing sequence converging to \( \infty \) such that the conditional law of \( M_{n,k} = (N_{f_1,t_k}, \ldots, N_{f_n,t_k}) \) given \( F_{e^{t_k}} \) converges in law towards a \( \mathbb{R}^n \)-valued process \( W^n = (W_1, \ldots, W_n) \).

**Lemma 4.16** \( W^n \) is a centered Gaussian process such that for all \( i \) and \( j \),
\[ \mathbb{E}[W^n_i(s)W^n_j(t)] = (s \wedge t)C(f_i, f_j). \]

**Proof:** We first prove that \( W^n \) is a martingale. For all \( k, M_{n,k} \) is a martingale. For all \( u \leq v \), Bürkholder-Davies-Gundy inequality (BDG inequality in the following) implies that \( (M_{n,k}^n(v) - M_{n,k}^n(u))^k \) is bounded in \( L^2 \).

Let \( l \geq 1, \varphi \in C(\mathbb{R}^1), 0 \leq s_1 \leq \cdots \leq s_l \leq u \) and \( (i_1, \ldots, i_l) \in \{1, \ldots, n\}^l \). Then for all \( k \) and \( i \in \{1, \ldots, n\} \), the martingale property implies that
\[ \mathbb{E}_{t_k}[M_{i,k}^{n,k}(v) - M_{i,k}^{n,k}(u)] = 0 \]
where \( Z_k \) is of the form
\[ Z_k = \varphi(M_{i_1,k}^{n,k}(s_1), \ldots, M_{i_l,k}^{n,k}(s_l)). \] (28)

Using the convergence of the conditional law of \( M_{n,k} \) given \( F_{e^{t_k}} \) towards the law of \( W^n \) and since \( (M_{i,k}^{n,k}(v) - M_{i,k}^{n,k}(u))^k \) is uniformly integrable (because it is bounded in \( L^2 \)), we prove that
\[ \mathbb{E}[(W^n_i(v) - W^n_i(u))Z] = 0. \]
where $Z$ is of the form

$$Z = \varphi(W^n_i(s_1), \ldots, W^n_i(s_t)).$$  \hfill (29)

This implies that $W^n$ is a martingale.

We now prove that for $(i, j) \in \{1, \ldots, n\}$ (with $C = C_{\mu^s}$),

$$\langle W^n_i, W^n_j \rangle_s = s \times C(f_i, f_j).$$

By definition of $\langle M^{n,k}_i, M^{n,k}_j \rangle$ (in the following $\langle \cdot, \cdot \rangle_u = \langle \cdot, \cdot \rangle_v - \langle \cdot, \cdot \rangle_u$)

$$E_t [((M^{n,k}_i(v) - M^{n,k}_i(u))(M^{n,k}_j(v) - M^{n,k}_j(u))$$

$$- \langle M^{n,k}_i, M^{n,k}_j \rangle_v Z_k] = 0$$

where $Z_k$ is of the form (28). Using the convergence in law and the fact that $(M^{n,k}(v) - M^{n,k}(u))^2$ is bounded in $L^2$ (still using BDG inequality), we prove that as $k \to \infty$,

$$E_t [((M^{n,k}_i(v) - M^{n,k}_i(u))(M^{n,k}_j(v) - M^{n,k}_j(u))Z_k]$$

converges towards

$$E[(W^n_i(v) - W^n_i(u))(W^n_j(v) - W^n_j(u))].$$

with $Z$ of the form (29). Now,

$$E_t [\langle M^{n,k}_i, M^{n,k}_j \rangle_v Z_k] - v \times E[Z] \times C(x_i, x_j)$$

$$= E_t [((M^{n,k}_i, M^{n,k}_j)_v - v \times C(f_i, f_j))Z_k]$$

$$+ v \times (E_t[Z_k] - E[Z]) \times C(f_i, f_j)$$

The convergence in $L^2$ of $\langle M^{n,k}_i, M^{n,k}_j \rangle_v$ towards $v \times C(f_i, f_j)$ shows that the first term converges towards 0. The convergence of the conditional law of $M^{n,k}$ with respect to $\mathcal{F}_{v+k}$ towards $W^n$ shows that the second term converges towards 0. Thus

$$E \left[ ((W^n_i(v) - W^n_i(u))(W^n_j(v) - W^n_j(u)) - (v - u)C(f_i, f_j)) Z \right] = 0.$$

This shows that $\langle W^n_i, W^n_j \rangle_s = s \times C(f_i, f_j)$. We conclude using Lévy’s theorem.  \hfill QED
4.8.2 Convergence in law of $M_{t+} - M_t$

In this section, we denote by $\mathcal{L}_t$ the conditional law of $M_{t+} - M_t$ knowing $\mathcal{F}_t$. Then $\mathcal{L}_t$ is a probability measure on $C(\mathbb{R}^+ \times M)$.

**Proposition 4.17** When $t \to \infty$, $\mathcal{L}_t$ converges weakly towards the law of a $C(M)$-valued Brownian motion of covariance $C_{\mu^*}$.

**Proof**: In the following, we will simply denote $M_{t+} - M_t$ by $M^t$. We first prove that

**Lemma 4.18** \{L_t : t \geq 0\} is tight.

**Proof**: For all $x \in M$, $t$ and $u$ in $\mathbb{R}^+$,

$$
\mathbb{E}(M^t_u(x))^2 = \mathbb{E}\left[\int_t^{t+u} d\langle M(x)\rangle_s\right] \leq Ku.
$$

This implies that for all $u \in \mathbb{R}^+$ and $x \in M$, $(M^t_u(x))_{t \geq 0}$ is tight.

Let $\alpha > 0$. We fix $T > 0$. Then for $(u, x)$ and $(v, y)$ in $[0, T] \times M$, using BDG inequality,

$$
\mathbb{E}(|M^t_u(x) - M^t_v(y)|^\alpha)^{1/\alpha} \leq \mathbb{E}(|M^t_u(x) - M^t_v(y)|^\alpha)^{1/\alpha} + \mathbb{E}(|M^t_v(y) - M^t_v(y)|^\alpha)^{1/\alpha} \leq K_\alpha \times (\sqrt{T}d(x, y) + \sqrt{|v - u|})
$$

where $K_\alpha$ is a positive constant depending only on $\alpha$, $\|V\|$ and Lip($V$) the Lipschitz constant of $V$.

We now let $D_T$ be the distance on $[0, T] \times M$ defined by

$$
D_T((u, x), (v, y)) = K_\alpha \times (\sqrt{T}d(x, y) + \sqrt{|v - u|}).
$$

The covering number $N([0, T] \times M, D_T, \epsilon)$ is of order $\epsilon^{-d-1/2}$ as $\epsilon \to 0$. Taking $\alpha > d + 1/2$, we conclude using corollary 5.3. **QED**

Let $(t_k)$ be an increasing sequence converging to $\infty$ and $N$ a $C(M)$-valued random process (or a $C(\mathbb{R}^+ \times M)$ random variable) such that $\mathcal{L}_{t_k}$ converges in law towards $N$.

**Lemma 4.19** $N$ is a $C(M)$-valued Brownian motion of covariance $C_{\mu^*}$.

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Proof: Let $W$ be a $C(M)$-valued Brownian motion of covariance $C_{\mu^*}$. Using lemma 4.16, we prove that for all $(x_1, \ldots, x_n) \in M^n$, $(N(x_1), \ldots, N(x_n))$ has the same distribution as $(W(x_1), \ldots, X(x_n))$. This implies the lemma.

QED

Since $\{\mathcal{L}_t\}$ is tight, this lemma implies that $\mathcal{L}_t$ converges weakly towards the law of a $C(M)$-valued Brownian motion of covariance $C_{\mu^*}$. QED

4.8.3 Convergence in law of $(N^g, M)_t + \cdot - (N^g, M)_t$

In this section, we fix $g = (g_1, \ldots, g_n) \in C(M)^n$ and we denote by $\mathcal{L}^g_t$ the conditional law of $(N^g, M)_t + \cdot - (N^g, M)_t$ knowing $\mathcal{F}_t$. Then $\mathcal{L}^g_t$ is a probability measure on $C(\mathbb{R}^+ \times M \cup \{1, \ldots, n\})$. In the following we will denote $(N^g(t), M(t))$ the process $(N^g, M)_t + \cdot - (N^g, M)_t$.

Let $(W^g_{t}(t,x))_{t \in \mathbb{R}^+}$ be a $X(M)$-valued Brownian motion of covariance $\hat{C}_{\mu^*}$. Denoting $W_t(x) = W^V_{x,t}$, then $W = (W_t(x))_{t \in \mathbb{R}^+}$ is a $C(M)$-valued Brownian motion of covariance $C_{\mu^*}$. For $g = (g_1, \ldots, g_n) \in C(M)^n$, $W^g$ will denote $(W^{g_1}, \ldots, W^{g_n})$. In the following we will simply denote $(W^g, W)$ the process $(W^g_t, W_t(x))_{x \in M, t \geq 0}$.

Proposition 4.20 As $t$ goes to $\infty$, $\mathcal{L}^g_t$ converges weakly towards the law of $(W^g, W)$.

Proof: We first prove that

Lemma 4.21 $\{\mathcal{L}^g_t : t \geq 0\}$ is tight.

Proof: This is a straightforward consequence of the tightness of $\{\mathcal{L}_t\}$ and of the fact that for all $\alpha > 0$, there exists $K_\alpha$ such that for all nonnegative $u$ and $v$, $E_t[|N^g_u - N^g_v|^\alpha]^{\frac{1}{\alpha}} \leq K_\alpha \sqrt{|v - u|}$. QED

Let $(t_k)$ be an increasing sequence converging to $\infty$ and $(\tilde{N}^g, \tilde{M})$ a $\mathbb{R}^n \times C(M)$-valued random process (or a $C(\mathbb{R}^+ \times M \cup \{1, \ldots, n\})$ random variable) such that $\mathcal{L}^g_{t_k}$ converges in law towards $(\tilde{N}^g, \tilde{M})$. Then lemmas 4.13 and 4.10 imply that $(\tilde{N}^g, \tilde{M})$ has the same law as $(W^g, W)$. Since $\{\mathcal{L}^g_t\}$ is tight, $\mathcal{L}^g_t$ converges towards the law of $(W^g, W)$. QED

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4.9 Convergence in law of $D$

4.9.1 Convergence in law of $(D_{t+s} - e^{-sG_{\mu^*}} D_t)_{s \geq 0}$

We have

$$D'_{t+s} - e^{-sG_{\mu^*}} D'_t = L_{\mu^*}^{-1}(M^t)_s + \int_0^s e^{-(s-u)G_{\mu^*}} e'_{t+u} du.$$  

Since (using lemma 4.9)

$$\left\| \int_0^s e^{-(s-u)G_{\mu^*}} e'_{t+u} du \right\| \leq Ke^{-\kappa t}$$

and $\|D_t - D'_t\| \leq K(1+t)e^{-t/2}$, this proves that $(D_{t+s} - e^{-sG_{\mu^*}} D_t - L_{\mu^*}^{-1}(M_{t+s} - M_t))_{s \geq 0}$ converges towards 0. Since $L_{\mu^*}^{-1}$ is continuous, this proves that the law of $L_{\mu^*}^{-1}(M_{t+s} - M_t)$ converges weakly towards $L_{\mu^*}^{-1}(W)$. Since $L_{\mu^*}^{-1}(W)$ is an Ornstein-Uhlenbeck process of covariance $C_{\mu^*}$ and drift $-G_{\mu^*}$ started from 0, we have

**Theorem 4.22** The conditional law of $(D_{t+s} - e^{-sG_{\mu^*}} D_t)_{s \geq 0}$ given $\mathcal{F}_{t'}$ converges weakly towards an Ornstein-Uhlenbeck process of covariance $C_{\mu^*}$ and drift $-G_{\mu^*}$ started from 0.

4.9.2 Convergence in law of $D_{t+}$.

We can now prove theorem 4.3. We here denote by $P_t$ the semigroup of an Ornstein-Uhlenbeck process of covariance $C_{\mu^*}$ and drift $-G_{\mu^*}$, and we denote by $\pi$ its invariant probability measure.

We know that (as $t \to \infty$) $(D_{t+s} - e^{-sG_{\mu^*}} D_t)_{s \geq 0}$ converges in law towards $L_{\mu^*}^{-1}(W)$, where $W$ is a $C(M)$-valued Brownian motion of covariance $C_{\mu^*}$. Since $(D_t)_{t \geq 0}$ is tight, there exists $\nu \in \mathcal{P}(C(M))$ and an increasing sequence $t_n$ converging towards $\infty$ such that $D_{t_n}$ converges in law towards $\nu$. Then $D_{t_n+}$ converges in law towards $(L_{\mu^*}^{-1}(W)_s + e^{-sG_{\mu^*}} Z_0)$, with $Z_0$ independent of $W$ and distributed like $\nu$. This proves that $D_{t_n+}$ converges in law towards an Ornstein-Uhlenbeck process of covariance $C_{\mu^*}$ and drift $-G_{\mu^*}$.

We now fix $t > 0$. Let $s_n$ be a subsequence of $t_n$ such that $D_{s_n-t+}$ converges in law. Then $D_{s_n-t}$ converges towards a law we denote by $\nu_t$ and $D_{s_n-t+}$ converges in law towards an Ornstein-Uhlenbeck process of covariance $C_{\mu^*}$ and drift $-G_{\mu^*}$. Since $D_{s_n} = D_{s_n-t+t}$, $D_{s_n}$ converges in law towards $\nu_t P_t$. On the other hand $D_{s_n}$ converges in law towards $\nu$. Thus $\nu_t P_t = \nu$.  

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Let $\varphi$ be a Lipschitz bounded function on $C(M)$. Then
\[
|\nu_t P_t \varphi - \pi \varphi| = \left| \int (P_t \varphi(f) - \pi \varphi) \nu_t(df) \right|
\leq \int |P_t \varphi(f) - P_t \varphi(0)| \nu_t(df) + |P_t \varphi(0) - \pi \varphi|
\]
where the second term converges towards 0 (using (37)) and the first term is dominated by (using lemma 5.15)
\[
K e^{-\kappa t} \int \|f\| \nu_t(df).
\]

It is easy to check that
\[
\int \|f\| \nu_t(df) = \lim_{k \to \infty} \int (\|f\| \wedge k) \nu_t(df)
= \lim_{k \to \infty} \lim_{n \to \infty} E[\|D_{s_n-t}\| \wedge k]
\leq \lim_{n \to \infty} E[\|D_{s_n-t}\|]
\leq \sup_{\tau} E[\|D_{\tau}\|].
\]

Since
\[
\|D_{\tau}\| \leq \|D_{\tau}^1 + D_{\tau}^5\| + \|L_{\mu^-}^{-1}(M)_{\tau}\|
+ \left\| \int_0^t e^{(t-s)G_{\mu^*}} \epsilon_s' ds \right\|,
\]
using the estimates (22), the proof of lemma 4.10 and remark 4.13, we get that
\[
\sup_{\tau \geq 0} E[\|D_{\tau}\|] < \infty.
\]

Taking the limit, we prove $\nu \varphi = \pi \varphi$ for all Lipschitz bounded function $\varphi$ on $C(M)$. This implies $\nu = \pi$, which proves the theorem. \textbf{QED}

4.9.3 Convergence in law of $D^g$

We can also prove theorem 3.2.
For \( g = (g_1, \ldots, g_n) \in C(M)^n \), we set \( D_t^g = (\Delta_t g, D_t) \), and \( D_t'^g = (\Delta_t' g, D_t') \). Since \( \|D_t^g - D_t'^g\| \leq K(1 + t)e^{-t/2} \), instead of studying \( D_t^g \), we can only study \( D_t'^g \). Then

\[
D_{t+s}^g - e^{-sG^g} D_t^g = (L_{\mu^*}^g)^{-1}(N^{g,t}, M^t)_s + \int_0^s e^{-(s-u)G^g} (\epsilon_{t+u}^g, \epsilon_{t+u}^g)du.
\]

The norm of the second term of the right hand side (using the proof of lemma 4.11) is dominated by

\[
\leq K(1 + \|g\|) \int_0^s e^{-\kappa(s-u)} \|\epsilon_{t+u}\| du
\]

\[
\leq K \int_0^s e^{-\kappa(s-u)}(1 + t + u)^2 e^{-(t+u)/2} du 
\]

\[
\leq ke^{-\kappa t}
\]

Like in section 4.9.1, since \((L_{\mu^*}^g)^{-1}(W^g, W)\) is an Ornstein-Uhlenbeck process of covariance \( C_{\mu^*}^g \) and drift \(-G_{\mu^*}^g\), started from 0,

**Theorem 4.23** The conditional law of \(((\Delta^g, D)_{t+s} - e^{-sG^g} (\Delta^g, D)_t)_{s \geq 0}\) given \( F_t \) converges weakly towards an Ornstein-Uhlenbeck process of covariance \( C_{\mu^*}^g \) and drift \(-G_{\mu^*}^g\), started from 0.

From this theorem, like in section 4.9.2, we prove theorem 3.2. QED

5 Appendix : Random variables and Ornstein-Uhlenbeck processes on \( C(M) \)

5.1 \( C(M) \)-valued random variables

5.1.1 Generalities

Let \((M, d)\) be a compact metric space (note that there is no assumption here that \( M \) is a manifold), \( C(M) \) the space of real valued continuous functions on \( M \) equipped with the uniform norm \( \|f\| = \sup_{x \in M} |f(x)| \). By classical results, \( C(M) \) is a separable (see e.g. [16]) Banach space (see e.g. [14] or [13]) and its topological dual is the space \( \mathcal{M}(M) \) of bounded signed measures on \( M \).
For $\mu \in \mathcal{M}(M)$ and $f \in C(M)$ we use the notation $\mu f = \langle \mu, f \rangle = \int_M f d\mu$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A $C(M)$-valued random variable is a Borel map $F : \Omega \rightarrow C(M)$.

For $x \in M$, let $\pi_x : C(M) \rightarrow \mathbb{R}$, denote the projection defined by $\pi_x(f) = f(x)$.

**Lemma 5.1** The Borel $\sigma$-field on $C(M)$ is the $\sigma$-field generated by the maps $\{\pi_x\}_{x \in M}$. In particular

(i) A map $F : \Omega \rightarrow C(M)$ is a $C(M)$-valued random variable if and only if $\{\pi_x(F)\}_{x \in M}$ is a family of real valued random variables.

(ii) The law of a $C(M)$-valued random variable is determined by its finite dimensional distributions (i.e. the law of $\{\pi_x(F)\}_{x \in I}$ with $I \subset M$ finite).

**Proof:** Let $\mathcal{A} = \sigma\{\pi_x, x \in M\}$ and $\mathcal{B}$ the Borel $\sigma$-field on $C(M)$. The maps $\pi_x$ being continuous, $\mathcal{B}$ contains $\mathcal{A}$. Conversely, let $B_f(r) = \{g \in C(M) : \|g - f\| \leq r\}$ and let $S$ be a countable dense subset of $M$. Then $B_f(r) = \cap_{x \in S} \{g \in C(M) : |\pi_x(f) - \pi_x(g)| \leq r\}$ Hence $B_f(r) \in \mathcal{A}$. Since $C(M)$ is separable, $\mathcal{B}$ is generated by the sets $\{B_f(r), f \in C(M), r \geq 0\}$.

Q.E.D.

### 5.1.2 Tightness criteria

Let $\mathcal{P}(C(M))$ be the space of Borel probability measures on $C(M)$. An element $\nu$ of $\mathcal{P}(C(M))$ is the law of a $C(M)$-valued random variable $F$, and $\nu = \mathbb{P}_F$. Recall that a sequence $\{\nu^n\}$ in $\mathcal{P}(C(M))$ is said converging weakly towards $\nu \in \mathcal{P}(C(M))$ if $\int \varphi d\nu_n \rightarrow \int \varphi d\nu$ for every bounded and continuous function $\varphi : C(M) \rightarrow \mathbb{R}$. A sequence $\{F_n\}$ of $C(M)$-valued random variable is said converging in law towards $F$ a $C(M)$-valued random variable if $\{\mathbb{P}_{F_n}\}$ converges in law towards $\mathbb{P}_F$. A family $\mathcal{X} \subset \mathcal{P}(C(M))$ is said to be tight if for every $\epsilon > 0$ there exists some compact set $K \subset C(M)$ such that $\mathbb{P}(K) \geq 1 - \epsilon$ for all $\mathbb{P} \in \mathcal{X}$. A family of random variables is said to be tight if the family of their laws is tight.
Since $C(M)$ is a separable and complete, Prohorov theorem \cite{8} asserts that $\mathcal{X} \subset \mathcal{P}(C(M))$ is tight if and only if it is relatively compact.

The next proposition gives a useful criterium for a class of random variables to be tight. It follows directly from \cite{14} (Corollary 11.7 p. 307 and the remark following Theorem 11.2). A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Young function if it is convex, increasing and $\psi(0) = 0$. If $Z$ is a real valued random variable, we let

$$\|Z\|_\psi = \inf\{c > 0 : \mathbb{E}(\psi(|Z|/c)) \leq 1\}.$$  

For $\epsilon > 0$, we denote by $N(M, d, \epsilon)$ the covering number of $E$ by balls of radius less than $\epsilon$ (i.e. the minimal number of balls of radius less than $\epsilon$ that cover $E$), and by $D$ the diameter of $M$.

**Proposition 5.2** Let $(F_t)_{t \in I}$ be a family of $C(M)$-valued random variables and $\psi$ a Young function. Assume that

(i) There exists $x \in E$ such that $(F_t(x))_{t \in I}$ is tight;

(ii) $\|F_t(x) - F_t(y)\|_\psi \leq Kd(x, y)$;

(iii) $\int_0^D \psi^{-1}(N(M, d, \epsilon))d\epsilon < \infty$.

Then $(F_t)_{t \geq 0}$ is tight.

**Corollary 5.3** Suppose $M$ is a compact finite dimensional manifold of dimension $r$, $d$ the Riemannian distance, and

$$\mathbb{E}[|F_t(x) - F_t(y)|^\alpha]^{1/\alpha} \leq Kd(x, y)$$

for some $\alpha > r$. Then conditions (ii) and (iii) of Proposition 5.2 hold true.

**Proof:** One has $N(E, d, \epsilon)$ is of order $\epsilon^{-r}$; and for $\psi(x) = x^\alpha$, $\| \cdot \|_\psi$ is the $L^\alpha$ norm. Hence the result.  \[QED\]
5.1.3 \( C(M) \)-valued Gaussian variable

Recall that a (centered) real-valued random variable \( Y \) with variance \( \sigma^2 \) is said to be Gaussian if it has distribution

\[
P_Y(dx) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right)dx.
\]

Its characteristic function is then

\[
\Phi_Y(t) = \mathbb{E}[\exp(itY)] = \exp\left(-\frac{t^2\sigma^2}{2}\right).
\]

Here we adopt the convention that the zero function \( (Y = 0) \) is Gaussian with variance 0 and that all the Gaussian random variables are centered.

A family \( \{Y_i\}_{i \in I} \) of real-valued random variables is said to be Gaussian if for all finite set \( J \subset I \) and for all \( \alpha \in \mathbb{R}^J, \sum_{j \in J} \alpha_j Y_j \) is Gaussian.

A \( C(M) \)-valued random variable \( F \) is said to be Gaussian if for all \( \mu \in \mathcal{M}(M), \langle \mu, F \rangle \) is Gaussian.

Lemma 5.4 A \( C(M) \)-valued random variable \( F \) is Gaussian if and only if the family \( \{\pi_x(F)\} \) is Gaussian.

Proof: The direct implication is obvious. We prove the second. Assume that \( \{\pi_x(F)\} \) is a Gaussian family. Let \( \mu \) be a probability over \( M \). By the strong law of large number and the separability of \( C(M) \) there exists a nonempty set \( \Lambda \subset M^N \) (actually \( \Lambda \) has \( \mu^N \) measure 1) such that for all \( (x_i) \in \Lambda \) and all \( f \in C(M) \)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \langle \mu, f \rangle.
\]

In particular \( Y_n \to \langle \mu, F \rangle \) where

\[
Y_n = \frac{1}{n} \sum_{i=1}^{n} F(x_i).
\]

And, by Lebesgue theorem, \( \Phi_{Y_n}(t) \to \Phi_{\langle \mu, F \rangle}(t) \). Since, by assumption \( Y_n \) is Gaussian, \( \Phi_{Y_n}(t) = \exp(-t^2\sigma_n^2/2) \). Let \( \sigma \in [0, \infty) \) be a limit point of \( (\sigma_n) \). Then \( \Phi_{\langle \mu, F \rangle}(t) = \exp(-t^2\sigma^2/2) \). This proves that \( \sigma < \infty \) (a characteristic function being continuous) and that \( \langle \mu, F \rangle \) is Gaussian.
If now $\mu \in \mathcal{M}(M)$ by Jordan-Hann decomposition we may write $\mu = a_1 \mu_1 - b_2 \mu_2$ with $a, b \geq 0$ and $\mu_1, \mu_2$ probabilities. It follows from what precede that $\langle \mu, X \rangle$ is Gaussian. \textbf{QED}

Given a $C(M)$-valued Gaussian random variable $F$ we let $\text{Var}_F : \mathcal{M}(M) \to \mathbb{R}$ denote the variance function of $F$ defined by

$$\text{Var}_F(\mu) = \mathbb{E}(\langle \mu, F \rangle^2).$$

In view of lemma 5.1 (ii), the law of $F$ is entirely determined by its variance function.

A useful property of Gaussian variables is the following.

\textbf{Lemma 5.5} Let $M'$ be another compact metric space and $A : C(M) \to C(M')$ a bounded linear operator. Let $F$ be a $C(M)$-valued Gaussian random variable. Then $AF$ is a $C(M')$-valued Gaussian random variable with variance

$$\text{Var}_{AF} = \text{Var}_F \circ A^*$$

where $A^* : \mathcal{M}(M') \to \mathcal{M}(M)$ is the adjoint of $A$.

\textbf{Proof:} follows from the duality $\langle \mu, AF \rangle = \langle A^* \mu, F \rangle$ and the definitions. \textbf{QED}

\section{5.2 Brownian motions on $C(M)$.}

Let $C : M \times M \to \mathbb{R}$ be a continuous symmetric (i.e $C(x, y) = C(y, x)$) function such that

$$\sum_{ij} a_i a_j C(x_i, x_j) \geq 0$$

for every finite sequence $(a_i, x_i)$ with $a_i \in \mathbb{R}$ and $x_i \in M$. Such a function is sometimes called a \textit{Mercer kernel}.

A Brownian motion on $C(M)$ with covariance $C$ is a $C(M)$-valued stochastic process $W = \{W_t\}_{t \geq 0}$ such that $W_0 = 0$ and for each $T \geq 0$, $W^T = \{W_t(x) : t \leq T, x \in M\}$ is a $C([0, T] \times M)$-valued Gaussian random variable with variance

$$\text{Var}_{W^T}(\nu) = \int_{([0,T] \times M)^2} (s \wedge t) C(x, y) \nu(dx)\nu(dy).$$
or equivalently
\[ E(W_t(x)W_s(y)) = (s \land t)C(x, y). \]

Let
\[ d_C(x, y) = \sqrt{C(x, x) - 2C(x, y) + C(y, y)}. \]
The \( d_C \) is a pseudo-distance on \( M \). For \( \epsilon > 0 \), let
\[ \omega_{C}(\epsilon) = \sup \{ \eta > 0 : d(x, y) \leq \eta \Rightarrow d_C(x, y) \leq \epsilon \}. \]
Then \( N(M, d; \omega_{C}(\epsilon)) \geq N(M, d_C; \epsilon) \).

**Hypothesis 5.6**
\[ \int_{0}^{1} \log(N(d, M; \omega_{C}(\epsilon))) \, d\epsilon < \infty \]
where \( N(d, M; \eta) \) is the covering number of \( M \) by balls of radius less than \( \eta \).

**Remark 5.7** Assume that \( M \) is a compact finite dimensional manifold and that \( d_C(x, y) \leq Kd(x, y)^{\alpha} \) for some \( \alpha > 0 \). Then \( \omega_{C}(\epsilon) \leq \left( \frac{\epsilon}{K} \right)^{1/\alpha} \) and
\[ N(d, M; \eta) = O(\eta^{-\dim(M)}) \]
so that the preceding hypothesis holds.

**Proposition 5.8** Under hypothesis 5.6 there exists a Brownian motion on \( C(M) \) with covariance \( C \).

**Proof:** By Mercer Theorem (see e.g. [10]) there exists a countable family of functions \( \Psi_i \in C(M), i \in \mathbb{N} \), such that
\[ C(x, y) = \sum_i \Psi_i(x)\Psi_i(y) \]
and the convergence is uniform. Let \( B^i, i \in \mathbb{N} \), be a family of independent standard Brownian motions. Set
\[ W^n_t(x) = \sum_{i \leq n} B^i_t \Psi_i(x), n \geq 0. \]
Then, for each \((t, x) \in \mathbb{R}^+ \times M\), the sequence \((W^n_t(x))_{n \geq 1}\) is a Martingale. It is furthermore bounded in \( L^2 \) since
\[ E[(W^n_t(x))^2] = t \sum_{i \leq n} \Psi_i(x)^2 \leq tC(x, x). \]
Hence by Doob’s convergence theorem one may define

\[ W_t(x) = \sum_{i \geq 0} B_i^t \Psi_i(x). \]

Let now \( S \subset \mathbb{R}^+ \times M \) be a countable and dense set. It is easily checked that the family \( (W_t(x))_{(t,x) \in S} \) is a centered Gaussian family with covariance given by

\[ \mathbb{E}[W_s(x)W_t(y)] = (s \wedge t)C(x,y), \]

In particular, for \( t \geq s \)

\[ \mathbb{E}[(W_s(x) - W_t(y))^2] = sC(x,x) - 2sC(x,y) + tC(y,y) \]
\[ = sdC(x,y)^2 + (t-s)C(y,y) \]
\[ \leq K(t-s) + sdC(x,y)^2 \]

This later bound combined with classical results on Gaussian processes (see e.g. Theorem 11.17 in [14]) implies that \( (t,x) \mapsto W_t(x) \) admits a version uniformly continuous over \( S_T = \{(t,x) \in S : t \leq T\} \). By density it can be extended to a continuous (in \( (t,x) \)) process

\[ W = (W_t(x))_{(t,x) \in \mathbb{R}^+ \times M} \]

The process \( W \) can be viewed as a \( C(M) \)-valued continuous random process with the desired covariance. \( \text{QED} \)

### 5.3 Ornstein-Ulhenbeck processes

Let \( A : C(M) \to C(M) \) be a bounded operator and \( W \) a \( C(M) \)-valued Brownian motion with covariance \( C \) as defined in the preceding section.

An Ornstein-Ulhenbeck process with drift \( A \), covariance \( C \) and initial condition \( F_0 = f \in C(M) \) is defined to be a \( C(M) \) valued stochastic process continuous in \( t \), such that

\[ F_t - f = \int_0^t AF_s ds + W_t. \]  \hspace{1cm} (31)

Note that we may think of \( F \) as the solution to the “stochastic differential equation” on \( C(M) \):

\[ dF_t = AF_t dt + dW_t \]
with initial condition $F_0 = f \in C(M)$.

Our aim here is to construct such a solution and state some of its properties.

We let $(e^{tA})_{t \in \mathbb{R}}$ denote the linear flow induced by $A$. Recall that for each $t$, $e^{tA}$ is the bounded operator on $C(M)$ defined by

$$e^{tA} = \sum_{k \in \mathbb{N}} \frac{(tA)^k}{k!}.$$ 

Given $T > 0$ we let $L_A : C(\mathbb{R}^+ \times M) \to C(\mathbb{R}^+ \times M)$ be defined by

$$L_A(f)_t = f_t - f_0 - \int_0^t Af_s ds, \quad t \geq 0. \quad (32)$$

Given $T > 0$ we let $L_A^T : C([0, T] \times M) \to C([0, T] \times M)$ be defined by

$$L_A^T(f)_t = f_t - f_0 - \int_0^t Af_s ds, \quad 0 \leq t \leq T. \quad (33)$$

Note that if for $f \in C(\mathbb{R}^+ \times M)$, we let $f^T \in C([0, T] \times M)$ be defined by $f^T_t = f_t$ for $t \in [0, T]$.

**Lemma 5.9** $L_A^T$ is a bounded operator and its restriction to $C_0([0, T] \times M) = \{ f \in C([0, T] \times M) : f_0 = 0 \}$ is bijective with inverse $(L_A^T)^{-1}$ defined by

$$(L_A^T)^{-1}(g)_t = g_t + \int_0^t e^{(t-s)A}Ag_s ds, \quad t \in [0, T]. \quad (34)$$

**Proof:** Linearity of $L_A^T$ is obvious. Also

$$\|L_A^T(f)\| \leq (2 + T\|A\|)\|f\|.$$  

This proves that $L_A^T$ is bounded.

Observe that $L_A^T(f) = 0$ implies that $f_t = e^{tA}f_0$. Hence $L_A^T$ restricted to $C_0([0, T] \times M)$ is injective. Let $g \in C_0([0, T] \times M)$ and let $f_t$ be given by the left hand side of (34). Then

$$h_t = L_A^T(f)_t - g_t = \int_0^t e^{(t-s)A}Ag_s ds - \int_0^t Af_s ds.$$  

It is easily seen that $h$ is differentiable and that $\frac{d}{dt}h_t = 0$. This proves that $h_t = h_0 = 0$.  

**QED**

We also have
Lemma 5.10  The restriction of $L_A$ to $C_0(\mathbb{R}^+ \times M) = \{ f \in C(\mathbb{R}^+ \times M) : f_0 = 0 \}$ is bijective with inverse $(L_A)^{-1}$ defined by
\[ L_A^{-1}(g)_t = g_t + \int_0^t e^{(t-s)A}Ag_s ds. \] (35)

The next lemma easily follows.

Lemma 5.11  For all $f \in C(M)$ and $g \in C_0(\mathbb{R}^+ \times M)$ the solution to
\[ f_t = f + \int_0^t Af_s ds + g_t, \]
is given by
\[ f_t = e^{tA}f + L_A^{-1}(g)_t. \]

If now $W$ is a $C(M)$-valued Brownian motion as defined in the preceding section, one may define
\[ F_t = e^{tA}f + L_A^{-1}(W)_t. \]
Such a process is the unique solution to (31). Note that, by Lemma 5.5
\[(F_t - e^{tA}f)_{t \leq T}\] is a $C_0([0,T] \times M)$-valued Gaussian random variable. In particular

Proposition 5.12  Let $(F_t)$ be the solution to (31) with initial condition $F_0 = 0$. Then for each $t \geq 0$, $F_t$ is a $C(M)$-valued Gaussian random variable with variance
\[ \text{Var}_{F_t}(\mu) = \int_0^t \langle \mu, e^{sA}Ce^{sA^*} \mu \rangle ds. \]

where $C : \mathcal{M}(M) \to C(M)$ is the operator defined by $C \mu(x) = \int_M C(x,y)\mu(dy)$.

Proof: Fix $T > 0$. To shorten notation let $G : C_0([0,T] \times M) \to C(M)$ be the operator defined by
\[ G(g) = (L_A^T)^{-1}(g)_T. \]
Hence $F_T = G(W^T)$. By Lemma 5.3, $F_T$ is Gaussian with variance
\[ \text{Var}_{F_T} = \text{Var}_{W^T} \circ G^*. \]
Now, for all $\nu \in \mathcal{M}([0,T] \times M)$
\[ \text{Var}_{W^T}(\nu) = \langle \nu, C\nu \rangle \]
where
\[ C_\nu(s, x) = \int_{[0, T] \times M} (s \wedge u) C(x, y) \nu(dy) \]

Thus
\[ \text{Var}_{F_T}(\mu) = \langle \mu, GC\mu \rangle. \]

Our next goal is to compute \( GC\mu \). It easily follows from the definition of \( G \) that
\[ G^*\mu = \delta_T \otimes \mu + dsA^*e^{(T-s)A^*}\mu. \]
Thus (integrating by parts)
\[ CG^*\mu = C\nu_1 + C\nu_2 \]
with \( \nu_1 = \delta_T \otimes \mu \) and \( \nu_2 = dsA^*e^{(T-s)A^*}\mu \). One has
\[ C\nu_1(s, x) = s(C\mu)(x); \]
\[ C\nu_2(s, x) = \int_0^T (s \wedge u)(Cm_u(\mu))(x)du \]
with \( m_u(\mu) = -e^{(T-u)A^*}\mu \) and \( m_u(\mu) \) stands for the derivative of \( u \mapsto m_u(\mu) \).
Thus
\[ C\nu_2(s, x) = -sC\mu - \int_0^s Cm_u(\mu)(x)du \]
and
\[ CG^*\mu(s, x) = -\int_0^s Cm_u(\mu)(x)du. \]
Set
\[ h_s(x) = \int_0^s Cm_u(\mu)(x)du. \]
Then
\[ GC\mu = h_T + \int_0^T e^{(T-s)A}Ah_s ds = \int_0^T e^{(T-s)A}h_s ds \]
\[ = \int_0^T e^{(T-s)A}Ce^{(T-s)A^*}\mu ds = \int_0^T e^{sA}Ce^{sA^*}\mu ds. \]

QED
5.3.1 Asymptotic Behaviour

Let \( \lambda(A) = \lim_{t \to \infty} \frac{\log(\|e^{tA}\|)}{t} \) which exists by subadditivity. Then for some constant \( K < \infty, \|e^{tA}\| \leq Ke^{\lambda(A)t} \) for all positive \( t \). Let \((F_t)\) denote the solution to (31), with \( F_0 = f \in C(M) \).

**Corollary 5.13** Assume \( \lambda(A) < 0 \). Then for each \( \mu \in \mathcal{M}(M) \) \((\langle \mu, F_t \rangle)\) converges in law toward a Gaussian random variable with variance \( V(\mu) = \int_0^\infty \langle \mu, e^{sA}Ce^{sA^*} \mu \rangle \, ds \).

*Proof:* follows from proposition 5.12 and Lemma 5.11. QED

**Corollary 5.14** Assume that \( \lambda(A) < 0 \). Set

\[
d_V(x,y) = \sqrt{V(\delta_x - \delta_y)}
\]

and

\[
\omega_V(\varepsilon) = \sup\{ \eta > 0 : d(x,y) \leq \eta \Rightarrow d_V(x,y) \leq \varepsilon \}.
\]

Assume furthermore that \( \omega_V \) verifies the condition expressed by hypothesis 5.6. Then \((F_t)\) converges in law toward a \( C(M) \)-valued Gaussian random variable with variance \( V \).

*Proof:* Let \( \nu_t \) denote the law of \( F_t \). Corollary 5.13 and lemma 5.1 imply that every limit point of \( \{\nu_t\} \) (for the weak* topology) is the law of a \( C(M) \)-valued Gaussian variable with variance \( V \). The proof then reduces to show that \( (\nu_t) \) is relatively compact or equivalently that \( \{F_t\} \) is tight. We use Proposition 5.2. The first condition follows from Lemma 5.13. Let \( \psi(x) = e^{x^2} - 1 \). It is easily verified that for any real valued Gaussian random variable \( Z \) with variance \( \sigma^2, \|Z\|_\psi = \sigma \sqrt{8/3} \). Hence \( \|F_t(x) - F_t(y)\|_\psi \leq 2d_V(x,y) \) so that condition (ii) holds with the pseudo distance \( d_V \). By definition of \( \omega_V \), \( N(M,d;\omega_V(\varepsilon)) \geq N(M,d_V;\varepsilon) \) and since \( \psi^{-1}(u) = \sqrt{\log(u - 1)} \) condition (iii) is verified. QED

Denote by \( P_t \) the semigroup associated to an Ornstein-Uhlenbeck process of covariance \( C \) and drift \( A \). Then for all bounded measurable \( \varphi : C(M) \to \mathbb{R} \) and \( f \in C(M) \),

\[
P_t \varphi(f) = \mathbb{E}[\varphi(F_t)]. \tag{36}
\]
Denote by $\pi$ the law of a $C(M)$-valued Gaussian random variable with variance $V$. Then $\pi$ is the invariant probability measure of $P_t$, i.e. $\pi P_t = \pi$. Corollary 5.14 implies that, when $\lambda(A) < 0$, for all $f \in C(M)$ and all bounded continuous $\varphi : C(M) \to \mathbb{R}$,
\[
\lim_{t \to \infty} P_t \varphi(f) = \pi \varphi. \quad (37)
\]

Even though we don’t have the speed of convergence in the previous limit, we have

**Lemma 5.15** Assume that $\lambda(A) < 0$. For all bounded Lipschitz continuous $\varphi : C(M) \to \mathbb{R}$, all $f$ and $g$ in $C(M)$,
\[
|P_t \varphi(f) - P_t \varphi(g)| \leq K e^{\lambda(A)t} \|f - g\|.
\]

**Proof:** We have $P_t \varphi(f) = E[\varphi(L_A^{-1}(W) + e^{tA}f)]$. So, using the fact that $\varphi$ is Lipschitz,
\[
|P_t \varphi(f) - P_t \varphi(g)| \leq K \|e^{tA}(f - g)\| \leq K e^{\lambda(A)t} \|f - g\|.
\]

This proves the lemma.  \textbf{QED}

To conclude this section we give a set of simple sufficient conditions ensuring that the hypotheses of corollary 5.14 are satisfied.

For $f \in C(M)$ we let
\[
\text{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} \in \mathbb{R}^+ \cup \{\infty\}. \quad (38)
\]

A map $f$ is said to be Lipschitz provided $\text{Lip}(f) < \infty$.

**Proposition 5.16** Assume

(i) $N(d, M; \epsilon) = O(\epsilon^{-r})$ for some $r > 0$ (This holds in particular if $M$ is a finite dimensional manifold).

(ii) $x \mapsto C(x, y)$ is Lipschitz uniformly in $y$. That is
\[
\sup_{z \in M} |C(x, z) - C(y, z)| \leq K d(x, y)
\]

for some $K \geq 0$. 46
(iii) There exists $K > 0$ such that

$$\text{Lip}(Af) \leq K(\text{Lip}(f) + \|f\|).$$

(iv) $\lambda(A) < 0$

Then the hypotheses, hence the conclusion, of corollary 5.14 are satisfied.

We begin with the following lemma.

**Lemma 5.17** Under hypotheses (iii) and (iv) of proposition 5.16

$$\text{Lip}(e^{tA}f) \leq e^{Kt}(\text{Lip}(f) + K'\|f\|)$$

for some constants $K, K'$.

**Proof:** For all $x, y$

$$|e^{tA}f(x) - e^{tA}f(y)| = \left| \int_0^t [Ae^{sA}f(x) - Ae^{sA}f(y)]ds + f(x) - f(y) \right|$$

$$\leq K \left( \int_0^t [\text{Lip}(e^{sA}f) + \|e^{sA}f\|] ds + \text{Lip}(f) \right) d(x, y).$$

Since $\lambda = \lambda(A) < 0$, there exists $C > 0$ such that $\|e^{sA}\| \leq Ce^{-s\lambda}$. Thus

$$\text{Lip}(e^{tA}f) \leq K \int_0^t \text{Lip}(e^{sA}f)ds + \frac{KC}{\lambda} \|f\| + \text{Lip}(f)$$

and the result follows from Gronwall’s lemma. QED

We now pass to the proof of the proposition. In what follows the constants may change from line to line.

**Proof:** Set $\mu = \delta_x - \delta_y$ and $f_s = C e^{sA^*} \mu$ so that

$$\langle \mu, e^{sA}Ce^{sA^*} \mu \rangle = e^{sA}f_s(x) - e^{sA}f_s(y).$$

It follows from hypotheses (ii) and (iv) that

$$\text{Lip}(f_s) + \|f_s\| \leq Ke^{-s\lambda}$$

for some positive constants $K$ and $a$. Therefore, by the preceding lemma,

$$\text{Lip}(e^{sA}f_s) \leq Ke^{sa}$$
for some (other) positive constants $K, \alpha$. Thus

$$d_V(x, y)^2 \leq d(x, y) \int_0^T \text{Lip}(e^{sA} f_s) ds + \int_T^\infty (e^{sA} f(x) - e^{sA} f(y)) ds$$

$$\leq d(x, y) \int_0^T K e^{s\alpha} ds + 2 \int_T^\infty \|e^{sA} f_s\| ds$$

$$\leq K \left( d(x, y) e^{\alpha T} + \int_T^\infty e^{-s\lambda} ds \right)$$

$$\leq K(d(x, y) e^{\alpha T} + e^{-\lambda T}).$$

Let $\gamma = \frac{\alpha}{\lambda}$, $\epsilon > 0$, and $T = -\ln(\epsilon)/\lambda$. Then

$$d^\gamma_V(x, y) \leq K(e^{-\gamma} d(x, y) + \epsilon).$$

Therefore

$$d(x, y) \leq e^{\gamma + 1} \Rightarrow d^\gamma_V(x, y) \leq K \epsilon,$$

so that $N(d, M; \omega_V(\epsilon)) = O(\epsilon^{-2r(\gamma + 1)})$ and hypothesis 5.6 holds true. QED

Example 5.18 Let

$$Af(x) = \int f(y) k(x, dy)$$

with

$$k(x, dy) = k_0(x, y) \mu(dy) + \sum_{i=1}^n a_i(x) \delta_{b_i(x)}$$

where

(i) $\mu$ is a bounded measure on $M$,

(ii) $k_0(x, y)$ is bounded and uniformly Lipschitz in $x$,

(iii) $a_i : M \to \mathbb{R}$ and $b_i : M \to M$ are Lipschitz.

Then hypothesis (iii) of proposition 5.16 is verified.
References


