Combinatorial map as multiplication of combinatorial knots

Dainis Zeps

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Abstract

We show that geometrical map can be expressed as multiplication of combinatorial maps, i.e. map \( P \) is equal to multiplication of its knot, inner knot’s square and trivial knot (= \( \mu \cdot \nu^2 \cdot \pi_1 \)).

1 Introduction

We proceed with building combinatorial map theory that from different points of view and formulations is considered in from [1] to [35].

We multiply permutations from left to right. Geometrical combinatorial map is pair of permutations, vertex and face rotations, \((P, Q)\) acting on set of elements \( C \) if \( P \cdot Q^{-1} = \rho \) edge rotation or inner edge rotation \( \pi = Q^{-1}P \) is involution without fixed elements. We consider set of maps with fixed \( \pi \) calling them normalized maps. Mostly we use one particular choice of \( \pi \) equal to \((12) \ldots (2k-1 2k)\), \( k > 1 \). If so, map may be characterized with one permutation, say vertex rotation \( P \).

In [30] we saw that particular choice of \( \rho \) by fixed \( \pi \) induces partitioning of the set \( C \) into to subsets \( C_1 \) and \( C_2 \) [in general in several ways] so that the knot \( \mu \) is defined. Here, knot \( \mu \) as permutation has \( 2^k \) choices if \( k \) is number of cycles in it. Rightly, changing direction of some cycle of \( \mu \) we get another possible value for knot \( \mu \). Moreover, \( \rho \) with choice of particular \( \mu \) partitions \( \pi \) into \( \pi_1 \cdot \pi_2 \), where we call \( \pi_1 \) cut edges and \( \pi_2 \) cycle edges, so that \( P \cdot \pi_1 : C_1 \mapsto C_2 \) and \( P \cdot \pi_2 : C_1 \mapsto C_1 \). In [31] was shown that by fixing \( \mu \) map \( P \) may be expressed as multiplication \( \gamma_1 \cdot \gamma_2 \cdot \pi_2 \), where \( \gamma_1 \) acts within \( C_1 \) and \( \gamma_2 \) acts within \( C_2 \).

In [30] was shown that normalized map always may be expressed as \( P = \mu \cdot \alpha \), where \( \alpha \) is called knotting and it is selfconjugate map in sense that \( \alpha^\pi = \alpha \). In [32] we got formulas for \( \mu \) and \( \alpha \), i.e.,

\[ \mu = \gamma_2 \pi \gamma_1^{-1} \]

and

\[ \alpha = \gamma_1 \gamma_2^\pi. \]

From [30] we know that \( \alpha \)'s form a group \( K_\pi \) with respect to multiplication of maps. Moreover, classes of maps with fixed \( \rho \)'s, denoted as \( K_\rho \), are cosets (left and right) of \( K_\pi \).

2 Main part

We are going to regain main formulae from introduction.

Let us prove some theorems that leads us to the main result.

Theorem 1. \( \rho \cdot \pi \) [or \( \pi \cdot \rho \)] is equal to some combinatorial knot \( \mu \) squared and one or other color cycles induced from this knot reversed.

Proof. Let us write knot \( \mu \) in the form \( \left\{ \begin{array}{l} C_1 : \pi \\ C_2 : \rho \end{array} \right. \). Then square of \( \mu \) we would get applying \( \pi \cdot \rho \) for one color corners and \( \rho \cdot \pi \) for other color corners. \( \square \)

Theorem 2. By fixing the square of the knot it has \( 2^k \) knots in correspondence [in general for different maps] where \( k \) is the number of cycles in the knot.

Proof. Two joined cycles of square of knot may be combined in the cycle of knot in two ways, and thus, \( k \) independent operations give \( 2^k \) results. \( \square \)

* Author’s address: Institute of Mathematics and Computer Science, University of Latvia, 29 Rainis blvd., Riga, Latvia. dainiæ@mii.lu.lv
Theorem 3. \([\mu \cdot \pi \text{ is knot's half-square}]\)
1) \(\mu \cdot \pi\) expresses squared knot's cycles of only one color.
2) For vertex rotation \(\mu \cdot \pi\) corresponding face rotation and knot are equal to \(\mu\), and knotting equal to \(\pi\).
\(\gamma_1 = \text{id}, \gamma_2 = \mu \cdot \pi,\) and \(\pi_1 = \pi\), because all edges of this map are cut edges.

Proof. 1) \(\mu\) expressing as \(\{ C_1 : \pi, C_2 : \rho \}\), and multiplying by \(\pi\), we get \(\{ C_1 : \pi \cdot \pi, C_2 : \rho \cdot \pi \}\) and using theorem 1 what was to be proved.
2) Corresponding graph to this map is set of star graphs as many as cycles in \(\mu\). Direct calculation gives what is stated by theorem.

Theorem 4. Map \(P\) can be expressed as \(P\pi_1 = \gamma_1\gamma_2\pi = \gamma_2\gamma_1\pi\) with \(\mu(P) = \gamma_2\pi\gamma_1^{-1}[= \gamma_1\rho\gamma_2^{-1}]\) and \(\pi_1 = \pi\) as inner cut edge rotation and \(\pi_2\) inner cycle edge rotation.

Proof. Let knot \(\mu = \mu(P)\) be fixed. Then set of corners is partitioned into two sets \(C_1\) and \(C_2\). From form of \(\mu(= \gamma_2 \cdot \pi \cdot \gamma_1^{-1})\) we directly judge that \(\gamma_1 \) belongs to, say, \(C_1\) and \(\gamma_2\) to \(C_2\). Thus, \(\gamma_1\) and \(\gamma_2\) commute by multiplying. Let us choose vertex rotation with this fixed knot and \(\pi_1 = \text{id}\), i.e., with all edges being cycle edges. Then vertex rotation is alternation of corners from \(C_1\) and \(C_2\) respectively, and face rotation's cycles are correspondingly of one color. Then form of \(\mu = \gamma_2 \cdot \pi \cdot \gamma_1^{-1}\) shows directly that \(P \cdot \pi_1\) must be equal to \(\gamma_1 \cdot \gamma_2(= \gamma_2 \cdot \gamma_1)\). Finally, in general we get

\[
\mu = \{ C_1 : \pi, C_2 : \rho \} = \{ C_2 : \gamma_2\pi\gamma_1^{-1}, C_1 : \gamma_1\rho\gamma_2^{-1} \}.
\]

Theorem 5. Map \(P \cdot \pi_1\) can be expressed as \(\{ C_1 : \beta_1, C_2 : \beta_2 \}\), where involutions \(\beta_1\) and \(\beta_2\) are equal to \(\beta_1 = \pi^{-\gamma_1}\) and \(\beta_2 = \pi^{-\gamma_2}\). Moreover, \(\beta_1 = \gamma_1\gamma_2^{-1}\mu\) and \(\beta_2 = \gamma_2\gamma_1^{-1}\mu\). Moreover, \(\beta_1\beta_2\) is squared knot \(\mu(..., \beta)\) with one color cycles reversed. See theorem 1. \(\delta = \pi^{\gamma_1}\). \(P = \gamma_1\gamma_2\pi_2\).

Proof.

\[
P\pi_1 = \{ C_1 : \beta_1, C_2 : \beta_2 \} = \{ C_1 : \gamma_1\pi\gamma_1^{-1}, C_2 : \gamma_2\pi_2 \} = \{ C_1 : \gamma_1, C_2 : \gamma_2 \} \cdot \pi = \gamma_1\gamma_2\pi.
\]

Corollary 6. Map \(P\pi_1\) is a knot for inner edge rotation \(\beta_1\) and edge rotation \(\beta_2\).

Theorem 7. Let for some fixed knot the map \(P\) be equal to \(\mu\alpha\). Then \(\alpha\) is equal to \(\gamma_1\pi\gamma_1\pi_2\) or \(\gamma_1\gamma_1^{-1}\pi_1\).

Proof. \(\alpha = \mu^{-1}P\pi_1 = \gamma_1\pi_2^{-1}\gamma_2\gamma_1\pi = \gamma_1\pi_1\pi = \gamma_1\gamma_1^{-1}\). This \(\alpha\) is knotting for \(P\pi_1\). For map \(P\) knotting is \(\gamma_1\pi_1\pi_2\) or \(\gamma_1\gamma_1^{-1}\pi_1\).

Corollary 8. \(\alpha^\pi = \alpha\). Self-conjugate \(\alpha\)'s comprise group.

Proof. \(\alpha^\pi = (\gamma_1\gamma_1^{-1}\pi_1)^\pi = \gamma_1\gamma_1^{-1}\pi_1^\pi = \gamma_2\gamma_1\pi_1 = \alpha\).

Theorem 9. \(\gamma_1\gamma_1^{-1}\) is some knot's square.

Proof. Let us denote this knot by \(\nu\). Direct observation shows that theorem is correct. Then fixed knot \(\mu\) induces \(\alpha\) and it determines fixed \(\nu\) such that \(\nu^2 = \gamma_1\gamma_1^{-1}\).

Theorem 10. Every combinatorial \(P\) can be expressed as multiplication of knots in the form

\[
P = \mu \cdot \nu^2 \cdot \pi_1.
\]

Proof. It directly follows from previous theorems. Really, \(P = \mu\alpha = \gamma_2\pi_1^{-1}\gamma_1\gamma_1^{-1}\pi_1 = \gamma_2\gamma_1\pi_2 = \mu(\gamma_1\pi^2\pi_1) = \mu\nu^2\pi_1\).

It must be noted that \(\pi_1\) is some knot too. We call this knot trivial knot. Let us call knot \(\nu\) map's inner knot.

Corollary 11. Map is multiplication of its knot with its inner knot's square and with its trivial knot.
Theorem 12. For $P\pi_1\mu$ commutes with $\alpha$, i.e.,

$$P\pi_1 = \mu \cdot \alpha = \alpha \cdot \mu.$$ 

In general,

$$P = \mu \alpha = \alpha \mu \pi_1.$$ 

Proof. For $P\pi_1$, $\mu^\alpha = (\gamma_2 \pi \gamma_1^{-1})^\gamma_1 \pi \gamma_1$. Further, $\gamma_1^\pi \gamma_1 = \gamma_2^\pi = \gamma_2$, because corners of $\gamma_1$ and $\gamma_2$ do not intersect. The same is true for the member $\gamma_1^\pi$. Further, $\pi^\gamma \gamma_1 = \pi^\alpha = \pi$. Thus, we get $\mu^\alpha = \mu$. 

Theorem 13. For partial map $[P, \mu]$ its inner edge rotation is $\alpha$. 

Proof. Direct observation. 

3 Conclusions

There are four types of permutations that are used to build "all" in combinatorial map theory, i.e., knot-type, knot-square-type, knot-square-with-reversed-cycles-type, two-color-involutions. Comprehensive algebra of all these types should be ground for combinatorial map theory.

References


