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To cite this version:

Rachid Zarouf. Constrained Nevanlinna-Pick interpolation in Weighted Hardy and Bergman spaces. 2008. hal-00381193v1

HAL Id: hal-00381193
https://hal.archives-ouvertes.fr/hal-00381193v1
Submitted on 5 May 2009 (v1), last revised 3 Nov 2010 (v4)

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Constrained Nevanlinna-Pick interpolation in Weighted Hardy and Bergman spaces

Rachid Zarouf

Abstract

Given a finite set $\sigma$ of the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ and a holomorphic function $f$ in $D$ which belongs to a class $X$, we are looking for a function $g$ in another class $Y$ (smaller than $X$) which minimizes the norm $\|g\|_Y$ among all functions $g$ such that $g|_\sigma = f|_\sigma$. For $Y = H^\infty$, $X = l^p(w_k)$, $f = \sum_{k \geq 0} \hat{f}(k)z^k : \|f\|^p = \sum_{k \geq 0} |\hat{f}(k)|^pw_k^p < \infty$, with a weight $w$ satisfying $w_k > 0$ for every $k \geq 0$ and $\lim_k (1/w_k^{1/k}) = 1$, and for the corresponding interpolation constant $c(\sigma, X, H^\infty)$, we show that if $p = 2$, $c(\sigma, X, H^\infty) \leq a\varphi_X(1 - \frac{1-r}{n})$ where $n = \#\sigma$, $r = \max_{\lambda \in \sigma} |\lambda|$ and where $\varphi_X(t)$ stands for the norm of the evaluation functional $f \mapsto f(t)$ on the space $X$. The upper bound is sharp over sets $\sigma$ with given $n$ and $r$. For $X = l^p(w_k)$, $p \neq 2$ and $X = \bar{L}_p^\beta \left((1 - |z|^2)^\beta \, dx \, dy \right)$ (the weighted Bergman space), $\beta > -1$, $1 \leq p < 2$, we also found upper and lower bounds for $c(\sigma, X, H^\infty)$ (sometimes for special sets $\sigma$) but with some gaps between these bounds.

0. Introduction

We recall that the problem considered in [Z] is the following: given two Banach spaces $X$ and $Y$ of holomorphic functions on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$, $X \supset Y$, and a finite set $\sigma \subset D$, what is the best possible interpolation by functions of the space $Y$ for the traces $f|_\sigma$ of functions of the space $X$, in the worst case?

Looking at this problem, we are lead to define and compute/estimate the interpolation constant

$$c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf \left\{ \|g\|_Y : g|_\sigma = f|_\sigma \right\},$$

(which is nothing but the norm of the embedding operator $(X|_\sigma, \|\cdot\|_{X|_\sigma}) \rightarrow (Y|_\sigma, \|\cdot\|_{Y|_\sigma})$). Let us notice that the following question was especially stimulating (which is a part of a more complicated question arising in an applied situation in [BL1] and [BL2]): given a set $\sigma \subset D$, how to estimate $c(\sigma, H^2, H^\infty)$ in terms of $n = \text{card}(\sigma)$ and $\max_{\lambda \in \sigma} |\lambda| = r$ only? Here and everywhere below, $H^2$ is the standard Hardy space of the disc,

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k)z^k : \|f\|_{H^2} = \left( \sum_{k \geq 0} |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$
and $H^\infty$ stands for the space (algebra) of bounded holomorphic functions in the unit disc $\mathbb{D}$ endowed with the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$. The issue of estimating/computing $c(\sigma, H^2, H^\infty)$ has been treated in [Z].

More precisely, we have treated in [Z] the cases $X = H^p, L^2_a$, where $H^p$ ($1 \leq p \leq \infty$) stands for the standard Hardy space of the unit disc and where $L^2_a$ stands for the Bergman space of all holomorphic functions $f$ such that
\[ \int_{\mathbb{D}} |f(z)|^2 dA < \infty, \]
where $dA$ stands for the area measure, and proved the following result, through Theorems A, B & C (see the Introduction of [Z]).

**Theorems A, B & C.**

1. \[ \frac{1}{4\sqrt{2}} \frac{\sqrt{n}}{\sqrt{1-r}} \leq c(\sigma_{r,n}, H^2, H^\infty) \leq C_{n,r} (H^2, H^\infty) \leq \frac{\sqrt{n}}{\sqrt{1-r}}, \]
   for all $n \geq 1$, $0 \leq r < 1$, where $\sigma_{r,n}$ is the one set point $\{r, r, \ldots, r\}$ ($n$ times).

2. Let $1 \leq p \leq \infty$. Then
   \[ \frac{1}{32} \frac{n}{1-r} \leq c(\sigma_{r,n}, L^2_a, H^\infty) \leq C_{n,r} (L^2_a, H^\infty) \leq \sqrt{14} \frac{n}{1-r}, \]
   for all $n \geq 1$, $0 \leq r < 1$, where $A_p$ is a constant depending only on $p$ and the left hand side inequality is proved only for $p \in 2\mathbb{Z}_+$. 

In this paper, we extend those results to the case where $X$ is a weighted space $X = l^p_a(w_k)$,

\[ l^p_a(w_k) = \left\{ f = \sum_{k \geq 0} \hat{f}(k)z^k : \|f\|^p = \sum_{k \geq 0} |\hat{f}(k)|^p w_k^p < \infty \right\}, \]

with a weight $w$ satisfying $w_k > 0$ for every $k \geq 0$ and $\lim_{k} (1/w_k^{1/k}) = 1$. The latter condition implies that $l^p_a(w_k)$ is continuously embedded into the space of holomorphic functions $Hol(\mathbb{D})$ on the unit disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ (and not on a larger disc, i.e. $l^p_a(w_k)$ does not contained in $Hol(r\mathbb{D})$ for every $r > 1$).

As in [Z], in order to find an upper bound for $c(\sigma, X, H^\infty)$, we first use a linear interpolation:
\[ f \mapsto \sum_{k=1}^n \langle f, e_k \rangle e_k, \]
where $\langle ., . \rangle$ means the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k)\overline{g(k)}$, and $(e_k)_{k=1}^n$ is the explicitly known Malmquist basis of the space $K_B = H^2 \Theta B|H_0^2$, $B = \prod_{i=1}^n \theta_{b_i}$ being the corresponding Blaschke product, $b_{\lambda} = \frac{\lambda z}{1-\lambda z}$ (see N. Nikolski, [N1] p. 117). Next, we use the complex interpolation between Banach spaces, (see H. Triebel [Tr] Theorem 1.9.3 p.59). Among the technical tools used in order to find an upper bound for $\|\sum_{k=1}^n (f, e_k) e_k\|_p$ (in terms of $\|f\|_{X}$), the most important is a Bernstein-type inequality $\|f\|_{\infty} \leq c_p \|B\|_{\infty} \|f\|_p$ for a (rational) function $f$ in the star-invariant subspace $H^p \cap B|H_0^p$ generated by a (finite) Blaschke product $B$, (K. Dyakonov [Dy]). For $p = 2$, we give an alternative proof of the Bernstein-type estimate we need.

The lower bound problem (for $C_{n,r}(X, H^\infty)$) is treated by using the “worst” interpolation $n$-tuple $\sigma = \{\lambda, ..., \lambda\}$, a one-point set of multiplicity $n$ (the Carathéodory-Schur type interpolation). The “worst” interpolation data comes from the Dirichlet kernels $\sum_{k=0}^{n-1} e^k$ transplanted from the origin to $\lambda$. We notice that spaces $X = l^p_1(w_k)$ satisfy the condition $X \circ b_{\lambda} \subset X$ when $p = 2$, whereas this is not the case for $p \neq 2$ and this makes the problem of upper/lower bound harder.

Our principal case is $p = 2$, where $l^2_1(w_k)$ is a reproducing kernel Hilbert space on the disc $\mathbb{D}$. It is important to recall that

$$l^2_1 \left( \frac{1}{(k+1)^{\alpha-1}} \right) = L^2_1 \left( (1-|z|^2)^{2\alpha-3} \right) A, \alpha > 1,$$

where $L^2_1 \left( (1-|z|^2)^{\beta} \right)$, $\beta > -1$, stand for the Bergman weighted spaces of all holomorphic functions $f$ such that

$$\int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{\beta} dA < \infty.$$

**Theorem. 1.0** Let $\sigma$ be a sequence in $\mathbb{D}$. Then

$$c \left( \sigma, l^2_1 \left( \frac{1}{(k+1)^{\alpha-1}} \right) , H^\infty \right) \leq A \left( \frac{n}{1-r} \right)^{2\alpha-1}.$$

Otherwise,

$$C_{n,r} \left( l^2_1 \left( \frac{1}{(k+1)^{\alpha-1}} \right) , H^\infty \right) \leq A \left( \frac{n}{1-r} \right)^{2\alpha-1},$$

$$C_{n,r} \left( (1-|z|^2)^{\beta} \right) , H^\infty \right) \leq A' \left( \frac{n}{1-r} \right)^{\frac{\beta+2}{2}},$$

for all $n \geq 1$, $0 \leq r < 1$, $\alpha \geq 1$, $\beta > -1$, where $A = A(\alpha-1)$ is a constant depending only on $\alpha$ and $A' = A'(\beta)$ is a constant depending only on $\beta$.

Later on, in Section 7 we show that for $\alpha = \frac{N+1}{2}$, where $N \geq 1$ is an integer, the latter estimate is sharp.

**Theorem. 7.0** Let $N \geq 1$ be an integer and $\sigma_{\lambda,n} = \{\lambda, ..., \lambda\}$ (n times). Then,

$$c \left( \sigma_{\lambda,n}, l^2_1 \left( \frac{1}{(k+1)^{\frac{N+1}{2}}} \right) , H^\infty \right) \geq a \left( \frac{n}{1-|\lambda|} \right)^{\frac{N}{4}}.$$
for a positive constant \( a = a_N \) depending on \( N \) only. In particular,

\[
a \left( \frac{n}{1-r} \right)^{\frac{N}{2}} \leq C_{n,r} \left( \frac{1}{(k+1)^{n-1}} \right), \quad H^\infty \leq A \left( \frac{n}{1-r} \right)^{\frac{N}{2}},
\]

for all \( n \geq 1, 0 \leq r < 1 \), where \( A = A(\frac{N-1}{2}) \) is a constant defined in Theorem 1.0. Moreover, \( a \) and \( A \) are such that \( a \approx \frac{1}{2^{2M(2N)!}} \) and \( A \approx N^{2N} \), \( N \) standing for the integer part of \( \alpha \). (The notation \( x \asymp y \) means that there exists numerical constants \( c_1, c_2 > 0 \) such that \( c_1y \leq x \leq c_2y \).)

In Sections 2, 3 and 4, we deal with an upper estimate for \( C_{n,r}(X, H^\infty) \) in the scale of spaces \( X = L^p_A\left( \frac{1}{(k+1)^{p-r}} \right), \alpha \geq 1, 1 \leq p \leq +\infty \). (The case \( p = 2 \) is solved in Section 1 (for the upper bound) and in Section 7 (for sharpness)). We start giving a result for \( 1 \leq p \leq 2 \).

**Theorem 3.0** Let \( 1 \leq p \leq 2 \), \( \alpha \geq 1 \). Then

\[
B \left( \frac{1}{1-r} \right)^{\alpha - \frac{1}{p}} \leq C_{n,r} \left( L^p_A \left( \frac{1}{(k+1)^{\alpha-1}} \right), \ H^\infty \right) \leq A \left( \frac{n}{1-r} \right)^{\alpha - \frac{1}{p}},
\]

for all \( r \in [0, 1[, \ n \geq 1 \), where \( A = A(\alpha - 1, p) \) is a constant depending only on \( \alpha \) and \( p \) and \( B = B(p) \) is a constant depending only on \( p \).

It is very likely that the bounds of Theorem 3.0 are not sharp. The sharp one should be probably \( \left( \frac{n}{1-r} \right)^{\alpha - \frac{1}{p}} \). In the same way, for \( 2 \leq p \leq \infty \), we give the following theorem, in which we feel again that the upper bound \( \left( \frac{n}{1-r} \right)^{\alpha + \frac{1}{2} - \frac{2}{p}} \) is not sharp. The sharp one probably should be the lower bound \( \left( \frac{n}{1-r} \right)^{\alpha - \frac{1}{2} - \frac{2}{p}} \).

**Theorem 5.0** Let \( 2 \leq p \leq \infty \), \( \alpha \geq 1 \). Then

\[
B \left( \frac{1}{1-r} \right)^{\alpha - \frac{1}{p}} \leq C_{n,r} \left( L^p_A \left( \frac{1}{(k+1)^{\alpha-1}} \right), \ H^\infty \right) \leq A \left( \frac{n}{1-r} \right)^{\alpha + \frac{1}{2} - \frac{2}{p}},
\]

for all \( r \in [0, 1[, \ n \geq 1 \), where \( A = A(\alpha - 1, p) \) is a constant depending only on \( \alpha \) and \( p \) and \( B = B(p) \) is a constant depending only on \( p \).

In Section 6, we suppose that \( X \) is equal to \( L^p_A\left( (1 - |z|^2)^\beta \ dA \right), \beta > -1, 1 \leq p \leq 2 \), where \( dA \) stands for the area measure, the Bergman weighted spaces of all holomorphic functions \( f \) such that

\[
\int_D |f(z)|^p (1 - |z|^2)^\beta \ dA < \infty.
\]

Notice that for \( p = 2 \), the two latter series of spaces coincide:

\[
L^2_A \left( \frac{1}{(k+1)^{\alpha-1}} \right) = L^2_A \left( (1 - |z|^2)^{2\alpha - 3} \ dA \right), \quad \alpha > 1.
\]
Our goal in this section is to give an estimate for the constant for a generalized Carathéodory-Schur interpolation, (a partial case of the Nevanlinna-Pick interpolation),

\[ c(\sigma_{\lambda,n}, X, H^\infty) = \sup \left\{ \| f \|_{H^\infty / \nu_{\lambda}^n H^\infty} : f \in X, \| f \|_X \leq 1 \right\}, \]

where \( \| f \|_{H^\infty / \nu_{\lambda}^n H^\infty} = \inf \left\{ \| f + b_n^\lambda g \|_\infty : g \in X \right\} \), and \( \sigma_{\lambda,n} = \{ \lambda, \lambda, ..., \lambda \} \), \( \lambda \in \mathbb{D} \). The corresponding interpolation problem is: given \( f \in X \), to minimize \( \| h \|_\infty \) such that

\[ h^{(j)}(\lambda) = f^{(j)}(\lambda), \quad 0 \leq j < n. \]

For this partial case, we have the following generalization of the estimate from Theorem 1.0.

**Theorem 6.0** Let \( \lambda \in \mathbb{D}, \beta > -1 \) and \( 1 \leq p \leq 2 \). Then,

\[ c \left( \sigma_{\lambda,n}, L^p, \left( (1 - |z|^2)^{\frac{\beta + 2}{p}} dA \right), H^\infty \right) \leq A' \left( \frac{n}{1 - |\lambda|} \right)^{\beta + 2}, \]

where \( A' = A'(\beta, p) \) is a constant depending only on \( \beta \) and \( p \).

Before starting Section 1 and studying upper estimates for \( c(\sigma, X, H^\infty) \), we give the following lemma which is going to be useful throughout this paper, in particular in view of applying interpolation between Banach spaces.

**Lemma 0.** Let \( X \) be a Banach space of holomorphic functions in the unit disc \( \mathbb{D} \) and \( \sigma = \{ \lambda_1, \lambda_2, ..., \lambda_n \} \subset \mathbb{D} \) a finite subset of the disc. We define the Blaschke product

\[ B_\sigma = \prod_{\lambda=1}^n b_\lambda \]

where \( b_\lambda = \frac{\lambda - z}{1 - \lambda z} \). Let \( T : X \rightarrow H^\infty / B_\sigma H^\infty \) be the restriction map defined by

\[ Tf = \{ g \in H^\infty : f - g \in B_\sigma X \}, \]

for every \( f \in X \). Then,

\[ \| T \|_{X \rightarrow H^\infty / B_\sigma H^\infty} = c(\sigma, X, H^\infty). \]

**Proof.** The proof is obvious. \( \square \)

**1. An upper bound for** \( c \left( \sigma, l^2_a(w_k), H^\infty \right) \)

In this section, we generalize the upper bound obtained in [Z] for \( C_{n,r}(X, H^\infty) \) where \( X = H^2, L^2_a \) to the case of spaces \( X \) which contain \( H^2 \): \( X = l^2_a \left( \frac{1}{(k + 1)^{\alpha - 1}} \right), \alpha \geq 1 \), the Hardy weighted spaces of all \( f(z) = \sum_{k \geq 0} \hat{f}(k) z^k \) satisfying

\[ \sum_{k \geq 0} \left| \hat{f}(k) \right|^2 \frac{1}{(k + 1)^{2(\alpha - 1)}} < \infty. \]

It is also important to recall that

\[ l^2_a \left( \frac{1}{(k + 1)^{\alpha - 1}} \right) = L^2_a \left( (1 - |z|^2)^{2\alpha - 3} dA \right), \alpha > 1, \]
where $L^2_a \left( (1 - |z|^2)^\beta \ dA \right)$, $\beta > -1$, stand for the Bergman weighted spaces of all holomorphic functions $f$ such that
\[
\int_D |f(z)|^2 \ (1 - |z|^2)^\beta \ dA < \infty.
\]
Notice also that $H^2 = l^2_a(1)$ and $L^2_a(D) = l^2_a \left( 1 \left( \frac{1}{k+1} \right)^{\alpha-1} \right)$, where $L^2_a(D)$ stands for the Bergman space of the unit disc $D$.

**Theorem. 1.0** Let $\sigma$ be a sequence in $D$. Then
\[
c \left( \sigma, l^2_a \left( \frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right) \leq A \left( \frac{n}{1-r} \right)^{\frac{2(\alpha-1)}{2}}.
\]
Otherwise,
\[
C_{n,r} \left( l^2_a \left( \frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right) \leq A \left( \frac{n}{1-r} \right)^{\frac{2(\alpha-1)}{2}},
\]
\[
C_{n,r} \left( L^2_a \left( (1 - |z|^2)^\beta \ dA \right), H^\infty \right) \leq A' \left( \frac{n}{1-r} \right)^{\frac{\beta+2}{2}},
\]
for all $n \geq 1$, $0 \leq r < 1$, $\alpha \geq 1$, $\beta > -1$, where $A = A(\alpha - 1)$ is a constant depending only on $\alpha$ and $A' = A'(\beta)$ is a constant depending only on $\alpha$.

First, we recall the following lemma (see [Z]). In fact, **Lemma 1.1** below is a partial case ($p = 2$) of the following K. Dyakonov’s result [Dy] (which is, in turn, a generalization of M. Levin’s inequality [L] corresponding to the case $p = \infty$): for every $p$, $1 < p \leq \infty$ there exists a constant $c_p > 0$ such that
\[
\|f'\|_{H^p} \leq c_p \|f\|_{H^\infty}
\]
for all $f \in K_B$, where $B$ is a finite Blaschke product (of order $n$) and $\|\cdot\|_{\infty}$ means the norm in $L^\infty(\mathbb{T})$. For our partial case, our proof (in [Z]) is different and the constant is slightly better. We notice that in general, Bernstein type inequalities have already been the subject of a lot of papers. Among others, Chapter 7 of P. Borwein and T. Erdélyi’s book, see [BoEr], is devoted to such inequalities. This is also the case of A. Baranov’s work, see [B1], [B2] and [B3], and also of R. A. DeVore and G. G. Lorentz’s book, see [DeLo].

**Lemma. 1.1** Let $B = \prod_{j=1}^n b_{\lambda_j}$, be a finite Blaschke product (of order $n$), $r = \max_j |\lambda_j|$, and $f \in K_B := H^2 \Theta BH^2$. Then,
\[
\|f'\|_{H^2} \leq \frac{5}{2} \frac{n}{1-r} \|f\|_{H^2}.
\]

**Corollary. 1.2** Let $B = \prod_{j=1}^n b_{\lambda_j}$, be a finite Blaschke product (of order $n$), $r = \max_j |\lambda_j|$, and $f \in K_B := H^2 \Theta BH^2$. Then,
\[
\|f^{(k)}\|_{H^2} \leq k! \left( \frac{5}{2} \right)^k \left( \frac{n}{1-r} \right)^k \|f\|_{H^2},
\]
for every $k = 0, 1, \ldots$
Indeed, since $f^{(k-1)} \in K_B$, we obtain applying Lemma 1.1 for $B^k$ instead of $B$,
\[
\|f^{(k)}\|_{H^2} \leq \frac{5}{2(1-r)} \|f^{(k-1)}\|_{H^2},
\]
and by induction,
\[
\|f^{(k)}\|_{H^2} \leq k! \left(\frac{5}{2(1-r)}\right)^k \|f\|_{H^2}. \quad \Box
\]

**Corollary 1.3** Let $N \geq 0$ be an integer and $\sigma$ a sequence in $\mathbb{D}$. Then,
\[
c(\sigma, l_a^2 \left(\frac{1}{(k+1)^N}\right), H^\infty) \leq A \left(\frac{n}{1-r}\right)^{\frac{2N+1}{2}},
\]
where $A = A(N)$ is a constant depending on $N$ (of order $N^{2N}$ from the proof below).

Indeed, if $f \in l_a^2 \left(\frac{1}{(k+1)^N}\right) = H$ then $|P_B f(\zeta)| = |\langle P_B f, k_\zeta \rangle| = |\langle f, P_B k_\zeta \rangle|$, where $\langle \cdot, \cdot \rangle$ means the Cauchy pairing and $k_\zeta = (1-\zeta z)^{-1}$. Denoting $H^*$ the dual of $H$ with respect to this pairing, $H^* = l_a^2 ((k+1)^N)$, we get
\[
|P_B f(\zeta)| \leq \|f\|_H \|P_B k_\zeta\|_{H^*} \leq \|f\|_H K_N \left(\|P_B k_\zeta\|_{H^2} + \|(P_B k_\zeta)^{(N)}\|_{H^2}\right),
\]
where
\[
K_N = \max \left\{ N^N, \sup_{k \geq 2} \frac{(k+1)^N}{k(k-1)...(k-N+1)} \right\} = \max \left\{ N^N, \frac{(N+1)^N}{N!} \right\} = \left\{ \begin{array}{ll} N^N, & \text{if } N \geq 3 \\ \frac{(N+1)^N}{N!}, & \text{if } N = 1, 2 \end{array} \right. .
\]
(Indeed, the sequence $\left(\frac{(k+1)^N}{k(k-1)...(k-N+1)}\right)_{k \geq N}$ is decreasing since $(1+x)^{-N} \geq 1-Nx$ for all $x \in [0, 1]$, and $\left[N^N > \frac{(N+1)^N}{N!}\right] \iff N \geq 3$). Since $P_B k_\zeta \in K_B$, Corollary 1.2 implies
\[
|P_B f(\zeta)| \leq \|f\|_H \|P_B k_\zeta\|_{H^*} \leq \|f\|_H K_N \left(\|P_B k_\zeta\|_{H^2} + N! \left(\frac{5}{2(1-r)}\right)^N \|P_B k_\zeta\|_{H^2}\right) \leq \left(\frac{n}{1-r}\right)^{\frac{N+1}{2}} \|f\|_H ,
\]
where $A(N) = \sqrt{2}K_N \left(1 + N! \left(\frac{5}{2(1-r)}\right)^N\right)$, since $\|P_B k_\zeta\|_2 \leq \frac{\sqrt{2}}{\sqrt{1-r}}$. $\Box$

**Proof of Theorem 1.0.** Applying Lemma 0 with $X = l_a^2 \left(\frac{1}{(k+1)^{\alpha-r}}\right)$, we get
\[
\|T\|_{l_a^2 \left(\frac{1}{(k+1)^{\alpha-r}}\right) \rightarrow H^\infty/B_\sigma H^\infty} = c(\sigma, l_a^2 \left(\frac{1}{(k+1)^{\alpha-r}}\right), H^\infty),
\]
where $T$ and $B_\sigma$ are defined in Lemma 0. Moreover, there exists an integer $N$ such that $N \leq \alpha \leq N + 1$. In particular, there exists $0 \leq \theta \leq 1$ such that $\alpha - 1 = (1-\theta)(N-1) + \theta N$. And since (as
in Theorem 4.0 of [Z]), we use the notation of the interpolation theory between Banach spaces see [Tr] or [Be])

\[
\left[ l_a^2 \left( \frac{1}{(k+1)^{N-1}} \right), l_a^2 \left( \frac{1}{(k+1)^N} \right) \right]_{\theta,2} = l_a^2 \left( \frac{1}{(k+1)^{N-1}} \right)^{\frac{2+\theta}{2}} \left( \frac{1}{(k+1)^N} \right)^{\frac{2\theta}{2}} = l_a^2 \left( \frac{1}{(k+1)^{(1-\theta)(N-1)+\theta N}} \right) = l_a^2 \left( \frac{1}{(k+1)^{\alpha-1}} \right),
\]

this gives, using Corollary 1.3 and (again) [Tr] Theorem 1.9.3 p.59,

\[
\| T \|_{l_a^2 \left( \frac{1}{(k+1)^{\alpha-1}} \right) \rightarrow \mathcal{H}_\infty / B_{\sigma \mathcal{H}_\infty}} \leq \left( c \left( \sigma, l_a^2 \left( \frac{1}{(k+1)^{N-1}} \right), H_\infty \right) \right)^{1-\theta} \left( c \left( \sigma, l_a^2 \left( \frac{1}{(k+1)^N} \right), H_\infty \right) \right)^\theta \leq \left( A(N-1) \left( \frac{n}{1-r} \right)^{\frac{2N+1}{2}} \right)^{1-\theta} \left( A(N) \left( \frac{n}{1-r} \right)^{\frac{2N+1}{2}} \right)^\theta = A(N-1)^{1-\theta} A(N)^\theta \left( \frac{n}{1-r} \right)^{\frac{(2N-1)(1-\theta) + (2N+1)\theta}{2}}.
\]

It remains to use \( \theta = \alpha - N \) and set \( A(\alpha - 1) = A(N-1)^{1-\theta} A(N)^\theta \).

\[
\square
\]

2. An upper bound for \( c \left( \sigma, l_a^1 \left( w_k \right), H_\infty \right) \)

The aim of this section is to prove the following theorem, in which the upper bound \( \left( \frac{n}{1-r} \right)^{\alpha-\frac{1}{2}} \) is not as sharp as in Section 1. We suspect \( \left( \frac{n}{1-r} \right)^{\alpha-1} \) is the sharp bound for the quantity \( C_{n,r} \left( l_a^1 \left( \frac{1}{(k+1)^{\alpha-1}} \right), H_\infty \right) \).

Theorem 2.0 Let \( \alpha \geq 1 \). Then,

\[
C_{n,r} \left( l_a^1 \left( \frac{1}{(k+1)^{\alpha-1}} \right), H_\infty \right) \leq A_1 \left( \frac{n}{1-r} \right)^{\alpha-\frac{1}{2}},
\]

for all \( r \in [0, 1[ \), \( n \geq 1 \), where \( A_1 = A_1(\alpha - 1) \) is a constant depending only on \( \alpha \).

First, we prove the following lemma.

Lemma 2.1 Let \( B = \Pi_{j=1}^n b_{\lambda_j} \), be a finite Blaschke product (of order \( n \)), \( r = \max_j |\lambda_j| \), and \( f \in K_B \). Then,

\[
\| f^{(k)} \|_{H^1} \leq k! \left( \frac{2n}{1-r} \right)^k \| f \|_{H^1}
\]

for every \( k = 0, 1, ... \)
Proof. By A. Baranov (see [B1] Theorem 5.1 p.50),
\[ \|f'\|_{H^1} \leq \|B'\|_{\infty} \|f\|_{H^1} \]
for every \( f \in K_B \). (A private communication with A. Baranov shows that Theorem 5.1 of [B1] is also true for the Hardy spaces of the unit disc \( \mathbb{D} \); see also [B2] Corollary 1.4, and [B3]). Since \( f^{(k-1)} \in K_{B^k} \), we obtain, applying Baranov’s inequality for \( B^k \) instead of \( B \),
\[ \|f^{(k)}\|_{H^1} \leq \|kB'\|_{B^{k-1}} \|f^{(k-1)}\|_{H^1}, \]
and by induction,
\[ \|f^{(k)}\|_{H^1} \leq k! \|B'\|_{B}\|f\|_{H^1}. \]
On the other hand, \( |B'| = \left| - \sum_j \frac{1 - |\lambda|^2}{(1 - |\lambda|^2)^j} \cdot \frac{B}{b_j} \right| \leq \sum_j \frac{1 + |\lambda|}{1 - |\lambda|} \leq \frac{2n}{r} \), which completes the proof. \( \square \)

Corollary 2.2 Let \( N \geq 0 \) be an integer. Then,
\[ C_{n, r} \left( l^1_a \left( \frac{1}{(k+1)^N} \right), H^\infty \right) \leq A_1 \left( \frac{n}{1-r} \right)^{N+\frac{1}{2}}, \]
for all \( r \in [0, 1], n \geq 1 \), where \( A_1 = A_1(N) \) is a constant depending only on \( N \) (of order \( N^{2N} \) from the proof below).

Indeed, the proof is exactly the same as in Corollary 1.3: if \( f \in l^1_a \left( \frac{1}{(k+1)^N} \right) = H \) then \( |P_B f(\zeta)| = |\langle P_B f, k_\zeta \rangle| = |\langle f, B k_\zeta \rangle| \), where \( \langle ., . \rangle \) means the Cauchy pairing and \( k_\zeta = (1-\zeta^{-1})^{-1} \). Denoting \( H^* \) the dual of \( H \) with respect to this pairing, \( H^* = l^\infty_a ((k+1)^N) \), we get,
\[ |P_B f(\zeta)| \leq \|f\|_H \|P_B k_\zeta\|_{H^*} \leq \|f\|_H K_{N, \text{max}} \left\{ \sup_{0 \leq k \leq N-1} \left| \widehat{P_B k_\zeta}(k) \right|, \sup_{k \geq N} \left| \widehat{(P_B k_\zeta)^{(N)}}(k-N) \right| \right\} \leq \|f\|_H K_{N, \text{max}} \left\{ \|P_B k_\zeta\|_{H^1}, \left\| (P_B k_\zeta)^{(N)} \right\|_{H^1} \right\}, \]
where \( K_N \) is defined in the the proof of Corollary 1.3. Since \( P_B k_\zeta \in K_B \), Lemma 2.1 implies
\[ |P_B f(\zeta)| \leq \|f\|_H \|P_B k_\zeta\|_{H^*} \leq \|f\|_H K_N \left( \|P_B k_\zeta\|_{H^1} + N!2^N \left( \frac{n}{1-r} \right)^N \|P_B k_\zeta\|_{H^1} \right) \leq \|f\|_H \|P_B k_\zeta\|_2 \left( 1 + N!2^N \left( \frac{n}{1-|\lambda|} \right)^N \right) \leq K_N \|f\|_H \left( \frac{2n}{1-r} \right)^{\frac{1}{2}} \left( 1 + N!2^N \left( \frac{n}{1-r} \right)^N \right), \]
which completes the proof setting \( A_1(N) = \sqrt{2} \left( 1 + N!2^N \right) K_N \). \( \square \)
Proof of Theorem 2.0. This is the same reasoning as in Theorem 1.0. Applying Lemma 0 with 
\( X = l^1_a \left( \frac{1}{(k+1)^{\alpha-1}} \right) \), we get

\[
\| T \|_{L^1_a \left( \frac{1}{(k+1)^{\alpha-1}} \right) \to H^\infty / B_\sigma H^\infty} = c \left( \sigma, l^1_a \left( \frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right),
\]

where \( T \) and \( B_\sigma \) are defined in Lemma 0. It remains to use Corollary 2.2 and (again) [Tr] Theorem 1.9.3 p.59 to complete the proof.

\[ \square \]

3. An upper bound for \( c(\sigma, l^p_a (w_k), H^\infty) \), \( 1 \leq p \leq 2 \)

The aim of this section is to prove the following theorem, in which the upper bound \( (\frac{n}{1-r})^{\alpha - \frac{1}{p}} \) is not sharp as sharp as in Section 1. We suppose that the sharp upper (and lower) bound here should be of the order of \( (\frac{n}{1-r})^{\alpha - \frac{1}{p}} \).

\textbf{Theorem. 3.0} Let \( 1 \leq p \leq 2 \), \( \alpha \geq 1 \). Then

\[
B \left( \frac{1}{1-r} \right)^{\alpha - \frac{1}{p}} \leq C_{n,r} \left( l^p_a \left( \frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right) \leq A \left( \frac{n}{1-r} \right)^{\alpha - \frac{1}{p}},
\]

for all \( r \in [0, 1] \), \( n \geq 1 \), where \( A = A(\alpha - 1, p) \) is constant depending only on \( \alpha \) and \( p \) and \( B = B(p) \) is a constant depending only on \( p \).

\textit{Proof.} We first prove the right hand side inequality. The scheme of the proof is completely the same as in Theorem 1.0 and Theorem 2.0, but we simply use interpolation between \( l^1 \) and \( l^2 \) (the classical Riesz-Thorin theorem). Applying Lemma 0 with \( X = l^p_a \left( \frac{1}{(k+1)^{\alpha-1}} \right) \), we get

\[
\| T \|_{L^p_a \left( \frac{1}{(k+1)^{\alpha-1}} \right) \to H^\infty / B_\sigma H^\infty} = c \left( \sigma, l^p_a \left( \frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right),
\]

where \( T \) and \( B_\sigma \) are defined in Lemma 0. It remains to use both Theorem 1.0 & 2.0 and (again) [Tr] Theorem 1.9.3 p.59 to complete the proof of the right hand side inequality.

Now, we prove the left hand side one. Firstly, it is clear that

\[
C_{n,r} \left( l^p_a \left( \frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right) \geq \| \varphi_r \|_{L^p_a ((k+1)^{\alpha-1})} = 
\]

\[
= \left( \sum_{k \geq 0} (k+1)^{(\alpha-1)p'} \left( \frac{1}{p'} \right)^k \right)^{\frac{1}{p'}},
\]

where \( \varphi_r \) is the evaluation functional

\[
\varphi_r(f) = f(r), \quad f \in X,
\]

and \( p' \) is the conjugate of \( p \): \( \frac{1}{p} + \frac{1}{p'} = 1 \). Now, since

\[
\sum_{k \geq 1} k^s x^k \sim \int_1^\infty t^s x^s dt \sim \Gamma(s+1)(1-x)^{s-1}, \quad \text{as } x \to 1,
\]

\[
\sum_{k \geq 1} \frac{1}{(k+1)^{\alpha-1}} \sim \int_1 \frac{1}{x^{\alpha-1}} dx = \frac{1}{\alpha-1}, \quad \text{as } \alpha \to 1,
\]

we have

\[
\sum_{k \geq 1} (k+1)^{(\alpha-1)p'} \left( \frac{1}{p'} \right)^k \sim \Gamma((\alpha-1)p'+1)(\frac{1}{\alpha-1})^\frac{1}{p'},
\]

which completes the proof.

\[ \square \]
for all \( s > -1 \), we get
\[
\sum_{k \geq 0} (k + 1)^{(\alpha-1)p'} \left( r^p \right)^k \sim \int_{1}^{\infty} t^{(\alpha-1)p'} r^{p'} dt, \text{ as } r \to 1.
\]

But
\[
\int_{1}^{\infty} t^{(\alpha-1)p'} r^{p'} dt = \left( \frac{1}{p'} \right)^{1+(\alpha-1)p'} \int_{p'}^{\infty} t^{(\alpha-1)p'} r^{p'} dt \sim \left( \frac{1}{p'} \right)^{1+(\alpha-1)p'} \Gamma \left( (\alpha-1)p' + 1 \right) (1 - r)^{-(\alpha-1)p'-1}, \text{ as } r \to 1.
\]
This gives
\[
\left( \sum_{k \geq 0} (k + 1)^{(\alpha-1)p'} \left( r^p \right)^k \right)^{\frac{1}{p'}} \sim \left( \frac{1}{p'} \right)^{\frac{1}{p'}+(\alpha-1)} \Gamma \left( (\alpha-1)p' + 1 \right)^{\frac{1}{p'}} (1 - r)^{-(\alpha-1)-\frac{1}{p'}}, \text{ as } r \to 1.
\]
This completes the proof since \( \frac{1}{p'} = 1 - \frac{1}{p} \). \( \square \)

4. An upper bound for \( c(\sigma, l_a^{\infty}(w_k), H^{\infty}) \)

The aim of this section is the following theorem, in which again the upper bound \( \left( \frac{n}{1-r} \right)^{\alpha+\frac{1}{2}} \) is not as sharp as in Section 1. We can suppose here that the constant \( \left( \frac{n}{1-r} \right)^{\alpha} \) is the sharp bound for the quantity \( C_{n,r} \left( l_a^{\infty} \left( \frac{1}{(k+1)^{\alpha-1}} \right), H^{\infty} \right) \).

**Theorem 4.0** Let \( \alpha \geq 1 \). Then
\[
C_{n,r} \left( l_a^{\infty} \left( \frac{1}{(k+1)^{\alpha-1}} \right), H^{\infty} \right) \leq A_\infty \left( \frac{n}{1-r} \right)^{\alpha+\frac{1}{2}},
\]
for all \( r \in [0, 1], \ n \geq 1 \), where \( A_\infty = A_\infty(\alpha - 1) \) is a constant depending only on \( \alpha \).

First, we prove the following partial case of Theorem 4.0.

**Lemma 4.1** Let \( N \geq 0 \) be an integer. Then,
\[
C_{n,r} \left( l_a^{\infty} \left( \frac{1}{(k+1)^N} \right), H^{\infty} \right) \leq A_\infty \left( \frac{n}{1-r} \right)^{N+\frac{3}{2}},
\]
for all \( r \in [0, 1], \ n \geq 1 \), where \( A_\infty = A_\infty(N) \) is a constant depending on \( N \) (of order \( N^{2N} \) from the proof below).

**Proof.** We use literally the same method as in Corollary 1.3&2.2. Indeed, if \( f \in l_a^{\infty} \left( \frac{1}{(k+1)^N} \right) = H \) then \( |P_B f(\xi)| = |\langle P_B f, k_\xi \rangle| = |\langle f, P_B k_\xi \rangle| \), where \( \langle \cdot, \cdot \rangle \) means the Cauchy pairing and \( k_\xi = (1 - \xi z)^{-1} \). Denoting \( H^* \) the dual of \( H \) with respect to this pairing, \( H^* = l_a^{1}((k+1)^N) \), we get
\[
|P_B f(\xi)| \leq \|f\|_H \|P_B k_\xi\|_{H^*} \leq \|f\|_H K_N \left( \|P_B k_\xi\|_W + \|P_B k_\xi(\xi)^{(N)}\|_W \right),
\]
where \( W = \{ f = \sum_{k \geq 0} \hat{f}(k)z^k : \|f\|_W := \sum_{k \geq 0}|\hat{f}(k)| < \infty \} \) stands for the Wiener algebra, and \( K_N \) is defined in Corollary 1.3. Now, applying Hardy’s inequality (see [N2] p.370, 8.7.4 (c)),

\[
|P_Bf(\zeta)| \leq \|f\|_H K_N \left( \pi \left( \|P_Bk\zeta\|_{H^1} \right) + \|P_Bk\zeta(0)\| + \pi \left( \|P_Bk\zeta(0)\| + \|P_Bk\zeta(N+1)\| \right) \right),
\]

which gives using Lemma 2.1,

\[
|P_Bf(\zeta)| \leq \|f\|_H K_N \pi \left( \left( \frac{2n}{1-r} \right)^N \|P_Bk\zeta\|_{H^1} + \|P_Bk\zeta(0)\| \right) + (N+1)! \left( \frac{2n}{1-r} \right)^{N+1} \|P_Bk\zeta\|_{H^2} + \|P_Bk\zeta\|_{H^2} + \|P_Bk\zeta\|_{H^2} \right).}
\]

This completes the proof since \( \|P_Bk\zeta\|_{H^2} \leq \left( \frac{2n}{1-r} \right)^2 \). \( \square \)

**Proof of Theorem 4.0.** This is the same application of interpolation between Banach spaces, as before (Theorem 1.0&2.0) excepted that this time we apply Lemma 0 with \( X = W^\infty \left( \frac{1}{(k+1)\alpha-1} \right) \) to get

\[
\| T \|_{W^\infty \left( \frac{1}{(k+1)\alpha-1} \right)} \to H^\infty / B_\sigma H^\infty = c \left( \sigma, \|P_a\| w_k, H^\infty \right),
\]

where \( T \) and \( B_\sigma \) are defined in Lemma 0.

Applying Lemma 4.1 and using (again) [Tr] Theorem 1.9.3 p.59, we can complete the proof. \( \square \)

### 5. An upper bound for \( c(\sigma, \|P_a\| w_k, H^\infty), \ 2 \leq p \leq \infty \)

The aim of this section is to prove the following theorem.

**Theorem 5.0** Let \( 2 \leq p \leq \infty \), \( \alpha \geq 1 \). Then

\[
B \left( \frac{1}{1-r} \right)^{\alpha - \frac{1}{p}} \leq C_{n,r} \left( \|P_a\| \left( \frac{1}{(k+1)\alpha-1} \right), H^\infty \right) \leq A \left( \frac{n}{1-r} \right)^{\alpha + \frac{1}{2} - \frac{2}{p}},
\]

for all \( r \in [0, 1[ \), \( n \geq 1 \), where \( A = A(\alpha - 1, p) \) is a constant depending only on \( \alpha \) and \( p \) and \( B = B(p) \) is a constant depending only on \( p \).

**Remark.** As before, the upper bound \( \left( \frac{n}{1-r} \right)^{\alpha + \frac{1}{2} - \frac{2}{p}} \) is not as sharp as in **Section 1.** We can suppose here the constant \( \left( \frac{n}{1-r} \right)^{\alpha - \frac{1}{p}} \) should be a sharp upper (and lower) bound for the quantity \( C_{n,r} \left( \|P_a\| \left( \frac{1}{(k+1)\alpha-1} \right), H^\infty \right), 2 \leq p \leq +\infty \).
Proof. We first prove the right hand side inequality. The proof repeats the scheme from previous theorems and from Theorem 3.0 in particular. We have already seen (in Theorem 3.0) that
\[ \| T \|_{L^p_{\lambda}} \rightarrow H^\infty / B_\sigma H^\infty = c \left( \sigma, p^\alpha \left( \frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right), \]
where \( T \) and \( B_\sigma \) are defined in Lemma 0. Now, using both Theorems 1.0 & 4.0, and [Tr] Theorem 1.9.3 p. 59, we complete the proof. The proof of the left hand side inequality is exactly the same as in Theorem 3.0. \( \square \)

6. Carathéodory-Schur Interpolation in weighted Bergman spaces

We suppose that \( X = L^p_a \left( (1 - |z|^2)^\beta \, dA \right), \beta > -1 \) and \( 1 \leq p \leq 2 \). Our aim in this section is to give an estimate for the constant for a generalized Carathéodory-Schur interpolation, (a partial case of the Nevanlinna-Pick interpolation),

\[ c(\sigma, n, X, H^\infty) = \sup \{ \| f \|_{H^\infty / b_n^\lambda H^\infty} : f \in X, \| f \|_X \leq 1 \}, \]

where \( \| f \|_{H^\infty / b_n^\lambda H^\infty} = \inf \{ \| f + b_n^\lambda g \| : g \in X \} \), and \( \sigma, n = \{ \lambda, \lambda, ..., \lambda \}, \lambda \in \mathbb{D} \). The corresponding interpolation problem is: given \( f \in X \), to minimize \( \| h \|_\infty \) such that \( h^{(j)}(\lambda) = f^{(j)}(\lambda), 0 \leq j < n \).

For this partial case, we have the following generalization of the estimate from Theorem 1.0.

**Theorem 6.0** Let \( \lambda \in \mathbb{D}, \beta > -1 \) and \( 1 \leq p \leq 2 \). Then,

\[ c \left( \sigma, n, X, H^\infty \right) \leq A' \left( \frac{n}{1 - |\lambda|} \right)^{\frac{\beta + 2}{p}}, \]

where \( A' = A'(\beta, p) \) is a constant depending only on \( \beta \) and \( p \).

We first need a simple equivalent to \( I_k(\beta) = \int_0^1 r^{2k+1}(1 - r^2)^\beta \, dr, \beta > -1 \).

**Lemma 6.1** Let \( k \geq 0, \beta > -1 \) and \( I_k(\beta) = \int_0^1 r^{2k+1}(1 - r^2)^\beta \, dr \). Then,

\[ I_k(\beta) \sim \frac{1}{2} \frac{\Gamma(\beta + 1)}{k^{\beta+1}}, \]

for \( k \rightarrow \infty \), where \( \Gamma \) stands for the usual Gamma function, \( \Gamma(z) = \int_0^{+\infty} e^{-s}s^{z-1}ds \).

**Proof.** Let \( a = \frac{1}{\sqrt{k+1}}, b = \max(1, a^{\beta}) \). Since \( 1 - e^{-u} \sim u \) as \( u \rightarrow 0 \), we have

\[ I_k(\beta) = \int_0^1 r^{2k+1}(1 - r^2)^\beta \, dr = \int_0^\infty e^{-(2k+1)t}(1 - e^{-2t})\beta e^{-t} \, dt = \int_a^\infty e^{-2(k+1)t}(1 - e^{-2t})^\beta e^{-t} \, dt = \int_a^\infty e^{-2(k+1)t}(1 - e^{-2t})^\beta e^{-t} \, dt = \]

The rest of the proof follows similarly as in the previous theorems.
\[ \int_0^a e^{-2(k+1)t} (1 - e^{-2t})^\beta dt + O \left( \frac{b}{k+1} e^{-2a(k+1)} \right) = \]
\[ = (1 + o(1)) \int_0^a e^{-2(k+1)t} (2t)^\beta dt + O \left( \frac{b}{k+1} e^{-2a(k+1)} \right) = \]
\[ = (1 + o(1)) \int_0^{2(k+1)a} e^{-s} \left( \frac{s}{k+1} \right)^\beta \frac{ds}{2(k+1)} + O \left( \frac{b}{k+1} e^{-2a(k+1)} \right) = \]
\[ = \frac{1}{2} \frac{1}{2(k+1)^{\beta+1}} (1 + o(1)) \int_0^{2(k+1)a} e^{-s} s^\beta ds + O \left( \frac{b}{k+1} e^{-2a(k+1)} \right) = \]
\[ = \frac{1}{2} \frac{\Gamma(\beta+1)}{2(k+1)^{\beta+1}} (1 + o(1)) + O \left( \frac{b}{k+1} e^{-2a(k+1)} \right) = \]
\[ = \frac{1}{2} \frac{\Gamma(\beta+1)}{2(k+1)^{\beta+1}} (1 + o(1)) \sim \frac{1}{2} \frac{\Gamma(\beta+1)}{k^{\beta+1}}, \]

which completes the proof. \(\Box\)

**Proof of Theorem 6.6.** **Step 1.** We start to prove the Theorem for \(p = 1\).

Let \(f \in X = L^1_a \left( (1 - |z|^2)^\beta \, dA \right)\) such that \(\|f\|_X \leq 1\). Since \(X \circ b_\lambda = X\), we have \(f \circ b_\lambda = \sum_{k \geq 0} a_k z^k \in X\). Let \(p_n = \sum_{k=0}^{n-1} a_k z^k\) and \(g = p_n \circ b_\lambda\). Then, \(f \circ b_\lambda - p_n \in \ell^n X\) and \(f - p_n \circ b_\lambda \in (\ell^n X) \circ b_\lambda = b_\lambda^n X\). Now, \(p_n \circ b_\lambda = \sum_{k=0}^{n-1} a_k b_\lambda^k\) and

\[ \|p_n \circ b_\lambda\|_\infty = \|p_n\|_\infty \leq A_n \|f \circ b_\lambda\|_X, \]

where \(A_n = \|\sum_{k \geq 0} a_k z^k \mapsto \sum_{k=0}^{n-1} a_k z^k\|_{X \rightarrow H^n}\). Now,

\[ \|f \circ b_\lambda\|_X \leq \int_{\mathbb{D}} |f(b_\lambda(z))| (1 - |z|^2)^\beta \, dA = \int_{\mathbb{D}} |f(w)| (1 - |b_\lambda(w)|^2)^\beta |b_\lambda(w)|^2 \, dA = \]
\[ \leq 2^\beta \int_{\mathbb{D}} |f(w)| \left( \frac{1 - |\lambda|^2}{1 - \lambda w|^2} \right)^\beta \left( \frac{1 - |\lambda|^2}{1 - \lambda w|^2} \right)^2 \, dA = \]
\[ = \int_{\mathbb{D}} |f(w)| (1 - |w|^2)^\beta \left( \frac{1 - |\lambda|^2}{1 - \lambda w|^2} \right)^{2+\beta} \, dA \leq \]
\[ \leq \sup_{w \in \mathbb{D}} \left( \frac{1 - |\lambda|^2}{1 - \lambda w|^2} \right)^{2+\beta} \int_{\mathbb{D}} |f(w)| (1 - |w|^2)^\beta \, dA \leq \left( \frac{1 - |\lambda|^2}{(1 - |\lambda|^2)^2} \right)^{2+\beta} \|f\|_X, \]

which gives,

\[ \|f \circ b_\lambda\|_X \leq \left( \frac{1 + |\lambda|}{1 - |\lambda|^2} \right)^{2+\beta} \|f\|_X. \]
We now give an estimation for \( A_n \). Let \( g(z) = \sum_{k \geq 0} \hat{g}(k)z^k \in X \), then
\[
\left\| \sum_{k=0}^{n-1} \hat{g}(k)z^k \right\|_\infty \leq \sum_{k=0}^{n-1} |\hat{g}(k)| .
\]
Now, noticing that
\[
\int_\mathbb{D} g(w) w^k (1 - |w|^2)^\beta dA = \int_0^1 \int_0^{2\pi} f(re^{it})r^k e^{-ikt}(1 - r^2)^\beta r dt dr = \int_0^1 (1 - r^2)^\beta r^{k+1} \int_0^{2\pi} f(re^{it})e^{-ikt} dt dr = \int_0^1 \hat{g}_r(k)r^{k+1}(1 - r^2)^\beta dr,
\]
where \( g_r(z) = g(rz) \), \( \hat{g}_r(k) = r^k \hat{g}(k) \). Setting \( I_k(\beta) = \int_0^1 r^{2k+1}(1 - r^2)^\beta dr \), we get
\[
\hat{g}(k) = \frac{1}{I_k(\beta)} \int_\mathbb{D} g(w) w^k (1 - |w|^2)^\beta dA.
\]
Then,
\[
|\hat{g}(k)| = \frac{1}{I_k(\beta)} \left| \int_\mathbb{D} g(w) w^k (1 - |w|^2)^\beta dA \right| \leq \frac{1}{I_k(\beta)} \|g\|_X ,
\]
which gives
\[
\left\| \sum_{k=0}^{n-1} \hat{g}(k)z^k \right\|_\infty \leq \left( \sum_{k=0}^{n-1} \frac{1}{I_k(\beta)} \right) \|g\|_X .
\]
Now using Lemma 6.1,
\[
\sum_{k=0}^{n-1} \frac{1}{I_k(\beta)} \sim_{n \to \infty} \frac{2}{\Gamma(\beta + 1)} \sum_{k=0}^{n-1} k^{\beta + 1} \sim \frac{2c_\beta}{\Gamma(\beta + 1)} n^{\beta + 2} ,
\]
where \( c_\beta \) is a constant depending on \( \beta \) only. This gives
\[
\left\| \sum_{k=0}^{n-1} \hat{g}(k)z^k \right\|_\infty \leq C_\beta n^{\beta + 2} \|g\|_X ,
\]
where \( C_\beta \) is also a constant depending on \( \beta \) only. Finally, we conclude that \( A_n \leq C_\beta n^{\beta + 2} \), and as a result,
\[
\|p_n \circ b_\lambda\|_\infty \leq C_\beta n^{\beta + 2} \left( \frac{1 + |\lambda|}{1 - |\lambda|} \right)^{2+\beta} \|f\|_X ,
\]
which proves the Theorem for \( p = 1 \).

**Step 2.** This step of the proof repetes the scheme from Theorems 3.0&5.0. Let \( T : L^p_a \left( (1 - |z|^2)^\beta dA \right) \to H^\infty/ b_\lambda^p H^\infty \) be the restriction map defined by
\[
Tf = \left\{ g \in H^\infty : f - g \in b_\lambda^p L^p_a \left( (1 - |z|^2)^\beta dA \right) \right\} ,
\]
for every \( f \). Then,
\[
\| T \|_{L^p_a \left( (1 - |z|^2)^\beta dA \right) \to H^\infty/ b_\lambda^p H^\infty} = c \left( \sigma, L^p_a \left( (1 - |z|^2)^\beta dA \right), H^\infty \right) .
\]
Now, let $\gamma > \beta$ and $P_\gamma : L^p \left( (1 - |z|^2)^\beta dA \right) \to L^p_a \left( (1 - |z|^2)^\beta dA \right)$ be the Bergman projection, (see [H], p.6), defined by

$$P_\gamma f = (\gamma + 1) \int_D \frac{(1 - |w|^2)^\gamma}{(1 - z \overline{w})^{\gamma+2}} f(w) dA(w),$$

for every $f$. $P_\gamma$ is a bounded projection from $L^p \left( (1 - |z|^2)^\beta dA \right)$ onto $L^p_a \left( (1 - |z|^2)^\beta dA \right)$ (see [H], Theorem 1.10 p.12), (since $1 \leq p \leq 2$). Moreover, since $L^p_a \left( (1 - |z|^2)^\beta dA \right) \subset L^p \left( (1 - |z|^2)^\gamma dA \right)$, we have $P_\gamma f = f$ for all $f \in L^p_a \left( (1 - |z|^2)^\beta dA \right)$, (see [H], Corollary 1.5 p.6). As a result,

$$\| T \|_{L^p_a \left( (1 - |z|^2)^\beta dA \right) \to H^\infty / b^\infty_a H^\infty} \leq \| TP_\gamma \|_{L^p \left( (1 - |z|^2)^\beta dA \right) \to H^\infty / b^\infty_a H^\infty},$$

for all $1 \leq p \leq 2$. We set

$$c_i(\beta) = \| P_\gamma \|_{L^1 \left( (1 - |z|^2)^\beta dA \right) \to L^1_a \left( (1 - |z|^2)^\beta dA \right)},$$

for $i = 1, 2$. Then,

$$\| TP_\gamma \|_{L^1 \left( (1 - |z|^2)^\beta dA \right) \to H^\infty / b^\infty_a H^\infty} \leq \| T \|_{L^1_a \left( (1 - |z|^2)^\beta dA \right) \to H^\infty / b^\infty_a H^\infty} \| P_\gamma \|_{L^1 \left( (1 - |z|^2)^\beta dA \right) \to L^1_a \left( (1 - |z|^2)^\beta dA \right)} =$$

$$= c \left( \sigma, L^1_a \left( (1 - |z|^2)^\beta dA \right), H^\infty \right) c_1(\beta) \leq A'(\beta, 1) \left( \frac{n}{1 - |\lambda|} \right)^{\beta+2} c_1(\beta),$$

using Step 1. In the same way,

$$\| TP_\gamma \|_{L^2 \left( (1 - |z|^2)^\beta dA \right) \to H^\infty / b^\infty_a H^\infty} \leq \| T \|_{L^2_a \left( (1 - |z|^2)^\beta dA \right) \to H^\infty / b^\infty_a H^\infty} c_2(\beta).$$

Now, we recall that

$$L^2_a \left( (1 - |z|^2)^\beta dA \right) = L^2 \left( \frac{1}{(k + 1)^{\frac{2\beta+1}{\beta+1}}} \right), \beta > -1.$$

As a consequence,

$$\| T \|_{L^2_a \left( (1 - |z|^2)^\beta dA \right) \to H^\infty / b^\infty_a H^\infty} = c \left( \sigma, L^2 \left( \frac{1}{(k + 1)^{\frac{2\beta+1}{\beta+1}}} \right), H^\infty \right),$$

and, applying Theorem 1.0,
\[ \|TP_\gamma\|_{L^2((1-|z|^2)^\beta\,dA)\to H^\infty/b_\lambda^pH^\infty} \leq c_2(\beta)A\left(\frac{\beta + 1}{2}, 2\right) \left(\frac{n}{1 - |\lambda|}\right)^{\frac{\beta+1}{2}+1} = \\
= c_2(\beta)A\left(\frac{\beta + 1}{2}, 2\right) \left(\frac{n}{1 - |\lambda|}\right)^{\frac{\beta+2}{2}}. \]

We finish the reasoning applying Riesz-Thorin Theorem, (see [Tr] for example), to the operator \(TP_\gamma\). If \(p \in [1, 2]\), there exists \(0 \leq \theta \leq 1\) such that
\[ \frac{1}{p} = (1 - \theta)\frac{1}{1} + \theta\frac{1}{2} = 1 - \frac{\theta}{2}, \]
and then,
\[ \left[ L^1_a\left((1 - |z|^2)^\beta\,dA\right), L^2_a\left((1 - |z|^2)^\beta\,dA\right) \right]_\theta = L^p_a\left((1 - |z|^2)^\beta\,dA\right), \]
and
\[ \|TP_\gamma\|_{L^p\left((1-|z|^2)^\beta\,dA\right)\to H^\infty/b_\lambda^pH^\infty} \leq \]
\[ \leq \left(\|TP_\gamma\|_{L^1((1-|z|^2)^\beta\,dA)\to H^\infty/b_\lambda^pH^\infty}\right)^{1-\theta} \left(\|TP_\gamma\|_{L^2((1-|z|^2)^\beta\,dA)\to H^\infty/b_\lambda^pH^\infty}\right)^{\theta} \leq \]
\[ \leq \left(c_1(\beta)A'(\beta, 1)\left(\frac{n}{1 - |\lambda|}\right)^{\beta+2}\right)^{1-\theta} \left(c_2(\beta)A\left(\frac{\beta + 1}{2}, 2\right) \left(\frac{n}{1 - |\lambda|}\right)^{\frac{\beta+2}{2}}\right)^{\theta} = \]
\[ = \left(c_1(\beta)A'(\beta, 1)\right)^{1-\theta} \left(c_2(\beta)A\left(\frac{\beta + 1}{2}, 2\right)\right)^{\theta} \left(\frac{n}{1 - |\lambda|}\right)^{(\beta+2)(1-\theta)+\theta\frac{\beta+2}{2}}. \]

Now, since \(\theta = 2(1 - \frac{1}{p})\), \((\beta + 2)(1 - \theta) + \theta\frac{\beta+2}{2} = \beta - (1 - \frac{1}{p})\beta + 2 - 2 + \frac{2}{p} = \frac{\beta+2}{p}\), and
\[ \|T\|_{L^p((1-|z|^2)^\beta\,dA)\to H^\infty/b_\lambda^pH^\infty} \leq \|TP_\gamma\|_{L^p\left((1-|z|^2)^\beta\,dA\right)\to H^\infty/b_\lambda^pH^\infty}, \]
we complete the proof. \(\Box\)

7. A lower bound for \(C_{n,r}(l^2_\alpha(w_k), H^\infty)\)
Here, we consider the weighted spaces $l^2_a(w_k)$ of polynomial growth and the problem of lower estimates for the one point special case $\sigma_{\lambda,n} = \{\lambda, \lambda, \ldots, \lambda\}$, $(n$ times) $\lambda \in \mathbb{D}$. Recall the definition of the semi-free interpolation constant

$$c(\sigma_{\lambda,n}, H, H^\infty) = \sup \{\|f\|_{H^\infty/\rho_k^\infty H^\infty} : f \in H, \|f\|_H \leq 1\},$$

where $\|f\|_{H^\infty/\rho_k^\infty H^\infty} = \inf \{\|f + b^\infty_k g\|_\infty : g \in H\}$. In particular, our aim is to prove the sharpness of the upper estimate for the quantity

$$C_{n, r} \left( l^2_a \left( \frac{1}{(k+1)^{\frac{N-1}{2}}} \right), H^\infty \right),$$

(where $N \geq 1$ is an integer), in Theorem 1.0.

**Theorem 7.0** Let $N \geq 1$ be an integer. Then,

$$c \left( \sigma_{\lambda,n}, l^2_a \left( \frac{1}{(k+1)^{\frac{N-1}{2}}} \right), H^\infty \right) \geq a_N \left( \frac{n}{1 - |\lambda|} \right)^{\frac{N}{2}}$$

for a positive constant $a_N$ depending on $N$ only. In particular,

$$a_N \left( \frac{n}{1 - r} \right)^{\frac{N}{2}} \leq C_{n, r} \left( l^2_a \left( \frac{1}{(k+1)^{\frac{N-1}{2}}} \right), H^\infty \right) \leq A \left( \frac{n}{1 - r} \right)^{\frac{N}{2}},$$

for all $n \geq 1$, $0 \leq r < 1$, where $A = A \left( \frac{N-1}{2} \right)$ is a constant defined in Theorem 1.0.

1. We first recall some properties of spaces $X = l^p_a(w_k)$. As it is mentioned in the Introduction,

$$l^p_a(w_k) = \left\{ f = \sum_{k \geq 0} \hat{f}(k)z^k : \|f\|^p = \sum_{k \geq 0} |\hat{f}(k)|^p w_k^p < \infty \right\},$$

with a weight $w$ satisfying $w_k > 0$ for every $k \geq 0$ and $\lim_k (1/w_k^{1/k}) = 1$. The latter condition implies that $l^p_a(w_k)$ is continuously embedded into the space of holomorphic functions $Hol(\mathbb{D})$ on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (and not on a larger disc, i.e. $l^p_a(w_k)$ does not contained in $Hol(r\mathbb{D})$ for every $r > 1$). In this section, we study the case $p = 2$, so that $l^2_a(w_k)$ is a reproducing kernel Hilbert space on the disc $\mathbb{D}$. The reproducing kernel of $l^2_a(w_k)$, by definition, is a $l^2_a(w_k)$-valued function $\lambda \mapsto k^w_\lambda$, $\lambda \in \mathbb{D}$, such that $(f, k^w_\lambda) = f(\lambda)$ for every $f \in l^2_a(w_k)$, where
(..., ...) means the scalar product \( (f, g) = \sum_{k \geq 0} \hat{h}(k) \overline{g(k)} w_k^2 \). Since one has
\[
f(\lambda) = \sum_{k \geq 0} \hat{f}(k) \lambda^k \frac{1}{w_k^2} w_k^2 \quad (\lambda \in \mathbb{D}),
\]
it follows that
\[
k^w_\lambda(z) = \sum_{k \geq 0} \frac{\lambda^k}{w_k^2} z^k, \quad z \in \mathbb{D}.
\]
In particular, for the Hardy space \( H^2 = l^2_\alpha(1) \), we get the Szegö kernel
\[
k_\lambda(z) = (1 - \overline{\lambda}z)^{-1},
\]
for the Bergman space \( L^2_\alpha(\mathbb{D}) = l^2_\alpha \left( \frac{1}{(k+1)^2} \right) \) - the Bergman kernel \( k_\lambda(z) = (1 - \overline{\lambda}z)^{-2} \).

(2) Conversely, following the Aronszajn theory of RKHS (see, for example [A] or [N2] p.317), given a positive definite function \((\lambda, z) \mapsto k(\lambda, z)\) on \( \mathbb{D} \times \mathbb{D} \) (i.e. such that \( \sum_{i,j} \pi_i \pi_j k(\lambda_i, \lambda_j) > 0 \) for all finite subsets \((\lambda_i) \subset \mathbb{D}\) and all non-zero families of complex numbers \((a_i)\) one can define the corresponding Hilbert spaces \( H(k) \) as the completion of finite linear combinations \( \sum_i \pi_i k(\lambda_i, \cdot) \) endowed with the norm
\[
\| \sum_i \pi_i k(\lambda_i, \cdot) \| = \sum_{i,j} \pi_i a_j k(\lambda_i, \lambda_j).
\]
When \( k \) is holomorphic with respect to the second variable and antiholomorphic with respect to the first one, we obtain a RKHS of holomorphic functions \( H(k) \) embedded into \( Hol(\mathbb{D}) \).

For functions \( k \) of the form \( k(\lambda, z) = K(\lambda z) \), where \( K \in Hol(\mathbb{D}) \), the positive definiteness is equivalent to \( \hat{K}(j) > 0 \) for every \( j \geq 0 \), where \( \hat{K}(j) \) stands for Taylor coefficients, and in this case we have \( H(k) = l^2_\alpha(w_j) \), where \( w_j = 1/\sqrt[1]{\hat{K}(j)}, j \geq 0 \). In particular, for \( K(w) = (1-w)^{-\beta} \), \( k_\lambda(z) = (1 - \overline{\lambda}z)^{-\beta} \), \( \beta > 0 \), we have \( \hat{K}(j) = (\beta j)_{\beta-1} \) (binomial coefficients), and hence \( w_j = \left( \frac{1}{\beta(\beta+1)...(\beta j-1)} \right)^{1/2} \). Indeed, deriving \( 1/(1-z)^{\beta-1} \), we get by induction
\[
(1 - z)^{-\beta} = \frac{1}{(\beta-1)!} \sum_{j \geq 0} (j + \beta - 1)...(j + 1) z^j = \sum_{j \geq 0} \beta_{\beta-1} j z^j.
\]
Clearly, \( w_j \simeq 1/j^{\beta-1} \), where \( a_j \simeq b_j \) means that there exist constants \( c_1 > 0, c_2 > 0 \) such that \( c_1 a_j \leq b_j \leq c_2 a_j \) for every \( j \). Therefore, \( H(k) = l^2_\alpha \left( \frac{1}{(k+1)^\beta} \right) \) (a topological identity: the spaces are the same and the norms are equivalent).

(3) Reproducing kernel Hilbert spaces containing \( H^2 \). We will use the previous observations for the following composed reproducing kernels (Aronszajn-deBranges, see [N2] p.320): given a reproducing kernel \( k \) and an entire function \( \varphi = \sum_{j \geq 0} \varphi(j) z^j \) with \( \varphi(j) \geq 0 \) for every \( j \geq 0 \), the function \( \varphi \circ k \) is also positive definite and the corresponding RKHS
\[
H(\varphi \circ k) =: \varphi(H(k))
\]
satisfies the following. For every \( f \in H(k) \) we have \( \varphi \circ f \in \varphi(H(k)) \) and \( \| \varphi \circ f \|_{\varphi(H(k))}^2 \leq \varphi(\|f\|_{H(k)}^2) \) (see [N2] p.320). In particular, if \( \varphi \) is a polynomial of degree \( N \) and \( k \) is the Szegö kernel then
ϕ \circ k_\lambda(z) = \sum_{j \geq 0} c_j \lambda^j z^j \text{ with } c_k \simeq (k + 1)^{N-1}, \text{ and hence }

\varphi(H^2) = l_a^2 \left( \frac{1}{(k + 1)^{\frac{2N}{2}}} \right)

(a topological identity: the spaces are the same and the norms are equivalent). The link between spaces of type \(l_a^2(1-\lambda z)^{1/2} \) (already mentioned in Section 1) and of type \(\varphi(H^2) = H_\varphi\) being established, we give the following result.

**Lemma 7.1** Let \(\varphi(z) = \sum_{k=0}^N a_k z^k\), \(a_k \geq 0\) \((a_N > 0)\), and \(H_\varphi = \varphi(H^2)\) be the reproducing kernel Hilbert space corresponding to the kernel \(\varphi\left(\frac{1}{1-\lambda z}\right)\). Then, there exists a constant \(a(\varphi) > 0\) such that

\[c(\sigma_{\lambda,n}, H_\varphi, H^\infty) \geq a(\varphi) \left( \frac{n}{1-|\lambda|} \right)^{\frac{N}{2}}.\]

**Proof.**

1) We set

\[Q_n = \sum_{k=0}^{n-1} b_{\lambda}^k \frac{(1-|\lambda|^2)^{1/2}}{1-\lambda z}, \quad H_n = \varphi \circ Q_n, \quad \Psi = b H_n.\]

Then \(\|Q_n\|_2^2 = n\), and hence by the Aronszajn-deBranges inequality, see [N2] p.320, point (k) of Exercise 6.5.2, with \(\varphi(z) = z^N\) and \(K(\lambda, z) = k_\lambda(z) = \frac{1}{1-\lambda z}\), and noticing that \(H(\varphi \circ K) = H_\varphi\),

\[\|\Psi\|_{H_\varphi}^2 \leq b^2 \varphi (\|Q_n\|_2^2) = b^2 \varphi(n).\]

Let \(b > 0\) such that \(b^2 \varphi(n) = 1\).

2) Since the spaces \(H_\varphi\) and \(H^\infty\) are rotation invariant, we have \(c(\sigma_{\lambda,n}, H_\varphi, H^\infty) = c(\sigma_{\mu,n}, H_\varphi, H^\infty)\) for every \(\lambda, \mu\) with \(|\lambda| = |\mu| = r\). Let \(\lambda = -r\). To get a lower estimate for \(\|\Psi\|_{H_\varphi/b^2 \varphi}^2\) consider \(G\) such that \(\Psi - G \in b_{\lambda}^2 H_\varphi(\mathbb{D})\), i.e. such that \(b H_n \circ b_{\lambda} - G \circ b_{\lambda} \in z^n H_\varphi(\mathbb{D})\).

3) First, we show that

\[\psi =: \Psi \circ b_{\lambda} = b H_n \circ b_{\lambda}\]

is a polynomial (of degree \(nN\)) with positive coefficients. Note that
\[
Q_n \circ b_\lambda = \sum_{k=0}^{n-1} z^k \left( \frac{1 - |\lambda|^2}{1 - \lambda b_\lambda(z)} \right) = \\
= (1 - |\lambda|^2)^{-\frac{1}{2}} \left( 1 + (1 - \lambda) \sum_{k=1}^{n-1} z^k - \lambda z^n \right) = \\
= (1 - r^2)^{-1/2} \left( 1 + (1 + r) \sum_{k=1}^{n-1} z^k + rz^n \right) =: (1 - r^2)^{-1/2} \psi_1.
\]

Hence, \( \psi = \Psi \circ b_\lambda = bH_n \circ b_\lambda = b \varphi \circ \left( (1 - r^2)^{-\frac{1}{2}} \psi_1 \right) \) and

\[
\varphi \circ \psi_1 = \sum_{k=0}^{N} a_k \psi_1^k(z).
\]

(In fact, we can simply assume that \( \varphi \circ \psi_1 = \psi_1^N(z) \) since \( H_\varphi = l_a^2 \left( \frac{1}{(k+1)^2} \right) = H_{N^N} \). Now, it is clear that \( \psi \) is a polynomial of degree \( Nn \) such that

\[
\psi(1) = \sum_{j=0}^{Nn} \hat{\psi}(j) = b \varphi \left( (1 - r^2)^{-1/2}(1 + r)n \right) = b \varphi \left( \sqrt{\frac{1 + r}{1 - r}} \frac{n}{N} \right) > 0.
\]

4) Next, we show that there exists a constant \( c = c(\varphi) > 0 \) (for example, \( c = \alpha/2^{2N} (N - 1)! \), \( \alpha \) is a numerical constant) such that

\[
\sum_{j=0}^{m} \hat{\psi}(j) =: \sum_{j=0}^{m} \hat{\psi}(j) \geq c \sum_{j=0}^{Nn} \hat{\psi}(j) = c \psi(1),
\]

where \( m \geq 1 \) is such that \( 2m = n \) if \( n \) is even and \( 2m - 1 = n \) if \( n \) is odd.

Indeed, setting

\[
S_n = \sum_{j=0}^{n} z^j,
\]

we have

\[
\sum_{k=0}^{m} \left( \psi_1^k \right) = \sum_{k=0}^{m} \left( \left( 1 + (1 + r) \sum_{k=1}^{n-1} z^k + rz^n \right)^k \right) \geq \sum_{k=0}^{m} \left( S_{n-1}^k \right).
\]

Next, we obtain

\[
\sum_{k=0}^{m} \left( S_{n-1}^k \right) = \sum_{k=0}^{m} \left( \left( \frac{1 - z^n}{1 - z} \right)^k \right) =
\]
\begin{align*}
= \sum_{j=0}^{m} \left( \sum_{j=0}^{k} C_{j}^{k} \frac{1}{(1-z)^{j}} \cdot \left( \frac{-z^{n}}{1-z} \right)^{k-j} \right) = \sum_{j=0}^{m} \left( \frac{1}{(1-z)^{k}} \right) = \\
= \sum_{j=0}^{m} \left( \sum_{j=0}^{C_{j}^{k+j-1} z^{j}} \right) = \sum_{j=0}^{m} C_{j}^{k+j-1} \geq \sum_{j=0}^{m} \frac{(j+1)^{k-1}}{(k-1)!} \geq \sum_{j=0}^{m} \frac{1}{(k-1)!} \\
\geq \frac{\alpha m^{k}}{(k-1)!},
\end{align*}

where \( \alpha > 0 \) is a numerical constant. Finally,

\[
\sum_{j=0}^{m} \left( \psi_{1}^{k} \right) \geq \frac{\alpha m^{k}}{(k-1)!} \geq \frac{\alpha (n/2)^{k}}{(k-1)!} = \\
= \frac{\alpha}{2^{k(k-1)!}} \cdot \frac{(1+r)^{k}}{(1+r)^{k}} \geq \frac{\alpha}{2^{k(1+r)^{k}(k-1)!}} \cdot (\psi_{1}(1))^{k} \geq \\
\geq \frac{\alpha}{2^{N(1+r)^{N}(N-1)!}} \cdot (\psi_{1}(1))^{k}.
\]

Summing up these inequalities in \( \sum_{m}^{m} (\psi) = b \sum_{m}^{m} (\varphi \circ \psi_{1}) = b \sum_{k=0}^{N} a_{k} (1-r^{2})^{-k/2} \sum_{m}^{m} (\psi_{1}^{k}) \) (or simply taking \( k = N \), if we already supposed \( \varphi = z^{N} \)), we obtain the result claimed.

5) Now, using point 4) and the preceding Fejer kernel argument and denoting \( F_{n} = \Phi_{m} + z^{m} \Phi_{m} \), where \( \Phi_{k} \) stands for the \( k \)-th Fejer kernel, we get

\[
\| \Psi \|_{H^{\infty}/\psi_{k}^{m} H^{\infty}} = \| \psi \|_{H^{\infty}/z^{n} H^{\infty}} \geq \frac{1}{2} \| \psi \|_{H^{\infty}/z^{n} H^{\infty}} \geq \frac{1}{2} \sum_{j=0}^{m} \hat{\psi}(j) \geq \\
\geq \frac{c}{2} \psi(1) = \frac{c}{2} b \varphi \left( \sqrt{\frac{1+r}{1-r}} \right) = \frac{c}{2} \cdot \frac{\varphi \left( \sqrt{\frac{1+r}{1-r}} \right)}{(\varphi(n))^{1/2}} \geq \\
(assuming that \( \varphi = z^{N} \))
\]

\[
\geq \alpha \left( \frac{n}{1-r} \right) ^{\frac{N}{2}}.
\]

Proof of Theorem 7.0. In order to prove the left hand side inequality, it suffices to apply Lemma 7.1 with \( \varphi(z) = z^{N} \). Indeed, in this case \( H_{\varphi} = l_{a}^{2} \left( \frac{1}{(k+1)^{2} r^{2}} \right) = H_{z^{N}} \). The right hand side inequality is a straightforward consequence of Theorem 1.0.

\( \square \)

Acknowledgement.

I would like to thank Professor Nikolai Nikolski for its invaluable help and its precious advices.
References


