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Jean-Luc Fouquet, Jean-Marie Vanherpe

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ON THE PERFECT MATCHING INDEX OF BRIDGELESS CUBIC GRAPHS

J.L. FOUQUET AND J.M. VANHERPE

Abstract. If $G$ is a bridgeless cubic graph, Fulkerson conjectured that we can find 6 perfect matchings $M_1, \ldots, M_6$ of $G$ with the property that every edge of $G$ is contained in exactly two of them and Berge conjectured that its edge set can be covered by 5 perfect matchings. We define $\tau(G)$ as the least number of perfect matchings allowing to cover the edge set of a bridgeless cubic graph and we study this parameter. The set of graphs with perfect matching index 4 seems interesting and we give some informations on this class.

1. Introduction

The following conjecture is due to Fulkerson, and appears first in [3].

Conjecture 1.1. If $G$ is a bridgeless cubic graph, then there exist 6 perfect matchings $M_1, \ldots, M_6$ of $G$ with the property that every edge of $G$ is contained in exactly two of $M_1, \ldots, M_6$.

If $G$ is 3-edge-colourable, then we may choose three perfect matchings $M_1, M_2, M_3$ so that every edge is in exactly one. Taking each of these twice gives us 6 perfect matchings with the properties described above. Thus, the above conjecture holds trivially for 3-edge-colorable graphs. There do exist bridgeless cubic graphs which are not 3-edge-colourable (for instance the Petersen graph), but the above conjecture asserts that every such graph is close to being 3-edge-colourable.

If Fulkerson’s conjecture were true, then deleting one of the perfect matchings from the double cover would result in a covering of the graph by 5 perfect matchings. This weaker conjecture was proposed by Berge (see Seymour [12]).

Conjecture 1.2. If $G$ is a bridgeless cubic graph, then there exists a covering of its edges by 5 perfect matchings.

Since the Petersen graph does not admit a covering by less that 5 perfect matchings (see section 3), 5 in the above conjecture can not be changed into 4 and the following weakening of conjecture 1.2 (suggested by Berge) is still open.

Conjecture 1.3. There exists a fixed integer $k$ such that the edge set of every bridgeless cubic graph can be written as a union of $k$ perfect matchings.

Another consequence of the Fulkerson conjecture would be that every bridgeless cubic graph has 3 perfect matchings with empty intersection (take any 3 of the 6 perfect matchings given by the conjecture). The following weakening of this (also suggested by Berge) is still open.
Conjecture 1.4. There exists a fixed integer $k$ such that every bridgeless cubic graph has a list of $k$ perfect matchings with empty intersection.

For $k = 3$ this conjecture is known as the Fan Raspaud Conjecture.

Conjecture 1.5. Every bridgeless cubic graph contains perfect matchings $M_1, M_2, M_3$ such that

$$M_1 \cap M_2 \cap M_3 = \emptyset$$

While some partial results exist concerning conjecture 1.5 (see [17]), we have noticed no result in the literature concerning the validity of Conjecture 1.1 or Conjecture 1.4 for the usual classes of graphs which are examined when dealing with the 5-flow conjecture of Tutte [15] or the cycle double conjecture of Seymour [11] and Szekeres [13]. Hence for bridgeless cubic graphs with oddness 2 (a 2-factor contains exactly two odd cycles) it is known that the 5-flow conjecture holds true as well as the cycle double conjecture (see Zhang [18] for a comprehensive study of this subject).

Let $G$ be a bridgeless cubic graph, we shall say that the set $M = \{M_1, \ldots, M_k\}$ ($k \geq 3$) of perfect matchings is a $k$-covering when each edge is contained in at least one of these perfect matchings. A Fulkerson covering is a 6-covering where each edge appears exactly twice. Since every edge of a bridgeless cubic graph is contained in a perfect matching (see [10]) the minimum number $\tau(G)$ of perfect matchings covering its edge set is well defined. We shall say that $\tau(G)$ is the perfect matching index of $G$. We obviously have that $\tau(G) = 3$ if and only if $G$ is 3-edge-colourable.

2. Preliminaries results

Proposition 2.1. let $G$ be a cubic graph with a $k$-covering $M = \{M_1, \ldots, M_k\}$ ($k \geq 3$) then $G$ is bridgeless.

Proof Assume that $e \in E(G)$ is an isthmus, then the edges incident to $e$ are not covered by any perfect matching of $G$ and $M$ is not a $K$-covering, a contradiction. □

2.1. 2-cut connection. Let $G_1, G_2$ be two bridgeless cubic graph and $e_1 = u_1v_1 \in E(G_1), e_2 = u_2v_2 \in E(G_1)$ be two edges. Construct a new graph $G = G_1 \bigodot G_2$

$$G = [G_1 \setminus \{e_1\}] \cup [G_2 \setminus \{e_2\}] \cup \{u_1u_2, v_1v_2\}$$

Proposition 2.2. Let $G_1$ be a cubic graph such that $\tau(G_1) = k \geq 3$ and let $G_2$ be any cubic bridgeless graph, then $\tau(G_1 \bigodot G_2) \geq k$

Proof Let $G = G_1 \bigodot G_2$. Assume that $k' = \tau(G) < k$ and let $M = \{M_1, \ldots, M_{k'}\}$ be a $k'$-covering of $G$. Any perfect matching of $G$ must intersect the 2-edge cut $\{u_1v_1, u_2v_2\}$ in two edges or has no edge in common with that set. Thus any perfect matching in $M$ leads to a perfect matching of $G_1$. Hence we should have a $k'$-covering of the edge set of $G_1$, a contradiction. □
2.2. 3—cut connection. Let $G_1$, $G_2$ be two bridgeless cubic graph and $u \in V(G_1)$, $v \in V(G_2)$ be two vertices with $N(u) = \{u_1, u_2, u_3\}$ and $N(v) = \{v_1, v_2, v_3\}$. Construct a new graph $G = G_1 \otimes G_2$

$$G = \left[G_1 \setminus \{u\}\right] \cup \left[G_2 \setminus \{v\}\right] \cup \{u_1v_1, u_2v_2, u_3v_3\}$$

It is well known that the resulting graph $G_1 \otimes G_2$ is bridgeless. The 3—edge cut $\{u_1v_1, u_2v_2, u_3v_3\}$ will be called the principal 3—edge cut.

**Proposition 2.3.** Let $G_1$ be a cubic graph such that $\tau(G_1) = k \geq 3$ and let $G_2$ be any cubic bridgeless graph. Let $k' = \tau(G_1 \otimes G_2)$ and let $\mathcal{M} = \{M_1, \ldots, M_{k'}\}$ be a $k'$—covering of $G_1 \otimes G_2$. Then one of the followings is true

1. $k' \geq k$
2. There is a perfect matching $M_i \in \mathcal{M}$ ($1 \leq i \leq k$) containing the principal 3—edge cut

**Proof** Assume that $k' < k$. Any perfect matching of $G_1 \otimes G_2$ must intersect the principal 3—edge cut in one or three edges. If none of the perfect matchings in $\mathcal{M}$ contains the principal 3—edge cut, then any perfect matching in $\mathcal{M}$ leads to a perfect matching of $G_1$ and any edge of $G_1$ is covered by one of these perfect matchings. Hence we should have a $k'$—covering of the edge set of $G_1$, a contradiction. $\square$

3. On graphs with perfect matching index 4

A natural question is to investigate the class of graphs for which the perfect matching index is 4.

**Proposition 3.1.** Let $G$ be a cubic graph with a 4—covering $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ then

1. Every edge is contained in exactly one or two perfect matchings of $\mathcal{M}$.
2. The set $M$ of edges contained in exactly two perfect matchings of $\mathcal{M}$ is a perfect matching.
3. If $\tau(G) = 4$ then $\forall i \neq j \in \{1, 2, 3, 4\} \hspace{1em} M_i \cap M_j \neq \emptyset$.

**Proof** Let $v$ be any vertex of $G$, each edge incident with $v$ must be contained in some perfect matching of $\mathcal{M}$ and each perfect matching must be incident with $v$. We have thus exactly one edge incident with $v$ which is covered by exactly two perfect matchings of $\mathcal{M}$ while the two other edges are covered by exactly one perfect matching. We get thus immediately Items 1 and 2.

When $\tau(G) = 4$, $G$ is not a 3—edge colourable graph. Assume that we have two perfect matchings with an empty intersection. These two perfect matchings lead to an even 2—factor and hence a a 3—edge colouring of $G$, a contradiction. $\square$

In the following the edges of the matching $M$ described in item 2 of Proposition 3.1 will be said to be covered twice.

**Proposition 3.2.** Let $G$ be a cubic graph such that $\tau(G) = 4$ then $G$ has at least 12 vertices

**Proof** Let $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ be a covering of the edge set of $G$ into 4 perfect matchings. From Proposition 3.1 we must have at least 6 edges in the perfect matching formed with the edges covered twice in $\mathcal{M}$. Hence, $G$ must have at least
12 vertices as claimed. □

From Proposition 3.2, we obviously have that the Petersen graph has a perfect matching index equal to 5.

**Proposition 3.3.** Let $G$ be a cubic graph such that $\tau(G) = 4$ and let $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ be a covering of its edge set into 4 perfect matchings then for each $j \ (j = 1 \ldots 4)$ $\mathcal{M} - M_j$ is a set of 3 perfect matchings satisfying the Fan Raspaud conjecture.

**Proof** Obvious since, by Item 2 of Proposition 3.1 any edge is contained in exactly one or two perfect matchings of $\mathcal{M}$.

Let $G$ be a cubic graph with 3 perfect matchings $M_1, M_2$ and $M_3$ having an empty intersection. Since such a graph satisfy the Fan Raspaud conjecture, when considering these three perfect matchings, we shall say that $(M_1, M_2, M_3)$ is an FR-triple. When a cubic graph has an FR-triple we define $T_i \ (i = 0, 1, 2)$ as the set of edges that belong to precisely $i$ matchings of the FR-triple. Thus $(T_0, T_1, T_2)$ is a partition of the edge set.

**Proposition 3.4.** Let $G$ be a cubic graph with 3 perfect matchings $M_1, M_2$ and $M_3$ having an empty intersection. Then the set $T_0 \cup T_2$ is a set of disjoint even cycles. Moreover, the edges of $T_0$ and $T_2$ alternate along these cycles.

**Proof** Let $v$ be a vertex incident to a edge of $T_0$. Since $v$ must be incident to each perfect matching and since the three perfect matchings have an empty intersection, one of the remaining edges incident to $v$ must be contained into 2 perfect matchings while the other is contained in exactly one perfect matching. The result follows. □

Let $G$ be a bridgeless cubic graph and let $C$ and $C'$ be distinct odd cycles of $G$. Assume that there are three distinct edges namely $xx'$, $yy'$ and $zz'$ such that $x, y$ and $z$ are vertices of $C$ while $x'$, $y'$, $z'$ are vertices of $C'$ which determine on $C$ and on $C'$ edge-disjoint paths of odd length then we shall say that $(xx', yy', zz')$ is a good triple and that the pair of cycles $(C, C')$ is a good pair.

**Theorem 3.5.** Let $G$ be a cubic graph which has a 2–factor $F$ whose odd cycles can be arranged into good pairs $\{C_1, D_1\}, \{C_2, D_2\}, \ldots, \{C_k, D_k\}$. Then $\tau(G) \leq 4$.

**Proof** For each good pair $\{C_i, D_i\}$ let $(c_i^1d_i^1, c_i^2d_i^2, c_i^3d_i^3)$ be a good triple of $C_i$ and $D_i$, $c_i^1, c_i^2, c_i^3$ being vertices of $C_i$ while $d_i^1$, $d_i^2$ and $d_i^3$ are on $D_i$. In order to construct a set $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$ of 4 perfect matchings covering the edge set of $G$ we let $M_1$ as the perfect matching of $G$ obtained by deleting the edges of the 2–factor.

Let $A_j$ be the set of edges $\{c_i^jd_i^j\} i = 1 \ldots k$. We construct a perfect matching $M_j \ (j = 2, 3, 4)$ of $G$ such that $M_j \cap M_j = A_j$. For each good pair $\{C_i, D_i\} \ (i = 1 \ldots k)$, we add to $A_j$ the unique perfect matching contained in $E(C_i) \cup E(D_i)$ when the two vertices $c_i^j$ and $d_i^j$ are deleted. We get hence 3 matchings $B_j \ (j = 2, 3, 4)$ where each vertex contained in a good pair is saturated. If the 2–factor contains some even cycles, we add first a perfect matching contained in the edge set of these even cycles to $B_2$. We obtain thus a perfect matching $M_2$ whose intersection with $M_1$ is reduced to $A_2$. The remaining edges of these even cycles are added to $B_3$ and to
\[ B_4 \text{, leading to the perfect matchings } M_3 \text{ and } M_4. \] Let us remark that each edge of these even cycles are contained in \( M_2 \cup M_3 \).

We claim that each edge of \( G \) is contained in at least one of \( M = \{ M_1, M_2, M_3, M_4 \} \). Since \( M_1 \) is the perfect matching which complements in \( G \) the 2-factor \( F \), the above remark says that we have just to prove that each edge of each good pair is covered by some perfect matching of \( M \). By construction, no edge is contained in \( M_1 \cup M_2 \cup M_3 \) which means that \((M_1, M_2, M_3)\) is an FR-triple. In the same way, \((M_1, M_3, M_4)\) and \((M_1, M_2, M_4)\) are FR-triples. The edges of \( T_0 \cup T_2 \) induced by the FR-triple \((M_1, M_2, M_3)\) on each good pair \((C_i, D_i)\) is the even cycle \( \Gamma_i \) containing \( c_i^1d_i^1 \) and \( c_i^2d_i^2 \), the odd path of \( C_i \) joining \( c_i^1 \) to \( c_i^2 \) and the odd path of \( D_i \) joining \( d_i^1 \) to \( d_i^2 \). In the same way, edges of \( T_0 \cup T_2 \) induced by the FR-triple \((M_1, M_3, M_4)\) on each good pair \((C_i, D_i)\) is the even cycle \( \Lambda_i \) containing \( c_i^2d_i^2 \) and \( c_i^3d_i^3 \), the odd path of \( C_i \) joining \( c_i^2 \) to \( c_i^3 \) and the odd path of \( D_i \) joining \( d_i^2 \) to \( d_i^3 \). It is an easy task to see that these two cycles \( \Gamma_i \) and \( \Lambda_i \) have the only edge \( c_i^2d_i^2 \) in common. Hence each edge of \( \Gamma_i \cap T_0 \) is contained into \( M_4 \) while each edge of \( \Lambda_i \cap T_0 \) is contained into \( M_2 \). The result follows.

3.1. **On balanced matchings.** A set \( A \subseteq E(G) \) is a balanced matching when we can find 2 perfect matchings \( M_1 \) and \( M_2 \) such that \( A = M_1 \cap M_2 \). Let \( B(G) \) be the set of balanced matchings of \( G \), we define \( b(G) \) as the minimum size of a any set \( A \in B(G) \), we have:

**Proposition 3.6.** Let \( G \) be a cubic graph such that \( \tau(G) = 4 \) then \( b(G) \leq \frac{n}{12} \).

**Proof** Let \( M = \{ M_1, M_2, M_3, M_4 \} \) be a covering of the edge set of \( G \) into 4 perfect matchings and let \( M \) be the perfect matching of edges contained in exactly two perfect matchings of \( M \) (Item 2 of Proposition ??). Since \( M_i \cap M_j = \emptyset \forall i \neq j \in \{1, 2, 3, 4\} \) by Proposition ??, these 6 balanced matchings partition \( M \). Hence, one of them must have at most \( \frac{|M|}{6} = \frac{n}{12} \) edges.

In [2] Kaiser, Král and Norine proved

**Theorem 3.7.** Any bridgeless cubic graph contains 2 perfect matchings whose union cover at least \( \frac{n}{6} \) edges of \( G \).

From Theorem 3.7 we can find two perfect matchings with an intersection having at most \( \frac{n}{4} \) edges in any cubic bridgeless graph. It can be proved (see [3]) that for any cyclically 4-edge connected cubic graph \( G \), either \( b(G) \leq \frac{n}{4} \) or any perfect matching contains an odd cut of size 5.

3.2. **On classical snarks.** As usual a **snark** is a non 3-edge colourable bridgeless cubic graph. In Figure 3 is depicted one of the two the Blanuša snarks on 18 vertices [1]. In bold we have drawn a 2-factor (each cycle has length 9) and the dashed edges connect the triple \((x, y, z)\) of one cycle to the triple \((x', y', z')\) of the second cycle. It is a routine matter to check that \((xx', yy', zz')\) is a good triple and Theorem 3.5 allows us to say that this graph has perfect matching index 4. In the same way the second Blanuša snark on 18 vertices depicted in Figure 3 can be covered by 4 perfect matchings by using Theorem 3.3.

For an odd \( k \geq 3 \) the Flower Snark \( F_k \) introduced by Isaac (see [3]) is the cubic graph on \( 4k \) vertices \( x_0, x_1, \ldots, x_{k-1}, y_0, y_1, \ldots, y_{k-1}, z_0, z_1, \ldots, z_{k-1}, t_0, t_1, \ldots, t_{k-1} \).
such that $x_0x_1\ldots x_{k-1}$ is an induced cycle of length $k$, $y_0y_1\ldots y_{k-1}$ $z_0z_1\ldots z_{k-1}$ is an induced cycle of length $2k$ and for $i = 0\ldots k-1$ the vertex $t_i$ is adjacent to $x_i$, $y_i$ and $z_i$. The set $\{t_i, x_i, y_i, z_i\}$ induces the claw $C_i$. In Figure 3 we have a representation of $F_5$, the half edges (to the left and to the right in the figure) with same labels are identified.

**Theorem 3.8.** $\tau(F_k) = 4$.

**Proof.** Let $k = 2p+1 \geq 3$ and let $C = x_0x_1\ldots x_{2p}$, $D = y_0y_1\ldots y_{2k}t_1t_2\ldots y_{2k}t_{2k+1}t_{2k+2}\ldots y_{2p}t_{2p+2p}$ $(0 \leq i \leq 3)$ be the odd cycles of lengths $2k+1$ and $3 \times (2k+1)$ respectively which partition $F_k$ (in bold in Figure 3). It is a routine matter to check that the edges $x_0t_0$, $x_1t_1$ and $x_2t_2$ form a good triple (dashed edges
in Figure 3. Hence \((C, D)\) is a 2-factor of \(G\) and it is a good pair. The result follows from Theorem 3.5.

Let \(H\) be the graph depicted in Figure 4.

\[\text{Figure 4. } H\]

Let \(G_k\) \((k\text{ odd})\) be a cubic graph obtained from \(k\) copies of \(H\) \((H_0 \ldots H_{k-1})\) where the name of vertices are indexed by \(i\) in adding edges \(a_i a_{i+1}, c_i c_{i+1}, e_i e_{i+1}, f_i f_{i+1}\) and \(h_i h_{i+1}\) (subscripts are taken modulo \(k\)).

If \(k = 5\), then \(G_k\) is known as the Goldberg snark. Accordingly, we refer to all graphs \(G_k\) as Goldberg graphs. The graph \(G_5\) is shown in Figure 5. The half edges (to the left and to the right in the figure) with same labels are identified.

\[\text{Figure 5. Goldberg snark } G_5\]

**Theorem 3.9.** \(\tau(G_k) = 4\).

**Proof** Let \(k = 2p + 1 \geq 3\) and let \(C = a_0 a_1 \ldots a_{2p}\), \(D = e_0 d_0 b_0 g_0 f_0, e_1 d_1 b_1 g_1 f_1, \ldots, e_i d_i b_i g_i f_i, \ldots, e_{2p} d_{2p} b_{2p} g_{2p} f_{2p}\) \((0 \leq i \leq 2p)\) be the odd cycles of lengths \(2k + 1\) and \(5 \times (2k + 1)\) respectively and \(E = c_0 h_0 c_1 h_1 \ldots, c_i h_i \ldots, c_{2p} h_{2p}\) the cycle of length \(4k\) of \(G_k\). This set of 3 cycles is a 2-factor of \(G_k\) (in bold in Figure 5). At last, \(a_0 b_0, a_1 b_1\) and \(a_2 b_2\) are edges of \(G\) (dashed edges in Figure 5). Then \((a_0 b_0, a_1 b_1, a_2 b_2)\) is a good triple. Hence \((C, D, E)\) is a 2-factor of \(G\) where \((C, D)\) is a good pair.

The result follows from Theorem 3.5. \(\Box\)

### 3.3. On permutation graphs

A cubic graph \(G\) is called a permutation graph if \(G\) has a 2-factor \(F\) such that \(F\) is the union of two chordless cycles \(C\) and \(C'\). Let \(M\) be the perfect matching \(G - F\). A subgraph homeomorphic to the Petersen graph with no edge of \(M\) subdivided is called a \(M - P_{10}\). Ellingham \(\Box\) showed that a permutation graph without any \(M - P_{10}\) is 3-edge colourable.
In general, we do not know whether a permutation graph distinct from the Petersen graph is 3-edge colourable or not. It is an easy task to construct a cyclically 4-edge connected permutation graph which is a snark (consider the two Blanusa snarks on 18 vertices for example) and Zhang [18] conjectured:

**Conjecture 3.10.** Let $G$ be a 3-connected cyclically 5-edge connected permutation graph. If $G$ is a snark, then $G$ must be the Petersen graph.

Let us consider a permutation graph $G$ with a 2-factor $F$ having two cycles $C$ and $C'$. Two distinct vertices of $C$ say $x$ and $y$ determine on $C$ two paths with $x$ and $y$ as end-points. In order to be unambiguous when considering those paths from their end-points we give an orientation to $C$. Thus $C(x, y)$ will denote in the following the path of $C$ that starts with the vertex $x$ and ends with the vertex $x$ according to the orientation of $C$. The notation $C'(x', y')$ is defined similarly when $x'$ and $y'$ are vertices of $C'$.

In order to determine which permutation graphs have a perfect matching index less than 4 we state the following tool (see Figure 6):

**Lemma 3.11.** Let $G$ be a permutation graph with a 2-factor containing precisely two odd cycles $C$ and $C'$. Assume that $\chi'(G) = 4$ and that $(C, C')$ is not a good pair. Let $ab$ be an edge of $C$ such that the odd path determined on $C'$ with the neighbors of $a$ and $b$, say $a'$ and $b'$ respectively, has minimum length. Assume that $C$ and $C'$ have an orientation such that $C(a, b)$ is an edge and $C'(a', b')$ has odd length.

Then there must exist 4 additional vertices $c$ and $d$ on $C$ and their neighbors on $C'$, say $c'$ and $d'$ respectively, verifying:

- the paths $C'(a', d')$, $C'(b', c')$ and $C(d, c)$ are edges.
- the path $C(b, d)$ is odd and the path $C'(d', b')$ is even.

**Proof** Observe first that $a'$ and $b'$ are not adjacent otherwise the cycle obtained with the paths $C(b, a)$ and $C'(b', a')$ together with the edges $aa'$ and $bb'$ would be hamiltonian, a contradiction since it is assumed that $\chi'(G) = 4$.

Since the path $C'(a', b')$ is odd there must be a neighbor of $b'$ on $C'(b', a')$, say $c'$. Let $c$ be the neighbor of $c'$ on $C$. The path $C(b, c)$ has even length, otherwise $(aa', bb', cc')$ would be a good triple and $(C, C')$ a good pair, a contradiction.

It follows that the vertex $c$ has a neighbor, say $d$ on $C(b, c)$ and $C(b, d)$ has odd length.

Let $d'$ be the neighbor of $d$ on $C'$. It must be pointed out that $d'$ is a vertex of $C'(c', a')$. As a matter of fact if on the contrary $d'$ belongs to $C'(c', a')$ we
would have a good triple with \((dd', cc', bb')\) when \(C'(c', d')\) has odd length and with 
\((aa', bb', dd')\) when \(C'(c', d')\) is an even path; a contradiction in both cases.

But now by the choice of the edge \(ab\) the length of \(C'(a', b')\) cannot be greater than \(C'(d', c')\), thus \(d'\) is adjacent to \(a'\) and the path \(C'(d', b')\) has even length. \(\square\)

We have:

**Theorem 3.12.** Let \(G\) be a permutation graph then \(\tau(G) \leq 4\) or \(G\) is the Petersen graph.

**Proof** Let \(C\) and \(C'\) the 2-factor of chordless cycles which partition \(V(G)\) and We can assume that \(G\) is not 3-edge colourable otherwise \(\tau(G) = 3\) and there is nothing to prove. Hence, \(C\) and \(C'\) have both odd lengths. In addition we assume that \((C, C')\) is not a good pair, otherwise we are done by Theorem 3.5.

Let \(x_1x_2\) be an edge of \(C\) such that the odd path determined on \(C'\) with the neighbors of \(x_1\) and \(x_2\), say \(y_1\) and \(y_2\) respectively, has minimum length.

We choose to orient \(C\) from \(x_1\) to \(x_2\) and to orient \(C'\) from \(y_1\) to \(y_2\). Thus \(C(x_1, x_2)\) is an edge and \(C'(y_1, y_2)\) is an odd path.

By Lemma 3.11 we must have two vertices \(x_3\) and \(x_4\) on \(C\) and their neighbors \(y_3\) and \(y_4\) on \(C'\) such that \(C(x_4, x_3), C'(y_1, y_4), C'(y_2, y_3)\) are edges, \(C(x_2, x_4)\) being an odd path while \(C'(y_4, y_2)\) has even length.

**Claim 1.** The vertices \(y_1\) and \(y_3\) are adjacent.

**Proof** Assume not.

The odd path \(C'(y_4, y_3)\) having the same length than \(C'(y_1, y_2)\) we may apply Lemma 3.11 on the edge \(x_4x_3\) \((x_4 = a, x_3 = b)\). Thus there is edges; say \(x_5y_5\) and \(x_6y_6, x_5\) and \(x_6\) being vertices of \(C, y_5\) and \(y_6\) vertices of \(C'\), the paths \(C(x_6, x_5), C'(y_3, y_6)\) and \(C'(y_1, y_5)\) having length 1. Moreover the paths \(C(x_3, x_6)\) and \(C'(y_6, y_2)\) are odd. Since it is assumed that \(y_1\) and \(y_3\) are independent we have \(y_5 \neq y_1\) and \(x_5 \neq x_1\).

Observe that the paths \(C'(y_1, y_2)\) and \(C'(y_6, y_5)\) have the same length, thus we apply Lemma 3.11 again with \(a = x_6\) and \(b = x_5\).

Let \(y_7\) be the neighbor of \(y_5\) on \(C'(y_5, y_1)\) and \(x_7\) be the neighbor of \(y_7\) on \(C\). We know that \(x_7\) is a vertex of \(C(x_5, x_1)\) at even distance of \(x_5\). The vertex \(x_7\) being the neighbor of \(x_7\) on \(C(x_5, x_7)\) and \(y_8\) the neighbor of \(x_8\) on \(C'\), we have that \(y_8\) is the neighbor of \(y_6\) on \(C'(y_6, y_2)\).

The path \(C'(y_8, y_2)\) has even length, hence there must be on this path a neighbor of \(y_8\) distinct from \(y_2\), say \(y_9\). Let \(x_9\) be the neighbor of \(y_9\) on \(C\).

The vertex \(x_9\) belongs to \(C(x_7, x_1)\). Otherwise when \(x_9\) is on \(C(x_2, x_4)\); if the path \(C(x_2, x_9)\) is odd we can find a good triple, namely \((x_8y_8, x_0y_9, x_2y_2)\) on the other case we have the good triple \((x_9y_9, x_4y_4, x_1y_1)\). A contradiction in both cases.

We get a similar contradiction if \(x_9\) belongs to \(C(x_3, x_6)\) by considering the triples \((x_5y_5, x_9y_9, x_8y_8)\) or \((x_3y_9, x_4y_4, x_2y_2)\).

Finally, when \(x_9\) is a vertex of \(C(x_5, x_8)\) a contradiction occurs with the triple \((x_5y_5, x_9y_9, x_7y_7)\) if \(C(x_5, x_9)\) is odd and with the triple \((x_8y_8, x_9y_9, x_6y_6)\) otherwise.
Observe that the path $C(x_7, x_9)$ must be odd or $(x_9y_9, x_7y_7, x_8y_8)$ would be a good triple, a contradiction.

But now $(x_9y_9, x_5y_5, x_4y_4)$ is a good triple, a contradiction which proves the Claim (see Figure 7). □

From now on we assume that $y_3y_1$ is an edge.

The path $C(x_1, x_3)$ being odd there must be a neighbor of $x_3$ on $C(x_3, x_1)$ distinct from $x_1$, let $x_5$ be this vertex. It’s neighbor on $C'$, say $y_5$, must be on $C'(y_1, y_5)$. Moreover the length of $C'(y_1, y_5)$ is odd otherwise the edges $x_5y_5, x_3y_3$ and $x_1y_1$ would form a good triple, a contradiction.

**Claim 2.** The paths $C'(y_4, y_5)$ and $C'(y_5, y_2)$ are reduced to edges.

**Proof** Assume in a first stage that the neighbor of $y_4$ on $C'(y_4, y_5)$ is distinct from $y_5$, let $y_6$ be this vertex and $x_6$ be its neighbor on $C$.

The vertex $x_6$ cannot belong to $C(x_5, x_1)$, otherwise we would have a good triple $(x_3y_1, x_6y_6, x_4y_4)$ when $C(x_5, x_6)$ is an even path and the good triple $(x_4y_4, x_6y_6, x_2y_2)$ if it’s an odd path, contradictions.

Similarly the vertex $x_6$ cannot belong to $C(x_2, x_4)$. On the contrary we would have a good triple with the edges $x_2y_2, x_6y_6$ and $x_1y_1$ when the path $C(x_2, x_6)$ is odd and another good triple with the edges $x_4y_4, x_6y_6$ and $x_1y_1$.

On the same manner we can prove that the path $C'(y_5, y_2)$ has length 1. □

It comes from Claim 2 that $C'$ has only 5 vertices. Since both cycles $C$ and $C'$ have the same length $C$ has 5 vertices too and $G$ is the Petersen graph. □

In [32] Watkins proposed two families of generalized Blanuša snarks using the blocks $B$, $A_1$ and $A_2$ described in Figure 2. The generalized Blanuša snarks of type 1 (resp. of type 2) are obtained by considering a number of blocks $B$ and one block $A_1$ (resp. $A_2$), these blocks are arranged cyclically, the semi-edges $a$ and $b$ of one block being connected to the semi-edges $a$, $b$ of the next one. Recently generalized Blanuša snarks were studied in terms of circular chromatic index (see [33, 34]).

The generalized Blanuša snarks are permutation graphs, hence:

**Corollary 3.13.** Let $G$ be a generalized Blanuša snarks then $\tau(G) = 4$. 

\[\begin{array}{c}
\text{Figure 7. Situation at the end of Claim 2.}
\end{array}\]
3-edge cut

\[ \tau \]

Let \( x \) be a vertex in \( G \) and let \( \tau \) be the edge set of the \( 3 \)-regular graph \( G \) such that \( \tau \) contains at least one edge from each of the \( 3 \) different \( 3 \)-regular graphs \( G_1, G_2, G_3 \) with \( \tau \). Take indeed the Petersen graph \( P \) and any bridgeless cubic graph \( G \) and apply the construction \( P \bigcirc G \).

**Proposition 4.1.** Let \( G \) be bridgeless cubic graph with perfect matching index at least \( 5 \) and let \( H \) be a connected bipartite cubic graph. Then \( G \bigcirc H \) is bridgeless cubic graph with perfect matching index at least \( 5 \).

**Proof** Assume that \( \tau(G \bigcirc H) = 4 \) and let \( \mathcal{M} = \{ M_1, M_2, M_3, M_4 \} \) be a covering of its edge set into \( 4 \) perfect matchings. Let \( \{ aa', bb', cc' \} \) (with \( a, b, c \) in \( G \) and \( a', b', c' \) in \( H \)) be the principal \( 3 \)-edge cut of \( G \bigcirc H \). From Item 2 of Proposition 2.3, there is perfect matching \( M_i \in \mathcal{M} \) such that \( \{ aa', bb', cc' \} \subseteq M_i \). This is clearly impossible since the set of vertices of \( H \) must be saturated by \( M_i \) is partitioned into \( 2 \) independent sets whose size differs by one unit.

Let us consider the following construction. Given four cubic graphs \( G_1^{x_1}, G_2^{x_2}, G_3^{x_3}, G_4^{x_4} \) together with a distinguished vertex \( x_i \) (\( i = 1, 2, 3, 4 \)) whose neighbors in \( G_i^{x_i} \) are \( a_i, b_i \), and \( c_i \), we get a \( 3 \)-connected cubic graph in deleting the vertices \( x_i \) (\( i = 1, 2, 3, 4 \)) and connecting the remaining subgraphs as described in Figure 9.

In other words we define the cubic graphs denoted \( K_4[G_1^{x_1}, G_2^{x_2}, G_3^{x_3}, G_4^{x_4}] \) whose vertex set is

\[
\bigcup_{i \in \{1,2,3,4\}} V(G_i^{x_i}) - \bigcup_{i \in \{1,2,3,4\}} \{x_i\}
\]

while the edge set is

\[
\bigcup_{i \in \{1,2,3,4\}} E(G_i^{x_i}) - \bigcup_{i \in \{1,2,3,4\}} \{a_i x_i, b_i x_i, c_i x_i\} \bigcup \{a_1 c_3, b_1 a_4, c_1 c_2, b_2 c_4, a_2 c_3, b_3 b_4\}.
\]

For convenience \( G_i \) (\( i \in \{1,2,3,4\} \)) will denote the induced subgraph of \( G_i^{x_i} \) where the vertex \( x_i \) has been deleted.

**Proposition 4.2.** Let \( G_1^{x_1}, G_2^{x_2}, G_3^{x_3}, G_4^{x_4} \) be \( 3 \)-connected cubic graphs such that \( \tau(G_1^{x_1}) \geq 5 \), \( \tau(G_2^{x_2}) \geq 5 \), \( G_4 \) is reduced to a single vertex, say \( x \). Then \( \tau(K_4[G_1^{x_1}, G_2^{x_2}, G_3^{x_3}, G_4^{x_4}]) \geq 5 \).

**Proof** Let us denote \( G = K_4[G_1^{x_1}, G_2^{x_2}, G_3^{x_3}, G_4^{x_4}] \). Observe that \( a_4 = b_4 = c_4 = x \).

If \( \tau(G) = 3 \) the graph \( G \) would be \( 3 \)-edge colourable, but in considering the \( 3 \)-edge cut \( \{a_1 a_3, b_1 a_4, c_1 c_2\} \) we would have \( \chi'(G_1^{x_1}) = 3 \), a contradiction. Hence

---

**Figure 8.** Blocks for the construction of generalized Blanuša snarks.
\[ \tau(G) \geq 4. \] Assume that \( \tau(G) = 4 \) and let \( \mathcal{M} = \{M_1, M_2, M_3, M_4\} \) be a covering of its edge set into 4 perfect matchings.

From Item 2 of Proposition 2.3 there is perfect matching \( M_i \in \mathcal{M} \) such that \( \{a_1a_3, b_1a_4, c_1c_2\} \subseteq M_i \). For the same reason, there is perfect matching \( M_j \in \mathcal{M} \) such that \( \{c_1c_2, xb_2c_3a_2\} \subseteq M_j \). We certainly have \( i \neq j \), otherwise the vertex \( x \) is incident twice to the same perfect matching \( M_i \). Without loss of generality, we suppose that \( i = 1 \) and \( j = 2 \). Hence \( c_1c_2 \in M_1 \cap M_2 \). If we consider the 3-edge cut \( \{a_1a_3, b_1a_4, c_1c_2\} \), since each perfect matching must intersect this cut in an odd number of edges we must have one of the edges \( a_1a_3 \) or \( b_1x \) in \( M_3 \) while the other must be in \( M_4 \). The same holds with the 3-edge cut \( \{c_1c_2, xb_2c_3a_2\} \) and the edges \( b_2x \) and \( a_2c_3 \). Hence, we can suppose that \( a_1a_3 \in M_1 \cap M_3 \) and \( b_1x \in M_1 \cap M_4 \) as well that \( b_2x \in M_2 \cap M_3 \) and \( a_2c_3 \in M_2 \cap M_4 \), a contradiction since the set of edges contained into 2 perfect matchings of \( \mathcal{M} \) is a perfect matching by Item 2 of Proposition 3.1 and \( x \) is incident to two such edges.

We do not know any cyclically 4-edge connected cubic graph, distinct from the Petersen graph, having a perfect matching index at least 5 and we propose as an open problem:

**Problem 4.3.** Is there any cyclically 4-edge connected cubic graph distinct from the Petersen graph with a perfect matching index at least 5?

5. **Technical tools.**

In fact Theorem 3.5 can be generalized. Let \( M \) be a perfect matching, a set \( A \subseteq E(G) \) is a \( M \)-balanced matching when we can find a perfect matchings \( M' \) such that \( A = M \cap M' \). Assume that \( \mathcal{M} = \{A, B, C\} \) are 3 pairwise disjoint
$M$–balanced matchings, we shall say that $M$ is a **good family** whenever the two following conditions are fulfilled:

i Every odd cycle $C$ of $G \setminus M$ has exactly one vertex incident with one edge of each subset of $M$ and the three paths determined by these vertices on $C$ are odd.

ii For every even cycle of $G \setminus M$ there are at least two matchings of $M$ with no edge incident to the cycle.

**Theorem 5.1.** Let $G$ be a bridgeless cubic graph together with a good family $M$. Then $\tau(G) \leq 4$.

**Sketch of the proof** Let us denote $M_A$ (resp. $M_B$, $M_C$) a perfect matching such that $M_A \cap M = A$ (resp. $M_B \cap M = B$, $M_C \cap M = C$).

Let $C$ be a cycle of the 2-factor $G - M$. When $C$ is an even cycle, there are precisely two matchings on $C$, namely $M_C$ and $M_C'$ such that $M_C \cup M_C'$ covers all the edge-set of $C$. Since there are at least two matchings in $\{M_A, M_B, M_C\}$ that are not incident to $C$, say $M_A$ and $M_B$, up to a redistribution of the edges in $M_A \cap C$ and $M_B \cap C$ we may assume that $M_C \subset M_A$ and $M_C' \subset M_B$.

If $C$ is an odd cycle we know that $C$ has precisely one vertex which is incident to $A$ say $a$, one vertex which is incident to $B$ say $b$, one vertex which is incident to $C$ say $c$. Without loss of generality we may assume that there is an orientation of $C$ such that the path $C(a, b)$ has odd length and the vertex $c$ in $C(b, a)$. We know that the path $C(b, c)$ is odd thus the edge-set of $C$ is covered with $M_A \cup M_B \cup M_C$. □

In the same manner we can obtain a theorem insuring the existence of a 5-covering.

Assume that $M = \{A, B, C, D\}$ are 4 pairwise disjoint $M$–balanced matchings, we shall say that $M$ is a **nice family** whenever the two following conditions are fulfilled:

i Every odd cycle $C$ of $G \setminus M$ has exactly one vertex incident with one edge of each subset of $M$ and at least two disjoint paths determined by these vertices on $C$ are odd.

ii For every even cycle of $G \setminus M$ there are at least two matchings of $M$ with no edge incident to the cycle.

**Theorem 5.2.** Let $G$ be a bridgeless cubic graph together with a nice family $M$. Then $\tau(G) \leq 5$.

**Proof** Let us denote $M_A$ (resp. $M_B$, $M_C$, $M_D$) a perfect matching such that $M_A \cap M = A$ (resp. $M_B \cap M = B$, $M_C \cap M = C$,$M_D \cap M = D$).

Let $C$ be a cycle of the 2-factor $G - M$. When $C$ is an even cycle, there is at least two matchings in $\{M_A, M_B, M_C, M_D\}$ that are not incident to $C$, say $M_1$ and $M_2$. As in Theorem 5.1 we may assume that the edge-set of $C$ is a subset of $M_1 \cup M_2$.

If $C$ is an odd cycle we know that $C$ has precisely one vertex which is incident to $A$ say $a$, one vertex which is incident to $B$ say $b$, one vertex which is incident to $C$ say $c$, one vertex which is incident to $D$ say $d$. Without loss of generality we may assume that there is an orientation of $C$ such that the path $C(a, b)$ has odd length and the vertices $c$ and $d$ are in this order in $(b, a)$. We can suppose that the path
(b, c) is even otherwise the edge-set of C would be covered with \( M_A \cup M_B \cup M_C \). But now, since C is an odd cycle the path C(d, a) has odd length and the edge-set of C is a subset of \( M_A \cup M_B \cup M_D \) and \( (M, M_A, M_B, M_C, M_D) \) is a 5-covering. □

In a forthcoming paper [3], we shall give an analogous theorem insuring the existence of a Fulkerson covering and some applications.

6. Odd or Even Coverings.

A covering of a bridgeless cubic graph being a set of perfect matchings such that every edge is contained in at least one perfect matching, we define an odd covering as a covering such that each edge is contained in an odd number of the members of the covering. In the same way, an even covering is a covering such that each edge is contained in an even number (at least 2) members of the covering. The size of an odd (or even) covering is its number of members. As soon as a covering is given an even covering is obtained by taking each perfect matching twice.

**Proposition 6.1.** Let \( G \) be bridgeless cubic graph such that \( \tau(G) = 4 \). Then \( G \) has an odd covering of size 5.

**Proof** Let \( G \) be a cubic graph such that \( \tau(G) = 4 \) and let \( \mathcal{M} = \{M_1, M_2, M_3, M_4\} \) be a covering of its edge set into 4 perfect matchings. Let \( M \) be the perfect matching formed with the edges contained in exactly two perfect matchings of \( \mathcal{M} \). Then we can check that \( \{M, M_1, M_2, M_3, M_4\} \) cover every edge of \( G \) either one time or three times. □

**Proposition 6.2.** Let \( G \) be a bridgeless cubic graph together with an odd covering \( \mathcal{M} \) of size \( k \). Then either \( G \) has an odd covering of size \( k-2 \) or \( \forall M, M' \in \mathcal{M} \) we have \( M \neq M' \).

**Proof** Assume that there are two identical perfect matchings \( M \) and \( M' \) in \( \mathcal{M} \). Each edge \( e \) covered by \( M \) (and thus \( M' \)) must be covered by at least another perfect matching \( M_e \) and the set \( \mathcal{M} - \{M, M'\} \) is still an odd covering. The result follows. □

**Proposition 6.3.** The Petersen graph has no odd covering.

**Proof** Let \( \mathcal{M} \) be an odd covering of the Petersen graph with minimum size. Then, by Proposition 6.2 \( \mathcal{M} \) must be a set of distinct perfect matchings. The Petersen graph has exactly 6 distinct perfect matchings (inducing a Fulkerson covering, that is an even covering) and it is an easy task to check that any subset of 5 perfect matchings is not an odd covering. Since \( \tau(Petersen) = 5 \), the result follows. □

Seymour ([12]) remarked that the edge set of the Petersen graph is not expressible as a symmetric difference (mod 2) of its perfects matchings.

**Problem 6.4.** Which bridgeless cubic graph can be provided with an odd covering?

We remark that 3−edge-colorable cubic graphs as well as bridgeless cubic graph with perfect matching index 4 have an odd covering (with size 3 and 5 respectively).

**Proposition 6.5.** Let \( G \) be bridgeless cubic graph without any odd covering and let \( H \) be a connected bipartite cubic graph. Then \( G \otimes H \) has no odd covering.
Proof Assume that $G \otimes H$ can be provided with an odd covering $\mathcal{M}$. Let $\{aa',bb',cc'\}$ (with $a,b,c$ in $G$ and $a',b',c'$ in $H$) be the principal 3-edge cut of $G \otimes H$. None of the perfect matchings of $\mathcal{M}$ can contain the principal 3-edge cut since the set of vertices of $H$ which must be saturated by such a perfect matching is partitioned into 2 independent sets whose size differs by one unit. Hence every perfect matching of $M \in \mathcal{M}$ contains exactly one edge in $\{aa',bb',cc'\}$ and leads to a perfect matching $M'$ of $G$. The set $M'$ of perfect matchings so obtained is an odd covering of $G$, a contradiction. □

Proposition 6.6. Let $G_{i}^{x_{i}}$ and $G_{j}^{y_{j}}$ be cubic graphs with distinguished vertices $x_{1}$ and $x_{2}$ such that $\tau(G_{i}^{x_{i}}) \geq 5$ ($i = 1,2$) and $\tau_{odd}(G_{i}^{x_{i}}) \neq 5$ ($i = 1,2$). Let $G_{i}'$ and $G_{j}'$ be two copies of the cubic graph on two vertices and $G = K_{4}[G_{i}^{x_{i}}, G_{j}^{y_{j}}, G_{i}^{x_{i}}, G_{j}^{y_{j}}]$, then $\tau(G) \geq 5$ and if $\tau_{odd}G$ is defined then $\tau_{odd}(G) \neq 5$.

Proof Let $x$ and $y$ be respectively the unique vertex of $G_{4}, G_{3}$ (see Figure 3 where $G_{3}$ is reduced to a single vertex $x$ and $G_{4}$ is reduced to $y$). We know by Proposition 4.2 that $\tau(G) \geq 5$. Assume that $\tau_{odd}(G) = 5$ and let $\mathcal{M} = \{M_{1}, M_{2}, M_{3}, M_{4}, M_{5}\}$ be an odd 5-covering. The perfect matchings of $\mathcal{M}$ are pairwise distinct otherwise by Proposition 6.4 either $G_{i}^{x_{i}}$ or $G_{j}^{y_{j}}$ would be 3-edge colorable, a contradiction. Observe that each vertex is incident to one edge that belongs to precisely three matchings of $\mathcal{M}$, the two other edges being covered only once. Moreover, the set of edges that belong to 3 matchings of $\mathcal{M}$ is a perfect matching itself.

The 3-edge cut $\{a_{1}y_{1}, b_{1}x_{1}, c_{1}c_{2}\}$ must be entirely contained in some matching of $\mathcal{M}$, say $M_{i}$ otherwise we would have a 5-odd covering of $G_{i}^{x_{i}}$, a contradiction. Similarly there is a perfect matching in $\mathcal{M}$, say $M_{j}$ that contains the edges $c_{1}c_{2}, b_{2}x_{1}, a_{2}y_{1}$. Thus the edge $c_{1}c_{2}$ must belong to 3 matchings of $\mathcal{M}$. Without loss of generality we assume that $i = 1, j = 2$ and $c_{1}c_{2} \in M_{1} \cap M_{2} \cap M_{5}$.

If $ya_{1} \in M_{3}$, since a perfect matching intersects any odd cut in an odd number of edges we have $xb_{1} \in M_{3}$, it follows that the edge $ya_{1}$ must be a member of a third matching of $\mathcal{M}$ as well as the edge $xb_{1}$. If for some $k$ we have $ya_{1} \in M_{k}$ and $xb_{1} \in M_{k}, k \in \{2, 4, 5\}$, $k$ being obviously distinct from 2 $K_{k}$ intersects the 3-edge cut in an even number of edges, a contradiction. Hence we may assume that $ya_{1} \in M_{4}$ and $xb_{1} \in M_{5}$. But now the edge $xy$ is covered by none of the matchings of $\mathcal{M}$, a contradiction. Consequently $ya_{1} \notin M_{3}$, similarly $xb_{1} \notin M_{3}$.

If $ya_{1} \in M_{4}$ this edge must belong to a third matching of $\mathcal{M}$ which is $M_{5}$. Since the set of edges that are covered 3 times is a perfect matching $xb_{1} \in M_{3} \cap M_{5}$. But in this case the edge $c_{1}c_{2}$ would belong to $M_{4}$ and $M_{5}$, a contradiction.

It follows that $ya_{1}$ as well as $xb_{2}$ are covered only once and the edge $xy$ belongs to 3 matchings of $\mathcal{M}$, that is $xy \in M_{3} \cap M_{4} \cap M_{5}$. But now, neither $M_{4}$ nor $M_{5}$ intersect the edge-cut $\{ya_{1}, xb_{1}, c_{1}c_{2}\}$ a contradiction since a perfect matching must intersect every odd edge-cut in an odd number of edges. □

The graph $G$ depicted in Figure 10 is an example of cubic graphs with a 7-odd covering and a perfect matching index equals to 5. We know by Proposition 5.3...
that \( \tau(G) \geq 7 \). As a matter of fact, this graph has 20 distinct perfect matchings and among all the 7-tuples of perfect matchings \((77520)\) 64 form an odd-covering. Let us give below such a 7-tuple.

\[
\{0 \rightarrow 10, 1 \rightarrow 5, 2 \rightarrow 9, 3 \rightarrow 13, 4 \rightarrow 8, 6 \rightarrow 7, 11 \rightarrow 15, 12 \rightarrow 19, 14 \rightarrow 18, 16 \rightarrow 17\}
\]
\[
\{0 \rightarrow 1, 2 \rightarrow 8, 3 \rightarrow 4, 5 \rightarrow 9, 6 \rightarrow 7, 10 \rightarrow 12, 11 \rightarrow 15, 13 \rightarrow 14, 16 \rightarrow 18, 17 \rightarrow 19\}
\]
\[
\{0 \rightarrow 1, 2 \rightarrow 10, 3 \rightarrow 13, 4 \rightarrow 5, 6 \rightarrow 8, 7 \rightarrow 9, 11 \rightarrow 15, 12 \rightarrow 19, 14 \rightarrow 18, 16 \rightarrow 17\}
\]
\[
\{0 \rightarrow 1, 2 \rightarrow 10, 3 \rightarrow 13, 4 \rightarrow 8, 5 \rightarrow 9, 6 \rightarrow 7, 11 \rightarrow 16, 12 \rightarrow 18, 14 \rightarrow 15, 17 \rightarrow 19\}
\]
\[
\{0 \rightarrow 11, 1 \rightarrow 5, 2 \rightarrow 9, 3 \rightarrow 13, 4 \rightarrow 8, 6 \rightarrow 7, 10 \rightarrow 12, 14 \rightarrow 15, 16 \rightarrow 17\}
\]
\[
\{0 \rightarrow 11, 1 \rightarrow 6, 2 \rightarrow 10, 3 \rightarrow 7, 4 \rightarrow 8, 5 \rightarrow 9, 12 \rightarrow 19, 13 \rightarrow 17, 14 \rightarrow 15, 16 \rightarrow 18\}
\]

Moreover the following perfect matchings form a 5-covering.

\[
\{0 \rightarrow 1, 2 \rightarrow 10, 3 \rightarrow 13, 6 \rightarrow 8, 7 \rightarrow 9, 4 \rightarrow 5, 12 \rightarrow 19, 14 \rightarrow 18, 11 \rightarrow 15\}
\]
\[
\{2 \rightarrow 9, 1 \rightarrow 6, 7 \rightarrow 9, 4 \rightarrow 5, 3 \rightarrow 13, 0 \rightarrow 11, 10 \rightarrow 12, 14 \rightarrow 15, 16 \rightarrow 18, 17 \rightarrow 19\}
\]
\[
\{1 \rightarrow 6, 7 \rightarrow 9, 2 \rightarrow 8, 5 \rightarrow 4, 0 \rightarrow 10, 12 \rightarrow 18, 17 \rightarrow 19, 14 \rightarrow 15, 11 \rightarrow 15, 3 \rightarrow 13\}
\]
\[
\{0 \rightarrow 1, 2 \rightarrow 8, 6 \rightarrow 7, 5 \rightarrow 9, 3 \rightarrow 4, 10 \rightarrow 12, 13 \rightarrow 17, 14 \rightarrow 18, 15 \rightarrow 19, 11 \rightarrow 16\}
\]
\[
\{1 \rightarrow 6, 5 \rightarrow 9, 4 \rightarrow 8, 3 \rightarrow 7, 2 \rightarrow 10, 0 \rightarrow 11, 12 \rightarrow 18, 13 \rightarrow 14, 15 \rightarrow 19, 16 \rightarrow 17\}
\]
We do not know any example of graph $G$ for which $\tau_{odd}$ is defined and with $\tau(G) = \tau_{odd}(G) = 5$. We just observe that in such a graph every vertex would be incident to an edge belonging to 3 perfect matchings and to precisely two edges covered only once. The set of edges covered by 3 perfect matchings being a perfect matching itself.

**Problem 6.7.** Is it true that every bridgeless cubic graph has an even covering where each edge appears twice or 4 times?

The answer is yes for 3—edge-colorable cubic graphs and for bridgeless cubic graphs with perfect matching index 4 since such graphs have an even covering of size 8.

**References**


L.I.F.O., Faculté des Sciences, B.P. 6759 Université d’Orléans. 45067 Orléans Cedex 2, FR