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Verifiable conditions of $\ell_1$-recovery of sparse signals with sign restrictions

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Abstract

We propose necessary and sufficient conditions for a sensing matrix to be “s-semigood” – to allow for exact $\ell_1$-recovery of sparse signals with at most $s$ nonzero entries under sign restrictions on part of the entries. We express the error bounds for imperfect $\ell_1$-recovery in terms of the characteristics underlying these conditions. Furthermore, we demonstrate that these characteristics, although difficult to evaluate, lead to verifiable sufficient conditions for exact sparse $\ell_1$-recovery and to efficiently computable upper bounds on those $s$ for which a given sensing matrix is s-semigood. We concentrate on the properties of proposed verifiable sufficient conditions of s-semigoodness and describe their limits of performance.

1 Introduction

In this paper we address the recovery problem as follows: given an observation $y \in \mathbb{R}^m$,

$$y = Aw + e,$$

(1)

where $A \in \mathbb{R}^{m \times n}$ (in this context $m < n$), is a given matrix, $e \in \mathbb{R}^m$ is the observation error, assess a sparse signal $w \in \mathbb{R}^n$. We suppose that the a priori information about $w$ amounts to the sign restrictions as follows: we are given the subsets $P_+$ and $P_-$ of $\{1, \ldots, n\}$, $P_+ \cap P_- = \emptyset$, such that $w_i \geq 0$ for $i \in P_+$ and $w_i \leq 0$ for $i \in P_-$. A celebrated solution to the problem is given by the $\ell_1$-recovery, which amounts to taking, as an estimate of $w$, an optimal solution $\hat{w}$ to the optimization problem

$$\hat{w} \in \text{Argmin}_x \{\|x\|_1 : \|Ax - y\| \leq \varepsilon, x_i \geq 0 \text{ for } i \in P_+, x_i \leq 0 \text{ for } i \in P_-\}$$

(2)

(here $\varepsilon$ is an a priori bound on the norm $\|e\|$ of the observation disturbance, $\|\cdot\|$ being some norm on $\mathbb{R}^m$). When no sign restrictions are imposed on $w$ (i.e. $P_+ = P_- = \emptyset$), this problem reduces to the most commonly studied estimator in the existing Compressive Sensing theory. The central result in Compressive Sensing is that when signal $w$ is $s$-sparse (i.e. has no more than $s$ nonzero entries) and the matrix $A$ possesses a certain well-defined property, then the $\ell_1$-recovery $\hat{w}$ is close to $w$, provided the error bound $\varepsilon$ is small. Our goal here is to specify efficiently the properties of a given sensing matrix $A$ with respect to $\ell_1$-recovery in the case when sign constraints are present.

To be more precise, let us consider the problem of noiseless recovery (there is no observation error, i.e. $y = Aw$). Let $A$ be a given $m \times n$ matrix. We are interested to answer the question:

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Whether $A$ is such that whenever the true signal $w$ in (1) is $s$-sparse and satisfies the sign constraints $w_i \geq 0$, $i \in P_+$, $w_i \leq 0$, $i \in P_-$, the $\ell_1$-recovery
\[
\hat{w} \in \text{Argmin}_x \{\|x\|_1 : \ Ax \ = \ y, \ \ x_i \geq 0 \text{ for } i \in P_+, \ x_i \leq 0 \text{ for } i \in P_-\}
\] (3)
reverses $w$ exactly.

If the answer is positive, we say that $A$ is $s$-semigood$^1$.

The Compressive Sensing theory provides several sufficient/necessary and sufficient conditions of $s$-semigoodness in two special cases: the one of nonnegative $w$, i.e. $P_+ = \{1, \ldots, n\}$, and the one when no sign constraints are imposed. In the former case, the founding paper of Donoho and Tanner [6] provides characterizations of $s$-semigoodness in terms of neighboring properties of the polytope $AS$, $S$ being the standard simplex $S = \{x \in \mathbb{R}^n : \ x \geq 0, \ \sum_i x_i \leq 1\}$. This paper contains also several important examples of $m \times n$ matrices which are $\lfloor \frac{n}{s} \rfloor$-semigood (here $\lfloor a \rfloor$ stands for the integer part of $a$) and demonstrates that various types of randomly generated matrices share this property with overwhelming probability. An equivalent characterization of $s$-semigoodness has been provided in this case by Zhang in [13], where it is shown that $A$ is $s$-semigood if and only if the kernel $\text{Ker}A$ is strictly half $s$-balanced, meaning that for any set $I \subset \{1, ..., n\}$ of cardinality $\leq s$ it holds
\[
\sum_{i \in I} z_i < \sum_{i \notin I} |z_i| \text{ for any } z \in \text{Ker}A \text{ such that } z_i \leq 0, \text{ for all } i \notin I. \tag{4}
\]
This necessary and sufficient condition for $s$-semigoodness can be compared to the condition for $s$-goodness of the matrix $A$, as it is given in [12]: $A$ is $s$-good if and only if $\text{Ker}A$ is strictly $s$-balanced, meaning that for any set $I \subset \{1, ..., n\}$ of cardinality $\leq s$ it holds
\[
\sum_{i \in I} |z_i| < \sum_{i \notin I} |z_i| \text{ for any } z \in \text{Ker}A \tag{5}
\]
(note that the sufficiency of this condition can be traced back to the discussion in Section 3 of [5]).

It should be mentioned that the characterizations (4), (5) and geometric characterizations of $s$-(semi)goodness of $A$ from [6, 7] share an important drawback – they seemingly cannot be verified in a computationally efficient way. To the best of our knowledge, the only computationally tractable techniques available in the “classical” theory of Compressive Sensing which allow to certify $s$-(semi)goodness of a given sensing matrix are those based on the mutual incoherence
\[
\mu(A) = \max_{i \neq j} \frac{|A_i^T A_j|}{A_i^T A_i} \tag{6}
\]
where $A_i$ are columns of $A$ (assumed to be nonzero). Clearly, the mutual incoherence can be easily computed even for large matrices. Unfortunately, it turns out that that the estimates of “level of (semi)goodness” of a sensing matrix based on mutual incoherence usually are too conservative, in particular, they are provably dominated by the verifiable Linear Programming-based sufficient conditions for $s$-goodness proposed in the companion paper [10] and based on Zhang’s characterization of $s$-goodness (5). Another verifiable sufficient condition for $s$-goodness, which uses the Semidefinite Relaxation, has been recently proposed in [4].

The contributions of this paper, which follow the approach developed in [10], are as follows.

$^1$We use the term “$s$-semigoodness” to comply with the terminology of the companion paper [10], where we used the name $s$-goodness to indicate that $\ell_1$-recovery as in (3) without the sign restrictions is exact.
Taking Zhang’s characterizations of (semi)goodness \((4), (5)\) as a starting point, we develop in Section 2 several equivalent necessary and sufficient conditions for s-semigoodness of a matrix \(A\) in the case of general-type sign restrictions. Then in Section 3 we establish error bounds for inexact \(\ell_1\)-recovery (noisy observation \((1)\), imprecise optimization in \((2)\), nearly-sparse true signals); these bounds are expressed in the same terms as the necessary and sufficient conditions for s-semigoodness from Section 2. To the best of our knowledge, these bounds are new.

We use the LP-relaxation technique, introduced in [10], to provide novel efficiently verifiable sufficient conditions for s-semigoodness of a matrix \(A\); utilizing these conditions, one can build, in a computationally efficient manner, lower bounds on the “level of s-semigoodness of \(A\),” that is, the largest \(s = s_*(A)\) for which \(A\) is s-semigood with respect to given \(P_{\pm}\). Some properties of these verifiable conditions, same as limits of their performance, are studied in Sections 4, 5, where we provide also a computationally efficient scheme for upper bounding \(s_*(A)\). In Section 6 we develop another efficiently computable lower bound for \(s_*(A)\) by applying the Semidefinite Relaxation scheme, completely similar to the construction of [4] handling the “unsigned” case \(P_{\pm} = \emptyset\).

It turns out that our verifiable sufficient conditions for s-semigoodness can be expressed in terms of specific properties of the linear recovery \(\hat{\omega}^{\text{lin}} = Y^T y\) associated with an appropriate \(m \times n\) matrix \(Y\). In Section 7, we propose and justify a new Matching Pursuit algorithm associated with this linear recovery.

## 2 Necessary and sufficient conditions for s-semigoodness

Let \(A\) be an \(m \times n\) matrix, let \(s, 1 \leq s \leq m\), be an integer, and let \(P_+, P_-\) and \(P_n\) be a partition of \(\{1, \ldots, n\}\) into three non-overlapping subsets. We say that \(A\) is s-semigood, if for every vector \(w\) with at most \(s\) nonzero entries satisfying \(w_i \geq 0\) for \(i \in P_+\), and \(w_i \leq 0\) for \(i \in P_-\), \(w\) is the unique optimal solution to the problem

\[
\text{Opt} = \min_z \{\|z\|_1 : Az = Aw, \ z_i \geq 0 \ \forall i \in P_+, \ z_i \leq 0 \ \forall i \in P_-\}.
\]

Our primary goals are to find necessary and sufficient and verifiable sufficient conditions for \(A\) to be s-semigood.

Note that without loss of generality we may assume \(P_- = \emptyset\). Indeed, by replacing the partition \(P_+, P_-\), \(P_n\) with the partition \(P'_+ = P_+ \cup P_- = \emptyset, P'_- = P_n\) and matrix \(A\) with the matrix \(\overline{A}\) obtained from \(A\) by multiplying the columns with indices \(i \in P_-\) by \(-1\), s-semigoodness of \(A\) with respect to the original sign restrictions given by \(P_{\pm}\), \(P_n\) is equivalent to the s-semigoodness of the new matrix \(\overline{A}\) with respect to the new sign restrictions. By this reason, we assume from now on that \(P_- = \emptyset\). Besides this, we assume without loss of generality that \(P_+ = \{1, \ldots, p\}\) and \(P_n = \{p+1, \ldots, n\}\) for some \(p\). From now on, we denote by \(P_n\) the set of all signals satisfying the sign restrictions:

\[
P_n = \{w \in \mathbb{R}^n : w_i \geq 0 \ \forall i \in P_+\}.
\]

Note that since \(P_- = \emptyset\), \((7)\) simplifies to

\[
\text{Opt} = \min_z \{\|z\|_1 : Az = Aw, \ z_i \geq 0 \ \forall i \in P_+\}.
\]

Let us fix a norm \(\| \cdot \|\) on \(\mathbb{R}^n\), and let \(\| \cdot \|_s\) be the conjugate norm.
Proposition 2.1 Let \( m, n, s \) and \( P_+ \) be given. The following six conditions on an \( m \times n \) matrix \( A \) are equivalent to each other:

(i) \( A \) is \( s \)-semigood;

(ii) For every subset \( J \) of \( \{1, \ldots, n\} \) with \( \text{Card}(J) \leq s \), and any \( x \in \text{Ker}A \setminus \{0\} \) such that \( x_i \leq 0 \) for all \( i \in P_+ \setminus J \) one has

\[
\sum_{i \in J \cap P_+} x_i + \sum_{i \in J \cap P_n} |x_i| < \sum_{i \in P_+ \setminus J} |x_i|.
\]

(iii) There exists \( \xi \in (0, 1) \) such that for every subset \( J \) of \( \{1, \ldots, n\} \) with \( \text{Card}(J) \leq s \) and any \( x \in \text{Ker}A \) such that \( x_i \leq 0 \) for all \( i \in P_+ \setminus J \) one has

\[
\sum_{i \in J \cap P_+} x_i + \sum_{i \in J \cap P_n} |x_i| \leq \xi \sum_{i \in P_+ \setminus J} |x_i|.
\]

(iv) There exist \( \xi \in (0, 1) \) and \( \theta \in [1, \infty) \) such that \( A \) satisfies the condition \( \text{SG}_s(\xi, \theta) \) as follows: for every \( x \in \text{Ker}A \) and every subset \( J \) of \( \{1, \ldots, n\} \) with \( \text{Card}(J) \leq s \), one has

\[
\sum_{i \in J \cap P_+} x_i + \sum_{i \in J \cap P_n} |x_i| \leq \xi \left( \sum_{i \in P_+ \setminus J} |x_i| + \sum_{i \in P_+ \setminus J} \psi(x_i) \right), \quad \psi(t) = \max[-t, \theta t],
\]

or, equivalently: for all \( x \in \text{Ker}A \),

\[
\Theta(x) := \max_{J} \sum_{i \in J \cap P_+} \max\left[ (1 - \xi) x_i, (1 + \theta \xi) x_i \right] + \sum_{i \notin J \cap P_+} (1 + \xi) |x_i| \leq \xi \Psi(x) \tag{8}
\]

(v) There exist \( \xi \in (0, 1) \), \( \theta \in [1, \infty) \) and \( \beta \in [0, \infty) \) such that \( A \) satisfies the condition \( \text{SG}_{s, \beta}(\xi, \theta) \) as follows: for every \( x \in \mathbb{R}^n \) and every subset \( J \) of \( \{1, \ldots, n\} \) with \( \text{Card}(J) \leq s \), one has

\[
\sum_{i \in J \cap P_+} x_i + \sum_{i \in J \cap P_n} |x_i| \leq \beta \|Ax\| + \xi \left( \sum_{i \notin J \cap P_+} |x_i| + \sum_{i \notin J \cap P_+} \psi(x_i) \right), \quad \psi(t) = \max[-t, \theta t].
\]

(vi) There exist \( \xi \in (0, 1) \) and \( \beta \in [0, \infty) \) such that \( A \) satisfies the condition \( \text{SG}_{s, \beta}(\xi) \) as follows: for every \( J \subset \{1, \ldots, n\} \) with \( \text{Card}(J) \leq s \) and any \( x \in \mathbb{R}^n \) such that \( x_i \leq 0 \) for all \( i \in P_+ \setminus J \), one has

\[
\sum_{i \in J \cap P_+} x_i + \sum_{i \in J \cap P_n} |x_i| \leq \beta \|Ax\| + \xi \sum_{i \notin J} |x_i|.
\]

We provide the proof of Proposition 2.1 in Appendix A.

As we have already mentioned in Introduction, when \( P_n = \emptyset \) or \( P_+ = \emptyset \), the characterizations (i)–(iv) of \( s \)-semigoodness are not completely new. For instance, when \( P_n = \emptyset \), a necessary and sufficient condition for \( s \)-semigoodness of \( A \) in the form (ii) has been established in [13] (compare (ii) to the definition (4) of half \( s \)-balancedness of \( \text{Ker}A \)). On the other hand, the equivalent formulation of this characterization in terms of conditions \( \text{SG}_{s, \beta}(\xi, \theta) \) and \( \text{SG}_{s, \beta}(\xi) \) seems to be new. It appears that the same conditions allow to control the error of \( \ell_1 \)-recovery in the case when the vector \( w \in \mathbb{R}^n \) is not \( s \)-sparse and the problem (7) is not solved to exact optimality.
3 Error bounds for imperfect $\ell_1$-recovery

We have seen that the conditions provided in Proposition 2.1 are responsible for $s$-semigoodness of a sensing matrix $A$, that is, for the precise $\ell_1$-recovery of a signal $w \in \mathcal{P}_n$ in the “ideal case” when the signal $w$ is $s$-sparse, there is no measurement error and the optimization problem (7) is solved to exact optimality. In this section, we will show that these quantities also control the error of $\ell_1$-recovery in the case when the signal vector $w \in \mathcal{P}_n$ is not exactly $s$-sparse, there is an observation noise and problem (7) is not solved to exact optimality. The error bound is as follows:

**Proposition 3.1** Let $w \in \mathcal{P}_n$ be such that $\|w-w^s\|_1 \leq \mu$, where $w^s$ is the vector obtained from $w$ by replacing all but the $s$ largest in magnitude entries in $w$ with zeros, let $y$ be such that $\|Aw-y\| \leq \varepsilon$, and let, finally, $x$ be an approximate solution to the optimization problem

$$\text{Opt} = \min \{ \|z\|_1 : \|Az-y\| \leq \varepsilon, \ z_i \geq 0 \ \forall i \in P_+ \} ,$$

such that $\|x\|_1 \leq \text{Opt} + \nu$ and $\|Ax-y\| \leq \delta$.

1. Let $A$ satisfy the condition $\text{SG}_{s,\beta}(\xi, \theta)$ with certain $\xi \in (0,1)$, $\beta \in [0,\infty)$ and $\theta \in [1,\infty)$. Then

$$\|x-w\|_1 \leq \frac{1 + \xi}{1 - \xi} \nu + \frac{2(1 + \xi \theta)}{1 - \xi} \mu + \frac{2 \beta}{1 - \xi} (\varepsilon + \delta).$$

(10)

2. Let $A$ satisfy condition $\text{SG}_{s,\beta}(\xi)$ with certain $\xi \in (0,1)$ and $\beta \in [0,\infty)$. Then

$$\|x-w\|_1 \leq \frac{1 + \xi}{1 - \xi} \nu + \frac{2(1 + \beta \alpha)}{1 - \xi} \mu + \frac{2 \beta}{1 - \xi} (\varepsilon + \delta).$$

(11)

where $\alpha$ stands for the maximum of $\|\cdot\|$-norms of the columns in $A$.

**Proof.** Let $I$ be the support of $w^s$, $\bar{I}$ be the complement of $I$ in $\{1,\ldots,n\}$, and let $z = w-x$. We denote $I_+ = \{i \in I : z_i \geq 0\}$, $\bar{I}_+ = \{i \in \bar{I} : z_i \geq 0\}$, and $I_- = I \setminus I_+, \bar{I}_- = \bar{I} \setminus \bar{I}_+$. Observe that $w$ is a feasible solution to (9), so that

$$\|x\|_1 \leq \|w\|_1 + \nu.$$  

(12)

Obviously, $|x_i| - |w_i| \geq -|z_i|$ and $|x_i| - |w_i| \geq |z_i| - 2|w_i|$. Now using $x_i, w_i \geq 0 \ \forall i \in P_+$, and $z_i \geq 0 \ \forall i \in I_+$, we get

$$\nu \geq \sum_i (|x_i| - |w_i|) \ [\text{by (12)}]$$

$$\geq \sum_{i \in I_+ \cap P_+} (x_i - w_i) + \sum_{i \in I_- \cap P_+} (x_i - w_i) + \sum_{i \in I_+ \cap P_+} (x_i - w_i) + \sum_{i \in I_- \cap P_+} (x_i - w_i)$$

$$\sum_{i \in P_+} (|x_i| - |w_i|)$$

$$\geq - \sum_{i \in I_+ \cap P_+} z_i + \sum_{i \in I_- \cap P_+} |z_i| + \sum_{i \in I_+ \cap P_+} |z_i| - \sum_{i \in I_- \cap P_+} w_i$$

$$- \sum_{i \in I_- \cap P_+} |z_i| + \sum_{i \in \bar{I} \cap P_+} (|z_i| - 2|w_i|),$$

or, equivalently,

$$\sum_{i \in I_- \cap P_+} |z_i| + \sum_{i \in I_- \cap P_+} |z_i| + \sum_{i \in \bar{I} \cap P_+} |z_i| \leq \nu + \sum_{i \in I_+ \cap P_+} z_i + \sum_{i \in I_- \cap P_+} |z_i| + \sum_{i \in I_+ \cap P_+} w_i + 2 \sum_{i \in \bar{I} \cap P_+} |w_i|. \ (13)$$

5
On the other hand, we have
\[ \|Az\| = \|Aw - Ax\| \leq \|Aw - y\| + \|Ax - y\| \leq \varepsilon + \delta. \] (14)
Then by condition \( \mathbf{SG}_{s, \beta}(\xi, \theta) \) with \((I_+ \cap P_+) \cup (I \cap P_n)\) in the role of \( J \), we get
\[ \sum_{i \in I_+ \cap P_+} z_i + \sum_{i \in I \cap P_n} |z_i| \leq \beta \|Az\| + \xi \left[ \sum_{i \in I \cap P_+} |z_i| + \sum_{i \in (I \cap P_+) \cup (I \cap P_n)} \psi(z_i) \right] \]
\[ \kappa \leq \beta \|Az\| + \xi \left[ \sum_{i \in I \cap P_+} |z_i| + \sum_{i \in I \cap P_+} |z_i| + \sum_{i \in I \cap P_n} |z_i| + \theta \sum_{i \in I_+ \cap P_+} z_i \right] := \tau(\theta) \] (15)
Let us derive a bound on \( \tau(\theta) \). Now (13) implies, independently of whether \( \mathbf{SG}_{s, \beta}(\xi, \theta) \) is or is not true, the first inequality in the following chain:
\[ \tau(\theta) \leq \nu + \sum_{i \in I_+ \cap P_+} z_i + \sum_{i \in I \cap P_n} |z_i| + \sum_{i \in I_+ \cap P_+} w_i + 2 \sum_{i \in I \cap P_n} |w_i| + \theta \sum_{i \in I_+ \cap P_+} z_i \]
\[ \leq \nu + \kappa + (1 + \theta) \sum_{i \in I_+ \cap P_+} w_i + 2 \sum_{i \in I \cap P_n} |w_i| [\text{since } w_i \geq z_i \text{ for } i \in P_+] \]
\[ \leq \nu + \kappa + (1 + \theta) \mu, \quad [\text{since } \theta \geq 1 \text{ and } \sum_{i \in I} |w_i| \leq \mu], \] (16)
and, in particular,
\[ \tau(1) = \sum_{i \in I_+ \cap P_+} z_i + \sum_{i \in I \cap P_n} |z_i| \leq \nu + \kappa + 2\mu. \] (17)
Combining (14), (15) and (16), we obtain
\[ \kappa \leq \beta(\varepsilon + \delta) + \xi \left[ \nu + \kappa + (1 + \theta) \mu \right], \]
whence
\[ \kappa = \sum_{i \in I_+ \cap P_+} z_i + \sum_{i \in I \cap P_n} |z_i| \leq \frac{\beta(\varepsilon + \delta) + \xi(\nu + (\theta + 1) \mu)}{1 - \xi}. \]
Summing up the latter inequality and (17), we obtain
\[ \|z\|_1 = \sum_{i \in I_+ \cap P_+} |z_i| + \sum_{i \in I_+ \cap P_+} z_i + \left[ \sum_{i \in I_+ \cap P_+} |z_i| + \sum_{i \in I \cap P_n} |z_i| \right] \leq \nu + 2\mu + 2\kappa \]
\[ \leq \nu + 2\mu + \frac{2\beta(\varepsilon + \delta) + 2\xi(\nu + (\theta + 1) \mu)}{1 - \xi} \]
\[ = \frac{1 + \xi}{1 - \xi} \nu + \frac{2(1 + \xi \theta)}{1 - \xi} \mu + \frac{2\beta}{1 - \xi}(\varepsilon + \delta), \]
which is (10).
To show (11) observe that increasing \( \varepsilon \) to \( \varepsilon' = \varepsilon + \alpha \mu \), we can think that the true signal underlying the observation \( y \) is \( w' \) rather than \( w \); note that (12) implies that
\[ \|x\|_1 \leq \|w'\|_1 + \nu', \ \nu' = \nu + \mu. \] (18)
We can now repeat the reasoning which follows (12), with (18) in the role of (12), \( w^* \) in the role of \( w, \varepsilon' \) in the role of \( \varepsilon \) and 0 in the role of \( \mu \), thus arriving at the following analogy of the bound (10):

\[
\|x - w^*\|_1 \leq \frac{1 + \varepsilon}{1 - \varepsilon} \varepsilon' + \frac{2\beta}{1 - \varepsilon}(\varepsilon' + \delta),
\]

whence

\[
\|x - w\|_1 \leq \frac{1 + \varepsilon}{1 - \varepsilon} \varepsilon' + \frac{2\beta}{1 - \varepsilon}(\varepsilon' + \delta) + \mu,
\]

which is nothing but (11).

\[\square\]

4 **Verifiable conditions for \( s \)-semigoodness**

We are about to demonstrate that among the conditions listed in Proposition 2.1, \( \text{SG}_{s,\beta}(\xi, \theta) \) leads to efficiently computable “nontrivial” lower and upper bounds.

4.1 **Verifiable sufficient conditions for \( s \)-semigoodness by LP-relaxation**

Let

\[
\mathcal{U}_s = \{ u \in \mathbb{R}^n : \|u\|_1 \leq s, \|u\|_\infty \leq 1 \},
\]

so that \( \mathcal{U}_s \) is the convex hull of all \( \{-1, 0, 1\} \) vectors with at most \( s \) nonzero entries, and for \( x \in \mathbb{R}^n \), let \( \|x\|_{s,1} \) be the sum of the \( s \) largest in magnitude entries in \( x \), or equivalently,

\[
\|x\|_{s,1} = \max_{u \in \mathcal{U}_s} u^T x.
\]

Let

\[
(D_\theta[x])_i = \begin{cases} 
[1 + \theta \xi] \max[x_i, 0], & i \in P_+ \\
(1 + \xi) |x_i|, & i \notin P_+
\end{cases},
\]

and

\[\Phi(x) = \|D_\theta[x]\|_{s,1}.\]

Suppose \( \xi \in [0, 1), \theta \in [1, \infty) \) and \( \rho, \sigma \in [0, \infty) \) are given. Consider the following condition on an \( m \times n \) matrix \( A \):

\( \text{VSG}_{s}(\xi, \theta, \rho, \sigma) \): There exists \( m \times n \) matrix \( Y = [y_1, ..., y_n] \) and a vector \( v \in \mathbb{R}^m \) such that

\[
\Phi(-C_i[Y, A]) + (A^T v)_i \leq \xi, \ 1 \leq i \leq n \quad (a)
\]

\[
\Phi(C_i[Y, A]) - (A^T v)_i \leq \xi, \ i \notin P_+ \quad (b)
\]

\[
\Phi(C_i[Y, A]) - (A^T v)_i \leq \theta \xi, \ i \in P_+ \quad (c)
\]

\[
\|y_i\|_s \leq \sigma, \ 1 \leq i \leq n \quad (d)
\]

\[
\|v\|_s \leq \rho \quad (e)
\]

where \( C_i[Y, A] \) is the \( i \)-th column of the matrix \( I - Y^T A \).

Observe that this condition is verifiable, since (19) is a system of explicit convex constraints on \( Y \) and \( v \).

**Proposition 4.1** Let \( A \) satisfy \( \text{VSG}_{s}(\xi, \theta, \rho, \sigma) \) with some \( \xi \in [0, 1), \theta \in [1, \infty) \), and \( \rho, \sigma \in [0, \infty) \). Then \( A \) satisfies \( \text{SG}_{s,\beta}(\xi, \theta) \) with

\[
\beta = \rho + \sigma \max_{k_+, k_-} \left\{ k_+(1 + \theta \xi) + k_-(1 + \xi) : \begin{array}{l}
0 \leq k_+ \leq \text{Card}(P_+)
\vspace{1mm}
0 \leq k_- \leq \text{Card}(P_-)
\vspace{1mm}
k_+ + k_- \leq s
\end{array} \right\} \leq \rho + \sigma s(1 + \theta \xi).
\]

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Proof. Let $A$ satisfy $\text{VSG}_s(\xi, \theta, \rho, \sigma)$, and let $Y = [y_1, \ldots, y_n]$ and $v$ satisfy (19). Let, further, $I \subset \{1, \ldots, n\}$ be such that $\text{Card}(I) \leq s$, and let $x \in \mathbb{R}^n$. Let $u \in \mathbb{R}^n$ be given by

$$u_i = \begin{cases} 
1 + \theta \xi, & i \in P_+ \cap I, \ x_i \geq 0 \\
1 - \xi, & i \in P_+ \cap I, \ x_i < 0 \\
(1 + \xi) \text{sign}(x_i), & i \in P_+ \cap I \\
0, & i \in P_+ \cap I \\
\end{cases} \ .$$

Note that $u$ has at most $s$ nonzero entries, the entries of $u$ with indices from $P_+$ belong to $[0, 1 + \theta \xi]$, and the modulae of entries in $u$ with indices from $P_-$ are $\leq 1 + \xi$, so that $u^T z \leq \Phi(z)$ for all $z$. We have

$$u^T[I - Y^T A]x = \sum_i u^T C_i[Y, A]x_i = \sum_{i: x_i \geq 0} u^T C_i[Y, A]x_i + \sum_{i: x_i < 0} u^T [-C_i[Y, A]]x_i$$

$$\leq \sum_{i: x_i \geq 0} \Phi(C_i[Y, A])x_i + \sum_{i: x_i < 0} \Phi(-C_i[Y, A])x_i \quad \text{[since } u^T z \leq \Phi(z)]$$

$$\leq \sum_{i: x_i \geq 0, i \notin P_+} [\xi + (A^T v)_i]x_i + \sum_{i: x_i < 0, i \notin P_+} [\theta \xi + (A^T v)_i]x_i + \sum_{i: x_i < 0, i \in P_+} [\xi - (A^T v)_i]x_i \quad \text{[by (19)]}$$

$$= \xi \sum_{i: x_i \geq 0, i \notin P_+} x_i + \theta \sum_{i: x_i \geq 0, i \in P_+} x_i + \sum_{i: x_i < 0} |x_i| + x^T A^T v$$

$$= \xi \sum_{i: x_i \geq 0, i \notin P_+} \max[-x_i, \theta x_i] + \sum_{i: x_i \geq 0} |x_i| + x^T A^T v,$$

whence

$$u^T[I - Y^T A]x \leq \xi \left[ \sum_{i \in P_+} \max[-x_i, \theta x_i] + \sum_{i \in P_+} |x_i| \right] + \rho \|Ax\|$$

(21)

(recall that $\|v\|_* \leq \rho$). On the other hand, recalling the definition of $u$ and that $\|y_i\|_* \leq \sigma$, we have

$$u^T[I - Y^T A]x = u^T x - \sum_{i \in I} u_i y_i^T A x$$

$$= \sum_{i \in I \cap P_+} \max[(1 - \xi)x_i, (1 + \theta \xi)x_i] + (1 + \xi) \sum_{i \in I \cap P_+} |x_i| - \sum_{i \in I} u_i y_i^T A x$$

$$\geq \sum_{i \in I \cap P_+} \max[(1 - \xi)x_i, (1 + \theta \xi)x_i] + (1 + \xi) \sum_{i \in I \cap P_+} |x_i| - \sigma \left[ \sum_{i \in I \cap P_+} (1 + \theta \xi) + \sum_{i \in I \cap P_+} (1 + \xi) \right] \|Ax\| \leq \beta - \rho$$

Combining the resulting inequality with (21), we get

$$\sum_{i \in I \cap P_+} [x_i + \xi \max[-x_i, \theta x_i]] + (1 + \xi) \sum_{i \in I \cap P_+} |x_i| \leq \beta \|Ax\| + \xi \left[ \sum_{i \in P_+} \max[-x_i, \theta x_i] + \sum_{i \in P_+} |x_i| \right]$$

with $\beta$ given by (20), or, equivalently,

$$\sum_{i \in I \cap P_+} x_i + \sum_{i \in I \cap P_+} |x_i| \leq \beta \|Ax\| + \xi \left[ \sum_{i \in P_+ \setminus I} \max[-x_i, \theta x_i] + \sum_{i \in P_+ \setminus I} |x_i| \right] .$$

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The latter relation holds true for every \( x \in \mathbb{R}^n \) and for every set \( I \subset \{1, \ldots, n\} \) of cardinality \( \leq s \), so that \( A \) satisfies \( \text{SG}_{k, \beta}(\xi, \theta) \). \( \square \)

### 4.1.1 Origin of the verifiable sufficient condition

The condition \( \text{VSG}_s(\xi, \theta, \rho, \sigma) \) is yielded by a simple and general construction, and we believe it makes sense to present this construction in its general form. The essence of the matter is in building a verifiable sufficient condition for the validity of (8), see Proposition 2.1.iv. By positive homogeneity of degree 1 of the convex functions \( \Theta, \Psi \) participating in (8), the latter relation is equivalent to

\[
\text{Opt} := \max_x \{ \Theta(x) : Ax = 0, x \in X \} \leq \xi, \quad X = \{ x : \Psi(x) \leq 1 \}. \tag{22}
\]

A verifiable sufficient condition for the latter relation is essentially the same as an efficiently computable upper bound for \( \text{Opt} \); the sufficient condition for the validity of (22) associated with such a bound merely states that the bound is \( \leq \xi \). Now observe that from the origin of \( \Psi \) (see (8)) it is clear that \( X \) has a moderate number, \( N \), of readily available extreme points \( x^1, \ldots, x^N \) (in the case of (8), \( N = 2n \)), so that the only difficulty in computing \( \text{Opt} \) exactly comes from linear constraints \( Ax = 0 \). The standard way to circumvent this difficulty and to efficiently bound \( \text{Opt} \) from above is to use the Lagrange relaxation: for any \( v \in \mathbb{R}^m \),

\[
\text{Opt} = \max_{x \in X} \{ \Theta(x) + v^T A x : Ax = 0, x \in X \} \leq \max_{x \in X} \left\{ \Theta(x) + v^T A x : x \in X \right\} = \max_{1 \leq i \leq N} [\Theta(x^i) + v^T A x^i],
\]

whence the efficiently computable quantity \( \inf_v \max_{1 \leq i \leq N} [\Theta(x^i) + v^T A x^i] \) is an upper bound on \( \text{Opt} \). Unfortunately, in our situation the Lagrange relaxation bound can be very poor; e.g., when \( x \) is symmetric with respect to the origin and \( \Theta \) is even (as it happens in (8) when \( P_+ = \emptyset \)), it is immediately seen that the Lagrange relaxation bound becomes the trivial bound \( \text{Opt} \leq \max_{x \in X} \Theta(x) = \max_i \Theta(x^i) \).

In order to strengthen the relaxation, we pass to the Fenchel-type representation of \( \Theta \)

\[
\Theta(x) = \sup_u \left[ [Pu + q]^T x - \Theta_*(u) \right]
\]

with a proper convex function \( \Theta_* \); such a representation, even with \( Pu + p \equiv u \), exists whenever \( \Theta \) is a proper convex function (and can be easily found for \( \Theta \) we are interested in). We now have for any \( Y \in \mathbb{R}^{m \times n} \), \( v \in \mathbb{R}^m \),

\[
\text{Opt} = \max_{x} \{ \Theta(x) : Ax = 0, x \in X \} = \sup_{x, u} \left\{ [Pu + p]^T [x - Y^T A x] + v^T A x - \Theta_*(u) : Ax = 0, x \in X \right\} \leq \sup_{x, u} \left\{ [Pu + p]^T [x - Y^T A x] + v^T A x - \Theta_*(u) : x \in X \right\} \leq \max_{1 \leq i \leq N} \sup_{u} \left\{ [Pu + p]^T [x^i - Y^T A x^i] + v^T A x^i - \Theta_*(u) \right\}, \]

so that the condition

\[
\exists (Y \in \mathbb{R}^{m \times n}, v \in \mathbb{R}^m) : \Theta_i(Y, v) \leq \xi, \quad 1 \leq i \leq N,
\tag{23}
\]

is sufficient for the validity of (22). Note that the functions \( \Theta_i \), by their origin, are convex, so that the condition (23) is efficiently verifiable, provided that \( \Theta_i(\cdot) \) are efficiently computable.
In the case we are interested in, the extreme points of \( X \) are the \( 2n \) vectors \( -e_i, 1 \leq i \leq n, e_i, i \in P_n \), and \( \theta^{-1} e_i, i \in P_+, e_i \) being the basic orths. Implementing the outlined bounding scheme and adding additional restrictions \((19.d,e)\) to get a control over \( \beta \), we arrive at \((19)\). It should be stressed that the outlined scheme can be applied to bounding from above the optimal value of a whatever problem of the form \((22)\) with a convex polytope \( X \) and a proper convex objective \( \Theta \); all what matters is that \( X \) is given as \( \text{Conv}\{x^1, ..., x^N\} \) and \( \Theta \) is efficiently computable. Note also that when \( X \) is a polytope given by list of \( M \) linear inequalities, we can efficiently represent it as the intersection of \( M \)-dimensional standard simplex and an affine plane, so that the outlined scheme is applicable to a whatever problem of maximizing an efficiently computable proper convex function under a (finite) system of linear inequality and equality constraints.

### 4.1.2 Effect of increasing \( \beta, \theta, \xi \)

The condition \( \text{SG}_{s,\beta}(\xi, \theta) \) appearing in Proposition 2.1.v clearly is “monotone” in the parameters \( \beta, \theta, \xi \): whenever \( A \) satisfies this condition and \( \beta' \geq \beta, \theta' \geq \theta \) and \( \xi' \geq \xi \), \( A \) satisfies the condition \( \text{SG}_{s,\beta'}(\xi', \theta') \) as well. Proposition 4.1 offers a verifiable sufficient condition for the validity of \( \text{SG}_{s,\beta}(\xi, \theta) \), specifically,

\[
\text{VSG}_{s,\beta}(\xi, \theta): \text{ There exist } Y, v, \rho, \sigma \text{ satisfying } (19) \text{ and the relation } \rho + \sigma s(1 + \theta \xi) \leq \beta.
\]

A natural question is, whether this verifiable condition possesses the same monotonicity properties as the “target” condition \( \text{SG}_{s,\beta}(\xi, \theta) \). In the case of the affirmative answer, in order to conclude that \( A \) is \( s \)-semigood, we could check the validity of \( \text{VSG}_{s,\beta}(\xi, \theta) \) for appropriately large values of \( \beta, \theta \) and a close to one value of \( \xi < 1 \); if the condition is satisfied, \( A \) is \( s \)-semigood, and error bounds from Proposition 3.1 take place. Were the condition \( \text{VSG}_{s,\beta}(\xi, \theta) \) “not monotone”, to justify the \( s \)-semigoodness of \( A \) via this condition would require a problematic and time-consuming search in the space of parameters \( \beta, \theta, \xi \). Fortunately, the condition \( \text{VSG}_{s,\beta}(\xi, \theta) \) indeed is monotone:

**Proposition 4.2** Let \( A \) satisfy \( \text{VSG}_{s,\beta}(\xi, \theta) \), and let \( Y, v, \sigma, \rho \) be the corresponding certificate, that is, \( \rho + \sigma s(1 + \theta \xi) \leq \beta \) and \( Y, v, \sigma, \rho \) satisfy \((19)\). Then \( A \) satisfies \( \text{VSG}_{s,\beta'}(\xi', \theta') \) whenever \( \beta' \geq \beta, \theta' \geq \theta \) and \( \xi' \in (\xi, 1) \), the certificate being \((Y', v, \sigma, \rho)\), where the columns \( Y'_i \) of \( Y' \) are multiplies of the columns \( Y_i \) of \( Y \), namely,

\[
Y'_i = a_i Y_i; \ [0, 1] \ni a_i = \begin{cases} (1 + \xi \theta)/(1 + \xi' \theta'), & i \in P_+ \\ (1 + \xi)/(1 + \xi'), & i \in P_n \end{cases}
\]

For the proof, see Appendix B.

### 4.1.3 Relation to the sufficient condition for \( s \)-goodness from \([10]\) and the Restricted Isometry Property

The verifiable sufficient condition for \( s \)-goodness from \([10]\) requires from an \( m \times n \) matrix \( A \) the existence of \( \gamma < 1/2 \) and \( Y = [y_1, ..., y_n] \in \mathbb{R}^{m \times n} \) such that

\[
\|C_i[Y, A]\|_{s,1} \leq \gamma, \text{ for all } 1 \leq i \leq n,
\]

Setting \( \theta = 1 \) and \( \xi = \frac{\theta}{1+\gamma} \) (so that \( \xi < 1 \) and \( \gamma = \frac{\xi}{1+\xi} \)) and taking into account that in the case of \( \theta = 1 \) we have \( \Phi(z) \leq (1 + \xi)\|z\|_{s,1} \), the latter condition implies that

\[
\Phi(\pm C_i[Y, A]) \leq (1 + \xi)\gamma = \xi, \forall i,
\]

that is, it implies the validity of \( \text{VSG}_s(\xi, 1, 0, \sigma) \), provided that \( \sigma \) is large enough, specifically, \( \sigma \geq \|y_i\| \) for all \( i \).
As it was shown in the companion paper [10], when $A$ satisfies the Restricted Isometry Property $\text{RIP}(\delta, k)$ with parameters $\delta \in (0, 1), k > 1$, the above sufficient condition for $s$-goodness is satisfied with $\gamma = 1/3$ for $s$ as large as $O(1)(1 - \delta)\sqrt{k}$; as a result, a $\text{RIP}(\delta, k)$-matrix satisfies $\text{VSG}_s(\frac{1}{2}, 1, 0, \sigma)$ provided that $\sigma$ is large enough and $s \leq O(1)(1 - \delta)\sqrt{k}$. Since for large $m, n$, $m < n$, typical random matrices possess, with overwhelming probability, property $\text{RIP}(\frac{1}{2}, k)$ with $k$ as large as $O(1)m/\ln(n/m)$, we see that our verifiable sufficient condition for $s$-semigoodness can certify the latter property for $s$ as large as $O(1)\sqrt{m/\ln(n/m)}$, provided that the matrix in question is “good enough”.

4.2 Upper bounding the level of $s$-semigoodness

Here we address the issue of bounding from above the maximal $s = s_*(A)$ for which $A$ is $s$-semigood. The construction to follow is motivated by item (iv) of Proposition 2.1. A necessary and sufficient condition for the $s$-semigoodness of $A$ is the existence of $\xi < 1$ and $\theta \geq 1$ such that for all $x \in \text{Ker} A$ and any set $I$ of indices with $\text{Card}(I) \leq s$

$$
\sum_{i \in I \cap P_+} \max[(1 - \xi)x_i, (1 + \theta \xi)x_i] + \sum_{i \in I \cap P_n} (1 + \xi)|x_i| \leq \xi \Psi(x)
$$

where

$$
\Psi(x) = \sum_{i \in P_+} \max[-x_i, \theta x_i] + \sum_{i \in P_n} |x_i|,
$$

or, equivalently,

$$(1) \text{ for every } x \in \text{Ker} A \text{ and every vector } v \text{ with at most } s \text{ nonzero entries and nonzero entries } v_i \text{ belonging to } [1 - \xi, 1 + \xi \theta] \text{ if } i \in P_+ \text{ and belonging to } [-1 - \xi, 1 + \xi] \text{ if } i \in P_n, \text{ one has}
$$

$$
v^T x \leq \xi \Psi(x).
$$

Observe that the convex hull of the vectors $v$ in question is exactly the set

$$
\mathcal{U}^{\xi, \theta} = \left\{ v \in \mathbb{R}^n : 0 \leq v_i \leq 1 + \theta \xi, i \in P_+, |v_i| \leq 1 + \xi, i \in P_n, \sum_{i \in P_+} \frac{v_i}{1 + \xi \theta} + \sum_{i \in P_n} |v_i| \leq s \right\}.
$$

Recalling that $P_+ = \{1, ..., p\}$, setting $q = n - p = \text{Card}(P_n)$ and

$$
\mathcal{U} = \{ u \in \mathbb{R}^n : \|u\|_1 \leq s, \|u\|_\infty \leq 1, u_i \geq 0 \text{ for } i \in P_+ \},
$$

we see that

$$
\mathcal{U}^{\xi, \theta} = V^{\xi, \theta} \mathcal{U}, \text{ where } V^{\xi, \theta} = \begin{bmatrix} (1 + \xi \theta)I_p & 0 \\ 0 & (1 + \xi)I_q \end{bmatrix}.
$$

The condition (1) now reads

$$
\max_{v \in \mathcal{U}^{\xi, \theta}} v^T x \leq \xi \Psi(x) \text{ for all } x \in \text{Ker} A.
$$

Setting $\mathcal{X} = \{ x \in \text{Ker} A : \Psi(x) \leq 1 \}$ the latter condition, by homogeneity reason, is the same as

$$
\text{Opt} = \text{Opt}(\xi, \theta) := \max_{v, x} \left\{ v^T x : v \in \mathcal{U}^{\xi, \theta}, x \in \mathcal{X} \right\} \leq \xi;
$$

(27)
recall that $A$ is $s$-semigood if and only if there exist $\theta \geq 1$ and $\xi < 1$ such that \((27)\) takes place.

We can use \((27)\) in order to bound $s_*(A)$ from above, as follows. In order to certify that $s_*(A) < s$ for a given $s$ ($s$ is the input to our algorithm), we fix a large $\theta$ and a close to one $\xi < 1$ (these are the parameters of the algorithm) and run the iterations

$$u_0 \in \mathcal{U}^{s,\theta} \mapsto x_1 \in \text{Argmax}_{x \in X} u_0^T x \mapsto u_1 \in \text{Argmax}_{u \in \mathcal{U}^{s,\theta}} xu^T x_1 \mapsto x_2 \in \text{Argmax}_{x \in X} u_1^T x \mapsto \ldots$$

initiating them by a picked at random vertex $u_0$ of $\mathcal{U}^{s,\theta}$. Note that the quantities $u_i^T x_i$, $i = 1, 2, \ldots$ clearly form a nondecreasing sequence of lower bounds on Opt. We terminate the outlined iterations when the progress in the bounds – the difference $u_i^T x_i - u_{i-1}^T x_{i-1}$ – falls below a given small threshold, and we run this process a predetermined number of times from different randomly chosen starting points. As a result, we get a bunch of lower bounds on Opt of the form $u^T x$, where $u$ is a vertex of $\mathcal{U}^{s,\theta}$ and $x \in X$. If our goal were merely to certify that \((33)\) is not valid for given $s, \theta, \xi$, we could terminate this process at the first step, if any, when the current lower bound $u^T x$ becomes $> \xi$ (cf. \([10, \text{Section 4.1}]\)). We, however, want to certify that $s > s_*(A)$, or, which is the same by Proposition 2.1.iv, that \((33)\) fails to be true for all $\theta$ and all $\xi < 1$, and not only for those $\theta, \xi$ we have selected for our test. To overcome this difficulty, we accompany every step $u \mapsto x \in \text{Argmax}_{x \in X} u^T x$ by an additional computation as follows. In our process, $u$ is an extreme point of $\mathcal{U}^{s,\theta}$, that is, a point with $s_u \leq s$ nonzero entries, let the set of indices of these entries be $I$. Setting $\epsilon_i = \text{sign}(u_i)$, we solve the following LP problem

$$\max_x \left\{ \sum_{i \in I \cap P_+} x_i + \sum_{i \in I \cap P_-} \epsilon_i x_i : \begin{cases} x_i \leq 0, \ i \in P_+ \setminus I \\ Ax = 0 \\ \sum_{i \in I} |x_i| \leq 1 \end{cases} \right\}.$$ 

If the optimal value in this problem is $\geq 1$, we terminate our test and claim that $A$ is not $s$-good; by Proposition 2.1.ii, this indeed is the case.

As applied to a given input $s$, the outlined test either terminates with a valid claim “$s > s_*(A)$”, or terminates with no conclusion at all, in which case we could pass to testing a larger value of $s$.

5 Limits of performance of LP-based sufficient conditions for $s$-semigoodness

Unfortunately, the condition in question, same as its predecessor from \([10]\), cannot certify $s$-semigoodness of an $m \times n$ matrix in the case of $s > O(1)/\sqrt{m}$, unless the matrix is “nearly square”. The precise statement is as follows (cf. \([10, \text{Proposition 4.2}]\)):

**Proposition 5.1** Let

$$n > 2(2\sqrt{2m} + 1)^2$$

and let $\xi < 1, \theta \geq 1, \sigma \geq 0, \rho \geq 0$, an integer $s$ and an $m \times n$ matrix $A$ be such that $A$ satisfies $\text{VSG}_s(\xi, \theta, \rho, \sigma)$. Then

$$s \leq 2\sqrt{2m} + 1.$$  

**Proof** is based on the following

**Lemma 5.1** Let $Z$ be a $\nu \times \nu$ matrix of rank $m$, $s > 1$ be a positive integer, and $\delta_i \in (0, 1], 1 \leq i \leq \nu$, be such that for the columns $C_i$ of the matrix $I_{\nu} - Z$ it holds $\|C_i\|_{s,1} \leq 1 - \delta_i$. Assume that

$$\nu > (2\sqrt{2m} + 1)^2.$$  

Then

$$s \leq 2\sqrt{2m} + 1.$$  

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Proof of the lemma. Let $\sigma_i = Z_{ii}$, and let $\gamma_i$ be the sum of $s - 1$ largest magnitudes of the entries in $C_i$ with indices different from $i$. We have

$$1 - \sigma_i + \gamma_i \leq \|C_i\|_{s,1} \leq 1 - \delta_i,$$

whence $\sigma_i \geq \delta_i + \gamma_i > 0$. Let us set $\lambda_i = \frac{1}{\sigma_i}$, and let $\bar{Z}$ be the matrix with the columns $\bar{Z}_i = \lambda_i Z_i$, where $Z_i$ is the $i$-th column in $Z$. Note that $\bar{Z}$ is of the same rank $m$ as $Z$, and that $\bar{Z}_{ii} = 1$ for all $i$. Recalling that $\gamma_i < \sigma_i$, we have also

$$\|\bar{Z}_i\|_{s-1,1} = \lambda_i\|Z_i\|_{s-1,1} \leq \lambda_i[\gamma_i + \sigma_i] \leq 2\lambda_i\sigma_i = 2.$$

Now let $\bar{s} = \min[s - 1, \lceil \nu^{1/2} \rceil]$, so that $\bar{s} \geq 1$ due to $s > 1$. We have $\|\bar{Z}_i\|_{s,1} \leq \|\bar{Z}_i\|_{s-1,1} \leq 2$ and $\bar{s}^2 \leq \nu$. From the latter inequality and due to $\|\bar{Z}_i\|_2^2 \leq \max\{1, \nu\bar{s}^{-2}\}\|\bar{Z}_i\|_2^2$ (cf. the proof of [10, Proposition 4.2]), it follows that $\|\bar{Z}_i\|_2^2 \leq 4\nu\bar{s}^{-2}$. We conclude that $\|\bar{Z}\|_2^2 \leq 4\nu^2\bar{s}^{-2}$, where for a matrix $B$, $\|B\|_2$ is the Frobenius norm of $B$. Setting $H = \frac{1}{2}[\bar{Z} + \bar{Z}^T]$, we have therefore $\|H\|_2^2 \leq 4\nu^2\bar{s}^{-2}$. On the other hand, $\text{Tr}(H) = \sum_{i=1}^{\nu} Z_{ii} = \nu$, while rank($H$) $\leq 2m$, whence, denoting by $\mu_i$, $1 \leq i \leq p \leq 2m$, the nonzero eigenvalues of $H$, we have

$$\|H\|_2^2 = \sum_{i=1}^{p} \mu_i^2 \geq \left(\sum_{i=1}^{p} \mu_i\right)^2 / p = (\text{Tr}(H))^2 / p \geq \nu^2 / (2m).$$

We arrive at the inequality

$$4\nu^2\bar{s}^{-2} \geq \|H\|_2^2 \geq \nu^2 / (2m),$$

whence

$$\bar{s}^2 \leq 8m. \quad (32)$$

Assuming that $\bar{s} = \lceil \nu^{1/2} \rceil$, (32) says that $\nu \leq (2\sqrt{2m} + 1)^2$, which is impossible. The only other option is that $\bar{s} = s - 1$, and we arrive at (31). □

Lemma 5.1 ⇒ Proposition 5.1: Let $Y, v$ satisfy (19). Consider first the case when $\nu := \text{Card}(P_n) \geq n/2$. Denoting by $\hat{C}_i$ the $\nu$-dimensional vector comprised of the last $\nu$ entries in $C_i = C_i[Y, A]$ (i.e., entries with indices from $P_n$). By (19), for every $i \in P_n$ and for every set $I \subset P_n$ with Card($I$) $\leq s$ we have

$$\sum_{j \in I} (1 + \xi)|C_{ij}| \leq \Phi(-C_i) \leq \xi - (A^T v)_i,$$

whence for any $i \in P_n$,

$$2(1 + \xi)|\hat{C}_i|_{s,1} \leq \Phi(-C_i) + \Phi(C_i) \leq 2\xi,$$

so that $\|\hat{C}_i\|_{s,1} < 1/2$. We see that the South-Eastern $\nu \times \nu$ submatrix $Z$ of $Y^T A$ satisfies the premise of Lemma 5.1, while the size $\nu$ of $Z$ satisfies (30) due to (28) and $\nu \geq n/2$. Applying the lemma, we arrive at (29).

Now consider the case when Card($P_n$) $< n/2$, that is, $\nu := \text{Card}(P_+) \geq n/2$. By (19), setting $C_i = C_i[Y, A]$, for every set $I \subset P_+$ with Card($I$) $\leq s$ and every $i \in P_+$ we have

$$\sum_{j \in I} (1 + \theta \xi) \max[-|C_{ij}|, 0] \leq \Phi(-C_i) \leq \xi - (A^T v)_i,$$

whence

$$\sum_{j \in I} |C_{ij}| \leq \frac{\xi (1 + \theta)}{1 + \theta \xi} < 1.$$
Since the latter inequality holds true for every subset $I$ of $P_+$ with $\text{Card}(I) \leq s$, when denoting by $\tilde{C}_i$ the part of $C_i$ comprised of the first $\nu$ entries (those with indexes from $P_+$), we have for all $i \in P_+$:

$$\|\tilde{C}_i\|_{s,1} < 1.$$ 

Now the proof can be completed exactly as in the previous case, with the North-Western $\nu \times \nu$ submatrix of $Y^T A$ in the role of $Z$. □

Proposition 5.1 brings a not so good news about our sufficient conditions for $s$-semigoodness. Another bad news is that while our condition in “good” cases, e.g., those of RIP-matrices, allows to certify $s$-semigoodness for “large” values of $s$, it can give a very poor impression on what is the largest $s = s_s(A)$ for which $A$ is $s$-semigood. An instructive example in this direction is as follows. Consider the case of $P_+ = \{1, \ldots, n\}$, let $m = 2d + 1$ be odd, and let the rows of $A$ be comprised of the values of basic trigonometric polynomials

$$p_0(\phi) \equiv 1, \quad p_{2i-1}(\phi) = \cos(i\phi), \quad p_{2i}(\phi) = \sin(i\phi), \quad 1 \leq i \leq d,$$

taken along the regular grid $\phi_j = 2\pi j/n$, $0 \leq j < n$, so that $A_{ij} = p_i(\phi_j)$, $0 \leq i < m$, $0 \leq j < n$ (we enumerate rows and columns starting with 0 rather than with 1). It is well known [3, 6] that in this case $A$ is $s$-semigood for $s = d$. In contrast to this, when $A$ is not “nearly square”, specifically, when $n \geq 4\pi d$, whatever large be $\theta, \sigma, \rho$ and whatever close to 1 be $\xi < 1$, $A$ can satisfy the condition $\text{VSG}_s(\xi, \theta, \sigma, \rho)$ only for $s \leq 2$.

To justify our claim, let $L$ be the $n \times n$ permutation matrix corresponding to the cyclic shift $e_j \mapsto e_{j^+}, \; j^+ = (j + 1) \bmod n$, of the standard basic orths $e_0, \ldots, e_{n-1}$ in $\mathbb{R}^n$, and $R$ be the $m \times m$ orthogonal block-diagonal matrix with the North-Western block 1 and $d$ additional $2 \times 2$ diagonal blocks

$$\begin{bmatrix}
\cos(2\pi i/n) & -\sin(2\pi i/n) \\
\sin(2\pi i/n) & \cos(2\pi i/n)
\end{bmatrix}, \quad 1 \leq i \leq d.$$

Denoting by $A_j$ the $j$-th column of $A$, $0 \leq j \leq n - 1$, we clearly have $RA_j = A_{j^+}$, whence $A = RAL^{-1}$ and therefore also $A = RL^{-1}A^{-1}$ for $1 \leq i \leq n$. Now assume that $Y, v$ satisfy (19) for certain $\xi < 1$, $\theta \geq 1, \rho, \sigma$. Then

$$\max_i [\Phi(-C_i[Y, A]) + \Phi(C_i[Y, A])] \leq \xi(1 + \theta),$$

whence, as it is immediately seen, $\max_i \|C_i[Y, A]\|_{s,1} \leq \kappa := \frac{\xi(1 + \theta)}{1 + \sigma \xi} < 1$, or, which is the same,

$$\Gamma(I - Y^T A) \leq \kappa < 1,$$

where $\Gamma(Z)$ is the maximum of the $\| \cdot \|_{s,1}$-norms of columns of $Z \in \mathbb{R}^{n \times n}$. Observe that $\Gamma$ is a convex function which is symmetric in the sense that $\Gamma(PZP^T) = \Gamma(Z)$ whenever $P$ is a permutation matrix. Now let $\bar{Y} = \frac{1}{n} \sum_{i=1}^n R^{-1}Y L_i$. Since $L^n = I_n$, $R^{-n} = I_m$, we have $R^{-1}Y L = \bar{Y}$. We claim that

$$\Gamma(I - \bar{Y}^T A) \leq \kappa.$$

Indeed, we have

$$\Gamma(I - \bar{Y}^T A) = \Gamma\left(\frac{1}{n} \sum_{i=1}^n [I - L^{-1}Y^T R^i A]\right)$$

$$\leq \frac{1}{n} \sum_{i=1}^n \Gamma(I - L^{-1}Y^T R^i A) \quad \text{[since $\Gamma$ is convex]}$$

$$= \frac{1}{n} \sum_{i=1}^n \Gamma(L^{-1} \left[I - Y^T [R^i AL^{-1}]\right] L)$$

$$= \frac{1}{n} \sum_{i=1}^n \Gamma(I - Y^T A) \quad \text{[since $\Gamma$ is symmetric and $R^i AL^{-1} = A$]}$$

$$= \Gamma(I - Y^T A)$$

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Now let
\[ y_j(\phi) = \bar{Y}_{0j} + \sum_{i=1}^{d} \bar{Y}_{2i-1,j} \cos(i\phi) + \bar{Y}_{2i,j} \sin(i\phi)]. \]

We have \( R^{-1}\bar{Y} L = \bar{Y} \), that is, \( R^{-1}\bar{Y} = \bar{Y} L^{-1} \). In other words, the columns \( \bar{Y}_j \) of \( \bar{Y} \) satisfy the relation \( \bar{Y}_j = R\bar{Y}_{j-1} \), where \( j = (j - 1) \mod n \). This is nothing but \( y_j(\phi) \equiv y_{j-1}(\phi - \delta) = 2\pi/n, \delta = 2\pi/n, \) whence \( y_j(\phi) = y_0(\phi - j\delta) \). Observe that the \( j \)-th column in \( \bar{Y}^T A \) has the entries
\[ \bar{Y}_i^T A_j = y_i(j\delta) = y_0((j - i)\delta), \quad 0 \leq i \leq n - 1, \]
meaning that the columns in the matrix \( I - \bar{Y}^T A \) are cyclic shifts of each other (so that the \( \| \cdot \|_{s,1} \)-norms of all columns are the same), and the zero column is comprised of the values of the trigonometric polynomial \( 1 - y_0(\phi) \) on the grid \( G = \{ \phi_j = \frac{2\pi j}{n} : 0 \leq j < n \} \). Assuming \( s > 1 \), when denoting by \( \gamma \) the sum of \( s - 1 \) largest magnitudes of entries in the \((n - 1)\)-dimensional vector \( \{y_0(\phi_i)\}_{i=1}^{n-1} \), we have
\[ 1 - y_0(0) + \gamma \leq \|C_0[\bar{Y}, A]\|_{s,1} \leq \kappa < 1, \]
whence \( \mu := y_0(0) > \gamma \). Now let \( M = \max_{0 \leq \phi \leq 2\pi} |y_0(\phi)| \), and let \( \tilde{\phi} \in \text{Argmax}_\phi |y_0(\phi)| \), so that \( y_0(\tilde{\phi}) = 0 \). By Bernstein theorem, we have \( |y_0''(\phi)| \leq d^2 M \) for all \( \phi \), whence \( |y_0(\phi)| \geq M/2 \) when \( |\phi - \tilde{\phi}| \leq 1/d \), so that
\[ \text{Card}\{j : |y_0(\phi_j)| \geq M/2\} > \frac{n}{\pi d} - 1. \]

It follows that \( \gamma \geq \min\{s - 1, \frac{n}{\pi d} - 2\} M/2, \) while \( \mu = y_0(0) \leq M \). Thus, the relation \( \mu > \gamma \) implies that
\[ \min\{s - 1, \frac{n}{\pi d} - 2\} < 2, \]
that is, \( s \leq 2 \) provided that \( n \geq 4\pi d \).

6 Verifiable sufficient conditions for \( s \)-semigoodness by Semidefinite relaxation

Following d’Aspremont and El Ghaoui [4], we are about to derive another verifiable sufficient condition for \( s \)-semigoodness, now via semidefinite relaxation. The construction to follow is motivated by the development in the beginning of Section 4.2, according to which \( s \)-semigoodness of \( A \) is implied by the validity of (27) for \( \theta > 1 \) and \( \xi < 1 \).

Let, as before,
\[ \mathcal{X} = \{x \in \text{Ker} A : \Psi(x) \leq 1\} \quad \text{and} \quad \mathcal{U}^{\xi,\theta} = \{V^{\xi,\theta} u : u \in \mathcal{U}\}, \]
where \( \Psi, \mathcal{U} \) and \( V^{\xi,\theta} \) are defined in, respectively, (24), (25) and (26). The condition (27) is equivalent to
\[ \max_{u,x} \left\{ (V^{\xi,\theta} u)^T x : u \in \mathcal{U}, x \in \mathcal{X} \right\} \leq \xi. \quad (33) \]
Observe that for \( u \in U, x \in X \) the matrices \( U = uu^T, P = ux^T \) and \( X = xx^T \) satisfy the relations

\[
\exists t \in \mathbb{R}^n, V \in \mathbb{S}^{2n}, \Lambda \in \mathbb{S}^{2n} : \\
(34) \quad \begin{bmatrix} U & P \\ P^T & X \end{bmatrix} \succeq 0; \\
(34.a) \quad U = \begin{bmatrix} I_n & -I_n \\ -I_n & V_{11} - V_{12} L^T, \\
0 & 0 \end{bmatrix}, \\
(34.b) \quad 0 \leq V_{ij} \leq \frac{1}{2}, \ V \succeq 0, \ V_{ij}^{12} = [V_{12}]^T, \ \text{Tr}(V) \leq s, \\
(34.c) \quad X = \begin{bmatrix} -I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} \Lambda F^T, \ 0 \leq \Lambda_{ij}, \ \Lambda \succeq 0, \ \sum_{i,j} \Lambda_{ij} \leq 1; \\
(34.d) \quad \sum_{j \in P_+} \max[-P_{ij}, \theta P_{ij}] + \sum_{j \in P_n} |P_{ij}| \leq t_i, \ \forall i \in P_+, \\
(34.e) \quad AXA^T = 0.
\]

Besides this,

\[
u^T(V_{i,j} \theta)^T x = \text{Tr}(V_{i,j} \theta P^T).
\]

The latter relation is evident. Let us verify (34). (34.a) is evident. To verify (34.b), let \( u_+ = \max[u, 0], u_- = \max[-u, 0] \), where max is acting coordinate-wise. Then

\[
U = L \begin{bmatrix} u_+ u_T^T & u_+ u_T \\ u_- u_T^T & u_- u_T \\ u_+ u_T^T & u_- u_T^T \end{bmatrix} L^T = L \begin{bmatrix} u_+ u_T^T & u_- u_T^T \\ u_+ u_T^T & u_- u_T \\ u_+ u_T^T & u_- u_T^T \end{bmatrix} L^T,
\]

and the matrix \( V \) we have just defined clearly satisfies all requirements from (34.b). To verify (34.c), observe that the extreme points of the set \( X^+ = \{ x : \Psi(x) \leq 1 \} \supset X \) are the vectors \( \pm e_i, \ i > p, \) and \( -e_i, \theta^{-1} e_i, \ i \leq p \), so that \( x = F \lambda \) with \( \lambda \in \mathbb{R}_{++}^n, \sum_i \lambda_i \leq 1; \) setting \( \Lambda = \lambda \lambda^T \), we satisfy (34.c). To satisfy (34.d), it suffices to set \( t_i = |u_i| \) for all \( i \) and to take into account that \( \max[-P_{ij}, \theta P_{ij}] \geq |P_{ij}| \) for all \( i, j \) due to \( \theta \geq 1 \), and that \( u_i \geq 0 \) for \( i \in P_+ \). (34.e) is evident.

It follows that a sufficient condition for (33) is

\[
\text{Opt} := \max_{U,P,V,A,X,T} \left\{ \text{Tr}(V_{i,j} \theta P^T) : X, U \in \mathbb{S}^n, \ V, \Lambda \in \mathbb{S}^{2n}, P \in \mathbb{R}^{n \times n}, t \in \mathbb{R}^n \text{ satisfy } (34) \right\} \leq \xi. \ (35)
\]

The optimization problem in (35) clearly reduces to a semidefinite maximization program \( S \); by weak duality, the optimal value in the semidefinite dual \( D \) to \( S \) is \( \geq \text{Opt} \). It follows that the efficiently verifiable condition

\[
\text{Opt}(D) \leq \xi
\]

is a sufficient condition for \( s \)-semigoodness of \( A \). Note that the above construction depends on \( \theta \geq 1 \) and \( \xi < 1 \) as parameters.

Consider the case of \( P_+ = \emptyset \), where \( X = \{ x \in \mathbb{R}^n : \|x\|_1 \leq 1, Ax = 0 \} \supset Z = \{ x \in \mathbb{R}^n : \|x\|_1 \leq 1 \} \). In this case, the standard semidefinite relaxation of the set \( C_* = \text{Conv}\{xx^T : x \in Z \} \) is

\[
C = \left\{ X : X \succeq 0, \sum_{i,j} |X_{ij}| \leq 1 \right\}
\]

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Set (cf. [4]). Note that (34.c) uses another semidefinite relaxation of $C_*$, namely,

$$C' = \left\{ X : \exists \Lambda \in S^{2n} : \Lambda \succeq 0, \Lambda_{i,j} \geq 0 \forall i,j, \sum_{i,j} \Lambda_{i,j} \leq 1 \right\}.$$ 

It is immediately seen that $C_* \subset C' \subset C$; a surprising fact is that the second of these inclusions is strict. Thus, the relaxation of $C_* \mapsto C'$ is less conservative than the standard one $C_* \mapsto C$.

As observed by A. d’Aspremont (private communication), the relaxation $C_* \mapsto C'$ can be further improved, namely, by using instead of $C'$

$$C^+ = \left\{ X : \exists \Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \in S^{2n} : \begin{array}{l} \Lambda^{\mu \nu} \in R^{n \times n}, \Lambda \succeq 0, \Lambda_{i,j} \geq 0 \forall i,j, \sum_{i,j} \Lambda_{i,j} \leq 1 \\ \Lambda_{11} = 0, 1 \leq i \leq n, X = [I_n, -I_n] \Lambda [I_n, -I_n]^T \end{array} \right\}.$$ 

Note that this idea can be used to improve the semidefinite relaxation in the general case as well. Specifically, the matrix $V$ as built in the justification of (34) clearly satisfies $(V^{12})_{ii} = 0, 1 \leq i \leq n$, and we can add these linear constraints on $V$ to (34.b). Similarly, when representing a vector $x \in X^+$ as $F \lambda$ with $\lambda \in R^{2n}$, $\sum \lambda_i \leq 1$, see the justification of (34), we clearly can ensure that $\lambda_i \lambda_{n+i} = 0, 1 \leq i \leq n$, that is, the matrix $\Lambda$ we have built in fact satisfies $\Lambda_{i,n+i} = \Lambda_{n+i,i} = 0, 1 \leq i \leq n$, and we can add these linear constraints on $\Lambda$ to (34.c).

7 Matching Pursuit algorithm

The Matching Pursuit algorithm for signal recovery has been first introduced in [11] and is motivated by the desire to provide a reduced complexity alternative to the $\ell_1$-recovery problem. Several implementations of Matching Pursuit has been proposed in the Compressive Sensing literature (see, e.g., the review [1]). All of them are based on successive Euclidean projections of the signal and the corresponding performance results rely upon the bounds on mutual incoherence $\mu(A)$ of the sensing matrix. We are about to show that the verifiable sufficient conditions from the previous section can be used to construct a specific version of the Matching Pursuit algorithm which we refer to Non-Euclidean Matching Pursuit (NEMP) algorithm.

Suppose that we have in our disposal $\tau, \tau_{\pm} \geq 0$ and a matrix $Y = [y_1, ..., y_n]$, such that

$$(a) \quad -\tau_+ \leq [I - Y^T A]_{ij} \leq \tau_+, \forall i \in P_+, \forall j,$$

$$(b) \quad -\tau \leq [I - Y^T A]_{ij} \leq \tau, \forall i \in P_-, \forall j,$$

$$(c) \quad \|y_j\|_s \leq \sigma, \forall j.$$ 

Consider a signal $w \in P_n$ such that $\|w - w^s\|_1 \leq \mu$, where $w^s$ is the vector obtained from $w$ by replacing all but $s$ largest magnitudes of entries in $w$ with zeros, and let $y$ and $\delta$ be such that $\|Ay - y\| \leq \delta$.

Suppose that

$$\rho = s \max \{\tau_+, \tau_-, \tau\} < 1.$$ 

(37)

To simplify notation, we denote $\max[a, b]$ by $a \vee b$. Consider the following iterative procedure:

Algorithm 7.1

1. **Initialization**: Set $v^{(0)} = 0$, 

$$\alpha_0 = \frac{\|Y^T y\|_{s,1} + s\sigma\delta + \mu}{1 - \rho}.$$ 

2. **Step** $k$, $k = 1, 2, ...$: Given $v^{(k-1)} \in R^n$ and $\alpha_{k-1} \geq 0$, compute
(a) \( u = Y^T(y - Av^{(k-1)}) \) and \( n \) segments

\[
S_i = \begin{cases} 
  [u_i - \tau_\cdot \alpha_{k-1} - \sigma \delta, u_i + \tau_\cdot \alpha_{k-1} + \sigma \delta], & i \in P_+, \\
  [u_i - \tau \cdot \alpha_{k-1} - \sigma \delta, u_i + \tau \cdot \alpha_{k-1} + \sigma \delta], & i \in P_-. 
\end{cases}
\]

Define \( \Delta \in \mathbb{R}^n \) by setting

\[
\Delta_i = \begin{cases} 
  [u_i - \tau \cdot \alpha_{k-1} - \sigma \delta]_+, & i \in P_+, \\
  [u_i - \tau \cdot \alpha_{k-1} - \sigma \delta]_+, & i \in P_n, \ u_i \geq 0, \\
  -[u_i - \tau \cdot \alpha_{k-1} - \sigma \delta]_+, & i \in P_n, \ u_i < 0
\end{cases}
\]

(here \([a]_+ = \max\{0, a\}\).

(b) Set \( v^{(k)} = v^{(k-1)} + \Delta \) and

\[
\alpha_k = s[2\tau_+ (\tau_+ + \tau_\cdot)] \alpha_{k-1} + 2s\sigma \delta + \mu. \tag{38}
\]

and loop to step \( k + 1 \).

3. The approximate solution found after \( k \) iterations is \( v^{(k)} \).

**Proposition 7.1** Assume that \( w_i \geq 0 \) for \( i \in P_+ \), (37) takes place, and that \( \|w - w^*\|_1 \leq \mu \) with a known in advance value of \( \mu \). Then the approximate solution \( v^{(k)} \) and the value \( \alpha_k \) after the \( k \)-th step of Algorithm 7.1 satisfy

\[
\begin{align*}
\text{(a)} & \quad \text{for all } i \ v_i^{(k)} \in \text{Conv}\{0; w_i\} \\
\text{(b)} & \quad \|w - v^{(k)}\|_1 \leq \alpha_k.
\end{align*}
\]

**Proof.** Let us proceed by induction. First, let us show that \( (a_{k-1}, b_{k-1}) \) implies \( (a_k, b_k) \). Thus, assume that \( (a_{k-1}, b_{k-1}) \) holds true. Let \( z^{(k-1)} = w - v^{(k-1)} \). By \( (a_{k-1}), z^{(k-1)} \) is supported on the support of \( w \) and is such that \( z_i^{(k-1)} \geq 0 \) for \( i \in P_+ \). Note that

\[
z^{(k-1)} - u = w - v^{(k-1)} - Y^T(y - Av^{(k-1)}) = (I - Y^T A)(w - v^{(k-1)}) - Y^T \epsilon
\]

\[
= (I - Y^T A)z^{(k-1)} - Y^T \epsilon,
\]

where \( \epsilon = y - Aw \) with \( \|Y^T \epsilon\|_\infty \leq \sigma \delta \) due to (36.c). Then by (36.a,b) for any \( i \in P_+ \),

\[
-\tau_\cdot \left[ \sum_{j \in P_+} z_j^{(k-1)} + \sum_{j \in P_0} |z_j^{(k-1)}| \right] - \sigma \delta \leq z_i^{(k-1)} - u_i \leq \tau_+ \left[ \sum_{j \in P_+} z_j^{(k-1)} + \sum_{j \in P_0} |z_j^{(k-1)}| \right] + \sigma \delta,
\]

whence

\[
-\gamma_- := -\tau_\cdot \alpha_{k-1} - \sigma \delta \leq z_i^{(k-1)} - u_i \leq \gamma_+ := \tau_+ \alpha_{k-1} + \sigma \delta. \tag{39}
\]

We conclude that for any \( i \in P_+ \) the interval \( S_i = [u_i - \gamma_-, u_i + \gamma_+] \) of the width

\[
\ell_+ = [\tau_- + \tau_+ \alpha_{k-1} + 2 \sigma \delta,
\]

covers \( z_i^{(k-1)} \). In the same way for any \( i \in P_0 \)

\[
-\gamma := -\tau_\cdot \alpha_{k-1} - \sigma \delta \leq z_i^{(k-1)} - u_i \leq \tau_\cdot \alpha_{k-1} + \sigma \delta = \gamma,
\]

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so that the interval \( S_i = [u_i - \gamma, u_i + \gamma] \) of the width
\[
\ell = 2\tau \alpha_{k-1} + 2\sigma \delta,
\]
covers \( z_i^{(k-1)} \) when \( i \in P_h \).

Recalling that \( z_i^{(k-1)} \geq 0 \) for \( i \in P_+ \), the closest to 0 point of \( S_i \) is
\[
\begin{align*}
\Delta_i & = [u_i - \gamma]_+ \quad \text{for } i \in P_+, \\
\bar{\Delta}_i & = [u_i - \gamma]_+ \quad \text{for } i \in P_h, \; u_i \geq 0, \\
\Delta_i & = -\|u_i - \gamma\|_+ \quad \text{for } i \in P_h, \; u_i < 0,
\end{align*}
\]
that is, \( \bar{\Delta}_i = \Delta_i \) for all \( i \). Since the segment \( S_i \) covers \( z_i^{(k-1)} \) and \( \Delta_i \) is the closest to 0 point in \( S_i \),
while the width of \( S_i \) is at most \( \ell \vee \ell_+ \), we clearly have
\[
\begin{align*}
(a) & \quad \Delta_i \in \text{Conv} \left\{ 0, z_i^{(k-1)} \right\}, \\
(b) & \quad |z_i^{(k-1)} - \Delta_i| \leq \ell \vee \ell_+.
\end{align*}
\]
Since \( (a_{k-1}) \) is valid, (40.a) implies that
\[
v_i^{(k)} = v_i^{(k-1)} + \Delta_i \in \left[ v_i^{(k-1)} + \text{Conv} \left\{ 0, w - v_i^{(k-1)} \right\} \right] \subseteq \text{Conv} \{0, w_i\},
\]
and \( (a_k) \) holds. Further, let \( I \) be the support of \( w^s \). Relation \( (a_k) \) clearly implies that \( |z_i^{(k)}| \leq |w_i| \),
and we can write due to (40.b):
\[
||w - v^{(k)}||_1 = \sum_{i \in I} |w - [v_i^{(k-1)} + \Delta_i]| + \sum_{i \notin I} |z_i^{(k)}| \leq \sum_{i \in I} |z_i^{(k-1)} - \Delta_i| + \sum_{i \notin I} |w_i| \leq s[\ell \vee \ell_+] + \mu = \alpha_k,
\]
which is \( (b_k) \). The induction step is justified.

It remains to show that \( (a_0, b_0) \) holds true. Since \( (a_0) \) is evident, all we need is to justify \( (b_0) \). Let
\[
\alpha_s = ||w||_1,
\]
and let \( u = Y^T u \). Same as above (cf. (39)), we have for all \( i \):
\[
|u_i - w_i| \leq \max \{ \tau_s, \tau_+, \tau \} \alpha_s + \sigma \delta = \frac{\rho}{s} \alpha_s + \sigma \delta.
\]
Then
\[
\alpha_s = \sum_{i \notin I} |u_i| + \sum_{i \notin I} |u_i| \leq \sum_{i \in I} |u_i| + \frac{\rho}{s} \alpha_s + \sigma \delta + \mu \leq ||u||_{s,1} + \rho \alpha_s + s \sigma \delta + \mu.
\]
Hence
\[
\alpha_s \leq \alpha_0 = \frac{||u||_{s,1} + s \sigma \delta + \mu}{1 - \rho},
\]
which implies \( (b_0) \). \( \square \)

Let
\[
\lambda = s[2\tau \vee (\tau_- + \tau_+)];
\]
if \( \lambda < 1 \), then also \( \rho < 1 \), so that Proposition 7.1 holds true. Furthermore, by (38) the sequence \( \alpha_k \)
converges exponentially fast to the limit \( \alpha_\infty := \frac{2s \sigma + \mu}{1 - \lambda} \):
\[
\alpha_k = \lambda^k [\alpha_0 - \alpha_\infty] + \alpha_\infty.
\]
Note that when \( P_+ = \emptyset \), we can set \( \tau_- = \tau_+ = 0 \) to obtain \( \lambda = 2s\tau \); in the case of \( P_n = \emptyset \), by setting \( \tau = 0 \), we have \( \lambda = s(\tau_- + \tau_+) \).

The bottom line is: if the optimal value in the convex program

\[
\begin{align*}
\text{Opt} = \min_{\tau, \tau_{\pm}, Y} & \left\{ s[2\tau \vee (\tau_- + \tau_+)] : \right. \\
& \left. -\tau_- \leq [I - Y^T A]_{ij} \leq \tau_+, \quad \forall i \in P_+, \forall j \right. \\
& \left. -\tau \leq [I - Y^T A]_{ij} \leq \tau, \quad \forall i \in P_n, \forall j \right. \\
& \tau, \tau_{\pm} \geq 0
\end{align*}
\]

is < 1, the above procedure, as yielded by an optimal solution to the latter problem, possesses the following properties:

1. All approximations \( v^{(k)}, k = 0, 1, \ldots \) of \( w \) are supported on the support of \( w \);
2. For \( i \in P_+ \), \( v^{(k)}_i \geq 0 \) are nondecreasing in \( k \) and are \( \leq w_i \) for all \( k \);
3. For \( i \in P_n \),
   - if \( w_i > 0 \), then \( 0 \leq v^{(k)}_i \leq w_i \) and \( v^{(k)}_i \) are nondecreasing in \( k \);
   - if \( w_i < 0 \), then \( w_i \leq v^{(k)}_i \leq 0 \) and \( v^{(k)}_i \) are nonincreasing in \( k \);
4. As \( k \) grows, the upper bound \( \alpha_k \) on the \( \ell_1 \)-error of approximating \( w \) by \( v^{(k)} \) goes exponentially fast to
   \[ \alpha_\infty = \frac{2s\sigma \delta + \mu}{1 - \text{Opt}}. \]

Let now \( \xi \in [0, 1) \), \( \sigma \geq 0 \) and \( \theta \geq 1 \) and suppose that an \( m \times n \) matrix \( A \) verifies the following condition:

\( \text{VSG}_s(\xi, \sigma, \theta) \): There exists \( m \times n \) matrix \( Y = [y_1, \ldots, y_n] \) such that \( \|y_i\|_s \leq \sigma \) for all \( i \) and

\[
\begin{align*}
-\frac{\xi}{1 + \xi \theta} & \leq [I - Y^T A]_{ij} \leq \frac{\xi}{1 + \xi \theta} \quad \forall i \notin P_+, \forall j, \\
-\frac{\xi}{1 + \xi \theta} & \leq [I - Y^T A]_{ij} \leq 0 \quad \forall i \in P_+, \forall j \notin P_+, \\
-\frac{\xi}{1 + \xi \theta} & \leq 0 \quad \forall i \notin P_+, \forall j \notin P_+.
\end{align*}
\]  

Observe that (41) is a system of convex inequalities in \( Y \). Further, \( \text{VSG}_s(\xi, \sigma, \theta) \) certainly implies \( \text{VSG}_s(\xi, \theta, 0, \sigma) \), and is therefore sufficient condition of the \( s \)-semigoodness of the matrix \( A \).

When \( \text{VSG}_s(\xi, \sigma, \theta) \) is satisfied with \( \xi \in (0, 1) \) and \( \theta > 1 \), by taking

\[
\tau_- = \frac{\xi}{(1 + \xi \theta)s}, \quad \tau_+ = \frac{\xi \theta}{(1 + \xi \theta)s} \quad \text{and} \quad \tau = \frac{\xi}{(1 + \xi)s},
\]

we obtain

\[
\lambda = \max \left( \frac{\xi + \xi \theta}{1 + \xi \theta}, \frac{2\xi}{1 + \xi} \right) < 1. \tag{42}
\]

Combining this condition with Proposition 7.1 gives:

**Corollary 7.1** Suppose that \( A \) satisfies the condition \( \text{VSG}_s(\xi, \sigma, \theta) \) with certain \( \xi \in (0, 1), \sigma \geq 0 \) and \( \theta \geq 1 \). Let \( w \in P_n \) be a vector with \( \|w - w^s\|_1 \leq \mu \) where \( w^s \) is the vector obtained from \( w \) by replacing all but \( s \) largest in magnitude entries in \( w \) with zeros, and let \( y \) be such that \( \|Ay - y\| \leq \delta \). Then the approximate solution \( v^{(t)} \) found by Algorithm 7.1 after \( t \) iterations satisfies \( v^{(t)}_i \geq 0 \) for all \( i \in P_+ \) and

\[
\|w - v^{(t)}\|_1 \leq \frac{2s\sigma \delta + \mu}{1 - \lambda} + \lambda^t \left[ \frac{\|Y^T y\|_{s,1} + s\sigma \delta + \mu}{1 - \rho} - \frac{2s\sigma \delta + \mu}{1 - \lambda} \right],
\]

where \( \lambda \) is given by (42) and \( \rho = \frac{s \theta}{1 + \xi \theta} \).
It should be noted the NEMP algorithm has several drawbacks as compared with the $\ell_1$-recovery. First, the pursuit algorithm requires a priori knowledge of several parameters ($\sigma, Y, \tau, \tau_-, \tau_+, s$ and $\mu$). Second, the value $(1 - \lambda)^{-1}(2s\sigma\delta + \mu)$ is a conservative upper bound on the error of the $\ell_1$-recovery, but the error bound in Corollary 7.1 is exact. On the other hand, the NEMP algorithm can be an interesting option if the $\ell_1$-recovery is to be used repeatedly on the observations obtained with the same sensing matrix $A$; the numerical complexity of the pursuit algorithm for a given matrix $A$ may only be a fraction of that of the $\ell_1$-recovery, especially when used on high-dimensional data.

Our concluding remark is on the condition

$$\frac{\mu(A)}{1 + \mu(A)} < \frac{1}{2s}, \quad (43)$$

where $\mu(A)$ is the mutual incoherence of $A$ (see (6)). This condition is usually used in order to establish convergence results for the Matching Pursuit algorithms (see, e.g. [8, 9, 2]). As it is immediately seen, when $\mu(A)$ is well defined (i.e., all columns in $A$ are nonzero), the matrix $Y = [y_1, ..., y_n]$ with the columns

$$y_i = \frac{A_i}{(1 + \mu(A))A_i^TA_i}$$

satisfies for all $i = 1, ..., m$ and $j = 1, ..., n$ the relations

$$|[I - Y^TA]_{ij}| \leq \frac{\mu(A)}{1 + \mu(A)}.$$

In the case of (43), setting $\theta = 1$ and specifying $\xi$ from the relation $\frac{\xi}{1 + \xi} = \frac{\sigma_1(A)}{1 + \mu(A)}$, we get $0 < \xi < 1$ meet all inequalities in (41). It follows that $Y$ certifies the validity of the condition $\text{VSG}_s(\xi, \sigma, 1)$ with the outlined $\xi$ and with all $\sigma \geq \max_i \left\| \frac{A_i}{(1 + \mu(A))A_i^2} \right\|_2$, and thus the above $Y$ can be readily used in Matching Pursuit. Note that in the situation in question Corollary 7.1 recovers some results from [8, 9, 2].

References


A Proof of Proposition 2.1

(i)$\Rightarrow$(ii): Let $A$ be $s$-semigood, and let, in contrast to what is stated by (ii), $J$ be a subset of $\{1, ..., n\}$ with $\text{Card}(J) \leq s$ and $x \in \text{Ker}A \setminus \{0\}$ be such that $x_i \leq 0$ for all $i \in P_+ \setminus J$ and

$$\sum_{i \in J \cap P_+} x_i + \sum_{i \notin J} |x_i| \geq \sum_{i \notin J \cap P_+} |x_i|.$$  

Let $I = (J \cap P_+ \cup \{i \in J \cap P_+: x_i \geq 0\}$ so that $I \subseteq J$. From the construction of $I$, we have $x_i \leq 0$ for $i \in J \setminus I$ implying that $x_i \leq 0$ for $i \in P_+ \setminus I$. Further,

$$\sum_{i \in I \cap P_+} x_i + \sum_{i \notin I \cap P_+} |x_i| = \sum_{i \in J \cap P_+} x_i - \sum_{i \notin J \cap P_+} x_i + \sum_{i \in J \cap P_+} |x_i|$$

$$\geq \sum_{i \notin J \cap P_+} |x_i| - \sum_{i \notin J \cap P_+} x_i$$

$$= \sum_{i \notin J \cap P_+} |x_i| + \sum_{i \notin J \cap P_+} |x_i| = \sum_{i \notin I \cap P_+} |x_i|.$$  

Hence $I$ also violates the condition in (ii). Setting $u_i = x_i$ when $i \in I$ and $u_i = 0$ otherwise and setting $v = u - x$, we have $u_i \geq 0$ for any $i \in I \cap P_+$, $u_i = 0$ for any $i \in P_+ \setminus I$, and $v_i \geq 0$ for $i \in P_+ \setminus I$, $v_i = 0$ for $i \in I \cap P_+$ and $\sum_i |u_i| \geq \sum_i |v_i|$. In addition, $Au = Av$ due to $Ax = 0$, and $u$ is $s$-sparse; finally, $u \neq v$ due to $x \neq 0$. We see that the $s$-sparse vector $u \in P_n$ is not the unique solution to

$$\min \left\{ \sum_i |z_i| : Az = Au, \quad z_i \geq 0 \ \forall i \in P_+ \right\},$$

which is a desired contradiction.
(ii)⇒(iii): Let $A$ satisfy (ii). Let $J$ be the family of all subsets $J$ of $\{1,\ldots,n\}$ of cardinality $\leq s$. For $J \in J$, let

$$X_J = \{x \in \text{Ker}A : \|x\|_1 = 1, \; x_i \leq 0 \; \forall i \in P_+ \setminus J\}.$$ 

Assuming that $X_J \neq \emptyset$, let $x \in X_J$. By (ii), we have

$$\sum_{i \in J \cap P_+} x_i + \sum_{i \notin J} |x_i| < \sum_{i \in J} |x_i|.$$ 

We claim that $\sum_{i \notin J} |x_i| > 0$.

Indeed, otherwise $x_i \neq 0$ implies that $i \in J$. Let $I_+$ and $I_-$ be the subsets of $J$ such that $x_i > 0$ for $i \in I_+$ and $x_i < 0$ for $i \in I_-$. At least one of these sets is nonempty due to $x \neq 0$. W.l.o.g. we can assume that $\sum_{i \in I_+} x_i \geq \sum_{i \in I_-} |x_i|$ (otherwise we could replace $x$ with $-x$ and swap $I_+$ and $I_-$. Applying (ii) to $x$ and to $I_+$ in the role of $J$, we should have

$$\sum_{i \in I_+ \cap P_+} x_i + \sum_{i \notin I_+} |x_i| = \sum_{i \in I_+} x_i < \sum_{i \in I_-} |x_i| = \sum_{i \notin I_+} |x_i|,$$

which is not the case. This contradiction shows that $\sum_{i \notin J} |x_i| > 0$ whenever $x \in X_J$.

From our claim it follows that the function

$$\frac{\sum_{i \in J \cap P_+} x_i + \sum_{i \notin J} |x_i|}{\sum_{i \notin J} |x_i|}$$

is continuous on $X_J$ and is $\leq 1$ at every point of this set. Since $X_J$ is compact, we conclude that when $J \in J$ is such that $X_J \neq \emptyset$, there exists $\xi_J < 1$ such that

$$\sum_{i \in J \cap P_+} x_i + \sum_{i \notin J} |x_i| \leq \xi_J \sum_{i \notin J} |x_i|$$

for any $x \in X_J$.

Setting $\xi = \max_{J \in J \cap J \neq \emptyset} \xi_J$, we clearly ensure the validity of (iii). The implication (ii)⇒(iii) is proved.

(iii)⇒(i): Let (iii) take place; let us prove that $A$ is $s$-semigood. Thus, let $u$ with $u_i \geq 0$ for all $i \in P_+$ be $s$-sparse; we should prove that $u$ is the unique optimal solution to the problem

$$\min_z \left\{ \sum_i z_i : Az = Au, \; z_i \geq 0 \; \forall i \in P_+ \right\}.$$ 

Assume, on the contrary to what should be proved, that the latter problem has an optimal solution $v$ different from $u$, and let $x = u - v$, so that $x \in \text{Ker}A$ and $x \neq 0$. Setting $I = \{i : u_i \neq 0\}$, we have $\text{Card}(I) \leq s$ and $x_i \leq 0$ when $i \in P_+ \setminus I$, whence by (iii)

$$\sum_{i \in I \cap P_+} x_i + \sum_{i \notin I} |x_i| \leq \xi \sum_{i \notin I} |x_i| = \xi \sum_{i \notin I} |v_i|,$$

whence also

$$\sum_{i \in I \cap P_+} u_i + \sum_{i \notin I} |u_i| \leq \sum_{i \in I \cap P_+} v_i + \sum_{i \notin I} |v_i| + \xi \sum_{i \notin I} |v_i|. \tag{44}$$

Since $\sum_i |v_i| \leq \sum_i |u_i| = \sum_{i \notin I} |u_i|$ due to the origin of $v$, (44) implies that $\sum_{i \notin I} |v_i| = 0$, that is, both $u$ and $v$ are supported on $I$, so that $x$ is supported on $I$ as well. Now let $I_+ = \{i \in I \cap P_+ : x_i \geq 0\}$,
Let $P = I \cap P_+$. Replacing, if necessary, $x$ with $-x$ and swapping $I_+$ and $I_-$, we can assume that $\sum_{i \in I_+} x_i = \sum_{i \in I_-} x_i \geq \sum_{i \in I_+} x_i \geq |x_i|$. Applying (iii) to $x$ and to $I_+ \cup I_-$ in the role of $J$, we get

$$\sum_{i \in I_+} x_i + \sum_{i \in I_-} |x_i| \leq \xi \sum_{i \in I_-} |x_i|,$$

whence $\sum_{i \in I_+} x_i = \sum_{i \in I_-} x_i = \sum_{i \in I_-} |x_i| = 0$ due to $\sum_{i \in I_+} x_i \geq \sum_{i \in I_-} |x_i|$. Thus, $x = 0$, which is a desired contradiction.

We have proved that the properties (i) – (iii) of $A$ are equivalent to each other.

(iii)⇔(iv): The implication (iv)⇒(iii) is evident. Let us prove the inverse implication. Thus, let $A$ satisfy (iii) (and thus – (i) – (ii) as well), and let $\xi' \in (\xi, 1)$. Let, as above, $\mathcal{J}$ be the family of all subsets $J$ of $\{1, \ldots, n\}$ of cardinality $\leq s$. Let $X = \{x \in \text{Ker}A : \|x\|_1 = 1\}$, and let $J \in \mathcal{J}$. Let $x \in X$. We claim that there exists a neighborhood $U_x$ of $x$ in $X$ and $\theta_{J,x} \in [1, \infty)$ such that for any $u \in U_x$ and $\theta \geq \theta_{J,x}$ it holds

$$\sum_{i \in J \cap P_+} u_i + \sum_{i \in J \cap P_-} |u_i| \leq \xi' \left( \sum_{i \in P_-, J} |u_i| + \sum_{i \in P_+, J} \max[-u_i, \theta u_i] \right).$$

(45)

The claim is clearly true when there exists $i \in P_+ \setminus J$ such that $x_i > 0$. Now assume that $x_i \leq 0$ for $i \in P_+ \setminus J$. Then $\sum_{i \in J} |x_i| > 0$. Indeed, otherwise $x_i = 0$ for all $i \notin J$, which combines with $s$-semigoodness of $A$ and the relation $Ax = 0$ to imply that $x = 0$ (since assuming $x \neq 0$, we have $x = u - v$ with $s$-sparse $u \geq 0, v \geq 0$ with non-overlapping supports, and $Au = Av$ due to $Ax = 0$, which of course contradicts the $s$-semigoodness of $A$), while $x$ definitely is nonzero (since $\|x\|_1 = 1$ due to $x \in X$). Now, since $x \in \text{Ker}A$ and $x_i \leq 0, i \in P_+ \setminus J$, we have

$$\sum_{i \in J \cap P_+} x_i + \sum_{i \in J \cap P_-} |x_i| \leq \xi \sum_{i \in J \cap P_+} |x_i| < \xi' \sum_{i \notin J} |x_i|$$

where the first inequality is due to (iii), and the second – due to $\sum_{i \notin J} |x_i| > 0$. The concluding strict inequality clearly implies the validity of (45) with $\theta = 1$, provided that $U_x$ is a small enough neighborhood of $x$. Thus, our claim is true.

From the validity of our claim, extracting from the covering $\{U_x\}_{x \in X}$ of the compact set $X$ a finite subcovering, we conclude that there exists $\theta_{J} \in [1, \infty)$ such that

$$\forall (x \in X, \theta \geq \theta_{J}): \sum_{i \in J \cap P_+} x_i + \sum_{i \in J \cap P_-} |x_i| \leq \xi' \left( \sum_{i \in P_-, J} |x_i| + \sum_{i \in P_+, J} \max[-x_i, \theta x_i] \right).$$

Setting $\theta = \max_{J \in \mathcal{J}} \theta_{J}$, we see that $A$ satisfies $\text{SG}_s(\xi', \theta)$.

(iv)⇒(v): Let $A$ satisfy $\text{SG}_s(\xi, \theta)$ for certain $\xi \in (0, 1), \theta \in [1, \infty)$ and let $\|\cdot\|$ be a norm on $\mathbb{R}^m$. Let, further, $P$ be the orthogonal projector of $\mathbb{R}^n$ on $\text{Ker}A$. Then clearly with a properly chosen $C$ one has

$$\|Px - x\|_1 \leq C\|Ax\|,$$
for any \( x \in \mathbb{R}^n \). Now let \( J \) be a subset of \( \{1, \ldots, n\} \) of cardinality \( s \leq s, x \in \mathbb{R}^n \) and \( u = Px \). We have

\[
\sum_{i \in J \cap P_+} x_i + \sum_{i \in J \cap P_n} |x_i| \leq \sum_{i \in J \cap P_+} u_i + \sum_{i \in J \cap P_n} |u_i| + \sum_{i \in J} |u_i - x_i|
\]

\[
\leq \xi \left[ \sum_{i \in P_+ \setminus J} |u_i| + \sum_{i \in P_+ \setminus J} \max[-u_i, \theta u_i] \right] + \sum_{i \in J} |u_i - x_i|
\]

\[
\leq \xi \left[ \sum_{i \in P_+ \setminus J} |x_i| + \sum_{i \in P_+ \setminus J} \max[-x_i, \theta x_i] \right] + \sum_{i \in J} |u_i - x_i|
\]

\[
\leq \xi \left[ \sum_{i \in P_+ \setminus J} |x_i| + \sum_{i \in P_+ \setminus J} \max[-x_i, \theta x_i] \right] + \max[1, \theta \xi] \|x - u\|_1
\]

\[
\leq \xi \left[ \sum_{i \in P_+ \setminus J} |x_i| + \sum_{i \in P_+ \setminus J} \max[-x_i, \theta x_i] \right] + \max[1, \theta \xi] C \|Ax\|,
\]

so that \( A \) satisfies \( \text{SG}_{s, \beta}(\xi, \theta) \) with \( \beta = \max(1, \theta \xi)C \). The implication (iv) \( \Rightarrow \) (v) is proved.

(v) \( \Rightarrow \) (vi) \( \Rightarrow \) (iii): These implications are evident. \( \square \)

**B Proof of Proposition 4.2**

Let \( Y = [Y_1, \ldots, Y_n], v, \sigma, \rho \) certify the validity of \( \text{VSG}_{s, \beta}^*(\xi, \theta) \), and let \( \beta' \geq \beta, \theta' \geq \theta \) and \( \xi' \in [\xi, 1) \). Let us set

\[
\lambda = \frac{1 + \theta \xi}{1 + \theta' \xi'}, \quad \mu = \frac{1 + \xi}{1 + \xi'},
\]

so that \( \lambda, \mu \in [0, 1] \), and let \( Y' \) be as in the assertion to be proved, that is, the columns of \( Y' \) are multiples of those of \( Y \): \( Y'_i = \lambda Y_i \) when \( i \in P_+ \) and \( Y'_i = \mu Y_i \) otherwise. All we need to prove is that \( (Y', v, \sigma, \rho) \) certify the validity of \( \text{VSG}_{s, \beta'}^*(\xi', \theta') \), and this immediately reduces to verification of the following fact:

**Lemma B.1** Let \( i, 1 \leq i \leq n, \) be fixed, and let \( z \in \mathbb{R}^n \) for any \( I \subset \{1, \ldots, n\} \) of cardinality \( s \) satisfy the relations

\[
(a) \quad (1 + \theta \xi) \sum_{j \in P_+ \cap I} \max[z_j - \delta_{ij}, 0] + (1 + \xi) \sum_{j \in P_+ \cap I} |z_j - \delta_{ij}| + (Av)_i \leq \xi,
\]

\[
(b) \quad (1 + \theta \xi) \sum_{j \in P_+ \cap I} \max[\delta_{ij} - z_j, 0] + (1 + \xi) \sum_{j \in P_+ \cap I} |\delta_{ij} - z_j| - (Av)_i \leq \eta
\]

where \( \delta_{ij} = \begin{cases} 0, & j \neq i, \\ 1, & i = j. \end{cases} \)

Then for every set \( I \subset \{1, \ldots, n\} \) of cardinality \( s \) we have

\[
(a) \quad (1 + \theta' \xi') \sum_{j \in P_+ \cap I} \max[\lambda z_j - \delta_{ij}, 0] + (1 + \xi') \sum_{j \in P_+ \cap I} |\lambda z_j - \delta_{ij}| + (Av)_i \leq \xi',
\]

\[
(b) \quad (1 + \theta' \xi') \sum_{j \in P_+ \cap I} \max[\delta_{ij} - \lambda z_j, 0] + (1 + \xi') \sum_{j \in P_+ \cap I} |\delta_{ij} - \lambda z_j| - (Av)_i \leq \eta'
\]

where \( \eta' = \begin{cases} \theta' \xi', & i \in P_+, \\ \xi', & i \in P_n. \end{cases} \)

**Proof.** Taking into account the definition of \( \lambda, \mu, \) in the case of \( i \notin I \) the relations (47) are readily given by (46), hence we can assume \( i \in I \). Consider two possible cases: \( i \in P_+ \cap I \) and \( i \in P_n \cap I \).
The case of $i \in P_+ \cap I$. In this case (46) reads:

\[(a) \quad (1 + \theta \xi) \max[z_i - 1, 0] + (1 + \theta \xi) \sum_{j \in P_+ \cap I, j \neq i} \max[z_j, 0] + (1 + \xi) \sum_{j \in P_+ \cap I, j \neq i} |z_j| + (Av)_i \leq \xi,\]

\[(b) \quad (1 + \theta \xi) \max[-z_i, 0] + (1 + \theta \xi) \sum_{j \in P_+ \cap I, j \neq i} \max[-z_j, 0] + (1 + \xi) \sum_{j \in P_+ \cap I, j \neq i} |z_j| - (Av)_i \leq \theta \xi,\]

and our goal is to verify that then

\[(a) \quad (1 + \theta' \xi') \max[\lambda z_i - 1, 0] + (1 + \theta' \xi') \sum_{j \in P_+ \cap I, j \neq i} \max[z_j, 0] + (1 + \xi') \sum_{j \in P_+ \cap I, j \neq i} |z_j| + (Av)_i \leq \xi',\]

\[(b) \quad (1 + \theta' \xi') \max[1 - \lambda z_i, 0] + (1 + \theta' \xi') \sum_{j \in P_+ \cap I, j \neq i} \max[-z_j, 0] + (1 + \xi) \sum_{j \in P_+ \cap I, j \neq i} |z_j| - (Av)_i \leq \theta' \xi'.\]

We have $\lambda z_i - 1 \leq \lambda (z_i - 1)$ due to $\lambda \leq 1$, whence

$$\max[\lambda z_i - 1, 0] \leq \max[\lambda (z_i - 1), 0] = \lambda \max[z_i - 1, 0],$$

and therefore (49.a) follows from (48.a) due to $(1 + \theta' \xi') \lambda = 1 + \theta \xi$ and $\xi' \geq \xi$. It remains to verify (49.b). Assume, first, that $\lambda z_i \leq 1$. From (48.b) it follows that

$$(1 + \theta \xi)[1 - z_i] + R \leq (1 + \theta \xi) \max[1 - z_i, 0] + R \leq \theta \xi,$$

whence $z_i \geq \frac{1+R}{1+\theta \xi}$ and therefore

$$1 - \lambda z_i \leq 1 - \frac{1+R}{1+\theta \xi} = \frac{\theta' \xi' - R}{1+\theta' \xi'}.$$

Since we are in the case $1 - \lambda z_i \geq 0$, we arrive at

$$\left(1 + \theta' \xi'\right) \max[1 - \lambda z_i, 0] + R = \left(1 + \theta' \xi'\right)[1 - \lambda z_i] + R \leq (1 + \theta' \xi') \frac{\theta' \xi' - R}{1+\theta' \xi'} + R = \theta' \xi',$$

as required in (49.b). The case of $1 - \lambda z_i \leq 0$ is trivial, since here the left hand side in (49.b) clearly is $\leq$ the left hand side in (48.b), while $\theta' \xi' \geq \theta \xi$, so that (49.b) is readily given by (48.b). Thus, when $i \in P_+ \cap I$, (49) follows from (48).

The case of $i \in P_0 \cap I$. In this case (46) means that

\[(a) \quad (1 + \theta \xi) \sum_{j \in P_0 \cap I, j \neq i} \max[z_j, 0] + (1 + \xi)[1 - z_i] + (1 + \xi) \sum_{j \in P_0 \cap I, j \neq i} |z_j| + (Av)_i \leq \xi,\]

\[(b) \quad (1 + \theta \xi) \sum_{j \in P_0 \cap I} \max[-z_j, 0] + (1 + \xi)[1 - z_i] + (1 + \xi) \sum_{j \in P_0 \cap I} |z_j| - (Av)_i \leq \xi,\]

and our goal is to verify that then

\[(a) \quad (1 + \theta' \xi') \sum_{j \in P_0 \cap I, j \neq i} \max[\lambda z_j, 0] + (1 + \xi)[1 - \mu z_i] + (1 + \xi') \mu \sum_{j \in P_0 \cap I, j \neq i} |z_j| + (Av)_i \leq \xi',\]

\[(b) \quad (1 + \theta' \xi') \sum_{j \in P_0 \cap I} \max[-\lambda z_j, 0] + (1 + \xi)[1 - \mu z_i] + (1 + \xi') \mu \sum_{j \in P_0 \cap I, j \neq i} |\mu z_j| - (Av)_i \leq \xi'.\]
Comparing (50.a) with (51.a), and (50.b) with (51.b), we see that all we need in order to derive (51) from (50) is to verify the following statement: if \((1 + \xi)|1 - z| \leq \xi + a\), then \((1 + \xi')|1 - \mu z| \leq \xi' + a\). This is immediate: assuming \((1 + \xi)|1 - z| \leq \xi + a\), the premises in the following two implication chains hold true:

\[
(1 + \xi)[1 - z] \leq \xi + a \Rightarrow z \geq \frac{1 - a}{1 + \xi} \Rightarrow \mu z \geq \frac{1 - a}{1 + \xi} \Rightarrow 1 - \mu z \leq 1 - \frac{1 - a}{1 + \xi} = \frac{\xi' + a}{1 + \xi}
\]

\[
(1 + \xi')[1 - \mu z] \leq \xi' + a,
\]

\[
(1 + \xi)[z - 1] \leq \xi + a \Rightarrow z \leq 1 + \frac{\xi + a}{1 + \xi} \Rightarrow \mu z \leq \frac{1 + 2\xi + a}{1 + \xi} \Rightarrow \mu z - 1 \leq \frac{2\xi - \xi' + a}{1 + \xi}
\]

\[
(1 + \xi')[\mu z - 1] \leq 2\xi - \xi' + a \Rightarrow (1 + \xi')[\mu z - 1] \leq \xi' + a,
\]

while the resulting inequalities in these chains lead to the desired conclusion \((1 + \xi')|1 - \mu z| \leq \xi' + a\). □