Utility maximization in incomplete markets with default

Thomas Lim, Marie-Claire Quenez

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Thomas LIM * Marie-Claire QUENEZ†

Abstract

We address the expected utility maximization from terminal wealth. The special feature of this paper is that we consider a financial market with a stock exposed to a counterparty risk inducing a jump in the price, and which can still be traded after this default time. We use a default-density modeling approach. Using dynamic programming, we characterize the value function with a backward stochastic differential equation and the optimal portfolio policies. We separately treat the cases of exponential, power and logarithmic utility functions. We define the indifference price of a contingent claim and we study in particular the indifference price for the exponential utility function. We also generalize the results to case of several default times and to case of Poisson jumps.

Keywords Counterparty risk, density of default time, optimal investment, dynamic programming, backward stochastic differential equation, indifference pricing.

1 Introduction

We consider an incomplete financial model with one bond and one risky asset. The price process $(S_t)_{0\leq t\leq T}$ of the risky asset is assumed to be a local martingale driven by a Brownian motion and a default indicating process. In such a context, we solve the portfolio optimization problem when an investor wants to maximize the expectation of his utility from terminal wealth.

The utility maximization problem has been largely studied in the literature. Originally introduced by Merton (1971) in the context of constant coefficients and treated by markovian methods via Bellman equation of dynamic programming, it was developed for general process by martingal duality approach by Kramkov and Schachermayer (1999). For the case of complete markets, we refer to Karatzas *et al.* (1987), Cox and Huang (1989). For the case of incomplete and/or constrained markets, we refer to Karatzas *et al.* (1991), He and Pearson (1991) and Cvitanič and Karatzas (1992). Lukas (2001) considers the case of incomplete markets with a default in the markovian case. In contrast to these papers, in Hu *et al.* (2004), the authors do not use the duality approach, and they directly characterize the solution of the primal problem as the solution of a backward stochastic differential equation

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(BSDE) by using a verification theorem of the same spirit as El Karoui *et al.* (1997). Since they work in a Brownian filtration, they can use directly some results on quadratic BSDEs (see Kobylanski (2000)). For the case of a discontinuous framework, we refer to Morlais (2008). She supposes that the price process of stocks is modeled by a local martingale driven by an independent one dimensional Brownian motion and a Poisson point process. Using the same approach as in Hu *et al.* (2004), she obtains formally a BSDE for which there is no existence and uniqueness results. She proves the existence of a solution of this BSDE by using an approximation method but she does not obtain uniqueness result, which does not allow to characterize the value function as the solution of a BSDE. To be able to characterize completely the optimization problem, she restricts the admissible portfolios set to a compact set so that in this case the value function can be proved to be the unique solution of a BSDE.

The utility maximization problem is also used for the pricing in the incomplete markets, we refer to Hodges and Neuberger (1989) for the case of Brownian filtration or Bielecki and Jeanblanc (2008) for the case of a discontinuous filtration in which the authors compare the optimal strategy and the indifference price depending on the filtrations.

In this paper, we use dynamic programming technics to show directly that the value function is solution of a quadratic BSDE. This method allows to derive in a simpler way the results stated in Morlais (2008) and to improve some of them.

The outline of this paper is organized as follows. In Section 2, we present the market model and the maximization problem. In Section 3, we carry out the calculation of the value function and an optimal strategy for exponential utility and in Section 4, we study the indifference price for a contingent claim using the results of Section 3. In Section 5, we consider the case of a logarithmic utility, and in Section 6 we treat the power utility to complete the spectrum of important utility functions. In the final section, we generalize the results of Section 3 to the case of several assets and default times. At last, we extend the previous results to the case of Poisson jumps.

2 The market model

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space. We assume throughout that all stochastic processes are defined on a finite time horizon $[0, T]$. Suppose that this space is equipped with two stochastic processes : a unidimensional standard Brownian motion $(W_t)_{0\leq t\leq T}$ and a jump process $(N_t = \mathbb{1}_{\tau\leq t})_{0\leq t\leq T}$ where τ is a default time, we assume that $\mathbb{P}(\tau > t) > 0$ for all $t \in [0, T]$, that implies that the default can appear at any time. We denote by $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ the completed filtration generated by these processes :

$$\mathcal{G}_t = \sigma(W_s, 0 \leq s \leq t) \vee \sigma(N_s, 0 \leq s \leq t) \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the class of negligible subsets of Ω.

We denote by (M_t) the compensated martingale of the process (N_t) and by (Λ_t) its compensator. We assume that the compensator (Λ_t) is absolutely continuous with the

Lebesgue measure, so that there exists a process (λ_t) such that $\Lambda_t = \int_0^t \lambda_s ds$. We have that

$$M_t = N_t - \int_0^t \lambda_s ds. \tag{2.1}$$

is a $\mathbb{G}$-martingale. It should be noted that the construction of such process (N_t) is fairly standard; see, for example, Bielecki and Rutkowski (2004).

We introduce the classical sets $S^{+,\infty}$, $L^2(W)$, $L^2_{loc}(W)$, $L^2(M)$ and $L^2_{loc}(M)$:

- $S^{+,\infty}$ is the set of positive càd-làg $\mathbb{G}$-adapted $\mathbb{P}$-essentially bounded processes on $[0,T]$.
- $L^2(W)$ (resp. $L^2_{loc}(W)$) is the set of $\mathbb{G}$-predictable processes on $[0,T]$ under $\mathbb{P}$ with

$$E\left[\int_0^T |Z_t|^2 dt\right] < \infty. \quad \text{(resp. } \int_0^T |Z_t|^2 dt < \infty \ \ a.s. \text{).}$$

- $L^2(M)$ (resp. $L^2_{loc}(M)$) is the set of $\mathbb{G}$-predictable processes on $[0,T]$ under $\mathbb{P}$ with

$$E\left[\int_0^T |U_t|^2 \lambda_t dt\right] < \infty \ \text{ (resp. } \int_0^T |U_t|^2 \lambda_t dt < \infty \ \ a.s. \text{).}$$

We recall the useful martingale representation theorem (see Kusuoka (1999))

Proposition 2.1. *Let m be any $(\mathbb{P},\mathbb{G})$-(resp. locally) square integrable (resp. local) martingale with $m_0 = 0$. Then, there exist two valued $\mathbb{G}$-predictable processes ϕ and ψ such that $\phi \in L^2(W)$ and $\psi \in L^2(M)$ (resp. $\phi \in L^2_{loc}(W)$ and $\psi \in L^2_{loc}(M)$) and*

$$m_t = \int_0^t \phi_s dW_s + \int_0^t \psi_s dM_s, \ 0 \le t \le T.$$

We consider a financial market which consists of one risk-free asset, whose price process is assumed for simplicity to be equal to 1 at each date, and one risky asset with price process S which admits a jump at time τ. In the following, we consider that the price process S evolves according to the equation :

$$dS_t = S_{t^-}(\mu_t dt + \sigma_t dW_t + \beta_t dN_t) \tag{2.2}$$

with the classical assumptions :

Assumption 2.1. *(i) (μ_t), (σ_t) and (β_t) are $\mathbb{G}$-predictable stochastic processes.*

(ii) The process (β_t) satisfied $\beta_t(\omega) > -1$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$ (this assumption implies that the process S is positive).

A $\mathbb{G}$-predictable process $\pi = (\pi_t)_{0\le t\le T}$ is called trading strategy if $\int_0^T \frac{\pi_t}{S_{t-}} dS_t$ is well defined, e.g. $\int_0^T |\pi_t\sigma_t|^2 dt < \infty$ $\mathbb{P}$-a.s. and $\int_0^T |\pi_t\beta_t|^2\lambda_t dt < \infty$ $\mathbb{P}$-a.s. The process $(\pi_t)_{0\le t\le T}$ describes the amount of money invested in the risky asset S at time t. The wealth process $(X_t^{x,\pi})$ of a trading strategy π with initial capital x satisfies the equation :

$$X_t^{x,\pi} = x + \int_0^t \frac{\pi_s}{S_{s-}} dS_s$$

and under the assumption that the trading strategy is self-financing, we get :

$$dX_t^{x,\pi} = \pi_t(\mu_t dt + \sigma_t dW_t + \beta_t dN_t). \tag{2.3}$$

For a given initial time t and an initial capital x, the associated wealth process is denoted by $X_s^{t,x,\pi}$.

Our aim is to study the classical optimization problem

$$V(x) = \sup_{\pi\in\mathcal{A}} E\left[U(X_T^{x,\pi})\right]. \tag{2.4}$$

where U is a utility function and $\mathcal{A}$ is the admissible portfolios set, which will be specified in the following.

In the rest of this paper, we will give a characterization of the value function $V(x)$ and of the optimal strategy.

3 Exponential utility

In this section, we specify the sense of optimality for trading strategies by stipulating that the investor wants to maximize his expected utility from his terminal wealth in the case of an exponential utility function. Let us recall that the exponential utility function is defined as :

$$U(x) = -\exp(-\gamma x), \quad x \in \mathbb{R}$$

where $\gamma > 0$ is a given constant, which can be seen as a coefficient of absolute risk aversion.

In the following, we want to maximize the expectation of utility from terminal wealth over a set of admissible strategies defined by

Definition 3.1. *The set of admissible trading strategies $\mathcal{A}$ consists of all $\mathbb{G}$-predictable processes $\pi = (\pi_t)_{0\le t\le T}$ which satisfy $\int_0^T |\pi_t\sigma_t|^2 dt < \infty$ a.s. and $\int_0^T |\pi_t\beta_t|^2\lambda_t dt < \infty$ a.s. and such that there exists a constant K_π such that $X_t^{0,\pi} \ge K_\pi$ for all $t \in [0,T]$.*

Note that the amount π_t^0 invested in the risk-free asset does not need to be specified since it is determined by the amount π_t invested in the risky asset and the wealth $X_t^{x,\pi}$ through the equation $\pi_t^0 = X_t^{x,\pi} - \pi_t$.

We assume that the investor in this financial market faces some liability, which we model by a random variable ξ (for example, ξ may be a contingent claim written on a default event, which itself affects the price of the underlying asset). We suppose that

$\xi \in \mathcal{G}_T$ and is non-negative (note that all the results still hold under the assumption that ξ is only lower bounded). Our first goal is to solve the optimization problem for an agent who buys a contingent claim ξ. Then, in Section 4, we will study the indifference price of this contingent claim. To this end it suffices to find a strategy that maximizes the value function

$$V(x,\xi) = \sup_{\pi\in\mathcal{A}} E\left[-\exp(-\gamma\left(X_T^{x,\pi}+\xi\right))\right], \quad \gamma>0, \tag{3.5}$$

which can be clearly written as

$$V(x,\xi) = -e^{-\gamma x}J_0,$$

where $J_0 = \inf_{\pi\in\mathcal{A}} E\left[\exp\left(-\gamma\left(X_T^{0,\pi}+\xi\right)\right)\right]$.

Let us fix $\xi \in \mathcal{G}_T$ a non-negative contingent claim. To solve this optimization problem, we give a dynamic extension of the initial problem. For each initial time $t\in[0,T]$ we define the value function $J(t)$ by the following random variable :

$$J(t) = \operatorname*{ess\,inf}_{\pi\in\mathcal{A}_t} E\left[\exp(-\gamma(X_T^{t,0,\pi}+\xi))\middle|\,\mathcal{G}_t\right] \tag{3.6}$$

where the set $\mathcal{A}_t$ consists of all $\mathbb{G}$-predictable processes $\pi=(\pi_s)_{t\le s\le T}$ which satisfy $\int_t^T|\pi_s\sigma_s|^2ds<\infty$ a.s. and $\int_t^T|\pi_s\beta_s|^2\lambda_s ds<\infty$ a.s. and such that there exists a constant K_π such that $X_s^{t,0,\pi}\ge K_\pi$ for all $s\in[t,T]$. For the sake of brevity, we shall denote now X_s^π (resp. $X_s^{t,\pi}$) instead of $X_s^{0,\pi}$ (resp. $X_s^{t,0,\pi}$).

Note that the random variable $J(t)$ is defined uniquely only up to $\mathbb{P}$-almost sure equivalent. Also, note that the process $(J(t))$ is adapted but not necessarily progressive. First, recall the dynamic programming principle :

Proposition 3.2. *For each admissible strategy $\pi\in\mathcal{A}$, $(e^{-\gamma X_t^\pi}J(t))_{0\le t\le T}$ is a submartingale.*

Proof. According to Schachermayer (2001) Theorem 2.2, for each initial time $t\in[0,T[$, there exists a strategy $\hat\pi\in\mathcal{A}_t$ such that

$$J(t) = E\left[\exp\left(-\gamma\left(X_T^{t,\hat\pi}+\xi\right)\right)\middle|\,\mathcal{G}_t\right] \quad a.s.$$

Suppose that $0\le t\le T$. For each admissible strategy π, we have :

$$E\left[e^{-\gamma(X_t^\pi-X_s^\pi)}J(t)\middle|\,\mathcal{G}_s\right] = E\left[\exp\left(-\gamma(X_T^{s,\pi'}+\xi)\right)\middle|\,\mathcal{G}_s\right], \quad a.s.,$$

where the strategy π' is defined by

$$\pi'_u = \begin{cases} \pi_u & \text{if} \quad s\le u\le t \\ \hat\pi_u & \text{if} \quad t<u\le T \end{cases}$$

It is clear that $\pi'\in\mathcal{A}_s$. Hence, we have :

$$E\left[e^{-\gamma X_t^\pi}J(t)\middle|\,\mathcal{G}_s\right] \ge e^{-\gamma X_s^\pi}J(s) \quad a.s.,$$

which gives that $(e^{-\gamma X_t^\pi}J(t))$ is a submartingale for each admissible strategy π. □

Remark 3.1. *Note that it is possible to prove this proposition without using the existence of an optimal strategy (see Appendix).*

Also, the value function can be characterized as follows :

Proposition 3.3. $(J(t))_{0\leq t\leq T}$ *is the largest* $\mathbb{G}$*-adapted process such that for each admissible strategy* $\pi \in \mathcal{A}$*, the process* $(e^{-\gamma X_t^\pi} J(t))_{0\leq t\leq T}$ *is a submartingale and* $J(T) = \exp(-\gamma\xi)$*.*

Proof. Let $(\hat{J}_t)$ be a $\mathbb{G}$-adapted process such that for all $\pi \in \mathcal{A}$, the process $(e^{-\gamma X_t^\pi} \hat{J}_t)$ is a submartingale and $\hat{J}_T = \exp(-\gamma\xi)$. Hence for all $t \in [0,T]$ and for each $\pi \in \mathcal{A}$, we have :

$$E\left[e^{-\gamma X_T^\pi} \hat{J}_T \Big| \mathcal{G}_t \right] \geq e^{-\gamma X_t^\pi} \hat{J}_t \quad a.s.$$

Then for each admissible strategy $\pi \in \mathcal{A}$, we have :

$$E\left[\exp\left(-\gamma \left(X_T^{t,\pi} + \xi\right)\right) \Big| \mathcal{G}_t \right] \geq \hat{J}_t \quad a.s.$$

Therefore we get :

$$\operatorname*{ess\,inf}_{\pi\in\mathcal{A}_t} E\left[\exp\left(-\gamma \left(X_T^{t,\pi} + \xi\right)\right) \Big| \mathcal{G}_t \right] \geq \hat{J}_t \quad a.s.,$$

which implies that :

$$J(t) \geq \hat{J}_t \quad a.s.$$

□

We now show that there exists a càd-làg version of the value function $(J(t))_{0\leq t\leq T}$. More precisely,

Proposition 3.4. *There exists a càd-làg* $\mathbb{G}$*-adapted process* $(J_t)_{0\leq t\leq T}$ *such that for all* $t \in [0,T]$ *:*

$$J_t = J(t) \quad a.s.$$

Proof. Let $\mathbb{D} = [0,T] \cap \mathbb{Q}$ where $\mathbb{Q}$ is the set of rational numbers. Because $(J(t))$ is a submartingale, we have that for almost every $\omega \in \Omega$, the mapping $t \to J(t,\omega)$ defined on $\mathbb{D}$ has at each point t of $[0,T[$ a finite right limit :

$$J(t^+,\omega) = \lim_{s\in\mathbb{D}, s\downarrow t} J(s,\omega)$$

and at each point of $]0,T]$ a finite left limit ;

$$J(t^-,\omega) = \lim_{s\in\mathbb{D}, s\uparrow t} J(s,\omega)$$

(see Karatzas and Shreve (1991), Proposition 1.3.14 or Dellacherie and Meyer (1980), Chapter 6).
Note that it is possible to define $J(t^+,\omega)$ for each $(t,\omega) \in [0,T] \times \Omega$ by :

$$\begin{cases} J(t^+,\omega) & := \limsup_{s\in\mathbb{D}, s\downarrow t} J(s,\omega), \quad 0 \leq t < T \\ J(T^+,\omega) & := J(T,\omega) \end{cases}$$

From the right-continuity of the filtration $(\mathcal{G}_t)$, the process $(J(t^+))$ is $\mathbb{G}$-adapted.
We can show that $(J(t^+))$ is a $\mathbb{G}$-submartingale and even that for each $\pi \in \mathcal{A}$, the process $(\exp(-\gamma X_t^\pi)J(t^+))$ is a $\mathbb{G}$-submartingale. Indeed, for $s \leq t$ and for any sequence of rationals $(t_n)_{n\geq 1}$ converging down to t,

$$E[\exp(-\gamma X_{t_n}^\pi)J(t_n)|\mathcal{G}_s] \geq \exp(-\gamma X_s^\pi)J(s).$$

Hence, by the Lebesgue theorem for conditional expectation, by letting n tend to $+\infty$,

$$E[\exp(-\gamma X_t^\pi)J(t^+)|\mathcal{G}_s] \geq \exp(-\gamma X_s^\pi)J(s).$$

This inequality applied to $s = t$ gives

$$J(t^+) = E[J(t^+)|\mathcal{G}_t] \geq J(t) \ \ a.s.$$

On the other hand, by characterization of $(J(t))$ (see Proposition 3.3), and since for each $\pi \in \mathcal{A}$, the process $(\exp(-\gamma X_t^\pi)J(t^+))$ is a $\mathbb{G}$-submartingale, we have that for each $t \in [0, T]$,

$$J(t^+) \leq J(t) \ \ a.s.$$

Thus, for each $t \in [0, T]$

$$J(t^+) = J(t) \ \ a.s.$$

Furthermore, the process $(J(t^+))$ is càd-làg.
The result follows by taking $J_t = J(t^+)$. □

Remark 3.2. *Note that Proposition 3.3 can be written for the càd-làg process (J_t) under the form : (J_t) is the largest càd-làg $\mathbb{G}$-adapted process such that for each $\pi \in \mathcal{A}$, the process $(e^{-\gamma X_t^\pi} J_t)$ is a submartingale and $J_T = \exp(-\gamma\xi)$.*

We now show that the process (J_t) is bounded (which will be useful in the following). More precisely,

Lemma 3.1. *For all $0 \leq t \leq T$, the process $(J_t)_{0\leq t\leq T}$ verifies :*

$$0 < J_t \leq 1 \ \ a.s.$$

Proof. Fix $t \in [0, T]$. The first inequality is easy to prove, since according to Schachermayer (2001), there exists an admissible strategy $\hat{\pi} \in \mathcal{A}_t$ such that

$$J_t = E\left[\exp\left(-\gamma\left(X_T^{t,\hat{\pi}} + \xi\right)\right)|\mathcal{G}_t\right],$$

which implies that $0 < J_t$ for all $0 \leq t \leq T$.
The second inequality is due to the fact that the strategy $\pi_s = 0$ for all $s \in [t, T]$ is admissible according to Definition 3.1, hence $J_t \leq \exp(-\gamma\xi)$ for all t. As we suppose that the contingent claim ξ is non negative, we have that $J_t \leq 1$ for all $0 \leq t \leq T$. □

Remark 3.3. *Note that if ξ is only lower bounded by a constant K, then (J_t) is still upper bounded but by $\exp(-\gamma K)$ instead of 1.*

Recall that the dynamic programming principle gives the following classical characterization of the optimal strategy :

Proposition 3.5. *Let $\hat{\pi} \in \mathcal{A}$, the two following assertions are equivalent :*

(i) $\hat{\pi} \in \mathcal{A}$ is an optimal strategy, that is $J_0 = E\left[\exp\left(-\gamma\left(X_T^{\hat{\pi}} + \xi\right)\right)\right]$.

(ii) The process $(e^{-\gamma X_t^{\hat{\pi}}} J_t)_{0 \le t \le T}$ is a martingale.

Proof. Suppose (i). Hence we have

$$J_0 = \inf_{\pi \in \mathcal{A}} E\left[\exp(-\gamma(X_T^{\pi} + \xi))\right] = E\left[\exp(-\gamma(X_T^{\hat{\pi}} + \xi))\right]. \tag{3.7}$$

As the process $(e^{-\gamma X_t^{\hat{\pi}}} J_t)$ is a submartingale from Remark 3.2 and as $J_0 = E\left[\exp(-\gamma(X_T^{\hat{\pi}} + \xi))\right]$, it follows that $(e^{-\gamma X_t^{\hat{\pi}}} J_t)$ is a martingale.

To show the converse, suppose that the process $(e^{-\gamma X_t^{\hat{\pi}}} J_t)$ is a martingale. Then we have $E\left[e^{-\gamma X_T^{\hat{\pi}}} J_T\right] = J_0$. Also, recall that by Remark 3.2, the process $(e^{-\gamma X_t^{\pi}} J_t)$ is a submartingale for each $\pi \in \mathcal{A}$ and since $J_T = \exp(-\gamma\xi)$, we have $J_0 \le \inf_{\pi \in \mathcal{A}} E\left[\exp(-\gamma(X_T^{\pi} + \xi))\right]$. Consequently,

$$J_0 = \inf_{\pi \in \mathcal{A}} E\left[\exp(-\gamma(X_T^{\pi} + \xi))\right] = E\left[\exp(-\gamma(X_T^{\hat{\pi}} + \xi))\right],$$

thus $\hat{\pi}$ is an optimal strategy. □

Remark 3.4. *Note that we can obtain a quite* general verification *theorem for the value function, which gives a* sufficient condition *for a process to be the value function : let $(\hat{J}_t)$ be a $\mathbb{G}$-adapted process which is equal to $\exp(-\gamma\xi)$ at T, such that for each strategy $\pi \in \mathcal{A}$, the process $\left(\exp\left(-\gamma \int_0^t \frac{\pi_s}{S_{s^-}} dS_s\right) \hat{J}_t\right)$ is a submartingale and there exists a strategy $\hat{\pi} \in \mathcal{A}$ satisfying $\left(\exp\left(-\gamma \int_0^t \frac{\hat{\pi}_s}{S_{s^-}} dS_s\right) \hat{J}_t\right)$ is a martingale, then we have $\hat{J}_t = J_t$ a.s. for each $t \in [0, T]$.*

Remark 3.5. *In the Brownian filtration case, see Hu et al. (2004), the authors use a similar verification theorem but they look for the process (J_t) under the form $(\exp(\gamma Y_t))$ where (Y_t) is a process defined as the solution of a BSDE of the form*

$$Y_t = -\xi - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds$$

for which some existence and uniqueness results hold. They characterize easily the function $f(s, z)$ with the two properties of the verification theorem. In the case of a filtration with jumps, see Morlais (2008), the author uses the same approach as in Hu et al. (2004); she obtains formally a BSDE for which there is no existence and uniqueness results. She proves the existence of a solution of this BSDE with an approximation method but she does not obtain uniqueness result; so, the value function cannot be characterized as the solution of *a BSDE. In order to be able to solve completely the optimization problem, she restricts the admissible portfolios set to a compact set, so that in this case, the value function can be proved to be the unique solution of a BSDE.*

In this work, we will not use a verification theorem (which corresponds to a *sufficient condition* for a process to be equal to the value function). Instead, by using dynamic programming technics (more precisely see Remark 3.2 and Proposition 3.5), we will show directly that the value process (J_t) is solution of a BSDE. Note that it is a *necessary condition*.

Since (J_t) is a submartingale by Remark 3.2 and since (J_t) is bounded by Lemma 3.1 (and hence of class D), it admits a unique Doob-Meyer decomposition (see Dellacherie and Meyer (1980), Chapter 7) :

$$dJ_t = dm_t + dA_t$$

where (m_t) is a square integrable martingale and (A_t) is an increasing $\mathbb{G}$-predictable process with $A_0 = 0$. From the martingale representation theorem (see Proposition 2.1), the previous Doob-Meyer decomposition can be written under the form :

$$dJ_t = Z_t dW_t + U_t dM_t + dA_t \tag{3.8}$$

with $Z \in L^2(W)$ and $U \in L^2(M)$.
Using Remark 3.2 and Proposition 3.5, it is possible to characterize the process (A_t) of (3.8), which gives that the value function (J_t) is solution of a BSDE with a quadratic driver. More precisely,

Proposition 3.6. *The process $(J_t, Z_t, U_t)_{0\le t\le T} \in S^{+,\infty} \times L^2(W) \times L^2(M)$ is solution of the following BSDE :*

$$\left\{\begin{array}{rcl} dJ_t &=& \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \left\{-\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 J_t + \gamma\pi_t(\mu_t J_t + \sigma_t Z_t) + \lambda_t(1-e^{-\gamma\pi_t\beta_t})(J_t+U_t)\right\} dt \\ && +Z_t dW_t + U_t dM_t \\ J_T &=& \exp(-\gamma\xi) \end{array}\right. \tag{3.9}$$

Proof. The proof of this proposition is based on the following arguments : first, for each strategy $\pi \in \mathcal{A}$, the process $(e^{-\gamma X_t^\pi} J_t)$ is a submartingale (see Remark 3.2). Also, since there exists an optimal strategy $\hat{\pi} \in \mathcal{A}$ (see Schachermayer (2001), Theorem 2.2) and by the previous characterization of optimal strategies (see Proposition 3.5), the process $(e^{-\gamma X_t^{\hat{\pi}}} J_t)$ is a martingale. It follows that the finite variation part which appears in the decomposition of the semi-martingale $(e^{-\gamma X_t^\pi} J_t)$ (resp. $(e^{-\gamma X_t^{\hat{\pi}}} J_t)$) is an nondecreasing process (resp. null process).
More precisely, let us calculate the derivative of process $(e^{-\gamma X_t^\pi} J_t)$. By Itô's formula, we have :

$$\begin{array}{rcl} d\left(e^{-\gamma X_t^\pi}\right) &=& -\gamma e^{-\gamma X_{t^-}^\pi} dX_t^{\pi,c} + \dfrac{\gamma^2}{2} e^{-\gamma X_{t^-}^\pi} d <X^{\pi,c}>_t + \left(e^{-\gamma X_t^\pi} - e^{-\gamma X_{t^-}^\pi}\right) dN_t \\ &=& e^{-\gamma X_{t^-}^\pi}\left[\left(\dfrac{\gamma^2}{2}\pi_t^2\sigma_t^2 - \gamma\pi_t\mu_t\right) dt - \gamma\pi_t\sigma_t dW_t + \left(e^{-\gamma\pi_t\beta_t} - 1\right) dN_t\right] \end{array}$$

with $X^{\pi,c}$ the continuous part of X^π.
Then the product rule yields

$$\begin{aligned}
d\left(e^{-\gamma X_t^\pi} J_t\right) &= e^{-\gamma X_{t-}^\pi} dJ_t + J_{t-} d\left(e^{-\gamma X_t^\pi}\right) + d\left[e^{-\gamma X_t^\pi}, J_t\right] \\
&= e^{-\gamma X_{t-}^\pi}\left(Z_t dW_t + U_t dM_t + dA_t\right) + e^{-\gamma X_{t-}^\pi} J_{t-}\left[\left(\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 - \gamma\pi_t\mu_t\right) dt\right. \\
&\quad \left. -\gamma\pi_t\sigma_t dW_t + \left(e^{-\gamma\pi_t\beta_t} - 1\right) dN_t\right] - e^{-\gamma X_{t-}^\pi}\gamma\pi_t\sigma_t Z_t dt \\
&\quad + e^{-\gamma X_{t-}^\pi}\left(e^{-\gamma\pi_t\beta_t} - 1\right) U_t dN_t.
\end{aligned}$$

Since $dN_t = dM_t + \lambda_t dt$, we have :

$$\begin{aligned}
d\left(e^{-\gamma X_t^\pi} J_t\right) &= e^{-\gamma X_{t-}^\pi}\left[(Z_t - \gamma\pi_t\sigma_t J_t)dW_t + (U_t + (e^{-\gamma\pi_t\beta_t} - 1)(U_t + J_{t-}))dM_t + dA_t\right. \\
&\quad \left. + \left\{\left(\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 - \gamma\pi_t\mu_t\right) J_t + \lambda_t\left(e^{-\gamma\pi_t\beta_t} - 1\right)(U_t + J_t) - \gamma\pi_t\sigma_t Z_t\right\} dt\right] \\
&= dm_t^\pi + dA_t^\pi,
\end{aligned}$$

with A^π the finite variational part given by $A_0^\pi = 0$ and

$$dA_t^\pi = e^{-\gamma X_{t-}^\pi}\left[dA_t + \left\{\left(\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 - \gamma\pi_t\mu_t\right) J_t + \lambda_t\left(e^{-\gamma\pi_t\beta_t} - 1\right)(U_t + J_t) - \gamma\pi_t\sigma_t Z_t\right\} dt\right]$$

and m^π the local martingale part given by $m_0^\pi = J_0$ and

$$dm_t^\pi = e^{-\gamma X_{t-}^\pi}\left[(Z_t - \gamma\pi_t\sigma_t J_t)dW_t + (U_t + (e^{-\gamma\pi_t\beta_t} - 1)(U_t + J_{t-}))dM_t\right].$$

Since by Remark 3.2, the process $\left(e^{-\gamma X_t^\pi} J_t\right)$ is a submartingale for each strategy $\pi \in \mathcal{A}$, it follows that $dA_t^\pi \geq 0$ for each strategy $\pi \in \mathcal{A}$ and we get

$$dA_t \geq \operatorname*{ess\,sup}_{\pi\in\mathcal{A}}\left\{\left[\left(-\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 + \gamma\pi_t\mu_t\right) J_t - \lambda_t\left(e^{-\gamma\pi_t\beta_t} - 1\right)(U_t + J_t) + \gamma\pi_t\sigma_t Z_t\right] dt\right\}.$$

Recall that there exists an optimal strategy $\hat{\pi} \in \mathcal{A}$ for our optimization problem 3.5 (see Theorem 2.2, Schachermayer (2001)), that is such that $J_0 = E\left[\exp\left(-\gamma\left(X_T^{\hat{\pi}} + \xi\right)\right)\right]$. By the previous characterization of an optimal strategy (see Proposition 3.5), the process $(e^{-\gamma X_t^{\hat{\pi}}} J_t)$ is a martingale, which implies that $dA_t^{\hat{\pi}} = 0$ and hence, a.s.

$$dA_t = \left[\left(-\frac{\gamma^2}{2}\hat{\pi}_t^2\sigma_t^2 + \gamma\hat{\pi}_t\mu_t\right) J_t - \lambda_t\left(e^{-\gamma\hat{\pi}_t\beta_t} - 1\right)(U_t + J_t) + \gamma\hat{\pi}_t\sigma_t Z_t\right] dt.$$

Therefore we have

$$\begin{aligned}
dA_t &= \left(-\frac{\gamma^2}{2}\hat{\pi}_t^2\sigma_t^2 + \gamma\hat{\pi}_t\mu_t\right) J_t - \lambda_t\left(e^{-\gamma\hat{\pi}_t\beta_t} - 1\right)(U_t + J_t) + \gamma\hat{\pi}_t\sigma_t Z_t \\
&= \operatorname*{ess\,sup}_{\pi\in\mathcal{A}}\left\{\left(-\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 + \gamma\pi_t\mu_t\right) J_t - \lambda_t\left(e^{-\gamma\pi_t\beta_t} - 1\right)(U_t + J_t) + \gamma\pi_t\sigma_t Z_t\right\}.
\end{aligned}$$

Thus the Doob-Meyer decomposition (3.8) of (J_t) can be written as follows :

$$\begin{aligned} dJ_t &= Z_t dW_t + U_t dM_t \\ &\quad + \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \left\{ \left(-\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 + \gamma\pi_t\mu_t \right) J_t - \lambda_t \left(e^{-\gamma\pi_t\beta_t} - 1 \right) (U_t + J_t) + \gamma\pi_t\sigma_t Z_t \right\} dt \end{aligned}$$

and the essential supremum is attained at $\hat{\pi}$. □

The problem is that we can not prove that BSDE (3.9) admits a unique solution in $S^{+,\infty} \times L^2(W) \times L^2(M)$. On the other hand we can characterize the value function (J_t) with the notion of largest solution of a BSDE defined by : (J_t, Z_t, U_t) is called the largest solution if for all solution $(\bar{J}_t, \bar{Z}_t, \bar{U}_t)$ of the BSDE in $S^{+,\infty} \times L^2(W) \times L^2(M)$, we have for all $0 \leq t \leq T$: $\bar{J}_t \leq J_t$ a.s. More precisely,

Theorem 3.1. *$(J_t, Z_t, U_t)_{0\leq t\leq T}$ is the largest solution in $S^{+,\infty} \times L^2(W) \times L^2(M)$ of the BSDE :*

$$\left\{ \begin{array}{rcl} dJ_t &=& \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \left\{ -\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 J_t + \gamma\pi_t(\mu_t J_t + \sigma_t Z_t) + \lambda_t \left(1 - e^{-\gamma\pi_t\beta_t}\right)(J_t + U_t) \right\} dt \\ && +Z_t dW_t + U_t dM_t \\ J_T &=& \exp(-\gamma\xi) \end{array} \right. \tag{3.10}$$

The optimal strategy $\hat{\pi}$ of the optimization problem (3.5) is characterized by the fact that the essential supremum in (3.10) is attained at $\hat{\pi}_t$ $dt \otimes d\mathbb{P}$ a.s. for $t \in [0, T]$.

Proof. Let $(\bar{J}_t, \bar{Z}_t, \bar{U}_t)$ be a solution of (3.10) in $S^{+,\infty} \times L^2(W) \times L^2(M)$. Let us prove that for each strategy $\pi \in \mathcal{A}$, the process $(e^{-\gamma X_t^\pi}\bar{J}_t)$ is a submartingale.
We have :

$$\begin{aligned} d\left(e^{-\gamma X_t^\pi}\bar{J}_t\right) &= e^{-\gamma X_{t^-}^\pi} \Bigg[\left(\bar{Z}_t - \gamma\pi_t\sigma_t\bar{J}_{t^-}\right) dW_t + \left(\bar{U}_t + \left(e^{-\gamma\pi_t\beta_t} - 1\right)\left(\bar{U}_t + \bar{J}_{t^-}\right)\right) dM_t + d\bar{A}_t \\ &\quad + \left\{ \left(\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 - \gamma\pi_t\mu_t \right) \bar{J}_{t^-} + \lambda_t \left(e^{-\gamma\pi_t\beta_t} - 1 \right) \left(\bar{U}_t + \bar{J}_t\right) - \gamma\pi_t\sigma_t\bar{Z}_t \right\} dt \Bigg], \end{aligned}$$

where $d\bar{A}_t = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \left\{ -\frac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t + \gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) + \lambda_t\left(1 - e^{-\gamma\pi_t\beta_t}\right)\left(\bar{J}_t + \bar{U}_t\right) \right\} dt$.
Hence, $d\left(e^{-\gamma X_t^\pi}\bar{J}_t\right)$ can be written under the form :

$$d\left(e^{-\gamma X_t^\pi}\bar{J}_t\right) = d\bar{M}_t^\pi + d\bar{A}_t^\pi,$$

with $\bar{A}^\pi$ the finite variational part given by $\bar{A}_0^\pi = 0$ and

$$d\bar{A}_t^\pi = e^{-\gamma X_t^\pi} \left\{ d\bar{A}_t + \left[\left(\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 - \gamma\pi_t\mu_t \right) \bar{J}_t + \lambda_t \left(e^{-\gamma\pi_t\beta_t} - 1 \right) \left(\bar{U}_t + \bar{J}_t\right) - \gamma\pi_t\sigma_t\bar{Z}_t \right] dt \right\},$$

and $\bar{M}^\pi$ the local martingale part given by $\bar{M}_0^\pi = \bar{J}_0$ and

$$d\bar{M}_t^\pi = e^{-\gamma X_{t^-}^\pi} \left[(\bar{Z}_t - \gamma\pi_t\sigma_t\bar{J}_t) dW_t + \left(\bar{U}_t + \left(e^{-\gamma\pi_t\beta_t} - 1 \right) \left(\bar{U}_t + \bar{J}_{t^-}\right) \right) dM_t \right].$$

It is easy to show that $d\bar{A}^\pi_t \geq 0$ from the definition of the process $(\bar{A}_t)$. Thus we have the inequality

$$\bar{M}^\pi_t \leq e^{-\gamma X^\pi_t} \bar{J}_t.$$

By definition of an admissible strategy, there exists a constant K_π for each admissible strategy $\pi \in \mathcal{A}$ such that $X^\pi_t \geq K_\pi$ for all $0 \leq t \leq T$ and since $(\bar{J}_t)$ is a positive bounded process, we have that the process $(\bar{M}^\pi_t)$ is upper bounded by a constant. Hence, the process $(\bar{M}^\pi_t)$ is a submartingale.
Thus the process $(e^{-\gamma X^\pi_t} \bar{J}_t)$ is a submartingale, because it is the sum of a submartingale and a nondecreasing process.
Now recall that (J_t) is the largest process such that for each $\pi \in \mathcal{A}$, $(e^{-\gamma X^\pi_t} J_t)$ is a submartingale and $J_T = \exp(-\gamma\xi)$ (see Remark 3.2). Therefore, we get :

$$\forall\, t \in [0, T],\ \ \bar{J}_t \leq J_t \ \ a.s.$$

□

Remark 3.6. *If we suppose that the set there is no default (i.e. we consider the Brownian motion case), then this result corresponds to that obtained in Hu et al. (2004) by taking $Y_t = \frac{1}{\gamma}\ln(J_t)$ (since they consider the process (Y_t) instead of the process (J_t)), see also Pham (2007). Recall that this result had been first stated by Rouge and El Karoui (2000).*

In the rest of this section, we show that the value function (J_t) can also be characterized as the limit of a nondecreasing sequence of solutions of classical BSDEs. More precisely, for each $k \in \mathbb{N}$, we consider the value function (J^k_t) defined by :

$$J^k_t = \underset{\pi \in \mathcal{A}^k_t}{\operatorname{ess\,inf}}\, E\left[\exp\left(-\gamma(X^{t,\pi}_T + \xi)\right) |\mathcal{G}_t\right] \quad a.s.$$

with $\mathcal{A}^k_t = \left\{\pi \in \mathcal{A}_t,\ |\pi_s| \leq k \ \ \forall\, s \in [t, T]\ a.s.\right\}$.
We will now show that the value function (J_t) can be characterized as the nonincreasing limit of the sequence $(J^k_t)_{k\in\mathbb{N}}$. But for that we need to make the following assumptions :

Assumption 3.2. *The processes $(\mu_t)_{0\leq t\leq T}$, $(\sigma_t)_{0\leq t\leq T}$ and $(\beta_t)_{0\leq t\leq T}$ are uniformly bounded, like the compensator $(\lambda)_{0\leq t\leq T}$.*

For each $k \in \mathbb{N}$, since the subset $\mathcal{A}^k$ is bounded, the process (J^k_t) is characterized as the unique solution of a BSDE. More precisely,

Proposition 3.7. *The process $\left(J^k_t, Z^k_t, U^k_t\right)_{0\leq t\leq T}$ is the unique solution in $S^{+,\infty} \times L^2(W) \times L^2(M)$ of the following classical BSDE :*

$$\begin{cases} dJ^k_t &= \underset{\pi \in \mathcal{A}^k}{\operatorname{ess\,sup}} \left\{-\frac{\gamma^2}{2}\pi^2_t\sigma^2_t J^k_t + \gamma\pi_t(\mu_t J^k_t + \sigma_t Z^k_t) + \lambda_t\left(1 - e^{-\gamma\pi_t\beta_t}\right)\left(J^k_t + U^k_t\right)\right\} dt \\ & \quad + Z^k_t dW_t + U^k_t dM_t \\ J^k_T &= \exp(-\gamma\xi) \end{cases} \tag{3.11}$$

Proof. The proof is very similar to that of Proposition 3.6. To prove that the process $\left(J_t^k, Z_t^k, U_t^k\right)$ is solution of (3.11), we use that for each strategy $\pi \in \mathcal{A}^k$, the process $(e^{-\gamma X_t^\pi} J_t^k)$ is a submartingale and that, since the domain $\mathcal{A}^k$ is bounded, there exits a strategy $\hat{\pi}$ optimal for J_0^k, which gives that the process $(e^{-\gamma X_t^{\hat{\pi}}} J_t^k)$ is a martingale.
We now show the uniqueness of the solution of BSDE (3.11). In BSDE (3.11) the driver is equal $d\mathbb{P} \otimes dt$ a.s. to

$$f(t, y, z, u) = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}^k} \left\{ \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 y - \gamma \pi_t(\mu_t y + \sigma_t z) - \lambda_t \left(1 - e^{-\gamma \pi_t \beta_t}\right)(y + u) \right\}.$$

We can easily show that the driver is Lipschitz w.r.t. y, z, u. Indeed, the driver is written as an infimum of linear drivers w.r.t. (y, z, u) with uniformly bounded coefficients since $\mathcal{A}^k$ is bounded and the coefficients $\sigma_t, \mu_t, \beta_t, \lambda_t$ are bounded by Assumption 3.2. The Lipschitz property of the driver follows by classical analysis results (see Lemma B.3 in Appendix B). □

Remark 3.7. *Note that in the case where the domain is restricted to a compact set, we have derived in a simpler way the result stated in Morlais (2008) (in which she considers the process defined by* $Y_t = \frac{1}{\gamma} \ln(J_t)$ *instead of* J_t*).*

We now state that the value function (J_t) can be written as the limit of the sequence (J_t^k)

Proposition 3.8. $J_t = \lim_{k \to \infty} \downarrow J_t^k$ *a.s.* $\forall\, t \in [0, T]$.

Proof. It is obvious with the definition of sets $\mathcal{A}$ and $\mathcal{A}^k$ that $\mathcal{A}^k \subset \mathcal{A}$ and hence,

$$\forall\, k \in \mathbb{N} \quad J_t \leq J_t^k \quad a.s.$$

Also, since for each $k \in \mathbb{N}$, $\mathcal{A}^k \subset \mathcal{A}^{k+1}$, it follows that the sequence $\left(J_t^k\right)_{k \in \mathbb{N}}$ is nonincreasing. Since it is also lower bounded, we get the existence of the limit denoted by $\tilde{J}_t$. Note that $(\tilde{J}_t)$ is an adapted process. We have clearly that $J_t \leq \tilde{J}_t$ for all $0 \leq t \leq T$ a.s.
Let us now prove that the process $(\tilde{J}_t)$ is a submartingale. Fix $0 \leq s \leq t \leq T$. As (J_t^k) is a submartingale, we get :

$$\forall\, k \in \mathbb{N} \quad E[J_t^k | \mathcal{G}_s] \geq J_s^k \geq \tilde{J}_s \quad a.s.$$

By monotone convergence theorem for conditional expectation, we have :

$$E[\tilde{J}_t | \mathcal{G}_s] \geq \tilde{J}_s \quad a.s.$$

Hence, the process $(\tilde{J}_t)$ is a submartingale. Let us show that for each bounded strategy $\pi \in \mathcal{A}$, the process $(e^{-\gamma X_t^\pi} \tilde{J}_t)$ is a submartingale.
Let π be an admissible bounded strategy. Then, there exists $n \in \mathbb{N}$ such that π is uniformly bounded by n. For each $k \geq n$, since $\pi \in \mathcal{A}^k$, the process $(e^{-\gamma X_t^\pi} J_t^k)$ is a submartingale. Then, by the monotone convergence theorem for conditional expectation, it can be easily be proved that the process $(e^{-\gamma X_t^\pi} \tilde{J}_t)$ is a submartingale.
Note now that the process $(\tilde{J}_t)$ is a submartingale not necessarily càd-làg. Let $\mathbb{D} = [0, T] \cap \mathbb{Q}$.

Because $(\tilde{J}_t)$ is a submartingale, for almost every $\omega \in \Omega$, the mapping $t \to \tilde{J}_t(\omega)$ defined on $\mathbb{D}$ has at each point t of $[0,T]$ a finite right limit :

$$\tilde{J}_{t^+}(\omega) = \lim_{s\in\mathbb{D}, s\downarrow t} \tilde{J}_s(\omega).$$

The process $(\tilde{J}_{t^+})$ can be proved to be a $\mathbb{G}$-submartingale. Also, for each bounded strategy $\pi \in \mathcal{A}$, the process $(\exp(-\gamma X_t^\pi)\tilde{J}_{t^+})$ can be shown to be a $\mathbb{G}$-submartingale. Also, note that since $(\tilde{J}_t)$ is a submartingale and since the filtration $(\mathcal{G}_t)$ is right-continuous, we have clearly that

$$\tilde{J}_t \le E[\tilde{J}_{t^+}|\mathcal{G}_t] = E[\tilde{J}_{t^+}|\mathcal{G}_{t^+}] = \tilde{J}_{t^+} \quad a.s. \tag{3.12}$$

It follows that

$$J_t \le \tilde{J}_t \le \tilde{J}_{t^+} \quad a.s.$$

To simplify notation, the process $(\tilde{J}_{t^+})$ will now be denoted by $(\bar{J}_t)$. So, we have proved that $J_t \le \bar{J}_t$ a.s.

Let us show that $\bar{J}_t \le J_t$ a.s. Since $(\bar{J}_t)$ is a càd-làg submartingale of class D, it admits the following Doob-Meyer decomposition :

$$d\bar{J}_t = \bar{Z}_t dW_t + \bar{U}_t dM_t + d\bar{A}_t,$$

where $\bar{Z} \in L^2(W)$, $\bar{U} \in L^2(M)$ and $\bar{A}$ is a nondecreasing $\mathbb{G}$-predictable process with $\bar{A}_0 = 0$. As before, we will use the fact that for each bounded strategy $\pi \in \mathcal{A}$, the process $(e^{-\gamma X_t^\pi}\bar{J}_t)$ is a submartingale to give some necessary conditions satisfied by the process $(\bar{A}_t)$.

Let $\pi \in \mathcal{A}$ a bounded strategy. We have

$$e^{-\gamma X_t^\pi}\bar{J}_t = \bar{M}_t^\pi + \bar{A}_t^\pi$$

with $\bar{A}^\pi$ the finite variational part given by $\bar{A}_0^\pi = 0$ and

$$d\bar{A}_t^\pi = e^{-\gamma X_t^\pi}\left\{d\bar{A}_t + \left[\frac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t + \lambda_t\left(e^{-\gamma\pi_t\beta_t} - 1\right)\left(\bar{U}_t + \bar{J}_t\right) - \gamma\pi_t\left(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t\right)\right]dt\right\} \tag{3.13}$$

and $\bar{M}^\pi$ the local martingale part given by $\bar{M}_0^\pi = \bar{J}_0$ and

$$d\bar{M}_t^\pi = e^{-\gamma X_{t^-}^\pi}\left[\left(\bar{Z}_t - \gamma\pi_t\sigma_t\bar{J}_t\right)dW_t + \left(\bar{U}_t + \left(e^{-\gamma\pi_t\beta_t} - 1\right)\left(\bar{U}_t + \bar{J}_{t^-}\right)\right)dM_t\right]. \tag{3.14}$$

Let $\bar{\mathcal{A}}$ be the set of essentially bounded admissible strategies. Since for each $\pi \in \bar{\mathcal{A}}$, the process $(e^{-\gamma X_t^\pi}\bar{J}_t)$ is a submartingale, we have that a.s. $d\bar{A}_t^\pi \ge 0$ and hence,

$$d\bar{A}_t \ge \operatorname*{ess\,sup}_{\pi\in\bar{\mathcal{A}}}\left\{-\frac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t + \gamma\pi_t\left(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t\right) + \lambda_t\left(1 - e^{-\gamma\pi_t\beta_t}\right)\left(\bar{J}_t + \bar{U}_t\right)\right\}dt \tag{3.15}$$

Let us now show that for each admissible (not necessarily essentially bounded) strategy $\pi \in \mathcal{A}$, the process $\left(e^{-\gamma X_t^\pi}\bar{J}_t\right)$ is a submartingale. Now, it is clear that $dt \otimes d\mathbb{P}$ a.s.

$$\operatorname*{ess\,sup}_{\pi\in\bar{\mathcal{A}}}\left\{-\frac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t + \gamma\pi_t\left(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t\right) + \lambda_t\left(1 - e^{-\gamma\pi_t\beta_t}\right)\left(\bar{J}_t + \bar{U}_t\right)\right\} =$$
$$\operatorname*{ess\,sup}_{\pi\in\mathcal{A}}\left\{-\frac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t + \gamma\pi_t\left(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t\right) + \lambda_t\left(1 - e^{-\gamma\pi_t\beta_t}\right)\left(\bar{J}_t + \bar{U}_t\right)\right\} \tag{3.16}$$

Fix $\pi \in \mathcal{A}$ (not necessarily bounded). We have

$$e^{-\gamma X_t^\pi} \bar{J}_t = \bar{M}_t^\pi + \bar{A}_t^\pi,$$

where $\bar{A}^\pi$ and $\bar{M}^\pi$ are given by (3.13) and (3.14).
Hence, from inequality (3.15) and equality (3.16), we have that $d\bar{A}_t^\pi \geq 0$ a.s. Then we get :

$$\bar{M}_t^\pi = e^{-\gamma X_t^\pi} \bar{J}_t - \bar{A}_t^\pi$$

and hence

$$\bar{M}_t^\pi \leq e^{-\gamma X_t^\pi} \bar{J}_t.$$

As $\bar{J}_t \leq 1$, we get :

$$\bar{M}_t^\pi \leq e^{-\gamma X_t^\pi} \leq e^{-\gamma K_\pi}.$$

where K_π is a constant such that $X_t^\pi \geq K_\pi$. Thus, the process $\left(\bar{M}_t^\pi\right)$ is an upper bounded local martingale, therefore it is a submartingale. As $\left(\bar{M}_t^\pi\right)$ is a submartingale and $\left(\bar{A}_t^\pi\right)$ is nondecreasing, the process $\left(e^{-\gamma X_t^\pi} \bar{J}_t\right)$ is a submartingale and this holds for any $\pi \in \mathcal{A}$. Also, the process $\left(\bar{J}_t\right)$ is càd-làg $\mathbb{G}$-adapted and $\bar{J}_T = \exp(-\gamma\xi)$. Since (J_t) is the largest process (see Remark 3.2) satisfying these properties, we have :

$$\bar{J}_t \leq J_t \quad a.s.$$

and the proof is ended. $\square$

Remark 3.8. *Note that we have derived in a simpler way the same approximation result as the one stated in Morlais (2008) (in which she considers the processes defined by* $Y_t^k = \frac{1}{\gamma}\ln(J_t^k)$ *instead of* J_t^k*).*

4 Indifference pricing

We present a general framework of the Hodges and Neuberger (1989) approach with some strictly increasing, strictly concave and continuously differentiable mapping u, defined on $\mathbb{R}$. We solve explicitly the problem in the case of exponential utility.

The Hodges approach to pricing of unhedgeable claims is a utility-based approach and can be summarized as follows : the issue at hand is to assess the value of some (defaultable) claim ξ as seen from the perspective of an investor who optimizes his behavior relative to some utility function, say u. The investor has two choices :

- he invests only in the risk-free asset and in the risky asset, in this case the associated optimization problem is
$$V(x,0) = \sup_{\pi \in \mathcal{A}} E[u(X_T^{x,\pi})],$$
- he invests also in the contingent claim, whose price is p at 0, in this case the associated optimization problem is
$$V(x-p,\xi) = \sup_{\pi \in \mathcal{A}} E[u(X_T^{x-p,\pi} + \xi)].$$

Definition 4.2. *For a given initial endowment x, the Hodges buying price of a defaultable claim ξ is the price p such that the investor's value functions are indifferent between holding and not holding the contingent claim, i.e.*

$$V(x,0) = V(x-p,\xi).$$

Remark 4.9. *We can define the Hodges selling price p_* of ξ by considering $-p$, where p is the buying price of $-\xi$, as specified in the previous definition.*

In the rest of this part, we give an explicit solution for the exponential utility function

$$u(x) = -\exp(-\gamma x), \quad \text{where } \gamma > 0 \text{ is fixed.}$$

The Hodges price p can be derived explicitly by applying the results of Section 3. If the investor buys the contingent claim at the price p and invests the rest of his wealth in the risk-free asset and in the risky asset, the value function is equal to

$$V(x-p,\xi) = -\exp(-\gamma(x-p))J_0^{\xi}.$$

If he invests all his wealth in the risk-free asset and in the risky asset, the value function is equal to

$$V(x,0) = -\exp(-\gamma x)J_0^0.$$

Recall that (J_t^{ξ}, Z_t, U_t) is the largest solution of

$$\begin{cases} dJ_t^{\xi} &= \underset{\pi\in\mathcal{A}}{\operatorname{ess\,sup}}\left\{-\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 J_t^{\xi} + \gamma\pi_t(\mu_t J_t^{\xi} + \sigma_t Z_t) + \lambda_t\left(1-e^{-\gamma\pi_t\beta_t}\right)\left(J_t^{\xi}+U_t\right)\right\}dt \\ & \quad +Z_t dW_t + U_t dM_t \\ J_T^{\xi} &= \exp(-\gamma\xi) \end{cases} \tag{4.17}$$

The Hodges price for the contingent claim ξ is clearly given by the formula :

$$p = \frac{1}{\gamma}\ln\left(\frac{J_0^0}{J_0^{\xi}}\right).$$

We can also define the Hodges price of the contingent claim ξ at time t by :

$$p_t = \frac{1}{\gamma}\ln\left(\frac{J_t^0}{J_t^{\xi}}\right).$$

Remark 4.10. *If we restrict the admissible strategies to the bounded set $\mathcal{A}^k$, the indifference price p^k can be also defined by the same method. More precisely,*

$$p^k = \frac{1}{\gamma}\ln\left(\frac{J_0^{k,0}}{J_0^{k,\xi}}\right) \quad (resp.\ p_t^k = \frac{1}{\gamma}\ln\left(\frac{J_t^{k,0}}{J_t^{k,\xi}}\right)),$$

where $J^{k,\xi}$ is the unique solution of (3.11) which is identical to BSDE (4.17) by substituting the bounded subset $\mathcal{A}^k$ for $\mathcal{A}$.
Note that

$$p = \lim_{k\to\infty} p^k \quad (resp.\ p_t = \lim_{k\to\infty} p_t^k).$$

That allows to approximate the indifference price by numerical computation.

5 Logarithmic utility

In this section, we calculate the value function and characterize the optimal strategy for the utility maximization problem with respect to

$$U(x) = \log(x).$$

This time, we shall use a somewhat different notion of trading strategy : p_t denotes the part of the wealth $X_t^{x,p}$ invested in stock S, that is advantageous for the calculus. The amount of money invested in stock S is given by the formula $\pi_t = p_t X_{t^-}^{x,p}$. A $\mathbb{G}$-predictable process $p = (p_t)_{0 \le t \le T}$ is said to be a trading strategy if the wealth process $(X_t^{x,p})$ given by

$$X_t^{x,p} = x + \int_0^t \frac{p_s X_{s^-}^{x,p}}{S_{s^-}} dS_s$$

is well defined where x is the initial capital. Under the assumption that the trading strategy is self-financing, we have the following relation :

$$dX_t^{x,p} = X_{t^-}^{x,p} p_t (\mu_t dt + \sigma_t dW_t + \beta_t dN_t)$$

and from Dolean's formula, we get the expression of the wealth process $(X_t^{x,p})$:

$$X_t^{x,p} = x \exp\left(\int_0^t p_s \mu_s ds + \int_0^t p_s \sigma_s dW_s - \frac{1}{2} \int_0^t |p_s \sigma_s|^2 ds + \int_0^t \log(1 + p_s \beta_s) dN_s \right). \tag{5.18}$$

In the following we want to maximize on a subset of strategies the expectation of utility from terminal wealth. For that we define the admissible strategies set :

Definition 5.3. *The set of admissible strategies $\mathcal{A}$ consists of all $\mathbb{G}$-predictable processes $(p)_{0 \le t \le T}$ satisfying $E\left[\int_0^T |p_t \sigma_t|^2 dt + \int_0^T |\log(1 + p_t \beta_t)|^2 \lambda_t dt\right] < \infty$ and $X_t^{x,p} > 0$ for all $0 \le t \le T$.*

Remark 5.11. *The condition $X_t^{x,p} > 0$ for all $0 \le t \le T$ is equivalent to $p_t \beta_t > -1$ for all $0 \le t \le \tau$.*

The optimization problem is given by

$$V(x) = \sup_{p \in \mathcal{A}} E\left[\log\left(X_T^{x,p}\right)\right]. \tag{5.19}$$

Let us define the value function $J_0 = \sup_{p \in \mathcal{A}} E\left[\log\left(\frac{X_T^{x,p}}{x}\right)\right]$. To solve this problem, we need few assumptions :

Assumption 5.3. *The processes $(\mu_t)_{0 \le t \le T}$, $(\sigma_t)_{0 \le t \le T}$ and $(\beta_t)_{0 \le t \le T}$ are uniformly bounded, like the processes (σ_t^{-1}) and (β_t^{-1}).*

Assumption 5.4. *The intensity $(\lambda_t)_{0 \le t \le T}$ is uniformly bounded.*

Contrary to the previous section, it is possible to characterize directly the value function without BSDE. More precisely,

Theorem 5.2. *The solution of the optimization problem (5.19) is given by* $V(x) = \log(x) + J_0$ *with :*

$$J_0 = E\left[\int_0^T \left(\hat{p}_t\mu_t - \frac{\hat{p}_t^2\sigma_t^2}{2} + \lambda_t \log(1+\hat{p}_t\beta_t)\right) dt\right]$$

where $\hat{p}$ *is the optimal trading strategy given by*

$$\hat{p}_t = \begin{cases} \frac{\mu_t}{2\sigma_t^2} - \frac{1}{2\beta_t} + \frac{\sqrt{(\mu_t\beta_t+\sigma_t^2)^2+4\lambda_t\beta_t^2\sigma_t^2}}{2\beta_t\sigma_t^2} & if\ t < \tau \\ \frac{\mu_t}{\sigma_t^2} \quad if\ t \geq \tau \end{cases}$$

Proof. With (5.18) and Definition 5.3, we get the following expression for J_0 :

$$J_0 = \sup_{p\in\mathcal{A}} E\left[\int_0^T \left(p_s\mu_s - \frac{|p_s\sigma_s|^2}{2} + \lambda_s\log(1+p_s\beta_s)\right) ds\right],$$

which implies that

$$J_0 \leq E\left[\int_0^T \operatorname*{ess\,sup}_{p_s\beta_s>-1} \left\{p_s\mu_s - \frac{|p_s\sigma_s|^2}{2} + \lambda_s\log(1+p_s\beta_s)\right\} ds\right]. \tag{5.20}$$

In the following, for each $s \in [0,T]$ and each $\omega \in \Omega$, we look for the value $\hat{p}_s(\omega)$ denoted also $\hat{p}_s$ which maximizes

$$f(x) = \mu_s x - \frac{\sigma_s^2 x^2}{2} + \lambda_s \log(1+\beta_s x),$$

with the unique condition that $\beta_s x > -1$ before the default. The derivative of this function f is

$$f'(x) = \mu_s - \sigma_s^2 x + \frac{\lambda_s\beta_s}{1+\beta_s x}. \tag{5.21}$$

After the default, since the process (λ_t) is null, the optimal value is clearly given by $\hat{p}_s = \frac{\mu_s}{\sigma_s^2}$. We now are interested by the optimal value before the default. For that, let $y = 1 + \beta_s x$:

$$f'(x) = 0 \Leftrightarrow \begin{cases} \mu_s y - \frac{\sigma_s^2}{\beta_s} y(y-1) + \lambda_s\beta_s = 0 \\ y = 1 + \beta_s x \end{cases}$$

Let y_- and y_+ be the roots of $\mu_s y - \frac{\sigma_s^2}{\beta_s}y(y-1) + \lambda_s\beta_s$ with $y_- \leq y_+$, then :

$$y_- y_+ = -\frac{\lambda_s\beta_s^2}{\sigma_s^2}.$$

Thus we have the inequality

$$y_- < 0 < y_+.$$

Hence, by taking $\hat{p}_s = \frac{y_+ - 1}{\beta_s}$ we have that $\hat{p}_s\beta_s > -1$ and for each $\omega \in \Omega$ we have the equality :

$$\hat{p}_s\mu_s - \frac{\hat{p}_s^2\sigma_s^2}{2} + \lambda_s\log(1+\hat{p}_s\beta_s) = \sup_{\beta_s x > -1}\left\{\mu_s x - \frac{\sigma_s^2 x^2}{2} + \lambda_s\log(1+\beta_s x)\right\}.$$

From (5.21) and the condition $\beta_t x > -1$ we obtain for each $s \in [0,T]$ and each $\omega \in \Omega$:

$$\hat{p}_s = \frac{\sigma_s^2 - \mu_s\beta_s - \sqrt{(\mu_s\beta_s + \sigma_s^2)^2 + 4\lambda_s\beta_s^2\sigma_s^2}}{-2\beta_s\sigma_s^2}.$$

Then from inequality (5.20), we have the following inequality :

$$J_0 \le E\left[\int_0^T \left(\hat{p}_s\mu_s - \frac{\hat{p}_s^2\sigma_s^2}{2} + \lambda_s \log(1+\hat{p}_s\beta_s)\right) ds\right].$$

It now is sufficient to show that the strategy $\hat{p}$, defined by $\hat{p}_s = \frac{\sigma_s^2-\mu_s\beta_s-\sqrt{(\mu_s\beta_s+\sigma_s^2)^2+4\lambda_s\beta_s^2\sigma_s^2}}{-2\beta_s\sigma_s^2}$, is admissible. That is clearly right with Assumptions 5.3 and 5.4. Thus the previous inequality is an equality

$$J_0 = E\left[\int_0^T \left(\hat{p}_s\mu_s - \frac{\hat{p}_s^2\sigma_s^2}{2} + \lambda_s \log(1+\hat{p}_s\beta_s)\right) ds\right]$$

and the strategy $\hat{p}$ is an optimal strategy. □

Note that if we substitute $\hat{p}_t$ by its value in the expression of the value function J_0, we get

$$\begin{aligned} J_0 \quad = \quad & E\left[\int_0^T \left(\frac{\mu_t^2}{4\sigma_t^2} - \frac{\mu_t^2}{2\beta_t^2} - \frac{\sigma_t^2}{4\beta_t^2} - \frac{\lambda_t}{2} + \frac{(\mu_t\beta_t+\sigma_t^2)^2\sqrt{(\mu_t\beta_t+\sigma_t^2)^2+4\lambda_t\beta_t^2\sigma_t^2}}{4\beta_t^2\sigma_t^2}\right.\right. \\ & \left.\left. +\lambda_t \log\left(\frac{1}{2} + \frac{\mu_t\beta_t}{2\sigma_t^2} + \sqrt{(\mu_t\beta_t+\sigma_t^2)^2 + 4\lambda_t\beta_t^2\sigma_t^2}\right)\right)\right]. \end{aligned}$$

Remark 5.12. *Assumptions 5.3 and 5.4 can be reduced to the fact that the strategy $\hat{p}$ is an admissible strategy.*

Remark 5.13. *Recall that in the case of no default, the optimal strategy is given by*

$$p_t^0 = \frac{\mu_t}{\sigma_t^2}.$$

Thus, in the case of default, the optimal strategy $\hat{p}$ can be written under the form

$$\hat{p}_t = p_t^0 - \epsilon_t$$

where ϵ_t is an additional term given by

$$\epsilon_t = \begin{cases} \frac{\mu_t}{2\sigma_t^2} + \frac{1}{2\beta_t} - \frac{\sqrt{(\mu_t\beta_t+\sigma_t^2)^2+4\lambda_t\beta_t^2\sigma_t^2}}{2\beta_t\sigma_t^2}, & \forall\ t < \tau \\ 0, \quad \forall\ t \ge \tau \end{cases}$$

Note that if we assume that $\beta_t \le 0$ (resp. $\beta_t \ge 0$), i.e. S has a negative jump (resp. a positive jump) at default, the additional term ϵ_t is positive (resp. negative), which is expected due to the default. After the default, the optimal strategy corresponds to the optimal strategy in a model without default.

Remark 5.14. *Note that if the process (β_t) converges to 0, then the optimal strategy converges to $\frac{\mu_t}{\sigma_t^2}$, which is expected because if (β_t) converges to 0, it is as if there is no default.*

6 Power utility

To complete the spectrum of important utility functions, in this section we calculate the value function and characterize the optimal strategy for the optimization problem with respect to

$$U(x) = x^\gamma, \quad x \geq 0, \quad \gamma \in (0,1).$$

Trading strategies and wealth process have the same meaning as in Section 5. Under the assumption that the trading strategy is self-financing, the investor's wealth equation is

$$\begin{cases} dX_t^{x,p} &= X_{t^-}^{x,p} p_t(\mu_t dt + \sigma_t dW_t + \beta_t dN_t), \\ X_0^{x,p} &= x. \end{cases}$$

Using Dolean's formula, we get an expression of the wealth process $(X_t^{x,p})$:

$$X_t^{x,p} = x \exp\left(\int_0^t p_s \mu_s ds + \int_0^t p_s \sigma_s dW_s - \frac{1}{2} \int_0^t |p_s \sigma_s|^2 ds + \int_0^t \log(1 + p_s \beta_s) dN_s \right).$$

The optimization problem consists in maximizing the expectation of utility from terminal wealth on the admissible strategies set defined by :

Definition 6.4. *The set of admissible strategies $\mathcal{A}$ consists of all $\mathbb{G}$-predictable processes $p = (p_t)_{0 \leq t \leq T}$ that satisfy $\int_0^T |p_t \sigma_t|^2 dt + \int_0^T |\log(1 + p_t \beta_t)|^2 \lambda_t dt < \infty$ and $X_t^{x,p} > 0$ for all $0 \leq t \leq T$.*

The investor faces the maximization problem

$$V(x) = \sup_{p \in \mathcal{A}} E[(X_T^{x,p})^\gamma]. \tag{6.22}$$

In order to find the value function and an optimal strategy we apply the same method as for the exponential utility function. Most of the proofs are identical to Section 3 and are given in Appendix. As in Section 3, we give a dynamic extension of the initial problem and define the value function for each time $t \in [0,T]$. More precisely, we denote

$$J_t = \operatorname*{ess\,sup}_{p \in \mathcal{A}} E\left[\frac{(X_T^{x,p})^\gamma}{(X_t^{x,p})^\gamma} \middle| \mathcal{G}_t \right] \quad a.s.$$

we assume that (J_t) is a càd-làg process, which is possible as in Section 3.
As in Section 3, we have a characterization for the process (J_t) by dynamic programming. More precisely,

Proposition 6.9. *$(J_t)_{0 \leq t \leq T}$ is the smallest càd-làg $\mathbb{G}$-adapted process such that for each $p \in \mathcal{A}$, the process $\left((X_t^{x,p})^\gamma J_t\right)_{0 \leq t \leq T}$ is a supermartingale and $J_T = 1$.*

And the dynamic programming principle also gives a characterization for the optimal strategy :

Proposition 6.10. *Let $\hat{p} \in \mathcal{A}$, the two following assertions are equivalent :*

(i) $\hat{p}$ *is an optimal strategy, i.e.* $J_0 = \sup_{p\in\mathcal{A}} E\left[\left(\frac{X_T^{x,p}}{x}\right)^{\gamma}\right] = E\left[\left(\frac{X_T^{x,\hat{p}}}{x}\right)^{\gamma}\right]$.

(ii) The process $((X_t^{x,\hat{p}})^{\gamma} J_t)_{0\le t\le T}$ *is a martingale.*

We now will characterize the process (J_t) as a solution of a BSDE. From Proposition 6.9, the process (J_t) is a supermartingale and we can write it under the following form with Doob-Meyer decomposition (see Dellacherie and Meyer (1980)) :

$$dJ_t = dm_t - dA_t$$

with (m_t) is a local martingale (since (J_t) is not necessarily of class D) and (A_t) is a nondecreasing $\mathbb{G}$-predictable process where $A_0 = 0$. Using a local martingale representation theorem (see Proposition 2.1), there exist two predictable processes (Z_t) and (U_t) such that $Z \in L^2_{loc}(W)$, $U \in L^2_{loc}(M)$ and :

$$dJ_t = Z_t dW_t + U_t dM_t - dA_t. \tag{6.23}$$

From Propositions 6.9 and 6.10, we can give a characterization of the process (J_t) with a BSDE. For that we define the notion of smallest solution of a BSDE by : (J_t, Z_t, U_t) is called the smallest solution of a BSDE if for all solution $(\bar{J}_t, \bar{Z}_t, \bar{U}_t)$ of the BSDE we have that $J_t \le \bar{J}_t$ a.s.

Theorem 6.3. $(J_t, Z_t, U_t)_{0\le t\le T}$ *is the smallest solution in* $S^{+,\infty} \times L^2_{loc}(W) \times L^2_{loc}(M)$ *of the following BSDE :*

$$\left\{\begin{array}{rcl} dJ_t &=& -\operatorname*{ess\,sup}_{p\in\mathcal{A}}\left\{\left(\gamma p_t\mu_t + \frac{\gamma(\gamma-1)}{2}p_t^2\sigma_t^2\right)J_t + \gamma p_t\sigma_t Z_t + \lambda_t((1+p_t\beta_t)^p - 1)(J_t + U_t)\right\}dt \\ && + Z_t dW_t + U_t dM_t \\ J_T &=& 1 \end{array}\right. \tag{6.24}$$

The optimal strategy $\hat{p}$ *of the optimization problem (6.22) is characterized by the fact that the essential supremum in (6.24) is attained at* $\hat{p}_t$ $dt \otimes d\mathbb{P}$ *a.s. for* $t \in [0,T]$.

As in the case of an exponential utility function, we can not say if BSDE (6.24) admits a unique solution. But again we have another characterization of the process (J_t) as the limit of a nonincreasing sequence of solutions of classical BSDEs, but for that we need to make Assumption 3.2.

Proposition 6.11. *For all* $k \in \mathbb{N}$*, there exists a bounded positive càd-làg* $\mathbb{G}$*-adapted process* $(J_t^k)_{0\le t\le T}$ *such that :*

$$J_t^k = \operatorname*{ess\,sup}_{p\in\mathcal{A}^k} E\left[\left(\frac{X_T^{x,p}}{X_t^{x,p}}\right)^{\gamma}\middle|\mathcal{G}_t\right] \quad a.s.$$

where $\mathcal{A}^k = \{\pi \in \mathcal{A},\ |\pi_t| \le k \text{ a.s.},\ \forall\, t \in [0,T]\}$.
The process $(J_t^k, Z_t^k, U_t^k)_{0\le t\le T}$ *is the unique solution in* $S^{+,\infty} \times L^2(W) \times L^2(M)$ *of the classical BSDE :*

$$\left\{\begin{array}{rcl} dJ_t^k &=& -\operatorname*{ess\,sup}_{p\in\mathcal{A}^k}\left\{\left(\gamma p_t\mu_t + \frac{\gamma(\gamma-1)}{2}p_t^2\sigma_t^2\right)J_t^k + \gamma p_t\sigma_t Z_t^k + \lambda_t((1+p_t\beta_t)^p - 1)(J_t^k + U_t^k)\right\}dt \\ && + Z_t^k dW_t + U_t^k dM_t \\ J_T^k &=& 1 \end{array}\right. \tag{6.25}$$

We now can state that the value function (J_t) can be written as the limit of the processes (J_t^k). More precisely,

Proposition 6.12. $J_t = \lim_{k\to\infty} \uparrow J_t^k$ *a.s.*

7 Generalizations

In this section, we give some generalizations of the previous results. The proofs are not given, but they are identical at the proofs of Section 3. It is also possible to generalize the results of Section 5 and Section 6, but it is not given in this paper.

7.1 Several default times

We consider a market defined on the complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$ equipped with two stochastic processes : a n-dimensional Brownian motion $(W_t)_{0\leq t\leq T}$ and a p-dimensional jump process $(N)_{0\leq t\leq T} = ((N^i)_{0\leq t\leq T}, 1 \leq i \leq p)$ with $N_t^i = \mathbb{1}_{\tau^i \leq t}$ where $(\tau^i)_{1\leq i\leq p}$ are p default times. We denote by $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ the completed filtration generated by these processes :

$$\mathcal{G}_t = \sigma(W_s, 0 \leq s \leq t) \vee \sigma(N_s, 0 \leq s \leq t) \vee \mathcal{N}.$$

Assumption 7.5. *We do the following assumptions on the default times*

(i) The defaults do not appear simultaneously : $\mathbb{P}(\tau^i = \tau^j) = 0$ *for* $i \neq j$.

(ii) Each default can appear at any time : $\mathbb{P}(\tau^i > t) > 0$.

We suppose that the $\mathbb{G}$-compensator (Γ_t^i) of (N_t^i) for each i is absolutely continuous with respect to the Lebesgue measure, so that there exists a process (λ_t^i) such that $\Gamma_t^i = \int_0^t \lambda_s^i ds$. Then the process (M_t^i) defined by :

$$M_t^i = N_t^i - \int_0^t \lambda_s^i ds$$

is a $\mathbb{G}$-martingale.
Introduce the classical sets

- $L^2(W)$ (resp. $L^2_{loc}(W)$) is the set of $\mathbb{G}$-predictable processes on $[0, T]$ under $\mathbb{P}$ with

$$E\left[\sum_{i=1}^n \int_0^T |Z_t^i|^2 dt\right] < \infty \quad (\text{resp. } \sum_{i=1}^n \int_0^T |Z_t^i|^2 dt < \infty \ a.s. \ ).$$

- $L^2(M)$ (resp. $L^2_{loc}(M)$) is the set of $\mathbb{G}$-predictable processes on $[0, T]$ under $\mathbb{P}$ with

$$E\left[\sum_{i=1}^p \int_0^T |U_t^i|^2 \lambda_t^i dt\right] < \infty \quad (\text{resp. } \sum_{i=1}^p \int_0^T |U_t^i|^2 \lambda_t^i dt < \infty \ a.s. \ ).$$

The martingale representation theorem given in Section 2 still holds in the multidimensional case (see Kusuoka (1999)) :

Proposition 7.13. *Let m be any $(\mathbb{P}, \mathbb{G})$-(resp. locally) square integrable (resp. local) martingale with $m_0 = 0$. Then, there are two valued $\mathbb{G}$-predictable processes $\phi = (\phi^i, 1 \leq i \leq n)$ and $\psi = (\psi^i, 1 \leq i \leq p)$ such that $\phi \in L^2(W)$ and $\psi \in L^2(M)$ (resp. $\phi \in L^2_{loc}(W)$ and $\psi \in L^2_{loc}(M)$) and*

$$m_t = \sum_{i=1}^{n} \int_0^t \phi^i_s dW^i_s + \sum_{i=1}^{p} \int_0^t \psi^i_s dM^i_s, \ 0 \leq t \leq T.$$

We consider a financial market which consists of one risk-free asset, whose price process is assumed for simplicity to be equal to 1 at each time, and n risky assets with price processes $(S^i_t)_{1 \leq i \leq n}$ which admit p jumps at time $(\tau^j)_{1 \leq j \leq p}$. In the following, we consider that the price processes $(S^i_t)_{1 \leq i \leq n}$ evolve according to the equation :

$$dS^i_t = S^i_{t^-}(\mu^i_t dt + \sum_{j=1}^{n} \sigma^{ij}_t dW^j_t + \sum_{j=1}^{p} \beta^{ij}_t dN^j_t) \tag{7.26}$$

with the classical assumption :

Assumption 7.6. *(i) (μ^i_t), (σ^{ij}_t) and (β^{ij}_t) are $\mathbb{G}$-predictable stochastic processes for each i and j.*

(ii) The process (β^{ij}_t) satisfied $\beta^{ij}_t > -1$ for all $t \in \mathbb{R}^+$ and for each i and j (it is necessary for that the processes $(S^i_t)_{1 \leq i \leq n}$ are positive).

A $\mathbb{G}$-predictable process $\pi = (\pi^i_t, 0 \leq t \leq T)_{1 \leq i \leq n}$ is called trading strategy if $\sum_{i=1}^{n} \int_0^T \frac{\pi^i_t}{S^i_{t^-}} dS^i_t$ is well defined. The process (π^i_t) describes the amount of money invested in stock S^i. Under the assumption that the trading strategy is self-financing, the wealth process $(X^{x,\pi}_t)$ of a trading strategy π with initial capital x satisfies the equation :

$$dX_t = \sum_{i=1}^{n} \frac{\pi^i_t}{S^i_{t^-}} dS^i_t.$$

In this part, we characterize the value function (J_t) for the maximization problem with an exponential utility function as in Section 3. For that, we first define the admissible strategies set on which we maximize the expectation of utility from terminal wealth :

Definition 7.5. *The admissible trading strategies set $\mathcal{A}$ consists of all $\mathbb{G}$-predictable processes $\pi = (\pi_t)_{0 \leq t \leq T}$ which satisfy $\int_0^T |\pi^\top_t \sigma_t|^2 dt < \infty$[1] a.s. and $\sum_{i=1}^{n} \sum_{j=1}^{p} \int_0^T |\pi^i_t \beta^{ij}_t|^2 \lambda^j_t dt < \infty$ a.s. and there exists a constant K_π such that $X^{0,\pi}_t \geq K_\pi$ for all $t \in [0, T]$.*

Recall that there exists a càd-làg process (J_t) such that for each $t \in [0, T]$:

$$J_t = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}_t} E\left[\exp(-\gamma X^{t,\pi}_T) | \mathcal{G}_t\right] \quad a.s.$$

[1] $x^\top$ denotes the transpose vector of the vector x

where the set $\mathcal{A}_t$ consists of all $\mathbb{G}$-predictable processes $\pi = (\pi_s)_{t\leq s\leq T}$ which satisfy $\int_t^T |\pi_t^\top \sigma_t|^2 dt < \infty$ a.s. and $\sum_{i=1}^{n}\sum_{j=1}^{p} \int_t^T |\pi_t^i \beta_t^{ij}|^2 \lambda_t^j dt < \infty$ a.s. and there exists a constant K_π such that $X_s^{t,0,\pi} \geq K_\pi$ for all $s \in [t,T]$. Using the same technics as in Section 3 and by noticing that $[M^i, M^j]_t = 0$ for each i and j, since the defaults do not appear simultaneously, we obtain :

Theorem 7.4. *$(J_t, Z_t, U_t)_{0\leq t\leq T}$ is the largest solution in $S^{+,\infty} \times L^2(W) \times L^2(M)$ of the BSDE :*

$$\begin{cases} dJ_t &= \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \Big\{ -\frac{\gamma^2}{2}|\pi_t^T \sigma_t|^2 J_t + \gamma \pi_t^T(\mu_t J_t + \sigma_t Z_t) + \sum_{j=1}^p \lambda_t^j \left(1 - e^{-\gamma \sum_{i=1}^n \pi_t^i \beta_t^{ij}}\right)(J_t + U_t^j) \Big\} dt \\ & \quad + Z_t dW_t + U_t dM_t \\ J_T &= 1 \end{cases} \tag{7.27}$$

The optimal strategy $\hat{\pi}$ is characterized by the fact that the essential supremum in (7.27) is attained at $\hat{\pi}_t$ $dt \otimes d\mathbb{P}$ a.s. for $t \in [0,T]$.

7.2 Poisson jumps

We consider a market defined on the complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$ equipped with two independent processes : a unidimensional Brownian motion (W_t) and a real-valued Poisson point process p defined on $[0,T] \times \mathbb{R}\backslash\{0\}$, we denote by $N_p(ds,dx)$ the associated counting measure, such that its compensator is $\hat{N}_p(ds,dx) = n(dx)ds$ and the Levy measure $n(dx)$ is positive and satisfies $n(\{0\}) = 0$ and $\int_{\mathbb{R}\backslash\{0\}} (1 \wedge |x|)^2 n(dx) < \infty$. We denote by $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ the completed filtration generated by the two processes (W_t) and (N_p). We denote by $\tilde{N}_p(ds,dx)$ ($\tilde{N}_p(ds,dx) = N_p(ds,dx) - \hat{N}_p(ds,dx)$) the compensated measure, which is a martingale random measure : in particular, for any predictable and locally square integrable process (U_t), the stochastic integral $U.\tilde{N}_p = \int U_s(x)\tilde{N}_p(ds,dx)$ is a locally square integrable martingale. We denote by $Z.W$ (resp. $U.\tilde{N}_p$) the stochastic integral of Z w.r.t. W (resp. the stochastic integral of U w.r.t. $\tilde{N}_p$). Introduce the classical sets

- $L^2(W)$ (resp. $L^2_{loc}(W)$) is the set of $\mathbb{G}$-predictable processes on $[0,T]$ under $\mathbb{P}$ with
$$E\left[\int_0^T |Z_t|^2 dt\right] < \infty \quad (\text{resp. } \int_0^T |Z_t|^2 dt < \infty \ a.s.).$$
- $L^2(\tilde{N}_p)$ (resp. $L^2_{loc}(\tilde{N}_p)$) is the set of $\mathbb{G}$-predictable processes on $[0,T]$ under $\mathbb{P}$ with
$$E\left[\int_0^T |U_t(x)|^2 n(dx)dt\right] < \infty \ (\text{resp. } \int_0^T |U_t(x)|^2 n(dx)dt < \infty \ a.s.).$$

The filtration $\mathbb{G}$ has the predictable representation property : for any (resp. local) martingale (K_t) of $\mathbb{G}$, there exist two predictable processes (Z_t) and (U_t) such that $Z \in L^2(W)$ and $U \in L^2(\tilde{N}_p)$ (resp. $Z \in L^2_{loc}(W)$ and $U \in L^2_{loc}(\tilde{N}_p)$) and

$$K_t = K_0 + (Z.W)_t + (U.\tilde{N}_p)_t \quad \forall\, t \in [0,T].$$

The financial market consists in one risk-free asset, whose price process is assumed to be equal to 1, and one single risky asset, whose price process is denoted by S. In particular, the stock price process is a one dimensional local martingale satisfying

$$dS_t = S_{t^-}\left(\mu_t dt + \sigma_t dW_t + \int_{\mathbb{R}\setminus\{0\}} \beta_t(x)\tilde{N}_p(dt,dx)\right).$$

All processes (μ_t),(σ_t) and (β_t) are assumed to be predictable and the process (β_t) satisfies : $\beta_t(x) > -1$. This last condition implies that the process (S_t) is almost surely positive. Recall that there exists a càd-làg process (J_t) such that for each $t \in [0,T]$:

$$J_t = \operatorname*{ess\,inf}_{\pi\in\mathcal{A}_t} E\left[\exp(-\gamma X_T^{t,\pi})\right] \quad a.s.$$

where the admissible trading strategies set $\mathcal{A}_t$ consists of all $\mathbb{G}$-predictable processes $\pi = (\pi_s)_{t\leq s\leq T}$ which satisfy $\int_t^T |\pi_s\sigma_s|^2 ds < \infty$ a.s. and $\int_t^T |\pi_s\beta_s(x)|^2 n(dx)ds < \infty$ a.s. and there exists a constant K_π such that $X_s^{t,\pi} \geq K_\pi$ for all $s \in [t,T]$.

Using the same technics as in Section 3, we obtain the following theorem which characterizes the solution to the maximization problem of exponential utility

Theorem 7.5. *$(J_t, Z_t, U_t)_{0\leq t\leq T}$ is the largest solution in $S^{+,\infty}\times L^2(W)\times L^2(\tilde{N}_p)$ of the BSDE :*

$$\begin{cases} dJ_t &= \operatorname*{ess\,sup}_{\pi\in\mathcal{A}}\left\{-\frac{\gamma^2}{2}|\pi_t\sigma_t|^2 J_t + \gamma\pi_t(\mu_t J_t + \sigma_t Z_t) + \int_{\mathbb{R}\setminus\{0\}}\left(1-e^{-\gamma x}\right)\left(J_t + U_t(x)\right)\right\} n(dx) \\ & \quad +Z_t dW_t + \int_{\mathbb{R}\setminus\{0\}} U_t(x)\tilde{N}_p(dt,dx) \\ J_T &= 1 \end{cases} \tag{7.28}$$

The optimal strategy $\hat{\pi}$ is characterized by the fact that the essential supremum in (7.28) is attained at $\hat{\pi}_t$ $dt\otimes d\mathbb{P}$ a.s. for $t\in[0,T]$.

Remark 7.15. *It is exactly the case treated by Morlais (2008). For the case $\mathcal{A}$ is compact, we prove as Morlais but in a simpler way that the BSDE has a unique solution. For the case $\mathcal{A}$ is no compact, we also prove in a simpler way that the BSDE has a solution.*

Appendix

A Proof of Proposition 3.2

In this section, we just note that it is possible to prove Proposition 3.2 without using the results of Schachermayer (2001). For that let us define the random variable $\Gamma(t,\pi)$ by the formula :

$$\Gamma(t,\pi) = E\left[\exp(-\gamma(X_T^{t,\pi}+\xi))|\mathcal{G}_t\right].$$

This family of random variables is stable by infimum. More precisely,

Lemma A.2. *For each $t \in [0,T]$, the set $\{\Gamma(t,\pi),\ \pi \in \mathcal{A}_t\}$ is stable by infimum, i.e. for every $\pi^0,\ \pi^1 \in \mathcal{A}_t$, there exists $\pi \in \mathcal{A}_t$ such that $\Gamma(t,\pi) = \Gamma(t,\pi^0) \wedge \Gamma(t,\pi^1)$.*
Furthermore, for each $t \in [0,T]$, there exists a sequence $(\pi^n)_{n\in\mathbb{N}} \in \mathcal{A}_t$, such that :

$$J(t) = \lim_{n\to\infty} \downarrow \Gamma(t,\pi^n) \quad a.s.$$

Proof. Let us fix $t \in [0,T]$ and define the set E :

$$E = \left\{\Gamma(t,\pi^0) \leq \Gamma(t,\pi^1)\right\}.$$

Thus by definition of the family $(\Gamma(t,\pi))_{\pi\in\mathcal{A}_t}$, we have that $E \in \mathcal{G}_t$. Let us define π by the formula :

$$\forall\ s \in [t,T],\ \ \pi_s = \pi_s^0 \mathbb{1}_E + \pi_s^1 \mathbb{1}_{E^c}.$$

Then the wealth process $(X_s^{t,\pi})$ associated at the strategy π is equal to :

$$X_s^{t,\pi} = X_s^{t,\pi^0}\mathbb{1}_E + X_s^{t,\pi^1}\mathbb{1}_{E^c},$$

therefore we have the inequality

$$X_s^{t,\pi} \geq K_{\pi^0} \wedge K_{\pi^1},$$

thus $\pi \in \mathcal{A}_t$. By construction of π, we have that

$$\Gamma(t,\pi) = \Gamma(t,\pi^0) \wedge \Gamma(t,\pi^1).$$

The second part of lemma follows by classical results (see Karatzas and Shreve (1999), Theorem A.3 in Appendix A). $\square$

Let us now give the proof of Proposition 3.2 without using the results of Schachermayer (2001). It is sufficient to show that :

$$E\left[e^{-\gamma X_t^\pi} J(t)|\mathcal{G}_s\right] \geq e^{-\gamma X_s^\pi} J(s), \quad \forall\ t \geq s$$

$$i.e.\ \ E\left[e^{-\gamma(X_t^\pi - X_s^\pi)} J(t)|\mathcal{G}_s\right] \geq J(s), \quad \forall\ t \geq s.$$

With (2.3) and Lemma A.2, we have by monotone convergence theorem :

$$E\left[\exp(-\gamma(X_t^\pi - X_s^\pi))J(t)|\,\mathcal{G}_s\right] = \lim_{n\to\infty} \downarrow E\left[\exp\left(-\gamma\left(\int_s^t \pi_u \frac{dS_u}{S_{u^-}} + \int_t^T \pi_u^n \frac{dS_u}{S_{u^-}} + \xi\right)\right)\Big|\,\mathcal{G}_s\right]$$

Let us define the strategy $\tilde{\pi}^n$ by

$$\tilde{\pi}_u^n = \begin{cases} \pi_u & \text{if} \quad s \leq u \leq t \\ \pi_u^n & \text{if} \quad t < u \leq T \end{cases}$$

We can easily show that $\tilde{\pi}^n \in \mathcal{A}_s$. By definition of $J(s)$, we have $\lim_{n\to\infty} \Gamma(s,\tilde{\pi}^n) \geq J(s)$ a.s. Therefore :

$$E\left[\exp(-\gamma X_t^{s,\pi})J(t)|\mathcal{G}_s\right] = \lim_{n\to\infty} \downarrow \Gamma(s,\tilde{\pi}^n) \geq J(s) \quad a.s.$$

Then, for each $\pi \in \mathcal{A}$, the process $(e^{-\gamma X_t^\pi} J(t))$ is a submartingale .

B Proof of Proposition 3.7

Lemma B.3. *The supremum of affine functions, whose coefficients are bounded by a constant $c > 0$, is Lipschitz with Lipschitz constant c.*
More precisely, let $\mathcal{A}$ be a subset of $[-c, c]^n \times [-c, c]$. Then the function f defined for each $y \in \mathbb{R}^n$ by

$$f(y) = \sup_{(a,b)\in\mathcal{A}} \{a.y + b\}$$

is Lipschitz with Lipschitz constant c.

Proof.

$$\sup_{(a,b)\in\mathcal{A}} \{a.y + b\} \leq \sup_{(a,b)\in\mathcal{A}} \{a.(y - y')\} + \sup_{(a,b)\in\mathcal{A}} \{a.y' + b\}.$$

Hence, we have

$$f(y) - f(y') \leq c||y - y'||.$$

By symmetry,

$$f(y') - f(y) \leq c||y - y'||,$$

which give the desired result. □

C Proofs of Propositions 6.9 and 6.10

The technics are similar to Section 3. We first want to show that for each strategy $p \in \mathcal{A}$, the process $((X_t^{x,p})^\gamma J_t)$ is a supermartingale. According to Theorem 2.2 of Kramkov and Schachermayer (1999) (we can prove this property without using this theorem by doing as the previous appendix), for each $t \in [0, T]$ there exists a strategy $\hat{p} \in \mathcal{A}$ such that :

$$J_t = E\left[\left(\frac{X_T^{x,\hat{p}}}{X_t^{x,\hat{p}}}\right)^\gamma \middle| \mathcal{G}_t\right].$$

For each admissible strategy $p \in \mathcal{A}$, we have :

$$E\left[\left(\frac{X_t^{x,p}}{X_s^{x,p}}\right)^\gamma J_t \middle| \mathcal{G}_s\right] = E\left[\left(\frac{X_T^{x,p'}}{X_s^{x,p'}}\right)^\gamma \middle| \mathcal{G}_s\right]$$

where the strategy p' is defined by

$$p'_u = \begin{cases} p_u & \text{if} \quad 0 \leq u \leq t \\ \hat{p}_u & \text{if} \quad t < u \leq T \end{cases}$$

It is easy to show that the strategy p' is admissible. Hence

$$E\left[\left(\frac{X_T^{x,p'}}{X_s^{x,p'}}\right)^\gamma \middle| \mathcal{G}_s\right] \leq J_s.$$

Thus we get the inequality

$$E\left[(X_t^{x,p})^\gamma J_t | \mathcal{G}_s\right] \leq (X_s^{x,p})^\gamma J_s.$$

Hence, the process $((X_t^{x,p})^\gamma J_t)$ is a supermartingale for each admissible strategy p.

We now prove that (J_t) is the smallest process such that for each strategy $p \in \mathcal{A}$, the process $((X_t^{x,p})^\gamma J_t)$ is a supermartingale and $J_T = 1$. Let $(\hat{J}_t)$ be a $\mathbb{G}$-adapted process such that for each $p \in \mathcal{A}$, the process $((X_t^{x,p})^\gamma \hat{J}_t)$ is a supermartingale and $\hat{J}_T = 1$. For all $t \in [0,T]$ and for each $p \in \mathcal{A}$, we have :

$$E\left[(X_T^{x,p})^\gamma \hat{J}_T | \mathcal{G}_t\right] \leq (X_t^{x,p})^\gamma \hat{J}_t,$$

which implies that

$$E\left[\frac{(X_T^{x,p})^\gamma}{(X_t^{x,p})^\gamma}\middle| \mathcal{G}_t\right] \leq \hat{J}_t.$$

Therefore we have :

$$\operatorname*{ess\,sup}_{p\in\mathcal{A}} E\left[\frac{(X_T^{x,p})^\gamma}{(X_t^{x,p})^\gamma}\middle| \mathcal{G}_t\right] \leq \hat{J}_t.$$

Then we get :

$$J_t \leq \hat{J}_t \quad a.s.$$

We now want to prove the equivalence for the optimal strategy : suppose that the strategy $\hat{p}$ is an optimal strategy, hence we have

$$J_0 = \sup_{p\in\mathcal{A}} E\left[\left(\frac{X_T^{x,p}}{x}\right)^\gamma\right] = E\left[\left(\frac{X_T^{x,\hat{p}}}{x}\right)^\gamma\right].$$

As the process $((X_t^{x,\hat{p}})^\gamma J_t)$ is a supermartingale and that we have $J_0 = E\left[\left(\frac{X_T^{x,\hat{p}}}{x}\right)^\gamma\right]$, we have that the process $((X_t^{x,\hat{p}})^\gamma J_t)$ is a martingale.
To show the converse, suppose that the process $((X_t^{\hat{p}})^\gamma J_t)$ is a martingale, then we have $E\left[\left(\frac{X_T^{\hat{p}}}{x}\right)^\gamma\right] = J_0$. Thus we have :

$$J_0 = \sup_{p\in\mathcal{A}} E\left[\left(\frac{X_T^{x,p}}{x}\right)^\gamma\right] = E\left[\left(\frac{X_T^{x,\hat{p}}}{x}\right)^\gamma\right].$$

D Proof of Theorem 6.3

We first characterize the process (A_t) of Doob-Meyer decomposition (6.23). For that we use the properties of process (J_t) (see Propositions 6.9 and 6.10). By Itô's formula we get :

$$\begin{aligned}
d((X_t^{x,p})^\gamma J_t) &= (X_{t^-}^{x,p})^\gamma J_{t^-}\left[\left(\gamma p_t\mu_t + \frac{\gamma(\gamma-1)}{2}p_t^2\sigma_t^2\right)dt + \gamma p_t\sigma_t dW_t + ((1+p_t\beta_t)^\gamma - 1)dN_t\right] \\
&\quad + (X_{t^-}^{x,p})^\gamma (Z_t dW_t + U_t dM_t - dA_t) + \gamma (X_{t^-}^{x,p})^\gamma p_t\sigma_t Z_t dt \\
&\quad + (X_{t^-}^{x,p})^\gamma ((1+p_t\beta_t)^\gamma - 1) U_t dt \\
&= \text{local martingale } - (X_t^{x,p})^\gamma \Big\{ dA_t - \left[\left(\gamma p_t\mu_t + \frac{\gamma(\gamma-1)}{2}p_t^2\sigma_t^2\right)J_t \right. \\
&\quad \left. + \gamma p_t\sigma_t Z_t + \lambda_t((1+p_t\beta_t)^\gamma - 1)(J_t + U_t)\right] dt \Big\}
\end{aligned}$$

From Proposition 6.9, the process $((X_t^{x,p})^\gamma J_t)$ is a supermartingale for each strategy $p \in \mathcal{A}$, thus :

$$dA_t - \left[\left(\gamma p_t \mu_t + \frac{\gamma(\gamma-1)}{2} p_t^2 \sigma_t^2\right) J_t + \gamma p_t \sigma_t Z_t + \lambda_t((1+p_t\beta_t)^\gamma - 1)(J_t + U_t)\right] dt \geq 0.$$

Then we have :

$$dA_t \geq \operatorname*{ess\,sup}_{p \in \mathcal{A}} \left[\left(\gamma p_t \mu_t + \frac{\gamma(\gamma-1)}{2} p_t^2 \sigma_t^2\right) J_t + \gamma p_t \sigma_t Z_t + \lambda_t((1+p_t\beta_t)^\gamma - 1)(J_t + U_t)\right] dt.$$

From Theorem 2.2 of Kramkov and Schachermayer (1999), there exists an optimal strategy $\hat{p}$ of the optimization problem (6.22). With Proposition 6.10, this optimal strategy $\hat{p} \in \mathcal{A}$ is such that the process $(\left(X_t^{x,\hat{p}}\right)^\gamma J_t)$ is a martingale, hence we get :

$$dA_t - \left[\left(\gamma \hat{p}_t \mu_t + \frac{\gamma(\gamma-1)}{2} \hat{p}_t^2 \sigma_t^2\right) J_t + \gamma \hat{p}_t \sigma_t Z_t + \lambda_t((1+\hat{p}_t\beta_t)^\gamma - 1)(J_t + U_t)\right] dt = 0.$$

Therefore we have :

$$dA_t = \operatorname*{ess\,sup}_{p \in \mathcal{A}} \left\{\left(\gamma p_t \mu_t + \frac{\gamma(\gamma-1)}{2} p_t^2 \sigma_t^2\right) J_t + \gamma p_t \sigma_t Z_t + \lambda_t((1+p_t\beta_t)^p - 1)(J_t + U_t)\right\} dt.$$

Thus the process (J_t) is solution of the BSDE

$$\begin{cases} dJ_t &= -\operatorname*{ess\,sup}_{p \in \mathcal{A}} \left\{\left(\gamma p_t \mu_t + \frac{\gamma(\gamma-1)}{2} p_t^2 \sigma_t^2\right) J_t + \gamma p_t \sigma_t Z_t + \lambda_t((1+p_t\beta_t)^p - 1)(J_t + U_t)\right\} dt \\ & \quad + Z_t dW_t + U_t dM_t \\ J_T &= 1 \end{cases}$$

We now want to show that the process (J_t) is the smallest solution of this BSDE : let $(\bar{J}_t, \bar{Z}_t, \bar{U}_t)$ be a solution of BSDE (6.24), we show that for each $p \in \mathcal{A}$ we have that the process $\left((X_t^{x,p})^\gamma \bar{J}_t\right)$ is a supermartingale :

$$\begin{aligned} d\left((X_t^{x,p})^\gamma \bar{J}_t\right) &= \bar{J}_{t^-} \left(X_{t^-}^{x,p}\right)^\gamma \left[\gamma p_t \mu_t dt + \gamma p_t \sigma_t dW_t + \tfrac{\gamma(\gamma-1)}{2} p_t^2 \sigma_t^2 dt + [(1+p_t\beta_t)^\gamma - 1]\, dN_t\right] \\ &\quad + \left(X_{t^-}^{x,p}\right)^\gamma (\bar{Z}_t dW_t + \bar{U}_t dM_t - d\bar{A}_t) + \gamma \bar{Z}_t \left(X_{t^-}^{x,p}\right)^\gamma p_t \sigma_t dt \\ &\quad + \bar{U}_t \left(X_{t^-}^{x,p}\right)^\gamma [(1+p_t\beta_t)^\gamma - 1]\, dN_t \end{aligned}$$

where

$$d\bar{A}_t = \operatorname*{ess\,sup}_{p \in \mathcal{A}} \left\{\left(\gamma p_t \mu_t + \frac{\gamma(\gamma-1)}{2} p_t^2 \sigma_t^2\right) \bar{J}_t + \gamma p_t \sigma_t \bar{Z}_t + \lambda_t((1+p_t\beta_t)^p - 1)(\bar{J}_t + \bar{U}_t)\right\} dt.$$

$d\left((X_t^{x,p})^\gamma \bar{J}_t\right)$ can be written under the form :

$$d\left((X_t^{x,p})^\gamma \bar{J}_t\right) = dm_t - (X_t^{x,p})^\gamma \left[d\bar{A}_t - d\bar{A}_t^p\right]$$

with

$$d\bar{A}_t^p = \left[\left(\gamma p_t \mu_t + \frac{\gamma(\gamma-1)}{2} p_t^2 \sigma_t^2\right) \bar{J}_t + \gamma p_t \sigma_t \bar{Z}_t + \lambda_t((1+p_t\beta_t)^p - 1)(\bar{J}_t + \bar{U}_t)\right] dt$$

and (m_t) is a local martingale given by $m_0 = (X_0^{x,p})^\gamma \bar{J}_0$ and

$$dm_t = (X_{t^-}^{x,p})^\gamma \left[\left(\gamma p_t \sigma_t \bar{J}_{t^-} + \bar{Z}_t \right) dW_t + [[(1 + p_t\beta_t)^\gamma - 1](\bar{U}_t + \bar{J}_{t^-}) + \bar{U}_t] dM_t \right].$$

By integrating, we get :

$$(X_t^{x,p})^\gamma \bar{J}_t - (X_0^{x,p})^\gamma \bar{J}_0 = m_t - m_0 - \int_0^t (X_s^{x,p})^\gamma \left(d\bar{A}_s - d\bar{A}_s^p \right).$$

As $d\bar{A}_s \geq d\bar{A}_s^p$, we have

$$m_t \geq (X_t^{x,p})^\gamma \bar{J}_t$$

and as $(X_t^{x,p})^\gamma \bar{J}_t \geq 0$, we get

$$m_t \geq 0.$$

Therefore the process (m_t) is a lower bounded local martingale, thus (m_t) is a supermartingale and the process $\left((X_t^{x,p})^\gamma \bar{J}_t\right)$ is a supermartingale for each $p \in \mathcal{A}$. From Proposition 6.9, we can affirm that $J_t \leq \bar{J}_t$ $a.s.$ for all $t \in [0, T]$.

E Proof of Proposition 6.11

The proof of existence of process (J_t^k) is similar to the proof of Proposition 3.4. Thus we only show that the process (J_t^k) is bounded

$$J_t^k = \operatorname*{ess\,sup}_{p \in \mathcal{A}^k} E\left[\exp\left(\int_t^T \gamma \mu_s p_s ds + \int_t^T \gamma \sigma_s p_s dW_s - \frac{1}{2} \int_t^T \gamma |\sigma_s p_s|^2 ds + \int_t^T \gamma \log(1 + p_s \beta_s) dN_s \right) \middle| \mathcal{G}_t \right].$$

Let Q^p be the equivalent probability measure to $\mathbb{P}$ defined by the formula :

$$\frac{dQ^p}{d\mathbb{P}} = \exp\left(\int_0^T \gamma \sigma_s p_s dW_s - \frac{1}{2} \int_0^T |\gamma \sigma_s p_s|^2 ds \right).$$

Then we have :

$$J_t^k = \operatorname*{ess\,sup}_{p \in \mathcal{A}^k} E_{Q^p}\left[\exp\left(\int_t^T \gamma \mu_s p_s ds + \frac{\gamma^2 - \gamma}{2} \int_t^T |\sigma_s p_s|^2 ds + \int_t^T \gamma \log(1 + p_s \beta_s) dN_s \right) \middle| \mathcal{G}_t \right].$$

As the processes $(\mu_t), (\sigma_t)$ and (β_t) are supposed bounded, the process (J_t^k) is bounded.
These processes (J_t^k) have the same properties as the process (J_t) : for each strategy $p \in \mathcal{A}^k$, the process $((X_t^p)^\gamma J_t^k)$ is a supermartingale and there exits a strategy $\hat{p}$ such that the process $\left(\left(X_t^{\hat{p}}\right)^\gamma J_t^k\right)$ is a martingale.
As the process (J_t^k) is a bounded supermartingale, so it admits a Doob-Meyer decomposition :

$$dJ_t^k = Z_t^k dW_t + U_t^k dM_t - dA_t^k$$

with $Z^k \in L^2(W)$, $U^k \in L^2(M)$ and A^k is a nondecreasing $\mathbb{G}$-predictable process with $A_0^k = 0$.
Using the properties of process (J_t^k), we can determine the form of process (A_t^k) as in the previous appendix and we obtain a BSDE for which the process (J_t^k) is a solution. Since the set $\mathcal{A}^k$ is bounded, this BSDE admits a unique solution.

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