HAL archives-ouvertes.fr

Control of the Continuity Equation with a Non Local Flow

Rinaldo Colombo, Michael Herty, Magali Lécureux-Mercier

To cite this version:

Rinaldo Colombo, Michael Herty, Magali Lécureux-Mercier. Control of the Continuity Equation with a Non Local Flow. 2009. <hal-00361393>

HAL Id: hal-00361393

https://hal.archives-ouvertes.fr/hal-00361393

Submitted on 14 Feb 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Control of the Continuity Equation with a Non Local Flow

Rinaldo M. Colombo*, Michael Herty†, Magali Mercier‡

February 14, 2009

Abstract

This paper focuses on the optimal control of weak (i.e. in general *non smooth*) solutions to the continuity equation with non local flow. Our driving examples are a supply chain model and an equation for the description of pedestrian flows. To this aim, we prove the well posedness of a class of equations comprising these models. In particular, we prove the differentiability of solutions with respect to the initial datum and characterize its derivative. A necessary condition for the optimality of suitable integral functionals then follows.

2000 Mathematics Subject Classification: 35L65, 49K20, 93C20

Keywords: Optimal Control of the Continuity Equation; Non-Local Flows.

1 Introduction

We consider the continuity equation in N space dimensions

$$\begin{cases} \partial_t \rho + \operatorname{div}\big(\rho\, V(\rho)\big) = 0 \\ \rho(0,x) = \rho_o(x) \end{cases} \tag{1.1}$$

with a *non local* speed function V. This kind of equation appears in numerous examples, a first one being the supply chain model introduced in [3, 4], where $V(\rho) = v\left(\int_0^1 \rho(x)\,\mathrm{d}x\right)$. Besides, this equation is very similar to that obtained in a kinetic model of traffic, see [5]. Another example comes from pedestrian traffic, in which a reasonable model can be based on (1.1) with the functional $V(\rho) = v(\rho*\eta)\,\vec{v}(x)$. Throughout, our assumptions are modeled on these examples.

The first question we address is that of the well posedness of (1.1). Indeed, we show in Theorem 2.2 that (1.1) admits a unique local in time solution on a time interval I_{ex}. For all t in I_{ex}, we call S_t the nonlinear local semigroup that associates to the initial condition ρ_o the solution $S_t\rho_o$ of (1.1) at time t. As in the standard case, S_t turns out to be non expansive.

*Department of Mathematics, Brescia University, Via Branze 38, 25133 Brescia; Italy.

†RWTH Aachen University, Templergraben 55, 52056 Aachen, Germany

‡Université de Lyon, Université Lyon1, École Centrale de Lyon, INSA de Lyon, CNRS UMR 5208, Institut Camille Jordan; 43 blvd. du 11 Novembre 1918, F-69622 Villeurbanne Cedex, France

Then, we present a rigorous result on the Gâteaux differentiability of the map $\rho_o \mapsto S_t\rho_o$, in any direction r_o and for all $t \in I_{\mathrm{ex}}$. Moreover, the Gâteaux derivative is uniquely characterized as solution to the following linear Cauchy problem, that can be obtained by linearising formally (1.1):

$$\begin{cases} \partial_t r + \operatorname{div}\big(rV(\rho) + \rho\, \mathrm{D}V(\rho)(r)\big) = 0 \\ r(0,x) = r_o(x)\,. \end{cases} \tag{1.2}$$

The well posedness of (1.2) is among the results of this paper, see Proposition 2.9 below.

We stress here the difference with the well known standard (i.e. local) situation: the semigroup generated by a conservation law is in general *not* differentiable in $\mathbf{L^1}$, not even in the scalar 1D case, see [9, Section 1]. To cope with these issues, an entirely new differential structure was introduced in [9], and further developed in [6, 10], also addressing optimal control problems, see [11, 14]. We refer to [7, 8, 22, 28, 29] for further results and discussions about the scalar one–dimensional case. The presented theories, however, seem not able to yield a *"good"* optimality criteria. On the one hand, several results deal only with smooth solutions, whereas the rise of discontinuities is typical in conservation laws. On the other hand, the mere definition of the shift differential in the scalar 1D case takes alone about a page, see [14, p. 89–90]. Therefore, in the following we postulate assumptions on the function V which are satisfied in the cases of the supply chain model and of the pedestrian model, but not for general functions. To be more precise, we essentially require below that V is a *non local* function, see (2.3).

Then, based on the differentiability results, we state a necessary optimality condition. We introduce a cost function $\mathcal{J}\colon \boldsymbol{\mathcal{C}^0}\left(I_{\mathrm{ex}}, \mathbf{L^1}(\mathbb{R}^N;\mathbb{R})\right) \to \mathbb{R}$ and, using the differentiability property given above, we find a *necessary condition* on the initial data ρ_o in order to minimize $\mathcal{J}$ along the solutions to (1.1) associated to ρ_o.

We emphasize that all this is obtained within the framework of *non smooth* solutions, differently from many results in the current literature that are devoted to differentiability [21], control [18, 19] or optimal control [15] for conservation laws, but limited to smooth solutions. Furthermore, we stress that the present necessary conditions are obtained within the functional setting typical of scalar conservation laws, i.e. within $\mathbf{L^1}$ and $\mathbf{L^\infty}$. No reflexivity property is ever used.

The paper is organized as follows. In Section 2, we state the main results of this paper. The differentiability is proved in Theorem 2.11 and applied to a control in supply chain management in Theorem 3.2. The sections 3 and 4 provide examples of models based on (1.1), and in Section 5 we give the detailed proofs of our results.

2 Notation and Main Results

2.1 Existence of a Weak Solution to (1.1)

Denote $\mathbb{R}_+ = [0,+\infty[$, $\mathbb{R}_+^* =]0,+\infty[$ and by I, respectively I_* or I_{ex}, the interval $[0,T[$, respectively $[0,T_*[$ or $[0,T_{\mathrm{ex}}[$, for $T, T_*, T_{\mathrm{ex}} > 0$. Furthermore, we introduce the norms:

$$\begin{aligned} \|v\|_{\mathbf{L}^\infty} &= \sup_{x\in\mathbb{R}^N} \big\|v(x)\big\|, & \|v\|_{\mathbf{W^{1,1}}} &= \|v\|_{\mathbf{L^1}} + \|\nabla_x v\|_{\mathbf{L^1}}\,, \\ \|v\|_{\mathbf{W^{2,\infty}}} &= \|v\|_{\mathbf{L}^\infty} + \|\nabla_x v\|_{\mathbf{L}^\infty} + \left\|\nabla_x^2 v\right\|_{\mathbf{L}^\infty}, & \|v\|_{\mathbf{W^{1,\infty}}} &= \|v\|_{\mathbf{L}^\infty} + \|\nabla_x v\|_{\mathbf{L}^\infty}\,. \end{aligned}$$

Let $V\colon \mathbf{L^1}(\mathbb{R}^N;\mathbb{R}) \to \boldsymbol{\mathcal{C}^2}(\mathbb{R}^N;\mathbb{R}^N)$ be a functional, not necessarily linear. A straightforward extension of [25, Definition 1] yields the following definition of weak solutions for (1.1).

Definition 2.1 *A weak entropy solution to (1.1) on I_{ex} is a bounded measurable map ρ which is a Kružkov solution to*

$$\partial_t \rho + \operatorname{div}\big(\rho\, w(t,x)\big) = 0\,, \qquad \textit{where} \qquad w(t,x) = \Big(V\big(\rho(t)\big)\Big)(x)\,.$$

In other words, for all $k \in \mathbb{R}$ and for any test function $\varphi \in \boldsymbol{\mathcal{C}_c^\infty}(\mathring{I}_{\mathrm{ex}} \times \mathbb{R}^N;\mathbb{R}_+)$

$$\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\left[(\rho-k)\partial_t\varphi + (\rho-k)\,V\big(\rho(t)\big)(x)\cdot\nabla_x\varphi - \operatorname{div}\Big(k\,V\big(\rho(t)\big)(x)\Big)\,\varphi\right] \times \operatorname{sgn}(\rho-k)\;\mathrm{d}x\;\mathrm{d}t \geq 0$$

and there exists a set $\mathcal{E}$ of zero measure in $\mathbb{R}_+$ such that for all $t \in I_{\mathrm{ex}} \setminus \mathcal{E}$ the function ρ is defined almost everywhere in $\mathbb{R}^N$ and for any $\delta > 0$

$$\lim_{t\to 0,\, t\in I_{\mathrm{ex}}\setminus\mathcal{E}} \int_{B(0,\delta)} \big|\rho(t,x) - \rho_o(x)\big|\mathrm{d}x = 0\,.$$

The open ball in $\mathbb{R}^N$ centered at 0 with radius δ is denoted by $B(0,\delta)$. Introduce the spaces

$$\mathcal{X} = (\mathbf{L^1} \cap \mathbf{L^\infty} \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R}) \quad \text{and} \quad \mathcal{X}_\alpha = (\mathbf{L^1} \cap \mathbf{BV})\big(\mathbb{R}^N;[0,\alpha]\big) \text{ for } \alpha > 0$$

both equipped with the $\mathbf{L^1}$ distance. Obviously, $\mathcal{X}_\alpha \subset \mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R})$ for all $\alpha > 0$.

We pose the following assumptions on V, all of which are satisfied in the examples on supply chain and pedestrian flow as shown in Section 3 and Section 4, respectively.

(V1) There exists a function $C \in \mathbf{L^\infty_{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that for all $\rho \in \mathbf{L^1}(\mathbb{R}^N,\mathbb{R})$,

$$\begin{array}{rcl} V(\rho) & \in & \mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R}^N)\,, \\ \big\|\nabla_x V(\rho)\big\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R}^{N\times N})} & \leq & C(\|\rho\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R})})\,, \\ \big\|\nabla_x V(\rho)\big\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R}^{N\times N})} & \leq & C(\|\rho\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R})})\,, \\ \Big\|\nabla_x^2 V(\rho)\Big\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R}^{N\times N\times N})} & \leq & C(\|\rho\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R})})\,. \end{array}$$

There exists a function $C \in \mathbf{L^\infty_{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that for all $\rho_1,\rho_2 \in \mathbf{L^1}(\mathbb{R}^N,\mathbb{R})$

$$\begin{array}{rcl} \big\|V(\rho_1) - V(\rho_2)\big\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R}^N)} & \leq & C(\|\rho_1\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R})})\,\|\rho_1-\rho_2\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})}\,, \\ \big\|\nabla_x V(\rho_1) - \nabla_x V(\rho_2)\big\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R}^{N\times N})} & \leq & C(\|\rho_1\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R})})\,\|\rho_1-\rho_2\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})}\,. \end{array} \tag{2.3}$$

(V2) There exists a function $C \in \mathbf{L^\infty_{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that for all $\rho \in \mathbf{L^1}(\mathbb{R}^N,\mathbb{R})$,

$$\Big\|\nabla_x^2 V(\rho)\Big\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R}^{N\times N\times N})} \leq C(\|\rho\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R})})\,.$$

(V3) $V\colon \mathbf{L^1}(\mathbb{R}^N;\mathbb{R}) \to \boldsymbol{\mathcal{C}^3}(\mathbb{R}^N;\mathbb{R}^N)$ and there exists a function $C \in \mathbf{L^\infty_{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that for all $\rho \in \mathbf{L^1}(\mathbb{R}^N,\mathbb{R})$,

$$\Big\|\nabla_x^3 V(\rho)\Big\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R}^{N\times N\times N\times N})} \leq C(\|\rho\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R})})\,.$$

Condition (2.3) essentially requires that V be a *non local* operator. Note that **(V3)** implies **(V2)**. Existence of a solution to (1.1) (at least locally in time) can be proved under only assumption **(V1)**, see Theorem 2.2. The stronger bounds on V ensure additional regularity of the solution which is required later to derive the differentiability properties, see Proposition 2.5.

Theorem 2.2 *Let **(V1)** hold. Then, for all $\alpha, \beta > 0$ with $\beta > \alpha$, there exists a time $T(\alpha,\beta) > 0$ such that for all $\rho_o \in \mathcal{X}_\alpha$, problem (1.1) admits a unique solution $\rho \in \boldsymbol{\mathcal{C}}^{\mathbf{0}}\left([0,T(\alpha,\beta)];\mathcal{X}_\beta\right)$ in the sense of Definition 2.1. Moreover,*

1. $\left\|\rho(t)\right\|_{\mathbf{L}^\infty} \leq \beta$ *for all* $t \in [0, T(\alpha,\beta)]$.
2. *There exists a function $L \in \mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that for all $\rho_{o,1}, \rho_{o,2}$ in $\mathcal{X}_\alpha$, the corresponding solutions satisfy, for all $t \in [0,T(\alpha,\beta)]$,*
$$\left\|\rho_1(t) - \rho_2(t)\right\|_{\mathbf{L}^1} \leq L(t) \left\|\rho_{o,1} - \rho_{o,2}\right\|_{\mathbf{L}^1}$$
3. *There exists a constant $\mathcal{L} = \mathcal{L}(\beta)$ such that for all $\rho_o \in \mathcal{X}_\alpha$, the corresponding solution satisfies for all $t \in [0,T(\alpha,\beta)]$*
$$\mathrm{TV}\left(\rho(t)\right) \leq \left(\mathrm{TV}\left(\rho_o\right) + \mathcal{L}t\|\rho_o\|_{\mathbf{L}^\infty}\right) e^{\mathcal{L}t} \quad \textit{and} \quad \left\|\rho(t)\right\|_{\mathbf{L}^\infty} \leq \|\rho_o\|_{\mathbf{L}^\infty} e^{\mathcal{L}t}\,.$$

The above result is *local* in time. Indeed, as t tends to $T(\alpha,\beta)$, the total variation of the solution may well blow up. To ensure existence globally in time we need to introduce additional conditions on V:

(A) V is such that for all $\rho \in \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$ and all $x \in \mathbb{R}^N$, $\left(\operatorname{div} V(\rho)\right)(x) \geq 0$.

(B) The function C in **(V1)** is bounded, i.e. $C \in \mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R}_+)$.

Note that in the supply chain model discussed in Section 3, condition **(A)** applies.

Lemma 2.3 *Assume all assumptions of Theorem 2.2. Let also **(A)** hold. Then, for all $\alpha > 0$, the set $\mathcal{X}_\alpha$ is invariant for (1.1), i.e. if the initial datum ρ_o satisfies $\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})} \leq \alpha$, then, $\left\|\rho(t)\right\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})} \leq \alpha$ as long as the solution $\rho(t)$ exists.*

Condition **(B)**, although it does not guarantee the boundedness of the solution, does ensure the global existence of the solution to (1.1).

Theorem 2.4 *Let **(V1)** hold. Assume moreover that **(A)** or **(B)** holds. Then, there exists a unique semigroup $S\colon \mathbb{R}_+ \times \mathcal{X} \to \mathcal{X}$ with the following properties:*

(S1): For all $\rho_o \in \mathcal{X}$, the orbit $t \mapsto S_t\rho_o$ is a weak entropy solution to (1.1).

(S2): S is $\mathbf{L}^1$-continuous in time, i.e. for all $\rho_o \in \mathcal{X}$, the map $t \mapsto S_t\rho_o$ is in $\boldsymbol{\mathcal{C}}^{\mathbf{0}}(\mathbb{R}_+;\mathcal{X})$.

(S3): S is $\mathbf{L}^1$-Lipschitz with respect to the initial datum, i.e. for a suitable positive $L \in \mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}_+;\mathbb{R}_+)$, for all $t \in \mathbb{R}_+$ and all $\rho_1, \rho_2 \in \mathcal{X}$,
$$\|S_t\rho_1 - S_t\rho_2\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} \leq L(t)\, \|\rho_1 - \rho_2\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})}\,.$$

(S4): There exists a positive constant $\mathcal{L}$ such that for all $\rho_o \in \mathcal{X}$ and all $t \in \mathbb{R}_+$,

$$\mathrm{TV}\left(\rho(t)\right) \leq \Big(\mathrm{TV}\left(\rho_o\right) + \mathcal{L} t \left\|\rho_o\right\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})}\Big) e^{\mathcal{L} t}.$$

Higher regularity of the solutions of (1.1) can be proved under stronger bounds on V.

Proposition 2.5 *Let **(V1)** and **(V2)** hold. With the same notations as in Theorem 2.2, if $\rho_o \in \mathcal{X}_\alpha$, then*

$$\begin{array}{lllllllll} \rho_o & \in & (\mathbf{W^{1,1}} \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R}) & \Longrightarrow & \forall t & \in & [0, T(\alpha,\beta)], & \rho(t) & \in & \mathbf{W^{1,1}}(\mathbb{R}^N;\mathbb{R}), \\ \rho_o & \in & \mathbf{W}^{\mathbf{1},\infty}(\mathbb{R}^N;\mathbb{R}) & \Longrightarrow & \forall t & \in & [0, T(\alpha,\beta)], & \rho(t) & \in & \mathbf{W}^{\mathbf{1},\infty}(\mathbb{R}^N;\mathbb{R}), \end{array}$$

and there exists a positive constant $C = C(\beta)$ such that

$$\left\|\rho(t)\right\|_{\mathbf{W^{1,1}}} \leq e^{2Ct} \left\|\rho_o\right\|_{\mathbf{W^{1,1}}} \quad \text{and} \quad \left\|\rho(t)\right\|_{\mathbf{W}^{\mathbf{1},\infty}} \leq e^{2Ct} \left\|\rho_o\right\|_{\mathbf{W}^{\mathbf{1},\infty}}.$$

*Furthermore, if V also satisfies **(V3)**, then*

$$\rho_o \in (\mathbf{W^{2,1}} \cap \mathbf{L}^\infty)(\mathbb{R}^N;[\alpha,\beta]) \Longrightarrow \forall t \in [0, T(\alpha,\beta)], \quad \rho(t) \in \mathbf{W^{2,1}}(\mathbb{R}^N;\mathbb{R})$$

and for a suitable non–negative constant $C = C(\beta)$, we have the estimate

$$\left\|\rho(t)\right\|_{\mathbf{W^{2,1}}} \leq e^{Ct}(2e^{Ct} - 1)^2 \left\|\rho_o\right\|_{\mathbf{W^{2,1}}}.$$

The proofs are deferred to Section 5.

2.2 Differentiability

This section is devoted to the differentiability of the semigroup S (defined in Theorem 2.2) with respect to the initial datum ρ_o, according to the following notion. Recall first the following definition.

Definition 2.6 *A map $F: \mathbf{L^1}(\mathbb{R}^N;\mathbb{R}) \to \mathbf{L^1}(\mathbb{R}^N;\mathbb{R})$ is* strongly $\mathbf{L^1}$ Gâteaux differentiable in any direction *at $\rho_o \in \mathbf{L^1}(\mathbb{R}^N;\mathbb{R})$ if there exists a continuous linear map $DF(\rho_o): \mathbf{L^1}(\mathbb{R}^N;\mathbb{R}) \to \mathbf{L^1}(\mathbb{R}^N;\mathbb{R})$ such that for all $r_o \in \mathbf{L^1}(\mathbb{R}^N;\mathbb{R})$ and for any real sequence (h_n) with $h_n \to 0$,*

$$\frac{F(\rho_o + h_n r_o) - F(\rho_o)}{h_n} \overset{n\to\infty}{\to} DF(\rho_o)(r_o) \quad \text{strongly in } \mathbf{L^1}.$$

Besides proving the differentiability of the semigroup, we also characterize the differential. Formally, a sort of first order expansion of (1.1) with respect to the initial datum can be obtained through a standard linearization procedure, which yields (1.2). Now, we rigorously show that the derivative of the semigroup in the direction r_o is indeed the solution to (1.2) with initial condition r_o. To this aim, we need a forth and final condition on V.

(V4) V is Fréchet differentiable as a map $\mathbf{L^1}(\mathbb{R}^N;\mathbb{R}_+) \to \boldsymbol{\mathcal{C}}^{\mathbf{2}}(\mathbb{R}^N;\mathbb{R}^N)$ and there exists a function $K \in \mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that for all $\rho \in \mathbf{L^1}(\mathbb{R}^N;\mathbb{R}_+)$, for all $r \in \mathbf{L^1}(\mathbb{R}^N;\mathbb{R})$,

$$\begin{aligned} \left\|V(\rho + r) - V(\rho) - DV(\rho)(r)\right\|_{\mathbf{W}^{\mathbf{2},\infty}} &\leq K\left(\left\|\rho\right\|_{\mathbf{L}^\infty} + \left\|\rho + r\right\|_{\mathbf{L}^\infty}\right) \left\|r\right\|^2_{\mathbf{L^1}}, \\ \left\|DV(\rho)(r)\right\|_{\mathbf{W}^{\mathbf{2},\infty}} &\leq K\left(\left\|\rho\right\|_{\mathbf{L}^\infty}\right) \left\|r\right\|_{\mathbf{L^1}}. \end{aligned}$$

Consider now system (1.2), where $\rho \in \boldsymbol{\mathcal{C}}^{\mathbf{0}}(I_{\text{ex}}, \mathcal{X})$ is a given function. We introduce a notion of solution for (1.2) and give conditions which guarantee the existence of a solution.

Definition 2.7 *Fix $r_o \in \mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})$. A function $r \in \mathbf{L}^\infty\big(I_{\text{ex}}; \mathbf{L}^{\mathbf{1}}_{\mathbf{loc}}(\mathbb{R}^N;\mathbb{R}_+)\big)$ bounded, mesurable and right continuous in time, is a* weak solution *to (1.2) if for any test function $\varphi \in \boldsymbol{\mathcal{C}}^{\boldsymbol{\infty}}_{\boldsymbol{c}}(\mathring{I}_{\text{ex}} \times \mathbb{R}^N;\mathbb{R})$*

$$
\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}^N} \left[r\,\partial_t\varphi + r\,a(t,x)\cdot\nabla_x\varphi - \operatorname{div} b(t,x)\varphi\right] \mathrm{d}x\,\mathrm{d}t = 0\,, \qquad \textit{and}\\
&r(0) = r_o \qquad \textit{a.e. in } \mathbb{R}^N,
\end{aligned}
\tag{2.4}
$$

where $a = V(\rho)$ and $b = \rho DV(\rho)(r)$.

We now extend the classical notion of Kružkov solution to the present non local setting.

Definition 2.8 *Fix $r_o \in \mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}_+)$. A function $r \in \mathbf{L}^\infty\big(I_{\text{ex}}; \mathbf{L}^{\mathbf{1}}_{\mathbf{loc}}(\mathbb{R}^N;\mathbb{R}_+)\big)$ bounded, mesurable and right continuous in time, is a* Kružkov solution to the nonlocal problem (1.2) *if it is a Kružkov solution to*

$$
\left\{
\begin{array}{l}
\partial_t r + \operatorname{div}\big(r\,a(t,x) + b(t,x)\big) = 0\\
r(0,x) = r_o(x)
\end{array}
\right.
\tag{2.5}
$$

where $a = V(\rho)$ and $b = \rho DV(\rho)(r)$.

In other words, r is a Kružkov solution to (1.2) if for all $k \in \mathbb{R}$ and for any test function $\varphi \in \boldsymbol{\mathcal{C}}^{\boldsymbol{\infty}}_{\boldsymbol{c}}(\mathring{I}_{\text{ex}} \times \mathbb{R}^N;\mathbb{R}_+)$

$$
\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\Big[(r-k)\partial_t\varphi + (r-k)V(\rho)\cdot\nabla_x\varphi - \operatorname{div}\big(kV(\rho)+\rho DV(\rho)r\big)\,\varphi\Big]\operatorname{sgn}(r-k)\,\mathrm{d}x\,\mathrm{d}t \geq 0
$$

$$
\text{and}\quad \lim_{t\to 0^+}\int_{B(0,\delta)} \big|r(t)-r_o\big|\,\mathrm{d}x = 0 \qquad \text{for all } \delta > 0\,.
$$

Condition **(V4)** ensures that if $\rho \in \mathbf{W}^{\mathbf{1},\mathbf{1}}(\mathbb{R}^N;\mathbb{R})$, then $DV(\rho)(r) \in \boldsymbol{\mathcal{C}}^{\mathbf{2}}(\mathbb{R}^N;\mathbb{R}^N)$ and hence for all $t \geq 0$, the map $x \mapsto \rho(t,x)\,DV\big(\rho(t)\big)\,r(t,x)$ is in $\mathbf{W}^{\mathbf{1},\mathbf{1}}(\mathbb{R}^N;\mathbb{R})$, so that the integral above is meaningful.

Proposition 2.9 *Let **(V1)** and **(V4)** hold. Fix $\rho \in \boldsymbol{\mathcal{C}}^{\mathbf{0}}\big(I_{\text{ex}}; (\mathbf{W}^{\mathbf{1},\infty}\cap\mathbf{W}^{\mathbf{1},\mathbf{1}})(\mathbb{R}^N;\mathbb{R})\big)$. Then, for all $r_o \in (\mathbf{L}^{\mathbf{1}}\cap\mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$ there exists a unique weak entropy solution to (1.2) in $\mathbf{L}^\infty\big(I_{\text{ex}};\mathbf{L}^{\mathbf{1}}(\mathbb{R}^N;\mathbb{R})\big)$ continuous from the right in time, and for all time $t \in I_{\text{ex}}$, with $C = C\left(\|\rho\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^N;\mathbb{R})}\right)$ as in **(V1)** and $K = K\left(\|\rho\|_{\mathbf{L}^\infty([0,t]\times\mathbb{R}^N;\mathbb{R})}\right)$ as in **(V4)***

$$
\begin{aligned}
\big\|r(t)\big\|_{\mathbf{L}^{\mathbf{1}}} &\leq e^{Kt\|\rho\|_{\mathbf{L}^\infty(I;\mathbf{W}^{\mathbf{1},\mathbf{1}})}}\,e^{Ct}\,\|r_o\|_{\mathbf{L}^{\mathbf{1}}}\\
\big\|r(t)\big\|_{\mathbf{L}^\infty} &\leq e^{Ct}\,\|r_o\|_{\mathbf{L}^\infty} + K\,t\,e^{2Ct}\,e^{Kt\|\rho\|_{\mathbf{L}^\infty(I;\mathbf{W}^{\mathbf{1},\mathbf{1}})}}\,\|\rho\|_{\mathbf{L}^\infty(I;\mathbf{W}^{\mathbf{1},\infty})}\,\|r_o\|_{\mathbf{L}^{\mathbf{1}}}\,.
\end{aligned}
$$

*If **(V2)** holds, $\rho \in \mathbf{L}^\infty\big(I_{\text{ex}};(\mathbf{W}^{\mathbf{1},\infty}\cap\mathbf{W}^{\mathbf{2},\mathbf{1}})(\mathbb{R}^N;\mathbb{R})\big)$ and $r_o \in (\mathbf{W}^{\mathbf{1},\mathbf{1}}\cap\mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$, then for all $t \in I_{\text{ex}}$, $r(t) \in \mathbf{W}^{\mathbf{1},\mathbf{1}}(\mathbb{R}^N;\mathbb{R})$ and*

$$
\big\|r(t)\big\|_{\mathbf{W}^{\mathbf{1},\mathbf{1}}} \leq (1+C't)\,e^{2C't}\,\|r_o\|_{\mathbf{W}^{\mathbf{1},\mathbf{1}}} + Kt(1+Ct)\,e^{4C't}\,\|r_o\|_{\mathbf{L}^{\mathbf{1}}}\,\|\rho\|_{\mathbf{L}^\infty(I;\mathbf{W}^{\mathbf{2},\mathbf{1}})}\,.
$$

where $C' = \max\big\{C, K\|\rho\|_{\mathbf{L}^\infty(I_{\text{ex}};\mathbf{W}^{\mathbf{2},\mathbf{1}}(\mathbb{R}^N;\mathbb{R}))}\big\}$. Furthermore, full continuity in time holds: $r \in \boldsymbol{\mathcal{C}}^{\mathbf{0}}(I_{\text{ex}};\mathbf{L}^{\mathbf{1}}(\mathbb{R}^N;\mathbb{R}))$.

With these tools, we can now state a theorem about the weak Gâteaux differentiability.

Theorem 2.10 *Let **(V1)** and **(V4)** hold. Let $\rho_o \in (\mathbf{W^{1,\infty}} \cap \mathbf{W^{1,1}})(\mathbb{R}^N;\mathbb{R})$, and denote $T_{\rm ex}$ the time of existence for the solution of (1.1). Then, for all time $t \in I_{ex}$, for all $r_o \in \mathcal{X}$ and all sequences $(h_n)_{n\in\mathbb{N}}$ converging to 0, there exists a subsequence of $\left(\frac{1}{h_n}\left(S_t(\rho_o + h_n r_o) - S_t(\rho_o)\right)\right)_{n\in\mathbb{N}}$ that converges weakly in $\mathbf{L^1}$ to a weak solution of (1.2).*

This theorem does not guarantee the uniqueness of this kind of *weak* $\mathbf{L^1}$ *Gâteaux derivative*. Therefore, we consider the following stronger hypothesis, under which we derive a result of strong Gâteaux differentiability and uniqueness of the derivative.

(V5) There exists a function $K \in \mathbf{L^\infty_{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that $\forall \rho, \tilde\rho \in \mathbf{L^1}(\mathbb{R}^N;\mathbb{R})$

$$\left\|\mathrm{div}\left(V(\tilde\rho) - V(\rho) - DV(\rho)(\tilde\rho - \rho)\right)\right\|_{\mathbf{L^1}} \le K\left(\|\rho\|_{\mathbf{L^\infty}} + \|\tilde\rho\|_{\mathbf{L^\infty}}\right)\left(\|\tilde\rho - \rho\|_{\mathbf{L^1}}\right)^2$$

and the map $r \to \mathrm{div}\, DV(\rho)(r)$ is a bounded linear operator on $\mathbf{L^1}(\mathbb{R}^N;\mathbb{R}) \to \mathbf{L^1}(\mathbb{R}^N;\mathbb{R})$, i.e. $\forall \rho, r \in \mathbf{L^1}(\mathbb{R}^N;\mathbb{R})$

$$\left\|\mathrm{div}\left(DV(\rho)(r)\right)\right\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})} \le K\left(\|\rho\|_{\mathbf{L^\infty}(\mathbb{R}^N;\mathbb{R})}\right)\|r\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})}.$$

Theorem 2.11 *Let **(V1)**, **(V3)**, **(V4)** and **(V5)** hold. Let $\rho_o \in (\mathbf{W^{1,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R}^N;\mathbb{R})$, $r_o \in (\mathbf{W^{1,1}} \cap \mathbf{L^\infty})(\mathbb{R}^N;\mathbb{R})$, and denote $T_{\rm ex}$ the time of existence of the solution of (1.1) with initial condition ρ_o. Then, for all time $t \in I_{\rm ex}$ the local semigroup defined in Theorem 2.2 is strongly $\mathbf{L^1}$ Gâteaux differentiable in the direction r_o. The derivative $DS_t(\rho_o)(r_o)$ of S_t at ρ_o in the direction r_o is*

$$DS_t(\rho_o)(r_o) = \Sigma_t^{\rho_o}(r_o).$$

where Σ^{ρ_o} is the linear application generated by the Kružkov solution to (1.2), where $\rho = S_t\rho_o$, then for all $t \in I_{\rm ex}$.

2.3 Necessary Optimality Conditions for Problems Governed by (1.1)

Aiming at necessary optimality conditions for non linear functionals defined on the solutions to (1.1), we prove the following chain rule formula.

Proposition 2.12 *Let $T > 0$ and $I = [0, T[$. Assume that $f \in \boldsymbol{\mathcal{C}^{1,1}}(\mathbb{R};\mathbb{R}_+)$, $\psi \in \mathbf{L^\infty}(I \times \mathbb{R}^N;\mathbb{R})$ and that $S: I \times (\mathbf{L^1} \cap \mathbf{L^\infty})(\mathbb{R}^N;\mathbb{R}) \to (\mathbf{L^1} \cap \mathbf{L^\infty})(\mathbb{R}^N;\mathbb{R})$ is strongly $\mathbf{L^1}$ Gâteaux differentiable. For all $t \in I$, let*

$$J(\rho_o) = \int_{\mathbb{R}^N} f\left(S_t\rho_o\right)\psi(t,x)\,\mathrm{d}x\,. \tag{2.6}$$

Then, J is strongly $\mathbf{L^\infty}$ Gâteaux differentiable in any direction $r_o \in (\mathbf{W^{1,1}} \cap \mathbf{L^\infty})(\mathbb{R}^N;\mathbb{R})$. Morever,

$$DJ(\rho_o)(r_o) = \int_{\mathbb{R}^N} f'(S_t\rho_o)\,\Sigma_t^{\rho_o}(r_o)\psi(t,x)\,\mathrm{d}x\,.$$

Proof. Since $\left|f(\rho_h) - f(\rho) - f'(\rho)(\rho_h - \rho)\right| \leq \mathrm{Lip}(f')\,|\rho_h - \rho|^2$, we have

$$\begin{aligned}
&\left|\frac{J(\rho_o + hr_o) - J(\rho_o)}{h} - \int_{\mathbb{R}^N} f'(S_t\rho_o)\, DS_t(\rho_o)(r_o)\,\psi(t,x)\;\mathrm{d}x\right| \\
\leq\; & \int_{\mathbb{R}^N} \left|f'(S_t\rho_o)\right| \left|\frac{S_t(\rho_o + hr_o) - S_t(\rho_o)}{h} - DS_t(\rho_o)(r_o)\right| \left|\psi(t,x)\right|\;\mathrm{d}x \\
& + \mathrm{Lip}(f')\,\frac{1}{|h|} \int_{\mathbb{R}^N} \left|S_t(\rho_o + hr_o) - S_t(\rho_o)\right|^2 \left|\psi(t,x)\right|\;\mathrm{d}x\,.
\end{aligned}$$

The strong Gâteaux differentiability of S_t in $\mathbf{L^1}$ then yields

$$\int_{\mathbb{R}^N} \left|f'(S_t\rho_o)\right| \left|\frac{S_t(\rho_o + hr_o) - S_t(\rho_o)}{h} - DS_t(\rho_o)(r_o)\right| \left|\psi(t,x)\right|\;\mathrm{d}x = o(1) \qquad \text{as } h \to 0$$

thanks to $S_t\rho_o \in \mathbf{L}^\infty$ and to the local boundedness of f'. Furthermore,

$$\begin{array}{rcl}
S_t(\rho_o), S_t(\rho_o + hr_o) & \in & \mathbf{L}^\infty \\
\dfrac{1}{h}\left(S_t(\rho_o + hr_o) - S_t(\rho_o)\right) & \overset{h\to 0}{\longrightarrow} & DS_t(\rho_o)(r_o) \text{ pointwise a.e.} \\
S_t(\rho_o + hr_o) - S_t(\rho_o) & \overset{h\to 0}{\longrightarrow} & 0 \text{ pointwise a.e.}
\end{array}$$

the Dominated Convergence Theorem ensures that the higher order term in the latter expansion tend to 0 as $h \to 0$. $\square$

The above result can be easily extended. First, to more general (non linear) functionals $J(\rho_o) = \mathcal{J}(S_t\rho_o)$, with $\mathcal{J}$ satisfying

(J) $\mathcal{J}\colon \mathcal{X} \to \mathbb{R}_+$ is Fréchet differentiable: for all $\rho \in \mathcal{X}$ there exists a continuous linear application $D\mathcal{J}(\rho)\colon \mathcal{X} \to \mathbb{R}$ such that for all $\rho, r \in \mathcal{X}$:

$$\left|\frac{\mathcal{J}(\rho + hr) - \mathcal{J}(\rho)}{h} - D\mathcal{J}(\rho)(r)\right| \overset{h\to 0}{\longrightarrow} 0\,.$$

Secondly, to functionals of the type

$$J(\rho_o) = \int_0^T \int_{\mathbb{R}^N} f(S_t\rho_o)\,\psi(t,x)\;\mathrm{d}x\;\mathrm{d}t \qquad \text{or} \qquad J(\rho_o) = \int_0^T \mathcal{J}(S_t\rho_o)\;\mathrm{d}t\,.$$

This generalization, however, is immediate and we omit the details.

Once the differentiability result above is available, a necessary condition of optimality is straightforward.

Proposition 2.13 *Let $f \in \mathcal{C}^{\mathbf{1,1}}(\mathbb{R};\mathbb{R}_+)$ and $\psi \in \mathbf{L}^\infty(I_{\mathrm{ex}} \times \mathbb{R}^N;\mathbb{R})$. Assume that $S\colon I \times (\mathbf{L^1} \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R}) \to (\mathbf{L^1} \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$ is strongly $\mathbf{L^1}$ Gâteaux differentiable. Define J as in (2.6). If $\rho_o \in (\mathbf{L^1} \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$ solves the problem*

$$\textit{find } \min_{\rho_o} \mathcal{J}(\rho) \textit{ subject to } \{\rho \textit{ is solution to (1.1)}\}.$$

then, for all $r_o \in (\mathbf{L^1} \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$

$$\int_{\mathbb{R}^N} f'(S_t\rho_o)\,\Sigma_t^{\rho_o} r_o\,\psi(t,x)\;\mathrm{d}x = 0\,. \tag{2.7}$$

3 Demand Tracking Problems for Supply Chains

Recently, Armbruster et al. [4], introduced a continuum model to simulate the average behavior of highly re-entrant production systems at an aggregate level appearing, for instance, in large volume semiconductor production line. The factory is described by the density of products $\rho(x,t)$ at stage x of the production at a time t. Typically, see [1, 4, 24], the production velocity V is a given smooth function of the total load $\int_0^1 \rho(t,x)\,\mathrm{d}x$, for example

$$v(u) = v_{\max}/(1+u) \quad \text{and} \quad V(\rho) = v\left(\int_0^1 \rho(t,s)\,ds\right). \tag{3.8}$$

The full model, given by (1.1)–(3.8) with $N=1$, fits in the present framework.

Proposition 3.1 *Let $v \in \boldsymbol{C}^{\mathbf{1}}\left([0,1];\mathbb{R}\right)$. Then, the functional V defined as in (3.8) satisfies **(A)**, **(V1)**, **(V2)**, **(V3)**. Moreover, if $v \in \boldsymbol{C}^{\mathbf{2}}\left([0,1];\mathbb{R}\right)$, then V satisfies also **(V4)** and **(V5)**.*

The proof is deferred to Paragraph 5.4.

The supply chain model with V given by (3.8) satisfies **(V1)** to **(V5)** and **(A)**. Therefore, Theorem 2.4 applies and, in particular, the set $[0,1]$ is invariant yielding global well posedness. By Theorem 2.11, the semigroup $S_t\rho_o$ is Gâteaux differentiable in any direction r_o and the differential is given by the solution to (1.2).

Note that the velocity is constant across the entire system at any time. In fact, in a real world factory, all parts move through the factory with the same speed. While in a serial production line, speed through the factory is dependent on all items and machines downstream, in a highly re-entrant factory this is not the case. Since items must visit machines more than once, including machines at the beginning of the production process, their speed through factory is determined by the total number of parts both upstream and downstream from them. Such re-entrant production is characteristic for semiconductor fabs. Typically, the output of the whole factory over a longer timescale, e.g. following a seasonal demand pattern or ramping up or down a new product, can be controlled by prescribing the inflow density to a factory $\rho(t,x=0)=\lambda(t)$. The influx should be chosen in order to achieve either of the following objective goals [4]:

(1) Minimize the mismatch between the outflux and a demand rate target $d(t)$ over a fixed time period (demand tracking problem). This is modelled by the cost functional $\frac{1}{2}\int_0^T \left(d(t)-\rho(1,t)\right)^2\,dt$.

(2) Minimize the mismatch between the total number of parts that have left the factory and the desired total number of parts over a fixed time period $d(t)$. The backlog of a production system at a given time t is defined as the total number of items that have been demanded minus the total number of items that have left the factory up to that time. Backlog can be negative or positive, with a negative backlog corresponding to overproduction and a positive backlog corresponding to a shortage. This problem is modeled by $\frac{1}{2}\int_0^T \left(\int_0^t d(\tau)u - \rho(1,\tau)d\tau\right)^2 dt$.

In both cases we are interested in the influx $\lambda(t)$. A numerical integration of this problem has been studied in [26]. In order to apply the previous calculus we reformulate the optimization problem for the influx density $\lambda(t) = \rho_o(-t)$ where ρ_0 is the solution to a minimization problem for

$$\begin{aligned} J_1(\rho_o) &= \frac{1}{2}\int_0^1 (d(x) - S_T\rho_o(x))^2 \,\mathrm{d}x \\ J_2(\rho_o) &= \frac{1}{2}\int_0^1 \left(\int_0^x \bigl(d(\xi) - S_T\rho_o(\xi)\bigr)\,\mathrm{d}\xi\right)^2 \mathrm{d}x\,, \end{aligned} \tag{3.9}$$

respectively, where $S_t\rho_o$ is the solution to (1.1) and (3.8). Clearly, J_1 and J_2 satisfy the assumptions imposed in the previous section. The assertions of Proposition 2.13 then state necessary optimality conditions, which we summarize in the theorem below.

Theorem 3.2 *Let $T > 0$ be given. Let the assumptions of Proposition 3.1 hold. Let $\rho_o \in (\mathbf{W^{1,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R};\mathbb{R})$ be a minimizer of J_1 as defined in (3.9), with S being the semigroup generated by (1.1)–(3.8). Then, for all $r_o \in (\mathbf{W^{1,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R};\mathbb{R})$ we have*

$$\begin{aligned} \int_0^1 \bigl(d(x) - \rho(T,x)\bigr)\, r(T,x)\,\mathrm{d}x &= 0\,, \quad \textit{where} \\ \partial_t r + \partial_x \left(v_{\max} \frac{\left(r\int_0^1 \rho\,\mathrm{d}x + \rho\int_0^1 r\,\mathrm{d}x\right)}{\left(1+\int_0^1 \rho\,\mathrm{d}x\right)^2}\right) &= 0\,, \quad r(0,x) = r_o(x)\,. \end{aligned}$$

The latter Cauchy problem is in the form (1.2) and Proposition 2.9 proves its well posedness. The latter proof is deferred to Paragraph 5.4.

4 A Model for Pedestrian Flow

Macroscopic models for pedestrian movements are based on the continuity equation, see [17, 23, 27], possibly together with a second equation, as in [20]. In these models, pedestrians are assumed to instantaneously adjust their (vector) speed according to the crowd density at their position. The analytical construction in Section 2 allows to consider the more realistic situation of pedestrian deciding their speed according to the local mean density at their position. We are thus led to consider (1.1) with

$$V(\rho) = v(\rho * \eta)\,\vec{v} \tag{4.10}$$

where

$$\eta \in \boldsymbol{\mathcal{C}}^{\mathbf{2}}_{\mathbf{c}}\bigl(\mathbb{R}^2;[0,1]\bigr) \text{ has support } \operatorname{spt}\eta \subseteq B(0,1) \text{ and } \|\eta\|_{\mathbf{L^1}} = 1\,, \tag{4.11}$$

so that $(\rho * \eta)(x)$ is an average of the values attained by ρ in $B(x,1)$. Here, $\vec{v} = \vec{v}(x)$ is the given direction of the motion of the pedestrian at $x \in \mathbb{R}^2$. Then, the presence of boundaries, obstacles or other geometric constraint can be described through $\vec{v}$, see [13, 27].

Note here that condition **(A)** is unphysical, for it does not allow any increase in the crowd density. Hence, for this example we have only a local in time solution by Theorem 2.2.

As in the preceding example, first we state the hypotheses that guarantee assumptions **(V1)** to **(V5)**.

Proposition 4.1 *Let V be defined in (4.10) and η be as in (4.11).*

1. *If $v \in \boldsymbol{\mathcal{C}^2}(\mathbb{R};\mathbb{R})$ and $\vec v \in (\boldsymbol{\mathcal{C}^2} \cap \mathbf{W^{2,1}})(\mathbb{R}^2;\mathbb{S}^1)$, then V satisfies **(V1)** and **(V2)**.*
2. *If moreover $v \in \boldsymbol{\mathcal{C}^3}(\mathbb{R};\mathbb{R})$, $\vec v \in \boldsymbol{\mathcal{C}^3}(\mathbb{R}^2;\mathbb{R}^2)$ and $\eta \in \boldsymbol{\mathcal{C}^3}(\mathbb{R}^2;\mathbb{R})$ then V satisfies **(V3)**.*
3. *If moreover $v \in \boldsymbol{\mathcal{C}^4}(\mathbb{R};\mathbb{R})$, $\vec v \in \boldsymbol{\mathcal{C}^2}(\mathbb{R}^2;\mathbb{R}^2)$ and $\eta \in \boldsymbol{\mathcal{C}^2}(\mathbb{R}^2;\mathbb{R})$, then V satisfies **(V4)** and **(V5)**.*

The proof is deferred to Paragraph 5.4.

A typical problem in the management of pedestrian flows consists in keeping the crowd density $\rho(t,x)$ below a given threshold, say $\hat\rho$, in particular in a sensible compact region Ω. To this aim, it is natural to introduce a cost functional of the type

$$J(\rho_o) = \int_0^T \int_{\mathbb{R}^N} f\left(S_t\rho_o(x)\right)\, \psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \tag{4.12}$$

where

(f) $f \in \boldsymbol{\mathcal{C}^{1,1}}(\mathbb{R};\mathbb{R}_+)$, $f(\rho) = 0$ for $\rho \in [0,\hat\rho]$, $f(\rho) > 0$ and $f'(\rho) > 0$ for $\rho > \hat\rho$.

($\boldsymbol{\psi}$) $g \in \boldsymbol{\mathcal{C}^\infty}(\mathbb{R}^N;[0,1])$, with $\operatorname{spt} g = \Omega$, is a smooth approximation of the characteristic function of the compact set Ω, with $\mathring{\Omega} \neq \emptyset$.

Paragraph 2.3 then applies, yielding the following necessary condition for optimality.

Theorem 4.2 *Let $T > 0$ and the assumptions of 1.–3. in Proposition 4.1 hold, together with **(f)** and **($\boldsymbol{\psi}$)**. Let $\rho_o \in (\mathbf{W^{1,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R};\mathbb{R})$ be a minimizer of J as defined in (4.12), with S being the semigroup generated by (1.1)–(4.10). Then, for all $r_o \in (\mathbf{W^{1,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R};\mathbb{R})$, ρ_o satisfies (2.7).*

The proof is deferred to Paragraph 5.4.

5 Detailed Proofs

Below, we denote by W_N the Wallis integral

$$W_N = \int_0^{\pi/2} (\cos\alpha)^N \,\mathrm{d}\alpha\ . \tag{5.13}$$

5.1 A Lemma on the Transport Equation

The next lemma is similar to other results in recent literature, see for instance [2].

Lemma 5.1 *Let $T > 0$, so that $I = [0,T[$, and w be such that*

$$\begin{array}{rclcrcl} w & \in & \boldsymbol{\mathcal{C}^0}(I\times\mathbb{R}^N;\mathbb{R}^N) & & w(t) & \in & \boldsymbol{\mathcal{C}^2}(\mathbb{R}^N;\mathbb{R}^N) \quad \forall t \in I \\ w & \in & \mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R}^N) & & \nabla_x w & \in & \mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R}^{N\times N})\,. \end{array} \tag{5.14}$$

Assume that $R \in \mathbf{L}^\infty\left(I;\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})\right) \cap \mathbf{L}^\infty\left(I\times\mathbb{R}^N;\mathbb{R}\right)$. Then, for any $r_o \in (\mathbf{L^1} \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$, the Cauchy problem

$$\left\{\begin{array}{l} \partial_t r + \operatorname{div}\left(r\, w(t,x)\right) = R(t,x) \\ r(0,x) = r_o(x) \end{array}\right. \tag{5.15}$$

admits a unique Kružkov solution $r \in \mathbf{L}^\infty\big(I;\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})\big)$, *continuous from the right in time, given by*

$$
\begin{aligned}
r(t,x) \quad &= \quad r_o\left(X(0;t,x)\right)\,\exp\left(-\int_0^t \operatorname{div} w\left(\tau, X(\tau;t,x)\right)\mathrm{d}\tau\right) \\
&\quad + \int_0^t R\left(\tau, X(\tau;t,x)\right)\exp\left(-\int_\tau^t \operatorname{div} w\left(u, X(u;t,x)\right)\mathrm{d}u\right)\mathrm{d}\tau\ ,
\end{aligned}
\tag{5.16}
$$

where $t \mapsto X(t;t_o,x_o)$ *is the solution to the Cauchy problem*

$$
\begin{cases}
\dfrac{d\chi}{dt} = w(t,\chi) \\
\chi(t_o) = x_o\,.
\end{cases}
\tag{5.17}
$$

Note that the expression (5.16) is formally justified integrating (5.15) along the characteristics (5.17) and obtaining

$$
\frac{d}{dt}\Big(r\left(t,\chi(t)\right)\Big) + r\left(t,\chi(t)\right)\operatorname{div} w\left(t,\chi(t)\right) = R\left(t,\chi(t)\right).
$$

Recall for later use that the flow $X = X(t;t_o,x_o)$ generated by (5.17) can be used to introduce the change of variable $y = X(0;t,x)$, so that $x = X(t;0,y)$, due to standard properties of the Cauchy problem (5.17). Denote by $J(t,y) = \det\big(\nabla_y X(t;0,y)\big)$ the Jacobian of this change of variables. Then, J satisfies the equation

$$
\frac{dJ(t,y)}{dt} = \operatorname{div} w\left(t, X(t;0,y)\right)\ J(t,y)
\tag{5.18}
$$

with initial condition $J(0,y) = 1$. Hence $J(t,y) = \exp\left(\int_0^t \operatorname{div} w\left(\tau, X(\tau;0,y)\right)\mathrm{d}\tau\right)$ which, in particular, implies $J(t,y) > 0$ for all $t \in I$, $y \in \mathbb{R}^N$.

The natural modification to the present case of [25, Definition 1] is the following.

Definition 5.2 *Let* $T > 0$, *so that* $I = [0,T[$, *and fix the maps* $w \in \boldsymbol{\mathcal{C}^0}(I\times\mathbb{R}^N;\mathbb{R})$ *as in (5.14) and* $R \in \mathbf{L}^\infty\big(I;\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})\big) \cap \mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R})$. *Choose an initial datum* $r_o \in \mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})$. *A bounded mesurable map* $r \in \mathbf{L}^\infty\big(I;\mathbf{L^1_{loc}}(\mathbb{R}^N;\mathbb{R})\big)$, *continuous from the right in time, is a* Kružkov solution *to (5.15) if for all* $k \in \mathbb{R}$, *for all test function* $\varphi \in \boldsymbol{\mathcal{C}_c^\infty}(]0,T[\times\mathbb{R}^N;\mathbb{R}_+)$

$$
\int_0^{+\infty}\!\!\int_{\mathbb{R}^N}\big[(r-k)(\partial_t\varphi + w\cdot\nabla_x\varphi) + (R - k\operatorname{div} w)\varphi\big]\operatorname{sgn}(r-k)\,\mathrm{d}x\ \mathrm{d}t \geq 0
\tag{5.19}
$$

and there exists a set $\mathcal{E}$ *of zero measure in* $\mathbb{R}_+$ *such that for* $t \in \mathbb{R}_+\backslash\mathcal{E}$ *the function* r *is defined almost everywhere in* $\mathbb{R}^N$ *and for any* $\delta > 0$

$$
\lim_{t\to 0, t\in]0,T[\backslash\mathcal{E}} \int_{B(0,\delta)} \left|r(t,x) - r_o(x)\right|\mathrm{d}x = 0\,.
\tag{5.20}
$$

Proof of Lemma 5.1. The proof consists of several steps.

1. (5.19) holds.
Let $k \in \mathbb{R}$ and $\varphi \in \boldsymbol{\mathcal{C}_c^\infty}(\mathring{I}\times\mathbb{R}^N;\mathbb{R}_+)$. Then, according to Definition 5.2, we prove (5.19)

for r given as in (5.16). By (5.18), the semigroup property of X and denoting $\mathcal{R}(t,y) = \int_0^t R\left(\tau, X(\tau;0,y)\right) J(\tau,y)\,\mathrm{d}\tau$, we get

$$
\begin{aligned}
& \int_0^{+\infty}\!\!\int_{\mathbb{R}^N} \left[(r-k)(\partial_t\varphi + w\cdot\nabla_x\varphi) + (R - k\operatorname{div} w)\varphi\right]\operatorname{sgn}(r-k)\,\mathrm{d}x\;\mathrm{d}t \\
= & \int_0^{+\infty}\!\!\int_{\mathbb{R}^N} \Bigg[\left(\frac{r_o(y)}{J(t,y)} + \frac{\mathcal{R}(t,y)}{J(t,y)} - k\right) \\
& \times\left(\partial_t\varphi\left(t,X(t;0,y)\right) + w\left(t,X(t;0,y)\right)\cdot\nabla_x\varphi\left(t,X(t;0,y)\right)\right) \\
& + \left(R\left(t,X(t;0,y)\right) - k\operatorname{div}\left(w\left(t,X(t;0,y)\right)\right)\right)\varphi\left(t,X(t;0,y)\right)\Bigg] \\
& \qquad \times\operatorname{sgn}\left(\frac{r_o(y)}{J(t,y)} + \frac{\mathcal{R}(t,y)}{J(t,y)} - k\right) J(t,y)\,\mathrm{d}y\;\mathrm{d}t \\
= & \int_0^{+\infty}\!\!\int_{\mathbb{R}^N} \Bigg[r_o(y)\frac{d}{dt}\varphi\left(t,X(t;0,y)\right) - k\,J(t,y)\frac{d}{dt}\varphi\left(t,X(t;0,y)\right) \\
& -k\,\varphi\left(t,X(t;0,y)\right)\frac{d}{dt}J(t,y) + \frac{d}{dt}\left(\mathcal{R}(t,y)\varphi\left(t,X(t;0,y)\right)\right)\Bigg] \\
& \qquad \times\operatorname{sgn}\left(r_o(y) + \mathcal{R}(t,y) - k\,J(t,y)\right)\mathrm{d}y\;\mathrm{d}t \\
= & \int_0^{+\infty}\!\!\int_{\mathbb{R}^N} \frac{d}{dt}\left(\left(r_o(y) + \mathcal{R}(t,y) - k\,J(t,y)\right)\,\varphi\left(t,X(t;0,y)\right)\right) \\
& \times\operatorname{sgn}\left(r_o(y) + \mathcal{R}(t,y) - k\,J(t,y)\right)\mathrm{d}y\;\mathrm{d}t \\
= & \int_0^{+\infty}\!\!\int_{\mathbb{R}^N} \frac{d}{dt}\left(\left|r_o(y) + \mathcal{R}(t,y) - k\,J(t,y)\right|\varphi\left(t,X(t;0,y)\right)\right)\mathrm{d}y\;\mathrm{d}t \\
\geq & \; 0\,.
\end{aligned}
$$

2. $r \in \mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R})$.
Indeed, by (5.16) we have

$$
\|r\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R})} \leq \left(\|r_o\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})} + T\|R\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R})}\right) e^{T\|\operatorname{div} w\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R})}}\,. \tag{5.21}
$$

3. r is right continuous.
Consider first the case

$$
\left\{\begin{array}{l} \partial_t r + \operatorname{div}\left(r w(t,x)\right) = 0\,, \\ r(0,x) = r_o(x)\,; \end{array}\right. \tag{5.22}
$$

where we can apply Kružkov Uniqueness Theorem [25, Theorem 2]. Therefore, it is sufficient to show that (5.16) does indeed give a Kružkov solution. To this aim, it is now sufficient to check the continuity from the right at $t=0$. Since $r_o \in (\mathbf{L^1}\cap\mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$, there exists a sequence $(r_{o,n})$ in $\mathcal{C}^{\mathbf{1}}_{\mathbf{c}}(\mathbb{R}^N;\mathbb{R})$ converging to r_o in $\mathbf{L^1}$. Then, the corresponding sequence of solutions (r_n) converges uniformly in time to r as given by (5.16). Indeed, by the same change of variable used above, we get

$$
\begin{aligned}
\int_{\mathbb{R}^N}\left|r_n(t,x) - r(t,x)\right|\mathrm{d}x &= \int_{\mathbb{R}^N}\left|\frac{r_{o,n}(y)}{J(t,y)} - \frac{r_o(y)}{J(t,y)}\right| J(t,y)\,\mathrm{d}y \\
&= \left\|r_{o,n} - r_o\right\|_{\mathbf{L^1}}\,.
\end{aligned}
$$

Furthermore, by (5.16), r_n is continuous in time, in particular at time $t = 0$. Finally, for any $\delta > 0$,

$$
\begin{aligned}
\int_{B(0,\delta)} \left|r(t,x) - r_o(x)\right| \mathrm{d}x \;\; & \leq \;\; \int_{B(0,\delta)} \left|r(t,x) - r_n(t,x)\right| \mathrm{d}x + \int_{B(0,\delta)} \left|r_n(t,x) - r_{o,n}(x)\right| \mathrm{d}x \\
& \quad + \int_{B(0,\delta)} \left|r_{o,n}(x) - r_o(x)\right| \mathrm{d}x \\
& \leq \;\; \frac{\varepsilon}{2} + \int_{B(0,\delta)} \left|r_n(t,x) - r_{o,n}(x)\right| \mathrm{d}x \;,\; \text{ for n large enough} \\
& \leq \;\; \varepsilon \,,\; \text{ for t small enough.}
\end{aligned}
$$

Next, we consider the system

$$
\left\{
\begin{array}{l}
\partial_t r + \operatorname{div}\left(r w(t,x)\right) = R\,, \\
r(0,x) = 0\,,
\end{array}
\right.
\tag{5.23}
$$

and introduce the map

$$
\mathcal{F}(t,h,\tau,y) = \exp\left[\int_0^t \operatorname{div} w\left(u, X(u;0,y)\right) \mathrm{d}u - \int_\tau^{t+h} \operatorname{div} w\Big(u, X\left(u; t+h, X(t;0,y)\right)\Big) \mathrm{d}u\right]
$$

so that the solution to (5.23) satisfies by (5.16), for all $h > 0$,

$$
\begin{aligned}
& \left\| r(t+h) - r(t) \right\|_{\mathbf{L^1}} \\
= \;\; & \int_{\mathbb{R}^N} \left| \int_0^{t+h} R\Big(\tau, X\left(\tau; t+h, X(t;0,y)\right)\Big) \mathcal{F}(t,h,\tau,y) \,\mathrm{d}\tau \right. \\
& \qquad \left. - \int_0^t R\left(\tau, X(\tau;0,y)\right) J(\tau,y) \,\mathrm{d}\tau \right| \mathrm{d}y \\
\leq \;\; & \int_{\mathbb{R}^N} \int_0^t \left| R\Big(\tau, X\left(\tau; t+h, X(t;0,y)\right)\Big) \right| \left| \mathcal{F}(t,h,\tau,y) - J(\tau,y) \right| \mathrm{d}\tau \,\mathrm{d}y \\
& + \int_{\mathbb{R}^N} \int_0^t J(\tau,y) \left| R\Big(\tau, X\left(\tau; t+h, X(t;0,y)\right)\Big) - R\left(\tau, X(\tau;0,y)\right) \right| \mathrm{d}\tau \,\mathrm{d}y \\
& + \int_t^{t+h} \int_{\mathbb{R}^N} \mathcal{F}(t,h,\tau,y) \left| R\Big(\tau, X\left(\tau; t+h, X(t;0,y)\right)\Big) \right| \mathrm{d}y \,\mathrm{d}\tau \;.
\end{aligned}
$$

The former summand above vanishes as $h \to 0$ because the integrand is uniformly bounded in $\mathbf{L^1}$ and converges pointwise to 0, since $X\left(u; t+h, X(t;0,y)\right) \overset{h\to 0}{\longrightarrow} X(u;0,y)$ and also $\mathcal{F}(t,h,\tau,y) \overset{h\to 0}{\longrightarrow} J(\tau,y)$. The second one, in the same limit, vanishes by the Dominated Convergence Theorem, R being in $\mathbf{L^1}$ and by the boundedness of J. Indeed, if (R_n) is a sequence of functions in $\boldsymbol{\mathcal{C}^1_c}(\mathbb{R}^N;\mathbb{R})$ that converges to R in $\mathbf{L^1}$ we have

$$
\begin{aligned}
& \int_{\mathbb{R}^N} \int_0^t J(\tau,y) \left| R\Big(\tau, X\left(\tau; t+h, X(t;0,y)\right)\Big) - R\left(\tau, X(\tau;0,y)\right) \right| \mathrm{d}\tau \,\mathrm{d}y \\
\leq \;\; & \int_{\mathbb{R}^N} \int_0^t J(\tau,y) \left| R_n\Big(\tau, X\left(\tau; t+h, X(t;0,y)\right)\Big) - R_n\left(\tau, X(\tau;0,y)\right) \right| \mathrm{d}\tau \,\mathrm{d}y
\end{aligned}
$$

$$+\int_{\mathbb{R}^N}\int_0^t J(\tau,y)\left|R_n\left(\tau, X\left(\tau; t+h, X(t;0,y)\right)\right) - R\left(\tau, X(\tau; t+h, X(t;0,y)\right)\right| \mathrm{d}\tau\,\mathrm{d}y$$
$$+\int_{\mathbb{R}^N}\int_0^t J(\tau,y)\left|R\left(\tau, X\left(\tau;0,y\right)\right) - R_n\left(\tau, X(\tau;0,y)\right)\right| \mathrm{d}\tau\,\mathrm{d}y\ .$$

$J(\tau,y)$ is uniformly bounded on $[0,t]\times\mathbb{R}^N$. We can first fix n large enough so that the second and third terms will be small, independently of h. Then, taking h small enough, we know from Dominated Convergence Theorem that the first term will shrink to 0. The integrand in the latter summand is in $\mathbf{L}^\infty$ since R is in $\mathbf{L}^\mathbf{1}$.

In general, right continuity follows by linearity adding the solutions to (5.22) and (5.23).

4. $r\in\mathbf{L}^\infty\left(I;\mathbf{L}^\mathbf{1}(\mathbb{R}^N;\mathbb{R})\right)$.
Indeed, for all $t\in I$ we have

$$\left\|r(t)\right\|_{\mathbf{L}^\mathbf{1}} \le \left(\|r_o\|_{\mathbf{L}^\mathbf{1}} + t\|R\|_{\mathbf{L}^\infty(I;\mathbf{L}^\mathbf{1})}\right)\exp\left(t\|\mathrm{div}\,w\|_{\mathbf{L}^\infty}\right)\ .$$

5. The solution to (5.15) is unique.

First, assume that $w\in\boldsymbol{\mathcal{C}}^\mathbf{2}(I\times\mathbb{R}^N;\mathbb{R}^N)$ and $R\in\boldsymbol{\mathcal{C}}^\mathbf{2}(I\times\mathbb{R}^N;\mathbb{R})$. Then, Kružkov Uniqueness Theorem [25, Theorem 2] applies.

Second, assume that $w\in\boldsymbol{\mathcal{C}}^\mathbf{2}(I\times\mathbb{R}^N;\mathbb{R}^N)$ and R satisfies the present assumptions. Then, we use the same procedure as in the proof of [16, Theorem 2.6]. There, the general scalar balance law $\partial_t u + \mathrm{div}\, f(t,x,u) = F(t,x,u)$ is considered, under assumptions that allow first to apply Kružkov general result and, secondly, to prove stability estimates on the solutions. Remark that these latter estimates are proved therein under the only requirement that solutions are Kružkov solutions, according to [25, Definition 1] or, equivalently, Definition 5.2. Here, the existence part has been proved independently from Kružkov result and under weaker assumptions.

Let (R_n) be a sequence in $\boldsymbol{\mathcal{C}}^\mathbf{2}_\mathbf{c}$ that converges in $\mathbf{L}^\mathbf{1}$ to $R\in\boldsymbol{\mathcal{C}}^\mathbf{0}\left(I;\mathbf{L}^\mathbf{1}(\mathbb{R}^N,\mathbb{R})\right)$. Also with reference to the notation of [16, Theorem 2.6], consider the equations

$$\partial_t r_n + \mathrm{div}\left(r_n\, w(t,x)\right) = R_n(t,x) \quad \text{and let} \quad \left\{\begin{array}{rcl} f(t,x,r) &=& r\,w(t,x)\\ F(t,x,r) &=& R_n(t,x)\end{array}\right. \tag{5.24}$$

$$\partial_t r + \mathrm{div}\left(r\, w(t,x)\right) = R(t,x) \quad \text{and let} \quad \left\{\begin{array}{rcl} g(t,x,r) &=& r\,w(t,x)\\ G(t,x,r) &=& R(t,x)\end{array}\right. \tag{5.25}$$

with the same initial datum $r_o\in(\mathbf{L}^\mathbf{1}\cap\mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$.

Note that here the sources F and G do not depend on r, hence the proof of [16, Theorem 2.6] can be repeated with $G\in\boldsymbol{\mathcal{C}}^\mathbf{0}\left(I;\mathbf{L}^\mathbf{1}(\mathbb{R}^N,\mathbb{R})\right)$ instead of $\boldsymbol{\mathcal{C}}^\mathbf{0}\left(I\times\mathbb{R}^N;\mathbb{R}\right)$. Indeed, in the proof of [16, Theorem 2.6], it is sufficient to have the (t,x)–regularity in the source term F of the first equation and existence and continuity of the derivative $\partial_r(F-G)$, which here vanishes. Besides, here the two flows f and g are identical, hence we do not need the **BV** estimate provided by [16, Theorem 2.5].

Thus, to apply the stability estimate in [16, Theorem 2.6], we are left to check the following points:

- the derivatives $\partial_r f = w$, $\partial_r\nabla_x f = \nabla_x w$, $\nabla_x^2 f = r\nabla_x^2 w$, $\partial_r F$ and $\nabla_x F$ exist and are continuous;

- $\partial_r f = w$ and $F - \operatorname{div} f = F - r \operatorname{div} w$ are bounded in $I \times \mathbb{R}^N \times [-A, A]$ for all $A \geq 0$;
- $\partial_r (F - \operatorname{div} f)(t, x, r) = -\operatorname{div} w \in \mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R})$,
- $\nabla_x \partial_r f(t, x, r) = \nabla_x w \in \mathbf{L}^\infty(I \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R}^{N \times N})$.

Hence, if r is any solution to (5.25) and r_n is the solution to (5.24) in the sense of Definition 5.2, then for $t \in I$, $x_o \in \mathbb{R}^N$, $\delta \geq 0$, $M = \|w\|_{\mathbf{L}^\infty(\mathbb{R}_+ \times \mathbb{R}^N; \mathbb{R}^N)}$:

$$\left\|(r_n - r)(t)\right\|_{\mathbf{L}^1(B(x_o, \delta); \mathbb{R})} \leq \int_0^t e^{\kappa t} \left\|(R_n - R)(s)\right\|_{\mathbf{L}^1(B(x_o, \delta + M(t-s)); \mathbb{R})} \, \mathrm{d}s \,,$$

where $\kappa = 2N \|\nabla_x w\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N; \mathbb{R}^{N \times N})}$. Therefore, if (R_n) converges in $\mathbf{L}^1$ to R, then (r_n) is a Cauchy sequence in $\mathbf{L}^\infty\left(I; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})\right)$ and $r \in \mathbf{L}^\infty\left(I; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})\right)$ is uniquely characterized as its limit.

Third, we consider the general case. Again, we rely on the proof of [16, Theorem 2.6] extending it to the case of $w \in \boldsymbol{\mathcal{C}}^{\mathbf{0}}(I \times \mathbb{R}^N; \mathbb{R}^N)$. Indeed, therein the higher regularity in time of the flow is used to apply Kružkov Existence Theorem [25, Theorem 5], to prove the **BV** estimates in [16, Theorem 2.4] and to obtain the limit [16, (5.11)]. In the former case, our existence proof in the previous steps replaces the use of Kružkov result. **BV** estimates are here not necessary, for we keep here the flow fixed. In the latter case, a simple argument based on the Dominated Convergence Theorem allows to get the same limit. □

Remark that as an immediate corollary of Lemma 5.1 we obain that any solution to (5.15) in the sense of Definition 5.2 is represented by (5.16).

5.2 Proof of Theorem 2.2

Lemma 5.3 *Let $T > 0$, so that $I = [0, T[$, and w be as in (5.14) such that*

$$\operatorname{div} w \in \mathbf{L}^\infty\left(I; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})\right) \tag{5.26}$$

$$\nabla_x \operatorname{div} w \in \mathbf{L}^\infty\left(I; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}^N)\right) \,. \tag{5.27}$$

Then, for any $\rho_o \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N; \mathbb{R})$, the Cauchy problem (5.15) with $R = 0$ admits a unique solution $\rho \in \mathbf{L}^\infty\left(I; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})\right)$, right continuous in time and satisfying

$$\left\|\rho(t)\right\|_{\mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R})} \leq \|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R})} \exp\left(t \left\|\operatorname{div} w\right\|_{\mathbf{L}^\infty([0,t] \times \mathbb{R}^N; \mathbb{R})}\right) \tag{5.28}$$

for all $t \in I$. Moreover, this solution has the following properties:

1. $\rho_o \geq 0$ *a.e.* $\Rightarrow$ $\rho(t) \geq 0$ *a.e., for all $t \in I$*

   $\operatorname{div} w \geq 0$ *a.e.* $\Rightarrow$ $\left\|\rho(t)\right\|_{\mathbf{L}^\infty} \leq \|\rho_o\|_{\mathbf{L}^\infty}$*, for all $t \in I$.*

2. *If $\rho_o \in \mathcal{X}$ then, for all $t \in I$, we have $\rho(t) \in \mathcal{X}$ and setting $\kappa_o = NW_N(2N + 1)\|\nabla_x w\|_{\mathbf{L}^\infty(I \times \mathbb{R}^N; \mathbb{R}^{N \times N})}$, we also get*

   $$\begin{aligned} \mathrm{TV}\left(\rho(t)\right) \leq\; & \mathrm{TV}(\rho_o) e^{\kappa_o t} \\ & + NW_N \int_0^t e^{\kappa_o (t-s)} \int_{\mathbb{R}^N} e^{s \|\operatorname{div} w\|_{\mathbf{L}^\infty}} \left\|\nabla_x \operatorname{div} w(s, x)\right\| \, \mathrm{d}x \, \mathrm{d}s \, \|\rho_o\|_{\mathbf{L}^\infty} \,. \end{aligned}$$

   Furthermore, $\rho \in \boldsymbol{\mathcal{C}}^{\mathbf{0}}\left(I; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})\right)$.

3. *If ρ_1, ρ_2 are the solutions of (5.15) associated to w_1, w_2 with $R_1 = R_2 = 0$ and with initial conditions $\rho_{1,o}$, $\rho_{2,o}$ in $\mathcal{X}$, then for all $t \in I$*

$$\begin{aligned}
& \left\|(\rho_1 - \rho_2)(t)\right\|_{\mathbf{L^1}} \\
\leq\ & e^{\kappa t}\left\|\rho_{1,o} - \rho_{2,o}\right\|_{\mathbf{L^1}} + \frac{e^{\kappa_o t} - e^{\kappa t}}{\kappa_o - \kappa}\,\mathrm{TV}\,(\rho_{1,o})\|w_1 - w_2\|_{\mathbf{L^\infty}} \\
& + N W_N \int_0^t \frac{e^{\kappa_o(t-s)} - e^{\kappa(t-s)}}{\kappa_o - \kappa} \int_{\mathbb{R}^N} e^{s\|\mathrm{div} w\|_{\mathbf{L^\infty}}} \left\|\nabla_x \,\mathrm{div}\, w_1(s,x)\right\| \,\mathrm{d}x\,\mathrm{d}s \\
& \qquad \times \|\rho_o\|_{\mathbf{L^\infty}} \|w_1 - w_2\|_{\mathbf{L^\infty}} \\
& + \int_0^t e^{\kappa(t-s)} e^{s\|\mathrm{div} w\|_{\mathbf{L^\infty}}} \int_{\mathbb{R}^N} \left|\mathrm{div}\,(w_1 - w_2)(s,x)\right| \,\mathrm{d}x\,\mathrm{d}s\, \|\rho_o\|_{\mathbf{L^\infty}}\,,
\end{aligned}$$

where $\kappa = 2N\|\nabla_x w_1\|_{\mathbf{L^\infty}(I\times\mathbb{R}^N;\mathbb{R}^{N\times N})}$ and κ_o as in 2 above.

4. *If there exists $C \geq 0$ such that*

$$\left\|\nabla_x^2 w\right\|_{\mathbf{L^\infty}(I\times\mathbb{R}^N;\mathbb{R}^{N\times N\times N})} \leq C \quad \text{and} \quad \|\nabla_x w\|_{\mathbf{L^\infty}(I\times\mathbb{R}^N;\mathbb{R}^{N\times N})} \leq C \tag{5.29}$$

then

$$\rho_o \in \mathbf{W^{1,1}}(\mathbb{R}^N;\mathbb{R}) \quad \Rightarrow \quad \begin{cases} \rho(t) \in \mathbf{W^{1,1}}(\mathbb{R}^N;\mathbb{R}) & \text{for all } t \in I \\ \left\|\rho(t)\right\|_{\mathbf{W^{1,1}}} \leq e^{2Ct}\|\rho_o\|_{\mathbf{W^{1,1}}}\,, \end{cases}$$

$$\rho_o \in \mathbf{W^{1,\infty}}(\mathbb{R}^N;\mathbb{R}) \quad \Rightarrow \quad \begin{cases} \rho(t) \in \mathbf{W^{1,\infty}}(\mathbb{R}^N;\mathbb{R}) & \text{for all } t \in I \\ \left\|\rho(t)\right\|_{\mathbf{W^{1,\infty}}} \leq e^{2Ct}\,\|\rho_o\|_{\mathbf{W^{1,\infty}}}\,. \end{cases}$$

5. *If there exists $C \geq 0$ such that (5.29) holds together with $\left\|\nabla_x^3 w\right\|_{\mathbf{L^\infty}(I\times\mathbb{R}^N;\mathbb{R}^{N\times N\times N\times N})} \leq C$, then $\rho_o \in \mathbf{W^{2,1}}(\mathbb{R}^N;\mathbb{R})$ implies*

$$\rho(t) \in \mathbf{W^{2,1}}(\mathbb{R}^N;\mathbb{R}) \quad \text{for all } t \in I \quad \text{and} \quad \left\|\rho(t)\right\|_{\mathbf{W^{2,1}}} \leq (1+Ct)^2 e^{3Ct}\,\|\rho_o\|_{\mathbf{W^{2,1}}}\,.$$

Proof. The existence of a Kružkov solution follows from Lemma 5.1. But we can also refer to [25, theorems 2 and 5], the assumptions in [25, § 5] being satisfied thanks to (5.14). The $\mathbf{L}^\infty$ bound directly follows from (5.16), which now reads

$$\rho(t,x) = \rho_o\left(X(0;t,x)\right)\,\exp\left(-\int_0^t \mathrm{div}\, w\left(\tau, X(\tau;t,x)\right)\mathrm{d}\tau\right)\,. \tag{5.30}$$

The representation formula (5.30) also implies the bounds at 1.

The bound on the total variation at 2 follows from [16, Theorem 2.5], the hypotheses on w being satisfied thanks to (5.14) and (5.27). More precisely, we do not have here the $\boldsymbol{\mathcal{C}^2}$ regularity in time as required in [16, Theorem 2.5], but going through the proof of this result, we can see that only the continuity in time of the flow function $f(t,x,r) = rw(t,x)$ is necessary. Indeed, time derivatives of f appear in the proof of [16, Theorem 2.5] when we bound the terms J_t and L_t, see [16, between (4.18) and (4.19)]. However, the use of the Dominated Convergence Theorem allows to prove that J_t and L_t converge to zero when η goes to 0 without any use of time derivatives. The continuity in times follows from [16, Remark 2.4], thanks to (5.26) of w and the bound on the total variation.

Similarly, the stability estimate at 3 is based on [16, Theorem 2.6]. Indeed, we use once again a flow that is only $\boldsymbol{\mathcal{C}}^{\mathbf{0}}$ instead of $\boldsymbol{\mathcal{C}}^{\mathbf{2}}$ in time. Besides, in the proof of [16, Theorem 2.6], the $\mathbf{L}^\infty$ bound into the integral term in [16, Theorem 2.6] can be taken only in space, keeping time fixed. With this provision, the proof of 3 is exactly the same as that in [16], so we do not reproduce it here. The same estimate is thus obtained, except that the $\mathbf{L}^\infty$ bound of the integral term is taken only in space.

The proofs of the $\mathbf{W}^{\mathbf{1},\mathbf{1}}$ and $\mathbf{W}^{\mathbf{1},\infty}$ bounds at 4 are similar. They follow from the representation (5.30), noting that $\|\nabla_x X\|_{\mathbf{L}^\infty} \leq e^{Ct}$. Indeed,

$$\begin{aligned}
\nabla_x X(t;0,x) &= \mathbf{Id} + \int_0^t \nabla_x w(\tau; X(\tau;0,x)) \nabla_x X(\tau;0,x)\, \mathrm{d}\tau \ , \text{ hence} \\
\left\|\nabla_x X(t;0,x)\right\| &\leq 1 + \int_0^t \left\|\nabla_x w(\tau; X(\tau;0,x))\right\| \left\|\nabla_x X(\tau;0,x)\right\| \mathrm{d}\tau \\
&\leq 1 + \int_0^t C \left\|\nabla_x X(\tau;0,x)\right\| \mathrm{d}\tau
\end{aligned}$$

and a direct application of Gronwall Lemma gives the desired bound. Hence, we obtain

$$\left\|\nabla \rho(t)\right\|_{\mathbf{L}^\infty} \leq (e^{2Ct} - e^{Ct}) \left\|\rho_o\right\|_{\mathbf{L}^\infty} + e^{2Ct} \left\|\nabla \rho_o\right\|_{\mathbf{L}^\infty}$$

and consequently

$$\left\|\rho(t)\right\|_{\mathbf{W}^{\mathbf{1},\infty}} \leq e^{2Ct} \left\|\rho_o\right\|_{\mathbf{W}^{\mathbf{1},\infty}} .$$

The $\mathbf{L}^\mathbf{1}$ estimate is entirely analogous.

The $\mathbf{W}^{\mathbf{2},\mathbf{1}}$ bound at 5. also comes from the (5.30). Indeed, again thanks to Gronwall Lemma, we get $\left\|\nabla_x^2 X\right\|_{\mathbf{L}^\infty} \leq e^{2Ct} - e^{Ct}$. Using the estimates above, together with

$$\left\|\nabla^2 \rho(t)\right\|_{\mathbf{L}^\mathbf{1}} \leq (2e^{2Ct} - 3e^{Ct} + 1)e^{Ct} \left\|\rho_o\right\|_{\mathbf{L}^\mathbf{1}} + 3(e^{Ct} - 1)e^{2Ct} \left\|\nabla \rho_o\right\|_{\mathbf{L}^\mathbf{1}} + e^{3Ct} \left\|\nabla^2 \rho_o\right\|_{\mathbf{L}^\mathbf{1}} ,$$

we obtain

$$\left\|\rho(t)\right\|_{\mathbf{W}^{\mathbf{2},\mathbf{1}}} \leq (2e^{Ct} - 1)^2 e^{Ct} \left\|\rho_o\right\|_{\mathbf{W}^{\mathbf{2},\mathbf{1}}}$$

concluding the proof. □

We use now these tools in order to obtain the existence of a solution for (1.1).

Proof of Theorem 2.2. Fix $\alpha, \beta > 0$ with $\beta > \alpha$. Let $T_* = \big(\ln(\beta/\alpha)\big) / C(\beta)$, with C as in **(V1)**. Define the map

$$\mathcal{Q} \ : \ \begin{array}{ccc} \boldsymbol{\mathcal{C}}^{\mathbf{0}}\left(I_*; \mathcal{X}_\beta\right) & \to & \boldsymbol{\mathcal{C}}^{\mathbf{0}}\left(I_*; \mathcal{X}_\beta\right) \\ \sigma & \mapsto & \rho \end{array}$$

where $I_* = [0, T_*[$ and ρ is the Kružkov solution to

$$\left\{ \begin{array}{l} \partial_t \rho + \operatorname{div}(\rho\, w) = 0 \\ \rho(0,x) = \rho_o(x) \end{array} \right. \qquad \text{with} \qquad \begin{array}{rcl} w &=& V(\sigma) \\ \rho_o &\in& \mathcal{X}_\alpha . \end{array} \tag{5.31}$$

The assumptions **(V1)** imply the hypotheses on w necessary in Lemma 5.3. Therefore, a solution ρ to (5.31) exists and is unique. In particular, the continuity in time of ρ follows from 2 in Lemma 5.3, due to the boundedness of the total variation. Note that by (5.28), the choice of T_* and **(V1)**, $\left\|\rho(t)\right\|_{\mathbf{L}^\infty} \leq \beta$ and hence $\mathcal{Q}$ is well defined.

Fix σ_1, σ_2 in $\boldsymbol{\mathcal{C}}^{\mathbf{0}}(I_*; \mathcal{X}_\beta)$. Call $w_i = V(\sigma_i)$ and ρ_i the corresponding solutions. With the same notations of [16, Theorem 2.6], we let

$$\kappa_o = N\, W_N(2N+1)\, \|\nabla_x w_1\|_{\mathbf{L}^\infty(I_* \times \mathbb{R}^N; \mathbb{R}^{N\times N})}, \qquad \kappa = 2\,N\, \|\nabla_x w_1\|_{\mathbf{L}^\infty(I_* \times \mathbb{R}^N; \mathbb{R}^{N\times N})}.$$

Note that by (5.13)

$$\frac{\kappa_o}{\kappa} \geq \left(N + \frac{1}{2}\right) \int_0^{\pi/2} \left(1 - \frac{2}{\pi} x\right)^N \mathrm{d}x = \frac{\pi}{2} \left(1 - \frac{1}{2(N+1)}\right) \geq \frac{3\pi}{8} > 1$$

hence $\kappa_o > \kappa$. Then, by 4 of Lemma 5.3 and **(V1)**, we obtain a bound on κ_o. Indeed,

$$\left\|\nabla_x V(\sigma_1)\right\|_{\mathbf{L}^\infty(I_* \times \mathbb{R}^N; \mathbb{R}^{N\times N})} \leq C\left(\|\sigma_1\|_{\mathbf{L}^\infty(I_* \times \mathbb{R}^N; \mathbb{R})}\right),$$

and since $\sigma_1 \in \boldsymbol{\mathcal{C}}^{\mathbf{0}}(I_*; \mathcal{X}_\beta)$, finally $\kappa_o \leq N W_N (2N+1)\, C(\beta)$. Let us denote

$$C = C(\beta) \quad \text{and} \quad C' = N\, W_N\, (2N+1)\, C(\beta)\,. \tag{5.32}$$

Again, **(V1)** implies the following uniform bounds on all $\sigma_1, \sigma_2 \in \boldsymbol{\mathcal{C}}^{\mathbf{0}}(I_*; \mathcal{X}_\beta)$:

$$\begin{array}{rcl} \left\|\nabla_x^2 V(\sigma_1)\right\|_{\mathbf{L}^\infty(I_*; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}^{N\times N\times N}))} & \leq & C\,, \\ \left\|V(\sigma_1) - V(\sigma_2)\right\|_{\mathbf{L}^\infty(I_* \times \mathbb{R}^N; \mathbb{R}^N)} & \leq & C\, \|\sigma_1 - \sigma_2\|_{\mathbf{L}^\infty(I_*; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))}\,, \\ \left\|\mathrm{div}\left(V(\sigma_1) - V(\sigma_2)\right)\right\|_{\mathbf{L}^\infty(I_*; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))} & \leq & C\, \|\sigma_1 - \sigma_2\|_{\mathbf{L}^\infty(I_*; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))}\,. \end{array}$$

Thus, we can apply [16, Theorem 2.6]. We get, for all $t \in I_*$,

$$\begin{array}{rcl} \left\|(\rho_1 - \rho_2)(t)\right\|_{\mathbf{L}^1} & \leq & C t e^{C't}\, \mathrm{TV}\,(\rho_o) \|\sigma_1 - \sigma_2\|_{\mathbf{L}^\infty([0,t]; \mathbf{L}^1)} \\ & & + C^2 N W_N e^{Ct} \displaystyle\int_0^t (t-s) e^{C'(t-s)}\, \mathrm{d}s\, \left\|\rho_{o,1}\right\|_{\mathbf{L}^\infty} \|\sigma_1 - \sigma_2\|_{\mathbf{L}^\infty([0,t]; \mathbf{L}^1)} \\ & & + e^{Ct} \displaystyle\int_0^t C\, e^{C'(t-s)} \left\|(\sigma_1 - \sigma_2)(s)\right\|_{\mathbf{L}^1} \mathrm{d}s\, \left\|\rho_{o,1}\right\|_{\mathbf{L}^\infty}. \end{array}$$

Therefore, we obtain the following Lipschitz estimate:

$$\begin{array}{rl} & \left\|\mathcal{Q}(\sigma_1) - \mathcal{Q}(\sigma_2)\right\|_{\mathbf{L}^\infty(I; \mathbf{L}^1)} \\ \leq & C T e^{C'T} \left[\mathrm{TV}\,(\rho_o) + (N W_N C T + 1) e^{CT} \left\|\rho_{o,1}\right\|_{\mathbf{L}^\infty}\right] \|\sigma_1 - \sigma_2\|_{\mathbf{L}^\infty(I; \mathbf{L}^1)}. \end{array}$$

Here we introduce the strictly increasing function

$$f(T) = C T e^{C'T} \left[\mathrm{TV}\,(\rho_o) + (N W_N C T + 1) e^{CT} \left\|\rho_{o,1}\right\|_{\mathbf{L}^\infty}\right]$$

and we remark that $f(T) \to 0$ when $T \to 0$. Choose now $T_1 > 0$ so that $f(T_1) = 1/2$. Banach Contraction Principle now ensures the existence and uniqueness of a solution ρ^* to (1.1) on $[0, \bar{T}]$ in the sense of Definition 2.1, with $\bar{T} = \min\{T_*, T_1\}$. In fact, if $T_1 < T_*$, we can prolongate the solution until time T_*. Indeed, if we take $\rho^*(T_1)$ as initial condition, we remark that $\left\|\rho^*(T_1)\right\|_{\mathbf{L}^\infty} \leq \|\rho_o\|_{\mathbf{L}^\infty} e^{C(\beta) T_1}$. Consequently, the solution of (5.31) on $[T_1, T_*]$ instead of I_* satisfy, thanks to (5.28)

$$\left\|\rho(t)\right\|_{\mathbf{L}^\infty} \leq \left\|\rho^*(T_1)\right\|_{\mathbf{L}^\infty} e^{C(\beta)(t - T_1)} \leq \|\rho_o\|_{\mathbf{L}^\infty} e^{C(\beta) T_1} e^{C(\beta)(t - T_1)} \leq \|\rho_o\|_{\mathbf{L}^\infty} e^{C(\beta) T_*}\,,$$

which is less than β thanks to the definition of T_* and since $\rho_o \in \mathcal{X}_\alpha$.

Now, we have to show that $T_n \geq T_*$ for n sufficiently large. To this aim, we obtain the contraction estimate

$$\begin{aligned}
& \left\|\mathcal{Q}(\sigma_1) - \mathcal{Q}(\sigma_2)\right\|_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{L}^1)} \\
\leq \;& C(T_{n+1}-T_n)e^{C'(T_{n+1}-T_n)}\left[\mathrm{TV}\left(\rho(T_n)\right) + (NW_N C(T_{n+1}-T_n)+1)e^{CT_n}\|\rho_o\|_{\mathbf{L}^\infty}\right] \\
& \times\|\sigma_1-\sigma_2\|_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{L}^1)} \\
\leq \;& \left[\mathrm{TV}\left(\rho_o\right)e^{C'T_n} + C'T_n e^{C'T_n} + (NW_N C(T_{n+1}-T_n)+1)e^{CT_n}\|\rho_o\|_{\mathbf{L}^\infty}\right] \\
& \times C'(T_{n+1}-T_n)e^{C'(T_{n+1}-T_n)}\|\sigma_1-\sigma_2\|_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{L}^1)}
\end{aligned}$$

where we used the bounds on $\mathrm{TV}\left(\rho(T_n)\right)$ and $\left\|\rho(T_n)\right\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})}$ provided by Lemma 5.3 associated to the conditions **(A)** and **(V1)**. We may thus extend the solution up to time T_{n+1}, where we take $T_{n+1} > T_n$ such that

$$\begin{aligned}
& \left[\mathrm{TV}\left(\rho_o\right)e^{C'T_n} + CT_n e^{C'T_n} + \left(NW_N C(T_{n+1}-T_n)+1\right)e^{CT_n}\|\rho_o\|_{\mathbf{L}^\infty}\right] \times \\
& \qquad\qquad\qquad\qquad \times C(T_{n+1}-T_n)e^{C'(T_{n+1}-T_n)} \;=\; \frac{1}{2}\,.
\end{aligned}$$

If the sequence (T_n) is bounded, then the left hand side above tends to 0, whereas the right hand side is taken equal to $1/2 > 0$. Hence, the sequence (T_n) is unbounded. In particular, for n large enough, T_n is larger than T_*; thus the solution to (1.1) is defined on all I_*.

The Lipschitz estimate follows by applying the same procedure as above, in the case when the initial conditions are not the same.

The $\mathbf{L}^\infty$ and TV bounds follow from (5.28) and from point 2 in Lemma 5.3. □

The proof of Lemma 2.3 directly follows from the second bound in 1. of Lemma 5.3.

Proof of Theorem 2.4. We consider the assumptions **(A)** and **(B)** separately.

(A): Let $T > 0$, so that $I = [0,T[$, and fix a positive α. As in the proof of Theorem 2.2, we define the map

$$\mathcal{Q} \;:\; \begin{array}{ccc} \mathcal{C}^0(I;\mathcal{X}_\alpha) & \to & \mathcal{C}^0(I;\mathcal{X}_\alpha) \\ \sigma & \mapsto & \rho \end{array}$$

where ρ is the Kružkov solution to (5.31) with $\rho_o \in \mathcal{X}_\alpha$. The existence of a solution for (5.31) in $\mathbf{L}^\infty(I,\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}))$ is given by Lemma 5.3, the set of assumptions **(V1)** allowing to check the hypotheses on w. Note that furthermore **(A)** gives an $\mathbf{L}^\infty$ bound on ρ, thanks to Lemma 2.3, so that for all $t \in I$, $\rho(t) \in [0,\alpha]$, a.e. in x. Fix σ_1, σ_2 in $\mathcal{C}^0(I;\mathcal{X}_\alpha)$, call $w_i = V(\sigma_i)$ and let ρ_1, ρ_2 be the associated solutions. With the same notations of [16, Theorem 2.6], we let as in the proof of Theorem 2.2,

$$\kappa_o = N\,W_N(2N+1)\,\|\nabla_x w_1\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R}^{N\times N})}\,, \qquad \kappa = 2\,N\,\|\nabla_x w_1\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R}^{N\times N})}\,.$$

so that $\kappa_o > \kappa$. Then we use Lemma 5.3 and assumptions **(V1)** in order to find a bound on κ_o. Indeed, by **(V1)** we have:

$$\left\|\nabla_x V(\sigma_1)\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R}^{N\times N})} \leq C\left(\|\sigma_1\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R}^{N\times N})}\right),$$

and since $\sigma_1 \in \boldsymbol{\mathcal{C}}^{\mathbf{0}}(I; \mathcal{X}_\alpha)$, we have $\|\sigma_1\|_{\mathbf{L}^\infty} \leq \alpha$ so that $\kappa_o \leq N W_N (2N+1) C(\alpha)$. Denote

$$C' = N W_N (2N+1) C(\alpha) \quad \text{and} \quad C = C(\alpha)\,. \tag{5.33}$$

The following bounds are also available uniformly for all $\sigma_1, \sigma_2 \in \boldsymbol{\mathcal{C}}^{\mathbf{0}}(\mathbb{R}_+; \mathcal{X}_\alpha)$, by **(V1)**:

$$\begin{array}{rcl}
\left\|\nabla_x^2 V(\sigma_1)\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}^{N\times N\times N}))} & \leq & C\,, \\
\left\|V(\sigma_1) - V(\sigma_2)\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R})} & \leq & C\left\|\sigma_1 - \sigma_2\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}))}\,, \\
\left\|\operatorname{div}\left(V(\sigma_1) - V(\sigma_2)\right)\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}))} & \leq & C\left\|\sigma_1 - \sigma_2\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}))}\,.
\end{array}$$

Applying [16, Theorem 2.6], we get

$$\begin{array}{rcl}
\left\|(\rho_1 - \rho_2)(t)\right\|_{\mathbf{L}^1} & \leq & C t e^{C't}\,\mathrm{TV}\,(\rho_o) \|\sigma_1 - \sigma_2\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1)} \\
& & \displaystyle +C^2 N W_N \int_0^t (t-s) e^{C'(t-s)}\,\mathrm{d}s\,\left\|\sigma_1 - \sigma_2\right\|_{\mathbf{L}^\infty([0,t];\mathbf{L}^1)} \\
& & \displaystyle +\int_0^t C\, e^{C'(t-s)} \left\|(\sigma_1 - \sigma_2)(s)\right\|_{\mathbf{L}^1}\,\mathrm{d}s\,.
\end{array}$$

So that

$$\left\|\mathcal{Q}(\sigma_1) - \mathcal{Q}(\sigma_2)\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)} \leq C T e^{C'T}\left[\mathrm{TV}\,(\rho_o) + N W_N C T + 1\right]\left\|\sigma_1 - \sigma_2\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)}\,.$$

Here we introduce the function $f(T) = C T e^{C'T}\left[\mathrm{TV}\,(\rho_o) + N W_N C T + 1\right]$ and we remark that $f(T) \to 0$ when $T \to 0$. Choose now $T_1 > 0$ so that $f(T_1) = \frac{1}{2}$. Banach Contraction Principle now ensures the existence and uniqueness of a solution to (1.1) on $[0, T_1]$ in the sense of Definition 2.1.

Iterate this procedure up to the interval $[T_{n-1}, T_n]$ and obtain the contraction estimate

$$\begin{array}{rl}
& \left\|\mathcal{Q}(\sigma_1) - \mathcal{Q}(\sigma_2)\right\|_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{L}^1)} \\
\leq & C(T_{n+1} - T_n) e^{C'(T_{n+1}-T_n)}\left[\mathrm{TV}\left(\rho(T_n)\right) + N W_N C (T_{n+1} - T_n) + 1\right] \\
& \times\left\|\sigma_1 - \sigma_2\right\|_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{L}^1)} \\
\leq & \left[\mathrm{TV}\,(\rho_o)\, e^{C'T_n} + C' T_n e^{C'T_n} + N W_N C (T_{n+1} - T_n) + 1\right] \\
& \times C'(T_{n+1} - T_n) e^{C'(T_{n+1}-T_n)}\left\|\sigma_1 - \sigma_2\right\|_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{L}^1)}
\end{array}$$

where we used the bounds on $\mathrm{TV}\left(\rho(T_n)\right)$ and $\left\|\rho(T_n)\right\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})}$ provided by Lemma 5.3 associated to the conditions **(A)** and **(V1)**. We may thus extend the solution up to time T_{n+1}, where we take

$$\left[\mathrm{TV}\,(\rho_o)\, e^{C'T_n} + C T_n e^{C'T_n} + N W_N C (T_{n+1} - T_n) + 1\right] C (T_{n+1} - T_n) e^{C'(T_{n+1}-T_n)} \quad = \quad \frac{1}{2}\,.$$

If the sequence (T_n) is bounded, then the left hand side above tends to 0, whereas the right hand side is taken equal to $1/2 > 0$. Hence, the sequence (T_n) is unbounded and the solution to (1.1) is defined on all $\mathbb{R}_+$.

(S2) follows from Lemma 5.3 associated to the assumption **(V1)** on V that allows to satisfy the hypotheses on w.

(S3) is obtained in the same way as (S1). Note that the Lipschitz constant obtained by such a way is depending on time.

The bound (S4) follows from Lemma 5.3, point 2, that gives us

$$\mathrm{TV}\left(\rho(t)\right) \leq \mathrm{TV}\left(\rho_o\right) e^{C't} + N W_N C t e^{C't} \|\rho_o\|_{\mathbf{L}^\infty} .$$

(B): Repeat the proof of Theorem 2.2 and, with the notation therein, note that if we find a sequence (α_n) such that $\sum_n T(\alpha_n, \alpha_{n+1}) = +\infty$ where $T(\alpha, \beta) = \left[\ln\left(\beta/\alpha\right)\right] / C(\beta)$, then the solution is defined on the all $\mathbb{R}_+$. It is immediate to check that **(B)** implies that

$$\sum_{n=1}^{k} T(\alpha_n, \alpha_{n+1}) \geq \left(\|C\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R}_+)}\right)^{-1} \ln \alpha_k \to +\infty \quad \text{as } k \to +\infty$$

completing the proof. □

Proof of Proposition 2.5. The bounds of ρ in $\mathbf{W}^{1,\infty}$ and $\mathbf{W}^{1,1}$ follow from 4 in Lemma 5.3, the hypotheses being satisfied thanks to **(V2)**. The bound in $\mathbf{W}^{2,1}$ comes from 5 in Lemma 5.3, the hypotheses being satisfied thanks to **(V3)**. □

5.3 Weak Gâteaux Differentiability

First of all, if $r_o \in (\mathbf{L}^\infty \cap \mathbf{L}^1)(\mathbb{R}^N;\mathbb{R})$ and $\rho \in \mathbf{L}^\infty\left(I_{\mathrm{ex}}; (\mathbf{W}^{1,1} \cap \mathbf{W}^{1,\infty})(\mathbb{R}^N;\mathbb{R})\right)$, we prove that the equation (1.2) admits a unique solution $r \in \mathbf{L}^\infty\left(I_{\mathrm{ex}}; \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})\right)$ continuous from the right.

Proof of Proposition 2.9. We use here once again Lemma 5.1 in order to get an expression of the Kružkov solution for (5.15).

We assume now that $\rho \in \mathcal{C}^0\left(I_{\mathrm{ex}}; (\mathbf{W}^{1,\infty} \cap \mathbf{W}^{1,1})(\mathbb{R}^N;\mathbb{R})\right)$ and we define $w = V(\rho)$; we also set, for all $s \in \mathbf{L}^\infty\left(I_{\mathrm{ex}}; \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})\right)$, $R = \mathrm{div}\left(\rho DV(\rho)(s)\right)$. Thanks to the assumptions on ρ and **(V4)**, we obtain $R \in \mathbf{L}^\infty\left(I_{\mathrm{ex}}; \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})\right) \cap \mathbf{L}^\infty(I_{\mathrm{ex}} \times \mathbb{R}^N;\mathbb{R})$. Let $\varepsilon \in \mathring{I}_{\mathrm{ex}}$. Then, on $[0, T_{\mathrm{ex}} - \varepsilon]$ we can apply Lemma 5.1 giving the existence of a Kružkov solution to

$$\partial_t r + \mathrm{div}\,(r w) = R\,, \quad r(x,0) = r_o \in (\mathbf{L}^\infty \cap \mathbf{L}^1)(\mathbb{R}^N;\mathbb{R})\,.$$

Let $T \in [0, T_{\mathrm{ex}} - \varepsilon]$ and $I = [0, T[$. We denote Q the application that associates to $s \in \mathbf{L}^\infty\left(I; \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})\right)$ continuous from the right in time, the Kružkov solution $r \in \mathbf{L}^\infty\left(I; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}^N;\mathbb{R})\right)$ continuous from the right in time of (5.15) with initial condition $r_o \in (\mathbf{L}^\infty \cap \mathbf{L}^1)(\mathbb{R}^N;\mathbb{R})$, given by Lemma 5.1. That is to say

$$\begin{aligned} Q \quad : \quad s \mapsto r(t,x) = {} & r_o\left(X(0;t,x)\right) \exp\left(-\int_0^t \mathrm{div}\, V(\rho)\left(\tau, X(\tau;t,x)\right) \mathrm{d}\tau\right) \\ & - \int_0^t \mathrm{div}\left(\rho DV(\rho)(s)\right)\left(\tau, X(\tau;t,x)\right) \exp\left(-\int_\tau^t \mathrm{div}\, V(\rho)\left(u, X(u;t,x)\right) \mathrm{d}u\right) \mathrm{d}\tau\,. \end{aligned}$$

Let us give some bounds on r. The representation of the solution (5.16) allows indeed to derive a $\mathbf{L}^\infty$ bound on r. For all $t \in I$, thanks to **(V1)** and **(V4)** we get, with $C = C\left(\|\rho\|_{\mathbf{L}^\infty([0,T_{\mathrm{ex}}-\varepsilon]\times\mathbb{R}^N;\mathbb{R})}\right)$,

$$\left\|r(t)\right\|_{\mathbf{L}^\infty} \leq \|r_o\|_{\mathbf{L}^\infty} e^{Ct} + t e^{Ct} \|\rho\|_{\mathbf{L}^\infty([0,t];\mathbf{W}^{1,\infty})} \left\|DV(\rho)\right\|_{\mathbf{W}^{1,\infty}} \|s\|_{\mathbf{L}^\infty([0,t],\mathbf{L}^1)}\,.$$

The same expression allows also to derive a $\mathbf{L^1}$ bound on $r(t)$

$$\left\|r(t)\right\|_{\mathbf{L^1}} \leq \|r_o\|_{\mathbf{L^1}} e^{Ct} + te^{Ct} \|\rho\|_{\mathbf{L}^\infty([0,t];\mathbf{W^{1,1}})} \left\|DV(\rho)\right\|_{\mathbf{W^{1,\infty}}} \|s\|_{\mathbf{L}^\infty([0,t],\mathbf{L^1})} .$$

Now, we want to show that Q is a contraction. We use once again the assumption **(V4)**. For all $s_1, s_2 \in \mathbf{L}^\infty\big(I; (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})\big)$ continuous from the right, we have

$$\left\|\operatorname{div}\left(\rho DV(\rho)(s_1 - s_2)\right)\right\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})} \leq C\|\rho\|_{\mathbf{W^{1,1}}(\mathbb{R}^N;\mathbb{R})} \|s_1 - s_2\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})} .$$

Thus, we get:

$$\begin{aligned}
& \left\|Q(s_1) - Q(s_2)\right\|_{\mathbf{L}^\infty(I;\mathbf{L^1})} \\
\leq\ & C\|\rho\|_{\mathbf{L}^\infty(I;\mathbf{W^{1,1}})} \|s_1 - s_2\|_{\mathbf{L}^\infty(I;\mathbf{L^1})} \int_0^T \exp\left((T-\tau)\left\|\operatorname{div} V(\rho)\right\|_{\mathbf{L}^\infty}\right) \mathrm{d}\tau \\
\leq\ & (e^{CT} - 1)\|\rho\|_{\mathbf{L}^\infty([0,T_{\mathrm{ex}}-\varepsilon];\mathbf{W^{1,1}})} \|s_1 - s_2\|_{\mathbf{L}^\infty(I;\mathbf{L^1})} .
\end{aligned}$$

Then, for T small enough, can apply the Fixed Point Theorem, that gives us the existence of a unique Kružkov solution to the problem. Furthermore, as the time of existence does not depend on the initial condition, we can iterate this procedure to obtain existence on the interval $[0, T_{\mathrm{ex}} - \varepsilon]$. Finally, as this is true for all $\varepsilon \in \mathring{I}_{\mathrm{ex}}$, we obtain the same result on the all interval I_{ex}.

The $\mathbf{L^1}$ bound follows from (5.16). Let $T \in I_{\mathrm{ex}}$ and $t \in I$, then for a suitable $C = C\left(\|\rho\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R})}\right)$

$$\left\|r(t)\right\|_{\mathbf{L^1}} \leq \|r_o\|_{\mathbf{L^1}} e^{Ct} + \|\rho\|_{\mathbf{L}^\infty(I;\mathbf{W^{1,1}})} \left\|\operatorname{div} DV(\rho)\right\|_{\mathbf{L}^\infty} \int_0^t \left\|r(\tau)\right\|_{\mathbf{L^1}} \mathrm{d}\tau .$$

A use of **(V4)** and an application of Gronwall Lemma gives

$$\left\|r(t)\right\|_{\mathbf{L^1}} \quad \leq \quad e^{Ct} e^{K\|\rho\|_{\mathbf{L}^\infty(I,\mathbf{W^{1,1}})} t} \|r_o\|_{\mathbf{L^1}} ,$$

where $K = K\left(\|\rho\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R})}\right)$ is as in **(V4)**.

The $\mathbf{L}^\infty$ bound comes from the same representation formula. Indeed, for $T \in I_{\mathrm{ex}}$ and $t \in I$ we have

$$\left\|r(t)\right\|_{\mathbf{L}^\infty} \leq e^{Ct}\|r_o\|_{\mathbf{L}^\infty} + \|\rho\|_{\mathbf{L}^\infty(I;\mathbf{W^{1,\infty}})} \left\|\operatorname{div} DV(\rho)\right\|_{\mathbf{L}^\infty} \int_0^t \left\|r(\tau)\right\|_{\mathbf{L^1}} \mathrm{d}\tau .$$

Then, the last $\left\|r(\tau)\right\|_{\mathbf{L^1}}$ is bounded just as above. We get

$$\left\|r(t)\right\|_{\mathbf{L}^\infty} \leq e^{Ct}\|r_o\|_{\mathbf{L}^\infty} + Kte^{2Ct} e^{K\|\rho\|_{\mathbf{L}^\infty(I,\mathbf{W^{1,1}})} t} \|r_o\|_{\mathbf{L^1}} \|\rho\|_{\mathbf{L}^\infty(I,\mathbf{W^{1,\infty}})} .$$

Finally, we get a $\mathbf{W^{1,1}}$ bound using the expression of the solution given by Lemma 5.1. Indeed, assuming in addition **(V2)** and **(V4)**, we get

$$\begin{aligned}
\left\|\nabla r(t)\right\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})} \ \leq\ & e^{2Ct}\|\nabla r_o\|_{\mathbf{L^1}} + Cte^{2Ct}\|r_o\|_{\mathbf{L^1}} \\
& + K(1+Ct)e^{2Ct}\|\rho\|_{\mathbf{L}^\infty(I;\mathbf{W^{2,1}})} \int_0^t \left\|r(\tau)\right\|_{\mathbf{L^1}} \mathrm{d}\tau \\
\leq\ & e^{2Ct}\|\nabla r_o\|_{\mathbf{L^1}} + Cte^{2Ct}\|r_o\|_{\mathbf{L^1}} \\
& + Kt(1+Ct)e^{3Ct} e^{K\|\rho\|_{\mathbf{L}^\infty(I,\mathbf{W^{1,1}})} t} \|r_o\|_{\mathbf{L^1}} \|\rho\|_{\mathbf{L}^\infty(I;\mathbf{W^{2,1}})} .
\end{aligned}$$

Hence, denoting $C' = \max\{C, K\|\rho\|_{\mathbf{L}^\infty(I,\mathbf{W}^{1,1})}\}$, we obtain

$$\left\|r(t)\right\|_{\mathbf{W}^{1,1}} \leq \|r_o\|_{\mathbf{W}^{1,1}}(1+C't)e^{2C't} + Kt(1+Ct)e^{4C't}\|r_o\|_{\mathbf{L}^1}\|\rho\|_{\mathbf{L}^\infty(I;\mathbf{W}^{2,1})}\,.$$

The full continuity in time follows from [16, Remark 2.4] and **(V1)**, **(V4)**, since $r(t) \in \mathbf{W}^{1,1}(\mathbb{R}^N;\mathbb{R})$ implies that $r(t) \in \mathbf{BV}(\mathbb{R}^N;\mathbb{R})$ with $\mathrm{TV}\,(r(t)) = \left\|\nabla_x r(t)\right\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})}$. □

Now, we can address the question of weak Gâteaux differentiability of the semigroup giving the solution to (1.1).

Proof of Theorem 2.10. Let $\alpha, \beta > 0$ with $\beta > \alpha$ and $h \in [0, h^*]$ with h^* small enough so that $\beta > \alpha(1+h^*)$. Fix $\rho_o, r_o \in \mathcal{X}_\alpha$. Thanks to Theorem 2.2, we get the weak entropy solution $\rho \in \boldsymbol{\mathcal{C}}^{\mathbf{0}}([0, T(\alpha,\beta)]; \mathcal{X}_\beta)$ of (1.1) with initial condition ρ_o and $\rho_h \in \boldsymbol{\mathcal{C}}^{\mathbf{0}}([0, T(\alpha(1+h),\beta)]; \mathcal{X}_\beta)$ of (1.1) with initial condition $\rho_o + h r_o$. Note that

$$T(\alpha(1+h),\beta) = \frac{\ln(\beta/(\alpha(1+h)))}{C(\beta)} = T(\alpha,\beta) - \frac{\ln(1+h)}{C(\beta)} \leq T(\alpha,\beta)$$

and $T(\alpha(1+h),\beta)$ goes to $T(\alpha,\beta)$ when h goes to 0. In particular, both solutions are defined on the interval $[0, T(\alpha(1+h^*),\beta)]$.

By Theorem 2.2, point 2, the sequence $\left(\frac{\rho_h-\rho}{h}(t)\right)_{h\in[0,h^*]}$ is bounded in $\mathbf{L}^1$ for all $t \in [0, T(\alpha(1+h^*),\beta)]$. By Dunford–Pettis Theorem, it has a weakly convergent subsequence, see [12]. Thus, there exists $r \in \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$ such that

$$\frac{\rho_h - \rho}{h}(t) \rightharpoonup_{h\to 0} r(t) \text{ weakly in } \mathbf{L}^1.$$

Write now the definition of weak solution for ρ, ρ_h. Let $\varphi \in \boldsymbol{\mathcal{C}}^{\boldsymbol{\infty}}_{\boldsymbol{c}}([0, T(\alpha(1+h^*),\beta)] \times \mathbb{R}^N; \mathbb{R})$

$$\begin{aligned}
\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \Big(\rho\partial_t\varphi + \big(\rho V(\rho)\big)\cdot\nabla_x\varphi\Big)\,\mathrm{d}x\;\mathrm{d}t &= 0\,;\\
\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \Big(\rho_h\partial_t\varphi + \big(\rho_h V(\rho_h)\big)\cdot\nabla_x\varphi\Big)\,\mathrm{d}x\;\mathrm{d}t &= 0\,.
\end{aligned}$$

Now, use **(V4)** and write, for a suitable function $\varepsilon = \varepsilon(\rho, \rho_h)$,

$$V(\rho_h) = V(\rho) + \mathrm{D}V(\rho)(\rho_h - \rho) + \varepsilon(\rho,\rho_h)\,,$$

with $\left\|\varepsilon(\rho,\rho_h)\right\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})} \leq K(2\beta)\left(\|\rho_h-\rho\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})}\right)^2$. Then,

$$\rho V(\rho) - \rho_h V(\rho_h) = (\rho-\rho_h)V(\rho) + \rho\mathrm{D}V(\rho)(\rho-\rho_h) + (\rho-\rho_h)\mathrm{D}V(\rho)(\rho-\rho_h) - \rho_h\varepsilon(\rho,\rho_h).$$

Consequently,

$$\begin{aligned}
\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \Bigg[\frac{\rho-\rho_h}{h}\partial_t\varphi + \Bigg(\frac{\rho-\rho_h}{h}V(\rho) + \rho\mathrm{D}V(\rho)\left(\frac{\rho-\rho_h}{h}\right)&\\
+\frac{\rho-\rho_h}{h}\mathrm{D}V(\rho)(\rho-\rho_h) - \rho_h\frac{\varepsilon(\rho,\rho_h)}{h}\Bigg)\cdot\nabla_x\varphi\Bigg]\,\mathrm{d}x\;\mathrm{d}t &= 0\,.
\end{aligned}$$

Using **(V4)**, $\rho(t) \in \mathcal{X}_\beta$ and the estimate on ε we obtain for all $t \in [0, T(\alpha(1+h^*), \beta)]$:

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla_x \varphi| \left| \frac{\rho - \rho_h}{h} \mathrm{D}V(\rho)(\rho - \rho_h) - \rho_h \frac{\varepsilon(\rho, \rho_h)}{h} \right| \mathrm{d}x \\ \leq \quad & K(2\beta) \int_{\mathbb{R}^N} \left(\frac{|\rho - \rho_h|}{h} + \beta \frac{\|\rho - \rho_h\|_{\mathbf{L^1}}}{h} \right) \|\rho - \rho_h\|_{\mathbf{L^1}} \, |\nabla_x \varphi| \, \mathrm{d}x \,, \end{aligned}$$

and since $\frac{\rho_h - \rho}{h}$ is bounded in $\mathbf{L}^\infty\Big([0, T(\alpha(1+h^*), \beta)]; \mathbf{L^1}(\mathbb{R}^N; \mathbb{R})\Big)$ and $\frac{\rho_h - \rho}{h}(t) \rightharpoonup_{h \to 0} r(t)$ in $\mathbf{L^1}$, then we can apply the Dominated Convergence Theorem. We get:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^N} \Big[r \partial_t \varphi + \big(r V(\rho) + \rho \mathrm{D}V(\rho)(r) \big) \cdot \nabla_x \varphi \Big] \, \mathrm{d}x \; \mathrm{d}t = 0 \,.$$

That is to say that r is a weak solution to (1.2) with initial condition r_o. As this is true for all h^* small enough, finally we obtain a solution on the all interval $[0, T(\alpha, \beta)[$. Hence we conclude that $\rho \in \mathcal{C}^{\mathbf{0}}(I_{\mathrm{ex}} \times \mathbb{R}^N; \mathbb{R})$ implies r defined on I_{ex}. □

In the proof just above, we can not conclude to the uniqueness of the weak Gâteaux derivative as we do not know if the weak solution is unique In particular, we don't know if the derivative is continuous.

We assume now that the assumptions **(V4)** and **(V5)** are satisfied by V. We want to show that with these hypotheses, we have now strong convergence in $\mathbf{L^1}$ to the Kružkov solution of (1.2)

Proof of Theorem 2.11. Let $\alpha, \beta > 0$ with $\beta > \alpha$, and $h \in [0, h^*]$ with h^* small enough so that $\beta > \alpha(1 + h^*)$. Let us denote $T(h) = T(\alpha(1+h), \beta)$ for $h \in [0, h^*]$ the time of existence of the solution of (1.1) given by Theorem 2.2.

Fix $\rho_o \in (\mathbf{W^{1,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R}^N; [0, \alpha])$, $r_o \in (\mathbf{L}^\infty \cap \mathbf{W^{1,1}})(\mathbb{R}^N; [0, \alpha])$. Let ρ, respectively ρ_h, be the weak entropy solutions of (1.1) given by Theorem 2.2 with initial condition ρ_o, respectively $\rho_o + h r_o$. Note that these both solutions are in $\mathcal{C}^{\mathbf{0}}\big([0, T(h^*)]; \mathbf{L^1}(\mathbb{R}^N; \mathbb{R})\big)$. Furthermore, under these hypotheses for ρ_o and r_o, we get thanks to Proposition 2.5 that the corresponding solutions ρ and ρ_h of (1.1) are in $\mathcal{C}^{\mathbf{0}}\big([0, T(h^*)]; (\mathbf{W^{1,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R}^N; [0, \beta])\big)$, condition **(V3)** being satisfied. Hence, we can now introduce the Kružkoz solution $r \in \mathcal{C}^{\mathbf{0}}\big([0, T(h^*)[; \mathbf{L^1}(\mathbb{R}^N; \mathbb{R})\big)$ of (1.2), whose existence is given in this case by Proposition 2.9. Note that, r_o being in $\mathbf{W^{1,1}}(\mathbb{R}^N; \mathbb{R})$ and $\rho \in \mathbf{L}^\infty\Big([0, T(h^*)]; \mathbf{W^{2,1}}(\mathbb{R}^N; \mathbb{R})\Big)$ and **(V2), (V4)** being satisfied, $r(t)$ is also in $\mathbf{W^{1,1}}(\mathbb{R}^N; \mathbb{R})$ for all $t \in \big[0, T(h^*)\big[$ thanks to the $\mathbf{W^{1,1}}$ bound of Proposition 2.9.

Let us denote $z_h = \rho + h r$. We would like to compare ρ_h and z_h thanks to [16, Theorem 2.6]. A straightforward computation shows that z_h is the solution to the following problem,

$$\left\{ \begin{array}{l} \partial_t z_h + \operatorname{div}\Big(z_h \big(V(\rho) + h DV(\rho)(r) \big) \Big) = h^2 \operatorname{div}\big(r \, DV(\rho)(r) \big) \,, \\ z_h(0) = \rho_o + h r_o \in \mathcal{X}_{\alpha(1+h)} \,. \end{array} \right.$$

Note that the source term being in $\mathcal{C}^{\mathbf{0}}\big(\big[0, T(h^*)\big[; \mathbf{L^1}(\mathbb{R}^N; \mathbb{R})\big)$, and the flow being regular, we can apply to this equation Lemma 5.1 that gives existence of a Kružkov solution.

As in the proof of Lemma 5.1, we make here the remark that [16, Theorem 2.6] can be used with the second source term in $\mathcal{C}^{\mathbf{0}}\Big(\big[0, T(h^*)\big[; \mathbf{L^1}(\mathbb{R}^N; \mathbb{R})\Big)$ and the flow $\mathcal{C}^{\mathbf{2}}$ in space and only $\boldsymbol{\mathcal{C}}^{\mathbf{0}}$ in time. Besides, we also use the same *slight improvement* as in the proof of Lemma 5.3, taking the $\mathbf{L}^\infty$ norm in the integral term only in space, keeping the time fixed. We get, with $\kappa_o = NW_N(2N+1)\left\|\nabla_x V(\rho_h)\right\|_{\mathbf{L}^\infty([0,T(h^*)]\times\mathbb{R}^N;\mathbb{R})}$ and $\kappa = 2N\left\|\nabla_x V(\rho_h)\right\|_{\mathbf{L}^\infty([0,T(h^*)]\times\mathbb{R}^N;\mathbb{R})}$, for some $T \in [0, T(h^*)]$,

$$\begin{aligned}
& \|\rho_h - z_h\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)} \\
\leq\; & T e^{\kappa_o T}\, \mathrm{TV}\,(\rho_o + h r_o) \left\|V(\rho_h) - V(\rho) - hDV(\rho)(r)\right\|_{\mathbf{L}^\infty([0,T(h^*)]\times\mathbb{R}^N;\mathbb{R}^N)} \\
& + NW_N \int_0^T (T-t) e^{\kappa_o (T-t)} \int_{\mathbb{R}^N} \left\|\rho_h(t)\right\|_{\mathbf{L}^\infty} \left\|\nabla_x \operatorname{div} V(\rho_h)\right\| \mathrm{d}x\, \mathrm{d}t \\
& \qquad \times \left\|V(\rho_h) - V(\rho) - hDV(\rho)(r)\right\|_{\mathbf{L}^\infty([0,T(h^*)]\times\mathbb{R}^N;\mathbb{R}^N)} \\
& + h^2 \int_0^T e^{\kappa(T-t)} \int_{\mathbb{R}^N} \left|\operatorname{div}\left(r DV(\rho)(r)\right)\right| \mathrm{d}x\ \mathrm{d}t \\
& + \int_0^T e^{\kappa(T-t)} \int_{\mathbb{R}^N} \left|\operatorname{div}\left(V(\rho_h) - V(\rho) - hDV(\rho)(r)\right)\right| \mathrm{d}x\ \mathrm{d}t \\
& \qquad \times \max_{t\in[0,T]} \left\{ \left\|\rho_h(t)\right\|_{\mathbf{L}^\infty}, \left\|z_h(t)\right\|_{\mathbf{L}^\infty} \right\}.
\end{aligned}$$

Then, setting $C = C(\beta)$ and $K = K(2\beta)$, we use:

- the bound of ρ and ρ_h in $\mathbf{L}^\infty$ given by Lemma 5.3
$$\left\|\rho(t)\right\|_{\mathbf{L}^\infty} \leq \|\rho_o\|_{\mathbf{L}^\infty} e^{Ct} \leq \beta \quad \text{and} \quad \left\|\rho_h(t)\right\|_{\mathbf{L}^\infty} \leq \|\rho_o + h r_o\|_{\mathbf{L}^\infty} e^{Ct} \leq \beta\,;$$
- the properties of V given in **(V1)** to get
$$\left\|\nabla_x \operatorname{div} V(\rho_h)\right\|_{\mathbf{L}^\infty(\mathbb{R}_+\times\mathbb{R}^N;\mathbb{R})} \leq C \quad \text{and} \quad \left\|\nabla_x \operatorname{div} V(\rho_h)\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}))} \leq C\,;$$
- the property **(V4)**, respectively **(V5)**, to get
$$\begin{aligned}
& \left\|V(\rho_h) - V(\rho) - hDV(\rho)(r)\right\|_{\mathbf{L}^\infty(I\times\mathbb{R}^N;\mathbb{R})} \\
\leq\; & K \left(\|\rho_h - \rho\|^2_{\mathbf{L}^\infty(I;\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}))} + \|\rho_h - z_h\|_{\mathbf{L}^\infty(I;\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}))} \right), \quad \text{respectively} \\
& \left\|\operatorname{div}\left(V(\rho_h) - V(\rho) - hDV(\rho)(r)\right)\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}))} \\
\leq\; & K \left(\|\rho_h - \rho\|^2_{\mathbf{L}^\infty(I;\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}))} + \|\rho_h - z_h\|_{\mathbf{L}^\infty(I;\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}))} \right);
\end{aligned}$$
- the property **(V4)** to get
$$\left\|\operatorname{div}\left(r DV(\rho)(r)\right)\right\|_{\mathbf{L}^1} \leq K\, \|r\|_{\mathbf{W}^{1,1}} \|r\|_{\mathbf{L}^1}.$$

Gathering all these estimates, denoting $C' = NW_N(2N+1)C$, we obtain

$$\begin{aligned}
& \|\rho_h - z_h\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)} \\
\leq\; & T e^{C'T} \left(\mathrm{TV}\,(\rho_o + h r_o) + NW_N C T \beta\right) K \left(\|\rho_h - \rho\|^2_{\mathbf{L}^\infty(I;\mathbf{L}^1)} + \|\rho_h - z_h\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)} \right) \\
& + h^2 K T e^{C'T} \|r\|_{\mathbf{L}^\infty(I;\mathbf{W}^{1,1})} \|r\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)} \\
& + \left(\beta + h \sup_{t\in I} \left\|r(t)\right\|_{\mathbf{L}^\infty} \right) T e^{C'T} K \left(\|\rho_h - \rho\|^2_{\mathbf{L}^\infty(I;\mathbf{L}^1)} + \|\rho_h - z_h\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)} \right).
\end{aligned}$$

Then, dividing by h and introducing

$$F_h(T) \quad = \quad KTe^{C'T}\left[\mathrm{TV}\,(\rho_o) + h\,\mathrm{TV}\,(r_o) + NW_N CT\beta + \beta + h\left\|r(t)\right\|_{\mathbf{L}^\infty}\right],$$

we obtain

$$\begin{aligned}\left\|\frac{\rho_h - z_h}{h}\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)} \leq\ & F_h(T)\left[\|\rho_h-\rho\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)}\left\|\frac{\rho_h-\rho}{h}\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)} + \left\|\frac{\rho_h - z_h}{h}\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)}\right]\\ & + hKTe^{C'T}\|r\|_{\mathbf{L}^\infty(I;\mathbf{W}^{1,1})}\|r\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)}\,.\end{aligned}$$

Note that F_h is a function that vanishes in $T=0$ and that depends also on ρ_o, r_o and h. Hence, we can find $\bar T \leq T(h^*)$ small enough such that $F_{h^*}(\bar T) \leq 1/2$. Furthermore, $F_h(T)$ is increasing in h consequently, $h \leq h^*$ implies $F_h(T) \leq F_{h^*}(T)$. Noticing moreover that $\left\|\frac{\rho_h-\rho}{h}\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)}$ has a uniform bound M in h by 2. in Theorem 2.2, we get for $T \leq \bar T$

$$\begin{aligned}\frac{1}{2}\left\|\frac{\rho_h-\rho}{h} - r\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)} =\ & \frac{1}{2}\left\|\frac{\rho_h - z_h}{h}\right\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)}\\ \leq\ & \frac{M}{2}\|\rho_h-\rho\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)} + hKTe^{C'T}\|r\|_{\mathbf{L}^\infty(I,\mathbf{W}^{1,1})}\|r\|_{\mathbf{L}^\infty(I;\mathbf{L}^1)}\,.\end{aligned}$$

The right side above goes to 0 when $h \to 0$, so we have proved the Gâteaux differentiability of the semigroup S for small time. Finally, we iterate like in the proof of Theorem 2.2 in order to have existence on the all interval $[0, T(h^*)]$. Let T_1 be such that $F_{h^*}(T_1) = 1/2$ and assume $T_1 < T(h^*)$. If we assume the Gâteaux differentiability is proved until time $T_n \leq T(h^*)$, we make the same estimate on $[T_n, T_{n+1}]$, T_{n+1} being to determine. We get

$$\begin{aligned}& \|\rho_h - z_h\|_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{L}^1)}\\ \leq\ & (T_{n+1}-T_n)e^{C'(T_{n+1}-T_n)}\left(\mathrm{TV}\,(\rho_h(T_n)) + NW_N C(T_{n+1}-T_n)\beta\right)\\ & \quad\times K\left(\|\rho_h-\rho\|^2_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{L}^1)} + \|\rho_h - z_h\|_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{L}^1)}\right)\\ & + h^2K(T_{n+1}-T_n)e^{C'(T_{n+1}-T_n)}\|r\|_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{W}^{1,1})}\|r\|_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{L}^1)}\\ & + \left(\beta + h\sup_{[T_n,T_{n+1}]}\left\|r(t)\right\|_{\mathbf{L}^\infty}\right)(T_{n+1}-T_n)e^{C'(T_{n+1}-T_n)}\\ & \quad\times K\left(\|\rho_h-\rho\|^2_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{L}^1)} + \|\rho_h - z_h\|_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{L}^1)}\right)\,.\end{aligned}$$

Then, we divide by h and we introduce, for $T \geq T_n$

$$\begin{aligned}F_{h,n}(T) \quad = \quad & K(T-T_n)e^{C'(T-T_n)}\Big[(\mathrm{TV}\,(\rho_o) + h\,\mathrm{TV}\,(r_o))e^{CT_n} + \beta C'T_n e^{C'T_n}\\ & + NW_N C(T-T_n)\beta + \beta + h\sup_{[T_n,T_{n+1}]}\left\|r(t)\right\|_{\mathbf{L}^\infty}\Big]\,.\end{aligned}$$

We define $T_{n+1} > T_n$ such that $F_{h,n}(T_{n+1}) = \frac{1}{2}$. This is possible since $F_{h,n}$ vanishes in $T = T_n$ and increases to infinity when $T \to \infty$. Hence, as long as $T_{n+1} \leq T(h^*)$, we get

$$\begin{aligned}& \left\|\frac{\rho_h-\rho}{h} - r\right\|_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{L}^1)}\\ \leq\ & KM\|\rho_h-\rho\|_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{L}^1)}\\ & + 2hK(T_{n+1}-T_n)e^{C'(T_{n+1}-T_n)}\|r\|_{\mathbf{L}^\infty([T_n,T_{n+1}],\mathbf{W}^{1,1})}\|r\|_{\mathbf{L}^\infty([T_n,T_{n+1}];\mathbf{L}^1)}\,.\end{aligned}$$

The next question is to wonder if (T_n) goes up to $T(h^*)$. We assume that it is not the case: then necessarily, $F_{h,n}(T_{n+1}) \overset{n\to\infty}{\longrightarrow} 0$, since $T_{n+1} - T_n \to 0$. This is a contradiction to $F_{h,n}(T_{n+1}) = 1/2$.

Consequently, $T_n \overset{n\to\infty}{\longrightarrow} \infty$ and the Gâteaux differentiability is valid for all time $t \in [0, T(h^*)]$. Then, making h^* goes to 0, we obtain that the differentiability is valid on in the interval $[0, T(\alpha, \beta)[$.

It remains to check that the Gâteaux derivative is a bounded linear operator, for t and ρ_o fixed. The linearity is immediate. Additionally, due to the $\mathbf{L^1}$ estimate on the solution r of the linearized equation (1.2) given by Proposition 2.9, we obtain

$$\left\| DS_t(\rho_o)(r_o) \right\|_{\mathbf{L^1}} = \left\| r(t) \right\|_{\mathbf{L^1}} \leq e^{Kt\|\rho\|_{\mathbf{L}^\infty(I;\mathbf{W^{1,1}})}} e^{Ct} \|r_o\|_{\mathbf{L^1}} ,$$

so that the Gâteaux derivative is bounded, at least for $t \leq T < T_{\text{ex}}$. □

5.4 Proofs Related to Sections 3 and 4

Proof of Proposition 3.1. Note that $v(\rho)$ is constant in x, hence $\operatorname{div} V(\rho) = 0$, and **(A)** is satisfied. Besides, we easily obtain $\left\| \partial_x V(\rho) \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} = 0$, $\left\| \partial_x V(\rho) \right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} = 0$, $\left\| \partial_x^2 V(\rho) \right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} = 0$ and

$$\begin{aligned} \left\| V(\rho_1) - V(\rho_2) \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} &\leq \left\| v' \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \|\rho_1 - \rho_2\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} , \\ \left\| \partial_x V(\rho_1) - \partial_x V(\rho_2) \right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} &= 0 , \end{aligned}$$

so that **(V1)** is satisfied. Similarly, $\partial_x^2 V(\rho) = 0$ and $\partial_x^3 V(\rho) = 0$ imply easily that **(V2)** and **(V3)** are satisfied.

We consider now **(V4)**: is v is $\boldsymbol{\mathcal{C}^2}$ then, for all $A, B \in \mathbb{R}$,

$$v(B) = v(A) + v'(A)(B - A) + \int_0^1 v'' \left(sB + (1 - s)A \right) (1 - s)(B - A)^2 \, \mathrm{d}s \; .$$

Choosing $A = \int_0^1 \rho(\xi) \, \mathrm{d}\xi$ and $B = \int_0^1 \tilde\rho(\xi) \, \mathrm{d}\xi$, we get

$$\begin{aligned} &\left\| v \left(\int_0^1 \tilde\rho(\xi) \, \mathrm{d}\xi \right) - v \left(\int_0^1 \rho(\xi) \, \mathrm{d}\xi \right) - v' \left(\int_0^1 \rho(\xi) \, \mathrm{d}\xi \right) \int_0^1 (\tilde\rho - \rho)(\xi) \, \mathrm{d}\xi \right\|_{\mathbf{L}^\infty} \\ \leq\ & \frac{1}{2} \left\| v'' \right\|_{\mathbf{L}^\infty} \|\tilde\rho - \rho\|_{\mathbf{L^1}}^2 \end{aligned}$$

and we choose $K = \frac{1}{2} \left\| v'' \right\|_{\mathbf{L}^\infty}$, $DV(\rho)(r) = v' \left(\int_0^1 \rho(\xi) \, \mathrm{d}\xi \right) \int_0^1 r(\xi) \, \mathrm{d}\xi$. Condition **(V4)** is then satisfied since there is no x-dependance, so

$$\begin{aligned} \left\| V(\tilde\rho) - V(\rho) - DV(\rho)(\tilde\rho - \rho) \right\|_{\mathbf{W^{2,\infty}}} &= \left\| V(\tilde\rho) - V(\rho) - DV(\rho)(\tilde\rho - \rho) \right\|_{\mathbf{L}^\infty} \\ &\leq \frac{1}{2} \left\| v'' \right\|_{\mathbf{L}^\infty} \|\tilde\rho - \rho\|_{\mathbf{L^1}}^2 . \end{aligned}$$

Similarly, $\left\|DV(\rho)(r)\right\|_{\mathbf{W^{2,\infty}}} = \left\|DV(\rho)(r)\right\|_{\mathbf{L^\infty}} \leq \left\|v'\right\|_{\mathbf{L^\infty}} \|r\|_{\mathbf{L^1}}$. Finally, consider **(V5)**:

$$\left\|\mathrm{div}\left[v\left(\int_0^1 \tilde\rho(\xi)\,\mathrm{d}\xi\right) - v\left(\int_0^1 \rho(\xi)\,\mathrm{d}\xi\right) - v'\left(\int_0^1 \rho(\xi)\,\mathrm{d}\xi\right)\int_0^1 \left(\tilde\rho-\rho\right)(\xi)\,\mathrm{d}\xi\right]\right\|_{\mathbf{L^1}} = 0\,,$$

$$\left\|\mathrm{div}\left(v'\left(\int_0^1 \rho(\xi)\mathrm{d}\xi\right)\int_0^1 r(\xi)\mathrm{d}\xi\right)\right\|_{\mathbf{L^1}} = 0\,.$$

Concluding the proof. □

Proof of Proposition 4.1. The proof exploits the standard properties of the convolution.

Consider first **(V1)**:

$$\begin{aligned}
\left\|\nabla_x V(\rho)\right\|_{\mathbf{L^\infty}} &= \left\|v'\right\|_{\mathbf{L^\infty}} \|\rho\|_{\mathbf{L^\infty}} \|\nabla_x\eta\|_{\mathbf{L^1}} \|\vec v\|_{\mathbf{L^\infty}} + \|v\|_{\mathbf{L^\infty}} \|\nabla_x \vec v\|_{\mathbf{L^\infty}}\\
&\leq C(\|\rho\|_{\mathbf{L^\infty}})\,,\\
\left\|\nabla_x V(\rho)\right\|_{\mathbf{L^1}} &\leq \|v\|_{\mathbf{W^{1,\infty}}} \|\vec v\|_{\mathbf{W^{1,1}}} (1 + \|\rho\|_{\mathbf{L^\infty}} \|\nabla_x\eta\|_{\mathbf{L^1}})\,,\\
\left\|\nabla_x^2 V(\rho)\right\|_{\mathbf{L^1}} &\leq \|v\|_{\mathbf{W^{2,\infty}}} \|\vec v\|_{\mathbf{W^{2,1}}}\\
&\quad\times\left[1 + \|\rho\|^2_{\mathbf{L^\infty}} \|\nabla_x\eta\|^2_{\mathbf{L^1}} + \|\rho\|_{\mathbf{L^\infty}} \left\|\nabla_x^2\eta\right\|_{\mathbf{L^1}} + 2\|\rho\|_{\mathbf{L^\infty}} \|\nabla_x\eta\|_{\mathbf{L^1}}\right]\\
&\leq C(\|\rho\|_{\mathbf{L^\infty}})\,,\\
\left\|V(\rho_1) - V(\rho_2)\right\|_{\mathbf{L^\infty}} &\leq \left\|v'\right\|_{\mathbf{L^\infty}} \|\vec v\|_{\mathbf{L^\infty}} \|\eta\|_{\mathbf{L^\infty}} \|\rho_1 - \rho_2\|_{\mathbf{L^1}}\,,\\
\left\|\nabla_x\left(V(\rho_1) - V(\rho_2)\right)\right\|_{\mathbf{L^1}} &= \|v\|_{\mathbf{W^{2,\infty}}} \|\vec v\|_{\mathbf{W^{1,\infty}}} \|\eta\|_{\mathbf{W^{1,1}}} \left(2 + \|\nabla_x\eta\|_{\mathbf{L^1}} \|\rho_1\|_{\mathbf{L^\infty}}\right) \|\rho_1 - \rho_2\|_{\mathbf{L^1}}.
\end{aligned}$$

Then, we check **(V2)**:

$$\begin{aligned}
\left\|\nabla_x^2 V(\rho)\right\|_{\mathbf{L^\infty}} &\leq 2\|v\|_{\mathbf{W^{2,\infty}}} \|\vec v\|_{\mathbf{W^{2,\infty}}}\\
&\quad\times\left(1 + \|\rho\|^2_{\mathbf{L^\infty}} \|\nabla_x\eta\|^2_{\mathbf{L^1}} + \|\rho\|_{\mathbf{L^\infty}} \left\|\nabla_x^2\eta\right\|_{\mathbf{L^1}} + \|\rho\|_{\mathbf{L^\infty}} \|\nabla_x\eta\|_{\mathbf{L^1}}\right)\,.
\end{aligned}$$

Entirely analogous computations allow to prove also **(V3)**.

Consider **(V4)**. First we look at the Fréchet derivative of $V(\rho)$: v being $\boldsymbol{\mathcal{C}^2}$, we can write, for all $A, B \in \mathbb{R}$,

$$v(B) = v(A) + v'(A)(B-A) + \int_0^1 v''(sB + (1-s)A)(1-s)(B-A)^2\ \mathrm{d}s\ .$$

If we take $A = \rho * \eta$ and $B = \tilde\rho * \eta$, then we get, for $\rho, \tilde\rho \in \mathbf{L^1}(\mathbb{R}^N;\mathbb{R})$

$$\left\|\Big(v\left(\tilde\rho * \eta\right) - v\left(\rho * \eta\right) - v'\left(\rho * \eta\right)\left(\left(\tilde\rho - \rho\right) * \eta\right)\Big)\vec v\right\|_{\mathbf{L^\infty}} \leq \frac{1}{2}\left\|v''\right\|_{\mathbf{L^\infty}} \|\eta\|^2_{\mathbf{L^\infty}} \|\tilde\rho - \rho\|^2_{\mathbf{L^1}} \|\vec v\|_{\mathbf{L^\infty}}\,;$$

and

$$\left\|\nabla_x\left[\Big(v\left(\tilde\rho * \eta\right) - v\left(\rho * \eta\right) - v'\left(\rho * \eta\right)\left(\left(\tilde\rho - \rho\right) * \eta\right)\Big)\vec v\right]\right\|_{\mathbf{L^\infty}}$$

$$
\begin{aligned}
&\le \frac{3}{2}\left\|v''\right\|_{\mathbf{L}^\infty}\|\eta\|^2_{\mathbf{W}^{1,\infty}}\|\tilde\rho-\rho\|^2_{\mathbf{L}^1}\|\vec v\|_{\mathbf{W}^{1,\infty}}\\
&\quad+\frac{1}{2}\left\|v'''\right\|_{\mathbf{L}^\infty}\|\eta\|^2_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^\infty}\|\nabla_x\eta\|_{\mathbf{L}^1}\|\tilde\rho-\rho\|^2_{\mathbf{L}^1}\|\vec v\|_{\mathbf{L}^\infty}\,;\\
&\quad\left\|\nabla_x^2\left[\Big(v\left(\tilde\rho*\eta\right)-v\left(\rho*\eta\right)-v'\left(\rho*\eta\right)\big((\tilde\rho-\rho)*\eta\big)\Big)\vec v\right]\right\|_{\mathbf{L}^\infty}\\
&\le \left\|v^{(4)}\right\|_{\mathbf{L}^\infty}\|\tilde\rho-\rho\|^2_{\mathbf{L}^1}\|\eta\|^2_{\mathbf{L}^\infty}\|\nabla_x\eta\|^2_{\mathbf{L}^1}\left(\|\rho\|_{\mathbf{L}^\infty}+\|\tilde\rho\|_{\mathbf{L}^\infty}\right)^2\|\vec v\|_{\mathbf{L}^\infty}\\
&\quad+2\left\|v^{(3)}\right\|_{\mathbf{L}^\infty}\|\tilde\rho-\rho\|^2_{\mathbf{L}^1}\|\eta\|^2_{\mathbf{W}^{1,\infty}}\left(\|\rho\|_{\mathbf{L}^\infty}+\|\tilde\rho\|_{\mathbf{L}^\infty}\right)\|\nabla\eta\|_{\mathbf{L}^1}\|\vec v\|_{\mathbf{L}^\infty}\\
&\quad+\frac{1}{2}\left\|v^{(3)}\right\|_{\mathbf{L}^\infty}\|\tilde\rho-\rho\|^2_{\mathbf{L}^1}\|\eta\|^2_{\mathbf{L}^\infty}\left(\|\rho\|_{\mathbf{L}^\infty}\|\eta\|_{\mathbf{W}^{2,1}}+1\right)\|\vec v\|_{\mathbf{W}^{1,\infty}}\\
&\quad+6\left\|v''\right\|_{\mathbf{L}^\infty}\|\tilde\rho-\rho\|^2_{\mathbf{L}^1}\|\eta\|^2_{\mathbf{W}^{2,\infty}}\|\vec v\|_{\mathbf{W}^{2,\infty}}\,.
\end{aligned}
$$

Then, $DV(\rho)(r)=v'(\rho*\eta)r*\eta\,\vec v$.

In order to satisfy **(V4)**, we have also to check that the derivative is a bounded operator from $\mathcal{C}^2$ to $\mathbf{L}^1$. We have,

$$
\begin{aligned}
\left\|DV(\rho)(r)\right\|_{\mathbf{L}^\infty} &\le \left\|v'\right\|_{\mathbf{L}^\infty}\|\eta\|_{\mathbf{L}^\infty}\|\vec v\|_{\mathbf{L}^\infty}\|r\|_{\mathbf{L}^1}\,,\\
\left\|\nabla_x DV(\rho)(r)\right\|_{\mathbf{L}^\infty} &\le \|v\|_{\mathbf{W}^{2,\infty}}\|\vec v\|_{\mathbf{W}^{1,\infty}}\|\eta\|_{\mathbf{W}^{1,\infty}}\left(2+\|\rho\|_{\mathbf{L}^\infty}\|\eta\|_{\mathbf{W}^{1,1}}\right)\|r\|_{\mathbf{L}^1}\,,\\
\left\|\nabla_x^2 DV(\rho)(r)\right\|_{\mathbf{L}^\infty} &\le \|v\|_{\mathbf{W}^{3,\infty}}\|\eta\|_{\mathbf{W}^{2,\infty}}\|\vec v\|_{\mathbf{W}^{2,\infty}}\\
&\quad\times\left(4+5\|\rho\|_{\mathbf{L}^\infty}\|\eta\|_{\mathbf{W}^{2,1}}+\|\rho\|^2_{\mathbf{L}^\infty}\|\nabla\eta\|^2_{\mathbf{L}^1}\right)\|r\|_{\mathbf{L}^1}\,.
\end{aligned}
$$

Finally, we check that also **(V5)** is satisfied:

$$
\begin{aligned}
&\left\|\mathrm{div}\left(V(\tilde\rho)-V(\rho)-DV(\rho)(\tilde\rho-\rho)\right)\right\|_{\mathbf{L}^1}\\
\le\ &\frac{1}{2}\|v\|_{\mathbf{W}^{3,\infty}}\|\tilde\rho-\rho\|^2_{\mathbf{L}^1}\|\eta\|_{\mathbf{L}^1}\|\eta\|_{\mathbf{W}^{1,\infty}}\|\vec v\|_{\mathbf{W}^{1,\infty}}\left(3+\|\rho\|_{\mathbf{L}^\infty}\|\eta\|_{\mathbf{W}^{1,1}}\right)\,,\\
&\left\|\mathrm{div}\,DV(\rho)(r)\right\|_{\mathbf{L}^1}\\
=\ &\left\|\mathrm{div}\left(v'(\rho*\eta)r*\eta\,\vec v\right)\right\|_{\mathbf{L}^1}\\
\le\ &\|v\|_{\mathbf{W}^{2,\infty}}\|\eta\|_{\mathbf{W}^{1,1}}\|\vec v\|_{\mathbf{W}^{1,\infty}}\left(2+\|\rho\|_{\mathbf{L}^\infty}\|\nabla_x\eta\|_{\mathbf{L}^1}\right)\|r\|_{\mathbf{L}^1}
\end{aligned}
$$

completing the proof. □

Remark 5.4 *The above proof shows that condition **(B)** is not satisfied by (4.10). Indeed, here we have that C grows linearly: $C(\alpha)=1+\alpha$. Hence, with the notation used in the proof of Theorem 2.4, for $\alpha_1>0$, we have*

$$
\sum_{k=1}^n T(\alpha_k,\alpha_{k+1})\le\sum_{k=1}^n\frac{1}{1+\alpha_{k+1}}\int_{\alpha_k}^{\alpha_{k+1}}\frac{1}{t}\,\mathrm{d}t\le\sum_{k=1}^n\int_{\alpha_k}^{\alpha_{k+1}}\frac{1}{(1+t)t}\,\mathrm{d}t\le\int_{\alpha_1}^{+\infty}\frac{1}{(1+t)t}\,\mathrm{d}t
$$

and the latter expression is bounded. This shows that, in the case of (4.10), the technique used in Theorem 2.4 does not apply.

Acknowledgment. This work has been supported by the DFG SPP1253, DAAD D/06/19582, DAAD D/08/11076, and HE5386/6-1. The third author thanks Sylvie Benzoni-Gavage for useful discussions.

References

[1] C. E. Agnew. Dynamic modeling and control of congestion-prone systems. *Operations Res.*, 24(3):400–419, 1976.

[2] L. Ambrosio. Transport equation and Cauchy problem for non-smooth vector fields. In *Calculus of variations and nonlinear partial differential equations*, volume 1927 of *Lecture Notes in Math.*, pages 1–41. Springer, Berlin, 2008.

[3] D. Armbruster, P. Degond, and C. Ringhofer. A model for the dynamics of large queuing networks and supply chains. *SIAM J. Appl. Math.*, 66(3):896–920 (electronic), 2006.

[4] D. Armbruster, D. E. Marthaler, C. Ringhofer, K. Kempf, and T.-C. Jo. A continuum model for a re-entrant factory. *Oper. Res.*, 54(5):933–950, 2006.

[5] S. Benzoni-Gavage, R. M. Colombo, and P. Gwiazda. Measure valued solutions to conservation laws motivated by traffic modelling. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 462(2070):1791–1803, 2006.

[6] S. Bianchini. On the shift differentiability of the flow generated by a hyperbolic system of conservation laws. *Discrete Contin. Dynam. Systems*, 6(2):329–350, 2000.

[7] F. Bouchut and F. James. One-dimensional transport equations with discontinuous coefficients. *Nonlinear Anal.*, 32:891–933, 1998.

[8] F. Bouchut and F. James. Differentiability with respect to initial data for a scalar conservation law. In *Hyperbolic problems: theory, numerics, applications.* Internat. Ser. Numer. Math, Birkhuser, Basel, 1999.

[9] A. Bressan and G. Guerra. Shift-differentiability of the flow generated by a conservation law. *Discrete Contin. Dynam. Systems*, 3(1):35–58, 1997.

[10] A. Bressan and M. Lewicka. Shift differentials of maps in BV spaces. In *Nonlinear theory of generalized functions (Vienna, 1997)*, volume 401 of *Chapman & Hall/CRC Res. Notes Math.*, pages 47–61. Chapman & Hall/CRC, Boca Raton, FL, 1999.

[11] A. Bressan and W. Shen. Optimality conditions for solutions to hyperbolic balance laws. *Control methods in PDE-dynamical systems, Contemp. Math.*, 426:129–152, 2007.

[12] H. Brezis. *Analyse fonctionnelle.* Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[13] R. M. Colombo, G. Facchi, G. Maternini, and M. D. Rosini. On the continuum modeling of crowds. In *HYP2008*, June 2008. To appear.

[14] R. M. Colombo and A. Groli. On the optimization of the initial boundary value problem for a conservation law. *J. Math. Analysis Appl.*, 291(1):82–99, 2004.

[15] R. M. Colombo, G. Guerra, M. Herty, and V. Sachers. Modeling and optimal control of networks of pipes and canals. Technical report, Milano Bicocca University, 2008.

[16] R. M. Colombo, M. Mercier, and M. Rosini. Stability and total variation estimates on general scalar balance laws. *Communications in Mathematical Sciences*, 2009. To appear.

[17] R. M. Colombo and M. D. Rosini. Pedestrian flows and non-classical shocks. *Math. Methods Appl. Sci.*, 28(13):1553–1567, 2005.

[18] J.-M. Coron. *Control and nonlinearity.* American Mathematical Society, 2007.

[19] J.-M. Coron, G. Bastin, and B. d'Andrea Novel. Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. *SIAM J. Control. Optim.*, 47:1460–1498, 2008.

[20] V. Coscia and C. Canavesio. First-order macroscopic modelling of human crowd dynamics. *Math. Models Methods Appl. Sci.*, 18(suppl.):1217–1247, 2008.

[21] M. Gugat. Nodal control of networked hyperbolic systems. *AMO Advanced Modeling and Optimization*, 7:1, 2005.

[22] M. Herty, M. Gugat, A. Klar, and G. Leugering. Conservation law constrained optimization based upon front-tracking. *M2AN Math. Model. Numer. Anal.*, 40:939–960, 2006.

[23] R. L. Hughes. A continuum theory for the flow of pedestrians. *Transportation Research Part B*, 36:507–535, 2002.

[24] U. Karmarkar. Capacity loading and release planning in work-in-progess (wip) and lead-times. *J. Mfg. Oper.Mgt.*, 2:105–123, 1989.

[25] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[26] M. Marca, D. Armbruster, M. Herty, and C. Ringhofer. Control of continuum models of production systems. *preprint*, 2008.

[27] B. Piccoli and A. Tosin. Time-evolving measures and macroscopic modeling of pedestrian flow. *Preprint*, Nov 2008.

[28] S. Ulbrich. A sensitivity and adjoint calculus for discontinuous solutions of hyperbolic conservation laws with source terms. *SIAM J. Control Optim.*, 41:740, 2002.

[29] S. Ulbrich. Adjoint-based derivative computations for the optimal control of discontinuous solutions of hyperbolic conservation laws. *Systems & Control Letters*, 3:309, 2003.