A priori $L^\infty$-estimates for degenerate complex Monge-Ampère equations
Philippe Eyssidieux, Vincent Guedj, Ahmed Zeriahi

To cite this version:
Philippe Eyssidieux, Vincent Guedj, Ahmed Zeriahi. A priori $L^\infty$-estimates for degenerate complex Monge-Ampère equations. IF_PREPUB. 2007. <hal-00360763>
A priori $L^\infty$-estimates for degenerate complex Monge-Ampère equations

P. Eyssidieux, V. Guedj and A. Zeriahi

February 11, 2009

Abstract: We study families of complex Monge-Ampère equations, focusing on the case where the cohomology classes degenerate to a non big class. We establish uniform a priori $L^\infty$-estimates for the normalized solutions, generalizing the recent work of S. Kolodziej and G. Tian. This has interesting consequences in the study of the Kähler-Ricci flow.

1 Introduction

Let $\pi: X \to Y$ be a non degenerate holomorphic mapping between compact Kähler manifolds such that $n := \dim_C X \geq m := \dim_C Y$. Let $\omega_X, \omega_Y$ Kähler forms on $X$ and $Y$ respectively. Let $F: X \to \mathbb{R}^+$ be a non-negative function such that $F \in L^p(X)$ for some $p > 1$.

Set $\omega_t := \pi^*(\omega_Y) + t\omega_X$, $t > 0$. We consider the following family of complex Monge-Ampère equations

\[(\star)_t \begin{cases} (\omega_t + dd^c \varphi_t)^n = c_t t^{n-m} F \omega_X^n \\ \max_X \varphi_t = 0 = 0 \end{cases}\]

where $\varphi_t$ is $\omega_t$–plurisubharmonic on $X$ and $c_t > 0$ is a constant given by

\[c_t t^{n-m} \int_X F \omega_X^n = \int_X \omega_t^n.\]

It follows from the seminal work of S.T. Yau [Y] and S. Kolodziej [K1], [K2] that the equation $(\star)_t$ admits a unique continuous solution. (Observe that for $t \in [0, 1]$, $\omega_t$ is a Kähler form).

Our aim here is to understand what happens when $t \to 0^+$, motivated by recent geometrical developments [ST], [KT]. When $n = m$, the cohomology class $\omega_0$ is big and semi-ample and this problem has been addressed by several authors recently (see [CN], [EGZ], [TZ], [To]).
We focus here on the case $m < n$. This situation is motivated by the study of the Kähler-Ricci flow on manifolds $X$ of intermediate Kodaira dimension $1 \leq \text{kod}(X) \leq n - 1$. When $n = 2$ this has been studied by J. Song and G. Tian [ST].

In a very recent and interesting paper [KT], S. Kolodziej and G. Tian were able to show, under a technical geometric assumption on the fibration $\pi$, that the solutions $(\varphi_t)$ are uniformly bounded on $X$ when $t \searrow 0^+$. The purpose of this note is to (re)prove this result without any technical assumption and with a different method: we actually follow the strategy introduced by S. Kolodziej in [K] and further developed in [EGZ], [BGZ].

**THEOREM.** There exists a uniform constant $M = M(\pi, \|F\|_p) > 0$ such that the solutions to the Monge-Ampère equations $(\ast)_t$ satisfy

$$\|\varphi_t\|_{L^\infty(X)} \leq M, \forall t \in [0, 1].$$

It follows from our result that Theorems 1 and 2 in [KT] hold without any technical assumption on the fibration (see condition 0.2 in [KT]). This result has been announced by J-P. Demailly and N. Pali [DP].

## 2 Proof of the theorem

### 2.1 Preliminary remarks

**Uniform control of $c_t$.** Observe that $\omega^k_0 = 0$ for $m < k \leq n$, hence for all $t \in [0, 1]$,

$$\omega^n_t = \sum_{k=1}^{m} \binom{n}{k} t^{n-k} \omega^k_0 \wedge \omega^{n-k}_X.$$

Note that $[0, 1] \ni t \mapsto t^{m-n} \omega^n_t$ is increasing (hence decreases as $t \searrow 0^+$) and satisfies for $t \in [0, 1]$

$$0 < \binom{n}{m} \int_X \omega^m_0 \wedge \omega^{n-m}_X =: c_0 \leq c_t \leq c_1.$$

**Uniform control of densities.** Let $J_\pi$ denote the (modulus square) of the Jacobian of the mapping $\pi$, defined through

$$\omega^m_0 \wedge \omega^{n-m}_X = J_\pi \omega^n_X.$$
Let us rewrite the equation \((\omega_t + dd^c \varphi_t)^n = f_t \omega_t^n\)
where for \(t \in [0, 1]\)
\[0 \leq f_t := \alpha t^{n-m} F \frac{\omega_t^n}{\omega_t^n} \leq c_1 \frac{F}{\mathcal{F}^\pi}.

Observe that
\[
\int_X f_t \omega_t^n = c t^{n-m} \int_X F \omega_t^n = \int_X \omega_t^n =: Vol_{\omega_t}(X),
\]
hence \((f_t)\) is uniformly bounded in \(L^1(\omega_t/V_t), V_t := Vol_{\omega_t}(X)\). We actually need a slightly stronger information.

**Lemma 2.1** There exists \(p' > 1\) and a constant \(C = C(\pi, \|F\|_{L^p(X)}) > 0\) such that for all \(t \in [0, 1]\)
\[
\int_X f_t^{p'} \omega_t^n \leq C Vol_{\omega_t}(X).
\]

**Proof of the lemma.** Set \(V_t := Vol_{\omega_t} = \int_X \omega_t^n\) and observe that
\[
0 \leq f_t \frac{\omega_t^n}{V_t} \leq c_1 \frac{\omega_t^n}{\int_X \omega_0^m \wedge \omega_X^n} = C_2 F \omega_X^n,
\]
where \(C_2 := c_1 \int_X J^\pi \omega_X^n\).

This shows that the densities \(f_t\) are uniformly in \(L^1\) w.r.t. the normalized volume forms \(\omega_t^n/V_t\).

Since \(J^\pi\) is locally given as the square of the modulus of a holomorphic function which does not vanish identically, there exists \(\alpha \in [0, 1]\) such that \(J^\pi_\alpha \in L^1(X)\). Fix \(\beta \in [0, \alpha]\) satisfying the condition \(\beta/p + \beta/\alpha = 1\). It follows from Hölder’s inequality that
\[
\int_X f_t^\beta \omega_X^n \leq \left( \int_X F^p \omega_X^n \right)^{\beta/p} \left( \int_X J^\pi_\alpha \omega_X^n \right)^{\beta/\alpha}.
\]

Setting \(\varepsilon := \beta/q\) and using Hölder’s inequality again, we obtain
\[
\int_X f_t^{1+\varepsilon} \frac{\omega_t^n}{V_t} \leq C_2 \int_X f_t^\beta F \omega_X^n.
\]

Now applying again Hölder inequality we get
\[
\int_X f_t^{1+\varepsilon} \frac{\omega_t^n}{V_t} \leq C_2 \left( \int_X f_t^\beta \omega_X \right)^{1/q} \|F\|_{L^p(X)}.
\]
Therefore denoting by $p' := 1 + \varepsilon$, we have the following uniform estimate

$$\int_X f_t^{p'} \omega_t^n / V_t \leq C(\pi, \|F\|_{L^p(X)}), \forall t \in [0, 1],$$

where

$$C(\pi, \|F\|_{L^p(X)}) := C_2 \left( \int_X J_{\pi}^{-\alpha} \omega_t^n X \right)^{\beta/\alpha q} \|F\|_{L^p(X)}^{1+\beta/q}.$$  

\textbf{2.2 Uniform domination by capacity}

We now show that the measure $\mu_t := f_t \omega_t^n / Vol_{\omega_t}$ are uniformly strongly dominated by the normalized capacity $\text{Cap}_{\omega_t}/Vol_{\omega_t}(X)$. It actually follows from a careful reading of the no parameter proof given in [EGZ], [BGZ].

\textbf{Lemma 2.2} There exists a constant $C_0 = C_0(\pi, \|F\|_{L^p(X)}) > 0$ such that for any compact set $K \subset X$ and $t \in [0, 1],$

$$\mu_t(K) \leq C_0^\alpha \left( \frac{\text{Cap}_{\omega_t}(K)}{Vol_{\omega_t}(X)} \right)^2.$$  

Proof: Fix a compact set $K \subset X$. Set $V_t := Vol_{\omega_t}(X)$. Hölder’s inequality yields

$$\mu_t(K) \leq \left( \int_X f_t^{p'} \omega_t^n / V_t \right)^{1/p'} \left( \int_K \omega_t^n / V_t \right)^{1/q'}.$$  

It remains to dominate uniformly the normalized volume forms $\omega_t^n / V_t$ by the normalized capacities $\text{Cap}_{\omega_t}/V_t$. Fix $\sigma > 0$ and observe that for any $t \in [0, 1],$

$$\int_K \omega_t^n / V_t \leq \int_X e^{-\sigma(V_{K,\omega_t} - \max_X V_{K,\omega_t})} \omega_t^n / V_t \cdot T_{\omega_t}(K)^\sigma,$$

where

$$V_{K,\omega_t} := \sup\{\psi \in PSH(X, \omega_t); \psi \leq 0, \text{ on } K\}$$

is the $\omega_t$-extremal function of $K$ and $T_{\omega_t}(K) := \exp(-\sup_X V_{K,\omega_t})$ is the associated $\omega_t$-capacity of $K$ (see [GZ 1] for their properties).

Observe that $\omega_t^n / V_t \leq c_1 \omega_1^n$ and $\omega_t \leq \omega_1$, hence the family of functions $V_{K,\omega_t} - \max_X V_{K,\omega_t}$ is a normalized family of $\omega_1$-psh functions. Thus there exists $\sigma > 0$ which depends only on $(X, \omega_1)$ and a constant $B = B(\sigma, X, \omega_1)$ such that ([Z])

$$\int_X e^{-\sigma(V_{K,\omega_t} - \max_X V_{K,\omega_t})} \omega_t^n / V_t \leq B, \forall t \in [0, 1].$$  

4
The Alexander-Taylor comparison theorem (see Theorem 7.1 in [GZ 1]) now yields for a constant $C_3 = C_3(\pi, \|F\|_{L^p(X)})$

$$\mu_t(K) \leq C_3 \exp \left[ -\sigma \left( \frac{V_t}{\text{Cap}_{\omega_t}(K)} \right)^{1/n} \right], \forall t \in [0, 1].$$

We infer that there is a constant $C_4 = C_4(\pi, \|F\|_{L^p(X)})$ such that

$$\mu_t(K) \leq C_4 \left( \frac{\text{Cap}_{\omega_t}(K)}{V_t} \right)^2, \forall t \in [0, 1].$$

2.3 Uniform normalization

The comparison principle (see [K], [EGZ]) yields for any $s > 0$ and $\tau \in [0, 1]$

$$\tau^n \text{Cap}_{\omega_t}(\{ \varphi_t \leq -s - \tau \}) \leq \int_{\{\varphi_t \leq -s\}} \frac{(\omega_t + dd^c \varphi_t)^n}{V_t}.$$

It is now an exercise to derive from this inequality an a priori $L^\infty$-estimate,

$$\|\varphi_t\|_{L^\infty(X)} \leq C_5 + s_0(\omega_t),$$

where $s_0(\omega_t)$ (see [EGZ], [BGZ]) is the smallest number $s > 0$ satisfying the condition $e^n C_n^0 \text{Cap}_{\omega_t}(\{ \psi \leq -s \}) / V_t \leq 1$ for all $\psi \in PSH(X, \omega_t)$ such that $\sup_X \psi = 0$. Recall from ([GZ 1], Prop. 3.6) that

$$\frac{\text{Cap}_{\omega_t}(\{ \psi \leq -s - \tau \})}{V_t} \leq \frac{1}{s} \left( \int_X (-\psi) \frac{\omega^n}{V_t} + n \right).$$

Since $\frac{s^n}{V_t} \leq C_1 \omega^n_1$, it follows that

$$\frac{\text{Cap}_{\omega_t}(\{ \psi \leq -s - \tau \})}{V_t} \leq \frac{1}{s} \left( C_1 \int_X (-\psi) \omega^n_1 + n \right).$$

Since $\psi$ is $\omega_1-$psh and normalized, we know that there is a constant $A = A(X, \omega_1) > 0$ such that $C_1 \int_X (-\psi) \omega^n_1 \leq A$ for any such $\psi$. Therefore $s_0(\omega_t) \leq s_0 := e^n C_n^0 (A + n)$ for any $t \in [0, 1]$. Finally we obtain the required uniform estimate for all $t \in [0, 1]$.

References


