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ALMOST SURE CONVERGENCE OF A STOCHASTIC APPROXIMATION PROCESS IN A CONVEX SET

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Abstract : We consider a stochastic approximation process in a convex set \(K\) of \(\mathbb{R}^k\) : 
\[
X_{n+1} = \Pi (X_n - A_n Y_n),
\]
with \(E \left[ A_n Y_n \mid T_n \right] = a_n M_n(X_n)\), where \(\Pi\) is the projection operator on \(K\), \(A_n\) a random matrix, \(a_n\) a positive number, \(M_n\) a function from \(K\) into \(\mathbb{R}^k\) and \(T_n\) the sub-\(\sigma\)-algebra generated by the events before time \(n\). We prove two theorems of almost sure convergence in the case where the equation \(M_n(x) = 0\) has a set of solutions and give two applications.

AMS Subj. Classification : 62L20
Key Words : stochastic approximation, linear regression

1. Introduction

We define a stochastic approximation process \((X_n)\) in a non-empty closed convex subset \(K\) of \(\mathbb{R}^k\), named parameter space ; we consider :

. for \(n \geq 1\), an observable random variable \(Y_n\) in \(\mathbb{R}^p\), named observation space ; remark that the observation space may be different from the parameter space ;
. for \(n \geq 1\), a \((k, p)\) random matrix \(A_n\) ;
. the projection operator \(\Pi\) on \(K\) ;
. the process \((X_n)\) in \(K\) defined recursively by
\[
X_{n+1} = \Pi (X_n - A_n Y_n)
\]
All random variables are defined on a probability space \((\Omega, \mathcal{A}, P)\). Denote \(T_n\) the sub-\(\sigma\)-algebra of \(\mathcal{A}\) generated by the events before time \(n\); \(X_1, \ldots, X_n, A_1, \ldots, A_n, Y_1, \ldots, Y_{n-1}\) are measurable with respect to \(T_n\).

Suppose that, for \(n \geq 1\), there exists a measurable function \(M_n\) from \(K\) into \(\mathbb{R}^k\) and a positive number \(a_n\) such that

\[
E[A_n Y_n \mid T_n] = A_n E[Y_n \mid T_n] = a_n M_n(X_n) \text{ a.s.}
\]

Let \(B_n\) be a set of solutions of the equation \(M_n(x) = 0\). Define a distance \(d(x, B)\) from \(x\) in \(\mathbb{R}^k\) to a subset \(B\).

We give in Section 2 two almost sure convergence theorems of \(d(X_n, B_n)\) to 0. An application of each theorem is given in Section 3, concerning the estimation of a quantile interval of an unknown probability distribution and the estimation of a linear regression parameter under convex constraints.

In the following, \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|\) are respectively the usual inner product and norm in \(\mathbb{R}^k\); \(A'\) denotes the transposed matrix of \(A\), \(\lambda_{\min}(B)\) the smallest eigenvalue of \(B\); the abbreviation \(\text{a:s.}\) means almost surely.

### 2. Lemmas

Let \((X_n)\) be a stochastic process in a subset \(K\) of \(\mathbb{R}^k\). Let \((F_n)\) and \((\varphi_n)\) be two sequences of measurable functions from \(K\) into \(\mathbb{R}^+\) and \((a_n)\) a sequence in \(\mathbb{R}^+\). Suppose:

- (H1a) There exists a random variable \(T\) in \(\mathbb{R}^+\) such that \(F_n(X_n) \to T\) a.s.
- (H1b) \(\sum_1^\infty a_n \varphi_n(X_n) < \infty\) a.s.
- (H2a) Whatever \(0 < \epsilon < 1\), \(\sum_1^\infty a_n \inf_{\{x \in K, \epsilon < F_n(x) < \epsilon + \frac{1}{k}\}} \varphi_n(x) = +\infty\).

#### Lemma 1

Assume H1a, b and H2a hold; then \(F_n(X_n) \to 0\) a.s.

**Proof.** \(\omega \in \Omega\) is fixed throughout the proof, belonging to the intersection of the defined a.s. convergence sets. Suppose \(T(\omega) \neq 0\) and suppress \(\omega\) writing.

By H1a, there exist \(0 < \epsilon_1 < 1\) and an integer \(N(\epsilon_1)\) such that for \(n > N(\epsilon_1), \epsilon_1 < F_n(X_n) < \frac{1}{\epsilon_1}\).

This implies \(\varphi_n(X_n) \geq \inf_{\{x \in K, \epsilon_1 < F_n(x) < \epsilon_1 + \frac{1}{k}\}} \varphi_n(x)\); then by H2a,

\[
\sum_1^\infty a_n \varphi_n(X_n) = \infty,
\]

a contradiction with H1b. Thus \(T(\omega) = 0\). \(\blacksquare\)
Suppose now:

(H1c) \[ \|X_{n+1} - X_n\| \to 0 \text{ a.s.} \]

(H3a) For all \( 0 < \epsilon_1 < 1 \), for all \( \epsilon > 0 \), there exists \( \eta > 0 \) such that

\[
(\|x_1 - x_2\| < \eta) \Rightarrow \left( \sup_n \sup_{\epsilon_1 < F_n(x_1) < \frac{1}{\epsilon_1}} |\varphi_n(x_1) - \varphi_n(x_2)| < \epsilon \right)
\]

(H3b) There exist a positive integer \( r \), a sequence of integers \((n_l)\), for all \( 0 < \epsilon < 1 \) an integer \( L(\epsilon) \) such that for \( n_l \leq n_l + r \) and

\[
b(\epsilon) = \inf_{l > L(\epsilon)} \inf_{\{x \in K, \epsilon < F_{n_l}(x) < \frac{1}{\epsilon_1}\}} \sum_{j \in I_l} \varphi_j(x) > 0
\]

with \( I_l = \{n_l, n_l + 1, \ldots, n_{l+1} - 1\} \)

(H2b) \( \sum_i \min_{j \in I_l} a_j = \infty \).

Lemma 2 Assume H1a, b, c, H2b and H3a, b hold; then \( F_n(X_n) \to 0 \text{ a.s.} \)

Proof. \( \omega \in \Omega \) is fixed throughout the proof, belonging to the intersection of the defined a.s. convergence sets. Suppose \( T(\omega) \neq 0 \). Below \( \omega \) is omitted.

By H1a, there exist \( 0 < \epsilon_1 < 1 \) and an integer \( N(\epsilon_1) \) such that for \( n > N(\epsilon_1), \epsilon_1 < F_n(X_n) < \frac{1}{\epsilon_1}. \)

By H3b, there exists an integer \( L(\epsilon_1) \) such that for \( l > L(\epsilon_1), \)

\[
\sum_{j \in I_l} \varphi_j(X_{n_l}) > b(\epsilon_1).
\]

It follows that there exists \( m_l \in I_l \) such that

\[
\varphi_{m_l}(X_{n_l}) > \frac{b(\epsilon_1)}{r}.
\]

Consider the decomposition

\[
\varphi_{m_l}(X_{m_l}) = \varphi_{m_l}(X_{n_l}) + \varphi_{m_l}(X_{m_l}) - \varphi_{m_l}(X_{n_l}).
\]

\[
\varphi_{m_l}(X_{m_l}) > \frac{b(\epsilon_1)}{r} - |\varphi_{m_l}(X_{m_l}) - \varphi_{m_l}(X_{n_l})|.
\]

Let \( \epsilon > 0 \); by H3a, there exists \( \eta > 0 \) corresponding to \( \epsilon_1 \) and \( \epsilon \); by H1c, we have for \( l \) sufficiently large:

\[
\|X_{m_l} - X_{n_l}\| < \eta \quad \epsilon_1 < F_{m_l}(X_{m_l}) < \frac{1}{\epsilon_1}.
\]

By H3a, this implies:

\[
|\varphi_{m_l}(X_{m_l}) - \varphi_{m_l}(X_{n_l})| < \epsilon;
\]
Choose $\epsilon < \frac{b(\epsilon_1)}{r}$. By H2b, $\sum_i a_m \varphi_m(X_m) = +\infty$. Then
\[
\sum a_n \varphi_n(X_n) = +\infty,
\]
a contradiction with H1b. Thus $T(\omega) = 0$. 

\section*{3. Theorems of almost sure convergence}

Consider the process $(X_n)$ as defined in section 1:
\[
X_{n+1} = \Pi(X_n - A_n Y_n), \quad E[A_n Y_n | T_n] = a_n M_n(X_n) \text{ a.s.}
\]

Denote $d(x, B)$ a distance from $x \in \mathbb{R}^k$ to a subset $B$.

For all $n$, let $F_n$ be a function from $\mathbb{R}^k$ into $\mathbb{R}^+$ twice continuously differentiable, with gradient $G_n$ and hessian matrix $H_n$; by the Taylor formula, there exists $0 < \mu_n < 1$ such that
\[
F_n(X_n - A_n Y_n) = F_n(X_n) - \langle G_n(X_n), A_n Y_n \rangle + \frac{1}{2} \langle A_n Y_n, H_n(X_n - \mu_n A_n Y_n) A_n Y_n \rangle.
\]

Denote $V_n = \frac{1}{2} E \left[ \langle A_n Y_n, H_n(X_n - \mu_n A_n Y_n) A_n Y_n \rangle | T_n \right]$.

Suppose:

(H4a) For all $n$, $F_n$ is twice continuously differentiable

(H4b) For all $\epsilon > 0$, there exists $\nu(\epsilon) > 0$ and for all $n$, there exists a subset $B_n$ of $K$ such that
\[
\inf_n \inf_{d(x, B_n) > \epsilon} F_n(x) > \nu(\epsilon).
\]

(H4c) There exist two sequences of positive numbers $(\gamma_n)$ and $(\delta_n)$ such that $\sum_1^\infty \gamma_n < \infty$, $\sum_1^\infty \delta_n < \infty$ and for all $n$ and $x$,
\[
F_{n+1}(\Pi x) \leq (1 + \delta_n)F_n(x) + \gamma_n.
\]

(H5) For all $n$, there exist two random variables $D_n$ and $E_n$ in $\mathbb{R}^+$, measurable with respect to $T_n$, such that
\[
\sum_1^\infty D_n < \infty, \sum_1^\infty E_n < \infty, \quad V_n \leq D_n F_n(X_n) + E_n \text{ a.s.}
\]
(H6) $\sum_{i=1}^{\infty} \langle G_n(X_n), a_n M_n(X_n) \rangle^- < \infty \quad a.s.$

(H7) For all $0 < \epsilon < 1$, $\sum_{i=1}^{\infty} a_n \inf \{ x \in \mathbb{K}, \epsilon < F_n(x) < \frac{1}{\epsilon} \} \langle G_n(x), M_n(x) \rangle^+ < \infty$.

Remark that in the case where $B_n$ is reduced to a single element $\theta$ of $\mathbb{R}^k$ not depending on $n$, if we take $F_n(x) = d^2(x, \theta) = \| x - \theta \|^2$, then assumptions H4a, b, c hold and $G_n(x) = 2(x - \theta)$, $H_n(x) = 2I$ ($I$ : identity matrix), $V_n = E \left[ \| A_n Y_n \|^2 \mid T_n \right]$.

**Theorem 3** Assume H4a, b, c, H5, H6 and H7 hold; then $F_n(X_n) \longrightarrow 0$ and $d(X_n, B_n) \longrightarrow 0 \ a.s.$

We use in the proof the Robbins-Siegmund lemma [4]:

**Lemma 4** Let $(\Omega, A, P)$ be a probability space and $(T_n)$ an increasing sequence of sub-$\sigma$-algebras of $A$. For $n \geq 1$, let $z_n$, $\beta_n$, $\xi_n$ and $\zeta_n$ be non-negative $T_n$-measurable random variables such that $E[Z_{n+1} \mid T_n] \leq z_n(1 + \beta_n) + \xi_n - \zeta_n$. Suppose $\sum_{i=1}^{\infty} \beta_n < \infty$, $\sum_{i=1}^{\infty} \xi_n < \infty \ a.s$. Then $\lim_{n \rightarrow \infty} z_n$ exists and is finite and $\sum_{i=1}^{\infty} \zeta_n < \infty \ a.s.$

**Proof.** By H4a, c and H5, we have:

$$F_{n+1}(X_{n+1}) \leq (1 + \delta_n)F_n(X_n - A_n Y_n) + \gamma_n.$$  

$$E[F_{n+1}(X_{n+1}) \mid T_n] \leq (1 + \delta_n)(F_n(X_n) - \langle G_n(X_n), a_n M_n(X_n) \rangle + V_n) + \gamma_n.$$  

$$\leq (1 + \delta_n)(1 + D_n)F_n(X_n) + (1 + \delta_n)E_n$$  

$$+ (1 + \delta_n) \langle G_n(X_n), a_n M_n(X_n) \rangle^- + \gamma_n$$  

$$- (1 + \delta_n) \langle G_n(X_n), a_n M_n(X_n) \rangle^+ \quad a.s.$$  

By H4c, H5 and H6, the assumptions of the preceding lemma hold; then there exists a random variable $T$ in $\mathbb{R}^+$ such that $F_n(X_n) \longrightarrow T \ a.s.$ and $\sum_{i=1}^{\infty} \langle G_n(X_n), a_n M_n(X_n) \rangle^+ < \infty \ a.s.$

Let $\varphi_n(x) = \langle G_n(x), M_n(x) \rangle^+$. The assumptions H1a, b and H2a of lemma 1 hold. Then $F_n(X_n) \longrightarrow 0 \ a.s.$

By H4b, it follows that $d(X_n, B_n) \longrightarrow 0 \ a.s.$

Prove now a second theorem.

Suppose:

(H4d) For all $0 < \epsilon < 1$, $\sup_n \sup_{\epsilon < F_n(x) < \frac{1}{\epsilon}} \| G_n(x) \| < \infty$

(H4e) For all $\epsilon > 0$, there exists $\eta > 0$ such that

$$\left( \| x_1 - x_2 \| < \eta \right) \Rightarrow \left( \sup_n \| G_n(x_1) - G_n(x_2) \| < \epsilon \right)$$

(H8a) For all $0 < \epsilon < 1$, $\sup_n \sup_{\epsilon < F_n(x) < \frac{1}{\epsilon}} \| M_n(x) \| < \infty$
For all $\epsilon > 0$, there exists $\eta > 0$ such that 
\[ \|x_1 - x_2\| < \eta \Rightarrow (\sup_n \|M_n(x_1) - M_n(x_2)\| < \epsilon) \]

(H8c) There exist a positive integer $r$, a sequence of integers $(n_l)$, for all $0 < \epsilon < 1$ an integer $L(\epsilon)$ such that $n_{l+1} \leq n_l + r$ and 
\[ b(\epsilon) = \inf_{t > L(\epsilon)} \inf_{x \in K, \epsilon < F_n(x) < \frac{1}{2}} \sum_{j \in I_t} (G_j(x), M_j(x))^+ > 0 \]

with $I_t = \{n_l, n_l + 1, ..., n_{l+1} - 1\}$

(H2b) $\sum_{j \in I_t} a_j = \infty$.

Remark that in the case where $B_n = \{\theta\}$ and $F_n(x) = \|x - \theta\|^2$, assumptions H4d, e hold.

**Theorem 5** Assume H2b, H4a, b, c, d, e, H5, H6, H8a, b, c hold; then in the set $\{A_n Y_n \rightarrow 0\}$, $F_n(X_n) \rightarrow 0$ and $d(X_n, B_n) \rightarrow 0$ a.s.

Proof. Following the proof of theorem 3, we have by H4a, c, H5, H6 : $F_n(X_n) \rightarrow T$ and $\sum_1^\infty \langle G_n(x_n), a_n M_n(x_n) \rangle^+ < \infty$ a.s.

Apply lemma 2 with $\varphi_n(x) = \langle G_n(x), M_n(x) \rangle^+$.

H1a, b and H3b hold. H1c holds in the set $\{A_n Y_n \rightarrow 0\}$ as

\[ \|X_{n+1} - X_n\| = \|\Pi(X_n - A_n Y_n) - \Pi X_n\| \leq \|X_n - A_n Y_n - X_n\| = \|A_n Y_n\| \]

As $|a^+ - b^+| \leq |a - b|$, we have :

\[ |\varphi_n(x_1) - \varphi_n(x_2)| \leq |\langle G_n(x_1), M_n(x_1) \rangle - \langle G_n(x_2), M_n(x_2) \rangle| \]
\[ \leq |\langle G_n(x_1), M_n(x_1) - M_n(x_2) \rangle| \]
\[ + |\langle G_n(x_1) - G_n(x_2), M_n(x_2) - M_n(x_1) \rangle| \]
\[ + |\langle G_n(x_1) - G_n(x_2), M_n(x_1) \rangle| . \]

By H4d, e and H8a, b, assumption H3a holds.

Then $F_n(X_n) \rightarrow 0$ a.s. By H4b, $d(X_n, B_n) \rightarrow 0$ a.s. $\blacksquare$

4. Application to the estimation of a quantile interval

Let $Z$ be a real random variable whose distribution function $F(t) = P(Z < t)$ is unknown. Suppose that there exists an interval $(a, b)$, which is eventually reduced to a single point, such that : $F(t) = \alpha \Leftrightarrow t \in (a, b)$. 

Let $m \geq 1$ be an integer and $(Z_{nj}, n \geq 1, j = 1, ..., m)$ a set of mutually independent random variables which have the same law as $Z$. For all $x$, define the random variables $I_{nj}(x)$ and $F_{nm}(x)$ such that:

\[ I_{nj}(x) = 1 \text{ if } Z_{nj} < x, \quad 0 \text{ otherwise} \]

\[ F_{nm}(x) = \frac{1}{m} \sum_{j=1}^{m} I_{nj}(x). \]

Then $E[F_{nm}(x)] = E[I_{nj}(x)] = F(x)$.

Define the stochastic approximation process $(X_n)$ such that

\[ X_{n+1} = X_n - a_n (F_{nm}(X_n) - \alpha). \]

If $z_{nj}$ is the observed value of $Z_{nj}$ and $x_n$ the value of $X_n$, $F_{nm}(x_n)$ is the proportion of elements of \{\{z_{n1}, ..., z_{nm}\} which are smaller than $x_n$.

Suppose:

- $(H2b') \sum_{1}^{\infty} a_n = \infty$
- $(H2c) \sum_{1}^{\infty} a_n^2 < \infty$.

**Theorem 6** Let $d(x, (a, b)) = \inf_{y \in (a, b)} |x - y|$. Assume $H2b'$, $c$ hold; then $d(X_n, (a, b)) \to 0$ a.s.

**Proof.** Define the function $f$ such that

\[
\begin{align*}
    f(x) &= (x - a)^2 \text{ if } x < a \\
    f(x) &= 0 \text{ if } a \leq x \leq b \\
    f(x) &= (x - b)^2 \text{ if } x > b.
\end{align*}
\]

H4a, b, c hold for $F_n = f$ and $B_n = (a, b)$.

$|f''(x)| \leq 2, |F_{nm}(x) - \alpha| \leq 1$; then $V_n \leq a_n^2$; $H5$ holds.

$M_n(X_n) = E[F_{nm}(X_n) - \alpha | T_n] = F(X_n) - \alpha$;

$f'(x)(F(x) - \alpha) \geq 0, \inf_{\epsilon < f'(x)} f'(x)(F(x) - \alpha) > 0$; $H6$ and $H7$ hold.

Applying theorem 3 gives $d(X_n, (a, b)) \to 0$ a.s. $\blacksquare$

5. Application to linear regression under convex constraints

Consider a sequence $(Z_n)$ of observable mutually independent real random variables.

Suppose that there exist an unknown vector $\theta$ in $\mathbb{R}^k$, for all $n$ a known vector $b_n$ in $\mathbb{R}^k$ and a real random variable $R_n$ with $E[R_n] = 0$ such that
Z_n = b'_n \theta + R_n.

Suppose moreover that \( \theta \) belongs to a non-empty closed convex set \( K \) of \( \mathbb{R}^k \). For instance:
1) \( \| \theta \| \) is bounded;
2) the components of \( \theta \) are non-negative.

Consider the stochastic approximation process \( (X_n) \) such that:
\[
X_{n+1} = \Pi \left( X_n - a_n \frac{b_n}{\| b_n \|^2} (b'_n X_n - Z_n) \right).
\]

Suppose:

(H2b) \( \sum_1^\infty \min_{j \in I} a_j = \infty \)
(H2c) \( \sum_1^\infty a_n^2 < \infty \)
(H2d) \( \sum_1^\infty a_n^2 E[R_n^2/\|b_n\|^2] < \infty \)
(H9) \( \lambda = \inf_{j \in I} \lambda_{\min} \left( \sum_{j \in I} b_j b'_j / \| b_j \|^2 \right) > 0 \).

**Theorem 7** Assume H2b, c, d and H9 hold; then \( X_n \rightarrow \theta \) a.s.

This theorem completes in the case of linear regression results of Albert and Gardner [1] (p. 103, conjectured theorem).

**Proof.** Let \( Y_n = b'_n X_n - Z_n = b'_n (X_n - \theta) - R_n \) and \( A_n = a_n b_n / \| b_n \|^2 \).

As \( E[R_n | T_n] = E[R_n] = 0 \), \( M_n(X_n) = b_n b'_n / \| b_n \|^2 (X_n - \theta) \) a.s.

Remark that, for fixed \( n \), equation \( M_n(x) = 0 \) has an infinity of solutions.

Denote \( I \) an identity matrix. Define \( F_n(x) = \| x - \theta \|^2 \); then:
\[
G_n(x) = 2(x - \theta), H_n(x) = 2I, V_n = E\left[ a_n^2 \| Y_n \|^2 | T_n \right].
\]

Assumptions H4a, b, c, d, e, H6, H8a, b hold.

\[
V_n = E\left[ a_n^2 \| Y_n \|^2 | T_n \right] = a_n^2 \| X_n - \theta \|^2 + a_n^2 E[R_n^2/\|b_n\|^2].
\]

By H2d, assumption H5 holds.

By H9, assumption H8c holds as
\[
\sum_{j \in I} \langle G_j(x), M_j(x) \rangle^+ = 2 \sum_{j \in I} \left\langle x - \theta, \frac{b_j b'_j}{\| b_j \|^2} (x - \theta) \right\rangle \geq 2 \lambda \| x - \theta \|^2.
\]
Furthermore, as \( E[R_n | T_n] = 0 \):

\[
E \left[ \|X_{n+1} - \theta\|^2 \mid T_n \right] = \|X_n - \theta\|^2 + a_n^2 E \left[ \|Y_n\|^2 \mid T_n \right] - 2a_n \left< \frac{X_n - \theta}{\|b_n\|^2}, b_n b_n' (X_n - \theta) \right> \\
\leq (1 + a_n^2) \|X_n - \theta\|^2 + a_n^2 E \left[ R_n^2 \right].
\]

\[
E \left[ \|X_{n+1} - \theta\|^2 \right] \leq (1 + a_n^2) E \left[ \|X_n - \theta\|^2 \right] + a_n^2 E \left[ R_n^2 \right].
\]

By H2c, d, there exists \( t \geq 0 \) such that \( E \left[ \|X_n - \theta\|^2 \right] \to t \). Then :

\[
\sum_1^\infty E \left[ a_n^2 \|Y_n\|^2 \right] = \sum_1^\infty \left( a_n^2 E \left[ \|X_n - \theta\|^2 \right] + a_n^2 E \left[ R_n^2 \right] \right) < \infty ;
\]

\[
\sum_1^\infty a_n^2 \|Y_n\|^2 < \infty \ a.s. ; \ a_n Y_n \to 0 \ a.s.
\]

Applying theorem 5 gives \( X_n \to \theta \ a.s. \)

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