Closed loop observer-based parameter estimation of quantum systems with a single population measurement

Zaki Leghtas

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Abstract

An observer-based Hamiltonian identification algorithm for quantum systems has been proposed in [2]. In this paper we propose another observer enabling the identification of the dipole moments of a multi-level case, and having access to the population of the ground state only.

Keywords: Nonlinear systems, Quantum systems, Parameter identification, Asymptotic observers, Averaging, Feedback.

1 Introduction

This work is the result of a summer internship I did at INRIA Rocquencourt, with Mazyar Mirrahimi[1] and Pierre Rouchon[2]. Our goal was to improve the result given in [2] in order to enable an experimentator to estimate the dipole moments of a quantum system measuring continuously the population on the first state.
First, we will explain the problem and expose the model of the considered quantum system. In the next section, we explicit the observer algorithm and look at some simulations. Then, using some averaging arguments and neglecting some second order terms, we prove the convergence of the observer.
We finally come to the interesting conclusion that a feedback is required.

2 The three level system

2.1 The problem

We have a quantum system and we wish to estimate its **dipole moment matrix**.

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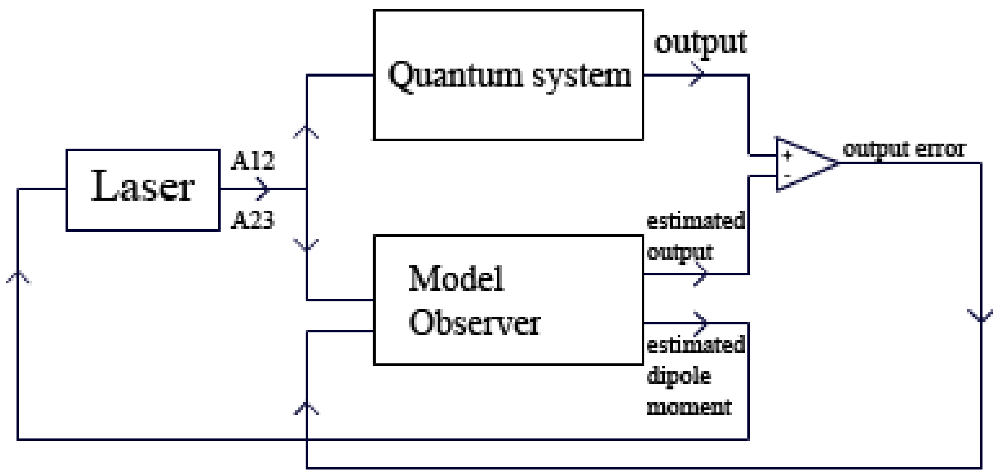

Figure 1: Diagram of the closed loop identification algorithm.

1. We **excite** the system with a laser. We are free to modulate its amplitude and it's phase.
2. We only measure continuously the population on it's ground state.

In this paper, we propose a method to extract the desired information (the dipole moments) from this output.

2.2 The model

We consider a three level quantum system, whose dynamics is described by the Schrödinger equation:

$$\frac{d}{dt}\Psi = \frac{-i}{\hbar}(H_0 + A(t)H_1)\Psi, \qquad y = \langle 1|\,\Psi\Psi^\star\,|1\rangle\,,$$

where Ψ is the wavefunction, $A(t)$ is the laser field, H_0 is the sum of the kinetic energy operator and the potential energy operator without the laser field, H_1 is the dipole moment matrix and y is the measurement output. Our aim is to estimate this matrix:

$$H_0 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad H_1 = \begin{pmatrix} 0 & \mu_{12} & 0 \\ \mu_{12} & 0 & \mu_{23} \\ 0 & \mu_{23} & 0 \end{pmatrix}$$

The goal is to estimate μ_{12} and μ_{23}. Within the density language matrix $\rho = |\Psi\rangle\langle\Psi|$, and taking $\hbar = 1$, we have:

$$\dot{\rho} = -i[H, \rho].$$

3 The observer

3.1 The laser field

We choose an electromagnetic field which is the sum of two lasers:

1. **Laser 1**: $\bar{A}_{12}$
   $cos((\lambda_2 - \lambda_1)t)$ with $\bar{A}_{12}$ constant and of order 1

2. **Laser 2**: A_{23}
   $cos((\lambda_3 - \lambda_2)t)$ with $A_{23} = \bar{A}_{23}$
   $cos(\hat{\theta})$ and $\dot{\hat{\theta}} = \frac{\bar{A}_{12}\hat{\mu}_{12}}{2}$, $\bar{A}_{23}$ constant of order 1 and $\hat{\mu}_{12}$ being the estimated value of μ_{12} and is explicitly defined below. This amplitude modulation can be as slow as desired by choosing a small $\bar{A}_{12}$ as long as $\bar{A}_{12} >> \epsilon$.

We then sum these two lasers, putting a small factor ϵ in front of the second one. This gives the following laser field:

$$A(t) = \bar{A}_{12}cos((\lambda_2 - \lambda_1)t) + \epsilon\bar{A}_{23}cos(\hat{\theta})cos((\lambda_3 - \lambda_2)t)$$

Which leads us to a dynamics written with the effective hamiltonian to:

$$\dot{\rho} = -i\frac{\bar{A}_{12}\mu_{12}}{2}[\sigma_x^{12}, \rho] - i\epsilon\frac{\bar{A}_{23}\mu_{23}}{2}\cos(\hat{\theta})[\sigma_x^{23}, \rho]$$

3.2 The proposed method

We use observers which make the estimated values converge to the real ones.
We suppose that H_1=$\hat{H}_1$, and see what output $\hat{y}$ it would give. The error: e=y-$\hat{y}$ is then used to correct the estimation of H_1. The difficulty is to find an adequate way to deal with this error in order to make $\hat{H}_1$ converge to H_1. The other difficulty is the choice of the laser amplitude $A(t)$ which enables us to extract the right information from the system.
The estimation method is the following: We design an observer such that estimator of μ_{12} converges rapidly towards μ_{12}. We add some small perturbation terms so that, once the estimation of μ_{12} is accurate enough, the second order terms will make the estimator of μ_{23} converge to μ_{23}.
We do this by using the estimated values of μ_{12} to modulate the amplitude of the second laser. The modulation is done with a frequency of $\frac{\bar{A}_{12}\mu_{12}}{2}$. Note that this frequency $\frac{\bar{A}_{12}\mu_{12}}{2}$ can be chosen as small as needed, through the choice of $\bar{A}_{12}$ (which is itself a control degree of freedom).
The notion of feedback can lead some difficulties in the realization of the experiment. However, to avoid suc complications, one can first apply the laser 1, and once the value of μ_{12} is known, he can switch on the laser 2 and modulate its amplitude with a signal of frequency $\frac{A_{12}\mu_{12}}{2}$. This will ensure the convergence of $\hat{\mu}_{23}$ to μ_{23}
Intuitively, one can predict that by measuring the population of state 1, one can identify the dipole moment between state 1 and any other state k ($k = 2$ or $k = 3$), by exciting the system with a resonant laser with the transition frequency between state 1 and k. On the other hand, identifying the dipole moment between any two states k and m (m and k different from 1), with only the population of state 1 seems difficult to achieve. Hence, we excite the system with a laser resonant with transition 1 and m, and m and k. We make sure the laser coupling 1 and m is much more powerful than the second one. We can therefore consider states 1 and m as being equivalent to a unified average state. And therefore, we are back to a situation where we have 2 states, knowing the population of the first one. The different scales of amplitudes between the two

lasers and the notion of averaged systems[4][3] are key notions in the design of the proposed observer.

3.3 The observer

We propose the following observer:

$$\begin{aligned}
\dot{\hat{\rho}} &= -i\frac{\bar{A}_{12}\hat{\mu}_{12}}{2}[\sigma_x^{12},\hat{\rho}] - i\epsilon\frac{\bar{A}_{23}\hat{\mu}_{23}}{2}\cos(\hat{\theta})[\sigma_x^{23},\hat{\rho}] \\
&\quad +\epsilon\Gamma_1(y-\hat{y})(\sigma_z^1\hat{\rho}+\hat{\rho}\sigma_z^1-2\mathrm{Tr}(\sigma_z^1\hat{\rho})\hat{\rho}) \\
&\quad +\epsilon^2\Gamma_2(y-\hat{y})[(1+2\cos(2\hat{\theta}))(Q_1\hat{\rho}+\hat{\rho}Q_1-2\mathrm{Tr}(Q_1\hat{\rho})\hat{\rho}) \\
&\qquad\qquad +(1-2\cos(2\hat{\theta}))(Q_2\hat{\rho}+\hat{\rho}Q_2-2\mathrm{Tr}(Q_2\hat{\rho})\hat{\rho}) \\
&\qquad\qquad -2(Q_3\hat{\rho}+\hat{\rho}Q_3-2\mathrm{Tr}(Q_3\hat{\rho})\hat{\rho})] \\
\dot{\hat{\theta}} &= \frac{\bar{A}_{12}\hat{\mu}_{12}}{2} \\
\dot{\hat{\mu}}_{12} &= -i\frac{2}{\bar{A}_{12}}\epsilon^2\gamma_1\mathrm{Tr}(\sigma_z^1[\sigma_x^{12},\hat{\rho}])(y-\hat{y}) \\
\dot{\hat{\mu}}_{23} &= -i\frac{2}{\bar{A}_{23}}\epsilon^4(y-\hat{y})\gamma_2[(1-2\cos(2\hat{\theta}))\mathrm{Tr}(Q_2[\Sigma_x^{23},\hat{\rho}])-2\mathrm{Tr}(Q_3[\Sigma_x^{23},\hat{\rho}])]
\end{aligned}$$

With: $\Gamma_1,\ \Gamma_2,\ \gamma_1,\ \gamma_2$ are constants of order 1. And ϵ is a small parameter. Explicit expressions of the matrices σ_x^{ij} and so on are given in the appendix. We have considered the notations:

$$\hat{\Omega}_{ij} = \frac{1}{2}\hat{\mu}_{ij}\bar{A}_{ij} \qquad y = \mathrm{Tr}(P_1\rho) \qquad \hat{y} = \mathrm{Tr}(P_1\hat{\rho})$$

and:

$$\Sigma_x^{12} = L^\dagger\sigma_x^{12}L \qquad \Sigma_z^{23} = L^\dagger\sigma_z^{23}L \qquad \Sigma_x^{23} = L^\dagger\sigma_x^{23}L$$
$$Q_1 = L^\dagger P_1 L \qquad Q_2 = L^\dagger P_2 L \qquad Q_3 = L^\dagger P_3 L$$

where:

$$L = e^{i\hat{\theta}\sigma_x^{12}}$$

3.4 Simulations

Before entering the details of the convergence proof of the observer, let us look at some simulations.

N.B: Ω^e and y^e stand respectively for $\hat{\Omega}$ and $\hat{y}$

The simulation was done with the following parameters:
$\epsilon = 1/10 \qquad \Gamma_1 = 2 \qquad \gamma_1 = 2 \qquad \Gamma_2 = 1.6 \qquad \gamma_2 = 1.28$

With an initial state: $\Psi_0 = \frac{1}{1.55}\begin{pmatrix} 1+i \\ .6+.1i \\ .2+.1i \end{pmatrix}$

We initialise the observer at: $\hat{\Psi}_0 = \frac{1}{1.7}\begin{pmatrix} 1.2+i \\ .6+.2i \\ .1+.2i \end{pmatrix}$

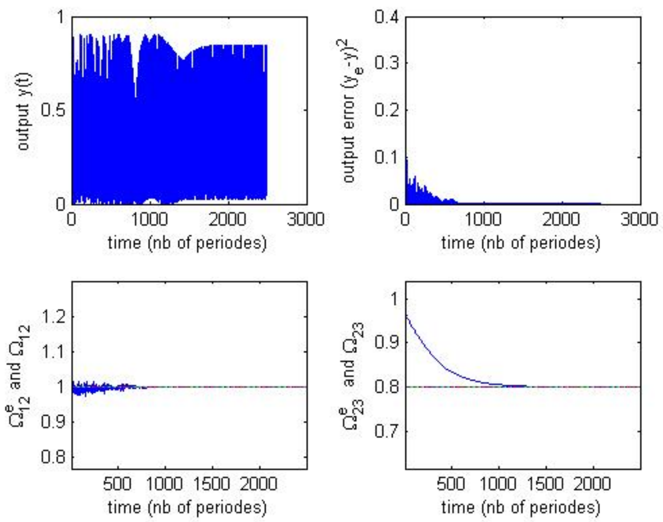

Figure 2: Output. Measured output error and parameter estimation

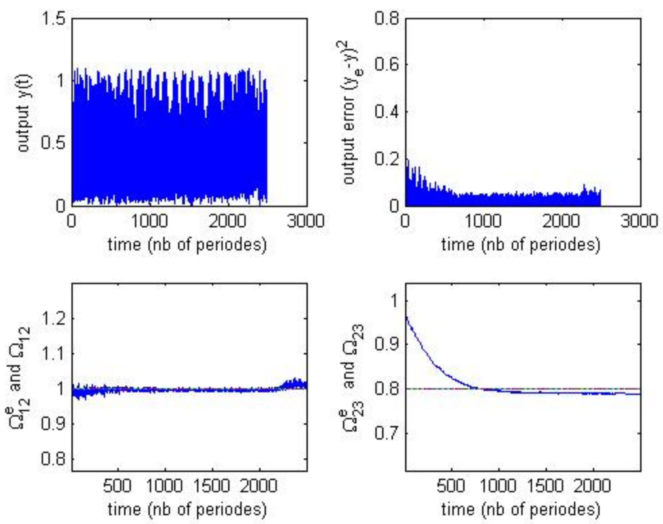

Figure 3: Output with a 20% gaussian noise. Measured output error and parameter estimation

The dipole moment matrix is: $H_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & .8 \\ 0 & .8 & 0 \end{pmatrix}$

We start the observer at $\hat{H}_1 = \begin{pmatrix} 0 & .83 & 0 \\ .83 & 0 & .96 \\ 0 & .96 & 0 \end{pmatrix}$

μ_{12} converges much faster (about $\frac{1}{\epsilon}$ times faster) than μ_{23}. We notice that the ouput error is about 10^{-2} which is the order of the noise.

3.5 Identifiability

It has been proved in [1] that if one has access to the population of all states of the quantum system, it is possible to identify the dipole moment matrix.
In this paper, we only measure the population of the ground state. But in fact, the needed information is coded in the averaged system. Indeed, by writing the dynamics of the system in a frame such as the terms of order 1 vanish, hence by considering: $\eta = e^{i\Omega_{12}t}\rho e^{-i\Omega_{12}t}$, we obtain the following output:

$$y(t) \quad = \quad \frac{1}{2}(\mathrm{Tr}(I_{12}\eta) + \cos(2\Omega_{12}t)\mathrm{Tr}(\sigma_z^1\eta) + \sin(2\Omega_{12}t)\mathrm{Tr}(\sigma_y^{12}\eta))$$

The first order terms in η are slow compared to $\cos(2\Omega_{12}t)$ and $\sin(2\Omega_{12}t)$.
Notice that in average:

$$\begin{aligned} y(t)(1 + 2\cos(2\Omega_{12}t)) &= \mathrm{Tr}(P_1\eta) \\ y(t)(1 - 2\cos(2\Omega_{12}t)) &= \mathrm{Tr}(P_2\eta) \\ 1 - 2y(t) &= \mathrm{Tr}(P_3\eta) \end{aligned}$$

Hence, we have transformed the problem from a system where we only measured the population of its first ground state, to a system where, in average, we know the populations of all the states. Considering that $\hat{\Omega}_{12}$ has converged to Ω_{12}, we obtain the following system:

$$\begin{aligned} \frac{d}{dt}\eta = &\quad -i\tfrac{\epsilon}{2}\Omega_{23}[\sigma_x^{23}, \eta] \\ y_j(t) = &\quad \mathrm{Tr}(P_j\eta) \qquad \text{is known in average for } j = 1, 2, 3. \end{aligned}$$

This, according to [1], is identifiable.

4 Convergence analysis

Let us write the observer as follows:

$$\begin{aligned} \dot{\hat{\rho}} &= -i\hat{\Omega}_{12}[\sigma_x^{12}, \hat{\rho}] - i\epsilon\hat{\Omega}_{23}\cos(\hat{\theta})[\sigma_x^{23}, \hat{\rho}] + \epsilon\Gamma_1(y - \hat{y})(\sigma_z^1\hat{\rho} + \hat{\rho}\sigma_z^1 - 2\mathrm{Tr}(\sigma_z^1\hat{\rho})\hat{\rho}) \\ &\quad + \epsilon^2\Gamma_2(y - \hat{y})\Big[(1 + 2\cos(2\hat{\theta}))(Q_1\hat{\rho} + \hat{\rho}Q_1 - 2\mathrm{Tr}(Q_1\hat{\rho})\hat{\rho}) \\ &\qquad + (1 - 2\cos(2\hat{\theta}))(Q_2\hat{\rho} + \hat{\rho}Q_2 - 2\mathrm{Tr}(Q_2\hat{\rho})\hat{\rho}) - 2(Q_3\hat{\rho} + \hat{\rho}Q_3 - 2\mathrm{Tr}(Q_3\hat{\rho})\hat{\rho})\Big] \\ \dot{\hat{\theta}} &= \hat{\Omega}_{12} \\ \dot{\hat{\Omega}}_{12} &= -i\epsilon^2\gamma_1\mathrm{Tr}(\sigma_z^1[\sigma_x^{12}, \hat{\rho}])(y - \hat{y}) \\ \dot{\hat{\Omega}}_{23} &= -i\epsilon^4\gamma_2(y - \hat{y})\gamma_2[(1 - 2\cos(2\hat{\theta}))\mathrm{Tr}(Q_2[\Sigma_x^{23}, \hat{\rho}]) - 2\mathrm{Tr}(Q_3[\Sigma_x^{23}, \hat{\rho}])]. \end{aligned}$$

As we can see in the simulation, and seeing the design of the observer, the convergence is done in two steps:

1. We consider the dynamics of the system neglecting the terms in $-i\epsilon\hat{\Omega}_{23}\cos(\hat{\theta})[\sigma_x^{23},\hat{\rho}]$ in front of $-i\hat{\Omega}_{12}[\sigma_x^{12},\hat{\rho}]$ and neglecting the term in $\epsilon^2\Gamma_2$ in front of $\epsilon\Gamma_1$. We then prove that
   (a) $\lim_{t\to+\infty}\hat{\Omega}_{12}=\Omega_{12}$
   (b) $\lim_{t\to+\infty}\mathrm{Tr}(\sigma_y^{12}\hat{\eta})=\mathrm{Tr}(\sigma_y^{12}\eta)$
   (c) $\lim_{t\to+\infty}\mathrm{Tr}(\sigma_z^{1}\hat{\eta})=\mathrm{Tr}(\sigma_z^{1}\eta)$
2. Once all these terms have converged, we consider the higher order terms of the system and prove that:
   (a) $\lim_{t\to+\infty}\hat{\Omega}_{23}=\Omega_{23}$
   (b) $\lim_{t\to+\infty}\mathrm{Tr}(P_1\hat{\zeta})=\mathrm{Tr}(P_1\zeta)$
   (c) $\lim_{t\to+\infty}\mathrm{Tr}(P_2\hat{\zeta})=\mathrm{Tr}(P_2\zeta)$
   (d) $\lim_{t\to+\infty}\mathrm{Tr}(P_3\hat{\zeta})=\mathrm{Tr}(P_3\zeta)$

η and ζ being equal to ρ in some adequately chosen frames. Their expressions will be given below.

4.1 The convergence of the first order system

We now write the system considering the approximations of step 1, mentioned above. Which leads us to the system:

$$\begin{aligned}
\dot{\rho} &= -i\Omega_{12}[\sigma_x^{12},\rho]\\
\dot{\hat{\rho}} &= -i\hat{\Omega}_{12}[\sigma_x^{12},\hat{\rho}]+\epsilon\Gamma_1(y-\hat{y})(\sigma_z^1\hat{\rho}+\hat{\rho}\sigma_z^1-2\mathrm{Tr}(\sigma_z^1\hat{\rho})\hat{\rho})\\
\dot{\hat{\Omega}}_{12} &= -i\epsilon^2\gamma_1\mathrm{Tr}(\sigma_z^1[\sigma_x^{12},\hat{\rho}])(y-\hat{y})
\end{aligned}$$

We are now tempted to write these equation in a rotating frame, such as the dominant term in Ω_{12} vanishes. We will then be able to see which terms have a first or higher order influence.
Let us define:

$$\eta=R^\dagger\rho R \text{ and } \hat{\eta}=R^\dagger\hat{\rho}R \quad \text{where} \quad R=e^{-i\Omega_{12}\sigma_x^{12}t}$$

Here are some results which will be useful throughout our calculations:

$$R^\dagger\sigma_z^1 R = \cos(2\Omega_{12}t)\sigma_z^1+\sin(2\Omega_{12}t)\sigma_y^{12} \tag{1}$$

$$LP_1L^\dagger = \frac{1}{2}(I_{12}+\cos(2\Omega_{12}t)\sigma_z^1+\sin(2\Omega_{12}t)\sigma_y^{12}) \tag{2}$$

Which leads us to:

$$
\begin{aligned}
&\frac{d}{dt}\eta = 0\\
&\frac{d}{dt}\hat{\eta} = -i(\hat{\Omega}_{12}-\Omega_{12})[\sigma_x^{12},\hat{\eta}]\\
&\qquad + \epsilon\frac{\Gamma_1}{2}\Big[\mathrm{Tr}(I_{12}(\eta-\hat{\eta})) + \cos(2\Omega_{12}t)\mathrm{Tr}(\sigma_z^1(\eta-\hat{\eta})) + \sin(2\Omega_{12}t)\mathrm{Tr}(\sigma_y^{12}(\eta-\hat{\eta}))\Big]\\
&\qquad + \cos(2\Omega_{12}t)\Big[\sigma_z^1\hat{\eta}+\hat{\eta}\sigma_z^1 - 2\mathrm{Tr}(\sigma_z^1\hat{\eta})\hat{\eta}\Big] + \sin(2\Omega_{12}t)\Big[\sigma_y^{12}\hat{\eta}+\hat{\eta}\sigma_y^{12} - 2\mathrm{Tr}(\sigma_y^{12}\hat{\eta})\hat{\eta}\Big]\\
&\frac{d}{dt}\hat{\Omega}_{12} = -i\epsilon^2\frac{\gamma_1}{2}\left[\mathrm{Tr}(I_{12}(\eta-\hat{\eta})) + \cos(2\Omega_{12}t)\mathrm{Tr}(\sigma_z^1(\eta-\hat{\eta})) + \sin(2\Omega_{12}t)\mathrm{Tr}(\sigma_y^{12}(\eta-\hat{\eta}))\right]\\
&\qquad + (\cos(2\Omega_{12}t)\mathrm{Tr}(\sigma_z^1[\sigma_x^{12},\hat{\eta}]) + \sin(2\Omega_{12}t)\mathrm{Tr}(\sigma_y^{12}[\sigma_x^{12},\hat{\eta}]))
\end{aligned}
$$

After neglecting the highly oscillating terms of average 0 in front of the first order secular ones, and rearranging the terms, we obtain the following equations:

$$
\begin{aligned}
\frac{d}{dt}\eta &= 0\\
\frac{d}{dt}\hat{\eta} &= -i(\hat{\Omega}_{12}-\Omega_{12})[\sigma_x^{12},\hat{\eta}] + \frac{\epsilon\Gamma_1}{4}\Big[\mathrm{Tr}(\sigma_z^1(\eta-\hat{\eta}))[\sigma_z^1\hat{\eta}+\hat{\eta}\sigma_z^1 - 2\mathrm{Tr}(\sigma_z^1\hat{\eta})\hat{\eta}]\\
&\qquad + \mathrm{Tr}(\sigma_y^{12}(\eta-\hat{\eta}))[\sigma_y^{12}\hat{\eta}+\hat{\eta}\sigma_y^{12} - 2\mathrm{Tr}(\sigma_y^{12}\hat{\eta})\hat{\eta}]\Big]\\
\frac{d}{dt}\hat{\Omega}_{12} &= -i\epsilon^2\frac{\gamma_1}{4}\left(\mathrm{Tr}(\sigma_z^1(\eta-\hat{\eta}))\mathrm{Tr}(\sigma_z^1[\sigma_x^{12},\hat{\eta}]) + \mathrm{Tr}(\sigma_y^{12}(\eta-\hat{\eta}))\mathrm{Tr}(\sigma_y^{12}[\sigma_x^{12},\hat{\eta}])\right).
\end{aligned}
$$

N.B: Note that $\hat{\Omega}_{12}-\Omega_{12}$ of order ϵ is an essential hypothesis. That is why it is important to choose, for $t=0$, $\hat{\Omega}_{12}$ close to Ω_{12}.
Let us use the following notations:

$$
\begin{array}{lll}
X = \mathrm{Tr}(\sigma_x^{12}\eta) & Y = \mathrm{Tr}(\sigma_y^{12}\eta) & Z = \mathrm{Tr}(\sigma_z^1\eta)\\
\hat{X} = \mathrm{Tr}(\sigma_x^{12}\hat{\eta}) & \hat{Y} = \mathrm{Tr}(\sigma_y^{12}\hat{\eta}) & \hat{Z} = \mathrm{Tr}(\sigma_z^1\hat{\eta}).
\end{array}
$$

We have

$$
\begin{aligned}
\dot{X} &= \dot{Y} = \dot{Z} = 0\\
\dot{\hat{X}} &= -\epsilon\frac{\Gamma_1}{4}[(Z-\hat{Z})(i\hat{Y}+\hat{Z}\hat{X}) + (Y-\hat{Y})(i\hat{Z}+\hat{Y}\hat{X})]\\
\dot{\hat{Y}} &= -(\hat{\Omega}_{12}-\Omega_{12})\hat{Z} + \epsilon\frac{\Gamma_1}{4}[(Z-\hat{Z})(i\hat{X}-\hat{Y}\hat{Z}) - (Y-\hat{Y})(1-\hat{Y}^2)]\\
\dot{\hat{Z}} &= (\hat{\Omega}_{12}-\Omega_{12})\hat{Y} + \epsilon\frac{\Gamma_1}{4}[(Z-\hat{Z})(1-\hat{Z}^2) - (Y-\hat{Y})(i\hat{X}+\hat{Y}\hat{Z})]\\
\dot{\hat{\Omega}}_{12} &= -i\epsilon^2\frac{\gamma_1}{4}((Z-\hat{Z})\hat{Y} - (Y-\hat{Y})\hat{Z})
\end{aligned}
$$

We are now going to prove that the proposed observer enables us to estimate Ω_{12}, X, Y and Z (we assume everywhere in this paper that Y and Z are both different from zero). Let's consider the Lyapounov function:

$$
\begin{aligned}
& \mathbb{R}^3 \to \mathbb{R}\\
V : & \begin{pmatrix}\hat{\Omega}_{12}\\ \hat{Y}\\ \hat{Z}\end{pmatrix} \to \frac{4}{\epsilon^2\gamma_1}(\hat{\Omega}_{12}-\Omega_{12})^2 + (\hat{Z}-Z)^2 + (\hat{Y}-Y)^2\\
& \frac{\dot{V}}{2} = \frac{4}{\epsilon^2\gamma_1}\dot{\hat{\Omega}}_{12}(\hat{\Omega}_{12}-\Omega_{12}) + \dot{\hat{Z}}(\hat{Z}-Z) + \dot{\hat{Y}}(\hat{Y}-Y)
\end{aligned}
$$

Let us detail some calculations which will help us compute the latter expression

$$\sigma_y^{12}\sigma_y^{12} = \sigma_z^1\sigma_z^1 = I_{12}$$

$$\sigma_y^{12}\sigma_z^1 = i\sigma_x^{12} = -\sigma_y^{12}\sigma_z^1$$

Notice that in a first approximation, we have $\mathrm{Tr}(I_{12}\hat{\eta}) = \mathrm{Tr}(I_{12}\hat{\eta}^2) = 1$ Which leads us to:

$$\frac{\dot{V}}{\Gamma} = -[(\hat{Z}-Z)^2 + (\hat{Y}-Y)^2] + ((\hat{Z}-Z)\hat{Z} + (\hat{Y}-Y)\hat{Y})^2.$$

According to the Cauchy Schwartz inequality, we have:

$$\begin{aligned}\frac{\dot{V}}{\Gamma} &\leq -\,[(\hat{Z}-Z)^2 + (\hat{Y}-Y)^2] + ((\hat{Z}-Z)^2 + (\hat{Y}-Y)^2)(\hat{Z}^2 + \hat{Y}^2)\\ &\leq -\,[(\hat{Z}-Z)^2 + (\hat{Y}-Y)^2][1 - \hat{Z}^2 - \hat{Y}^2].\end{aligned}$$

Obviously,$-[(\hat{Z}-Z)^2 + (\hat{Y}-Y)^2] \leq 0$ and since $\mathrm{Tr}(I_{12}\hat{\eta}^2) \leq 1$, we have $1 - \hat{Z}^2 - \hat{Y}^2 \geq 0$.

4.1.1 case of equality:

case where $1-\hat{Z}^2-\hat{Y}^2 = 0$ and $\exists K$ in $\mathbb{R}$ such that $\hat{Z} = KZ$ and $\hat{Y} = KY$: In this situation,

$$\hat{\eta} = \hat{\eta}_{\pm} = \frac{1}{2}(I \pm (\frac{Y}{\sqrt{Y^2+Z^2}}\sigma_y^{12} + \frac{Z}{\sqrt{Y^2+Z^2}}\sigma_z^1) + ...)$$

One can see that $(\hat{\eta}_{\pm}, \Omega_{12})$ are equilibrium points. But by choosing $|\hat{\eta}(0) - \eta| < |\hat{\eta}_{\pm} - \eta|$, $\hat{\eta}(0)$ being the initial state of the observer, the Lyapounov function will keep decreasing, and the observer will never reach this equilibrium point.

case where $(\hat{Z}-Z)^2 = (\hat{Y}-Y)^2 = 0$:
The dynamics of the system at this point become:

$$\begin{aligned}\frac{d}{dt}\eta &= 0\\ \frac{d}{dt}\hat{\eta} &= -i(\hat{\Omega}_{12} - \Omega_{12})[\sigma_x^{12}, \hat{\eta}]\\ \frac{d}{dt}\hat{\Omega}_{12} &= 0\end{aligned}$$

and:

$$\begin{aligned}\frac{d}{dt}\eta &= 0\\ \frac{d}{dt}(\hat{Z}-Z) &= (\hat{\Omega}_{12} - \Omega_{12})\hat{Y}\\ \frac{d}{dt}(\hat{Y}-Y) &= -(\hat{\Omega}_{12} - \Omega_{12})\hat{Z}\\ \frac{d}{dt}\hat{\Omega}_{12} &= 0\end{aligned}$$

Notice that by assuming Y and Z both non-zeros, we have excluded the case $\hat{Y} = \hat{Z} = 0$ and therefore necessarily $\hat{\Omega}_{12} = \Omega_{12}$. We conclude that the LaSalle invariant set is given by $\hat{Z} = Z$, $\hat{Y} = Y$ and $\hat{\Omega}_{12} = \Omega_{12}$. Hence

$$\begin{pmatrix} \hat{\Omega}_{12} \\ \hat{Y} \\ \hat{Z} \end{pmatrix} \quad \text{converges to} \quad \begin{pmatrix} \Omega_{12} \\ Y \\ Z \end{pmatrix}.$$

4.2 The identification of μ_{23}

The main difficulty here, compared to the solution proposed by [1] is that we only have access to the population on the ground state.
We now consider the averaged system with all the terms, in which we neglect the terms in $Z - \hat{Z}$, $Y - \hat{Y}$ and $\Omega_{12} - \hat{\Omega}_{12}$:

$$\begin{aligned}
\frac{d}{dt}\eta &= -i\Omega_{23}\epsilon[\sigma_x^{23}, \eta] \\
\frac{d}{dt}\hat{\eta} &= -i\hat{\Omega}_{23}\epsilon[\sigma_x^{23}, \hat{\eta}] \\
&\quad +\epsilon^2\Gamma_2[\mathrm{Tr}(P_1(\eta - \hat{\eta}))(P_1\hat{\eta} + \hat{\eta}P_1 - 2\mathrm{Tr}(P_1\hat{\eta})\hat{\eta}) \\
&\qquad\qquad +\mathrm{Tr}(P_2(\eta - \hat{\eta}))(P_2\hat{\eta} + \hat{\eta}P_2 - 2\mathrm{Tr}(P_2\hat{\eta})\hat{\eta}) \\
&\qquad\qquad +\mathrm{Tr}(P_3(\eta - \hat{\eta}))(P_3\hat{\rho} + \hat{\rho}P_3 - 2\mathrm{Tr}(P_3\hat{\rho})\hat{\rho})] \\
\frac{d}{dt}\hat{\Omega}_{23} &= -i\epsilon^4\gamma_2(\mathrm{Tr}(P_2(\eta - \hat{\eta}))\mathrm{Tr}(P_1[\sigma_x^{23}, \hat{\eta}] + \mathrm{Tr}(P_3(\eta - \hat{\eta}))\mathrm{Tr}(P_3[\sigma_x^{23}, \hat{\eta}])
\end{aligned}$$

We are now confronted to the problem solved in [2], where the populations on all the states are known. Please refer to the latter paper for an explanation about the convergence of this observer. The scheme of the explanation is:

1. We write the equations in the rotating frame by defining a new variable: $\zeta = e^{i\Omega_{23}\sigma_x^{23}t}\eta e^{-i\Omega_{23}\sigma_x^{23}t}$
2. We then neglect the highly oscillating terms
3. After defining a Lyapounov function and using the Lasalle invariance principle and the averaging theorem, we conclude that
   (a) $\lim_{t\to+\infty} \hat{\Omega}_{23} = \Omega_{23}$
   (b) $\lim_{t\to+\infty} \mathrm{Tr}(P_1\hat{\zeta}) = \mathrm{Tr}(P_1\zeta)$
   (c) $\lim_{t\to+\infty} \mathrm{Tr}(P_2\hat{\zeta}) = \mathrm{Tr}(P_2\zeta)$
   (d) $\lim_{t\to+\infty} \mathrm{Tr}(P_3\hat{\zeta}) = \mathrm{Tr}(P_3\zeta)$

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A Gell-Mann matrices

Let's recall the expressions of the Gell-Mann Matrices:

$$\sigma_x^{kl} = |k\rangle\langle l| + |l\rangle\langle k|, \qquad \sigma_y^{kl} = -i\,|k\rangle\langle l| + i\,|l\rangle\langle k|$$

and

$$\sigma_{z1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \sigma_{z2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

We also define

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \sigma_z^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$