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THE COMBINATORICS OF AL-SALAM-CHIHARA $q$-LAGUERRE POLYNOMIALS

ANISSE KASRAOUI, DENNIS ST ANTON, AND JIANG ZENG

Abstract. We describe various aspects of the Al-Salam-Chihara $q$-Laguerre polynomials. These include combinatorial descriptions of the polynomials, the moments, the orthogonality relation and a combinatorial interpretation of the linearization coefficients.

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1. INTRODUCTION

The monic simple Laguerre polynomials $L_n(x)$ may be defined by the explicit formula:

$$L_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} \binom{n}{k} x^k,$$

or by the three-term recurrence relation

$$L_{n+1}(x) = (x - (2n + 1))L_n(x) - n^2 L_{n-1}(x).$$

The moments are

$$\mu_n = L(x^n) = \int_{0}^{\infty} x^n e^{-x} dx = n!.$$

The linearization formula reads as follows:

$$L_{n_1}(x)L_{n_2}(x) = \sum_{n_3} C_{n_1 n_2}^{n_3} L_{n_3}(x),$$

where
\[ C^3_{n_1 n_2} = \sum_{s \geq 0} \frac{n_1! n_2! 2^{N_2 + n_3 - 2s} s!}{(s - n_1)!(s - n_2)!(s - n_3)!(N_2 + n_3 - 2s)! n_3!}. \]

Equivalently we have
\[ \mathcal{L}(L_{n_1}(x)L_{n_2}(x)L_{n_3}(x)) = \sum_{s \geq 0} \frac{n_1! n_2! n_3! 2^{N_2 + n_3 - 2s} s!}{(s - n_1)!(s - n_2)!(s - n_3)!(N_2 + n_3 - 2s)! n_3!}. \]

Given positive integers \( n_1, n_2, \ldots, n_k \) such that \( n = n_1 + \cdots + n_k \), let \( S_i \) be the consecutive integer segment \( \{n_1 + \cdots + n_{i-1} + 1, \ldots, n_1 + \cdots + n_i\} \) with \( n_0 = 0 \), then \( S_1 \cup \ldots \cup S_k = [n] \). A permutation \( \sigma \) of \([n]\) is said to be a generalized derangement if \( i \) and \( \sigma(i) \) do not belong to a same segment \( S_j \) for all \( i \in [n] \). Let \( D_n \) be the set of generalized derangements of \([n]\) then we have
\[ \mathcal{L}(L_{n_1}(x) \ldots L_{n_k}(x)) = \sum_{\sigma \in D_n} 1. \]

A \( q \)-version of (1) was studied by Garsia and Remmel [9] in 1980. Several \( q \)-analogues of the moments (2) and recurrence relation (3) were investigated in the last two decades (see [2, 18, 19]) in order to obtain new mahonian moments and recurrence (2) and recurrence relation (3) were investigated in the last two decades (see [2, 18, 19]). In particular, their machinery of the unified combinatorial interpretations of several aspects of Sheffer orthogonal polynomials (moments, polynomials, and the linearization coefficients) (see [14, 20, 22]) it is natural to seek for a \( q \)-version of this picture.

As one can expect, the first result in this direction was the linearization formula for \( q \)-Hermite polynomials due to Ismail, Stanton and Viennot [12], dated back to 1987. In particular, their formula provides a combinatorial evaluation of the Askey-Wilson integral. However, a similar formula for \( q \)-Charlier polynomials was discovered only recently by Anshelevich [1], who used the machinery of \( q \)-Levy stochastic processes. Short later, Kim, Stanton and Zeng [15] gave a combinatorial proof of Anshelevich’s result.

The object of this paper is to give a \( q \)-version of all the above formulas for simple Laguerre polynomials.

2. AL-SALAM-CHIHARA POLYNOMIALS REVISITED

The Al-Salam-Chihara polynomials \( Q_n(x) := Q_n(x; \alpha, \beta|q) \) may be defined by the recurrence relation [16, Chapter 3]:
\[
\begin{cases} 
Q_0(x) = 1, & Q_{-1}(x) = 0, \\
Q_{n+1}(x) = (2x - (\alpha + \beta)q^n)Q_n(x) - (1 - q^n)(1 - \alpha \beta q^{n-1})Q_{n-1}(x), & n \geq 0.
\end{cases}
\]

Let \( Q_n(x) = 2^n p_n(x) \) then
\[
 xp_n(x) = p_{n+1}(x) + \frac{1}{2}(\alpha + \beta)q^n p_n(x) + \frac{1}{4}(1 - q^n)(1 - \alpha \beta q^{n-1})p_{n-1}(x).
\]
They also have the following explicit expressions:

\[ Q_n(x; \alpha, \beta | q) = \frac{(\alpha \beta; q)_n}{\alpha^n} \phi_2 \left( q^{-n}, \alpha u, \alpha u^{-1} \left| q; q \right. \right) \]

\[ = (\alpha u; q)_n u^{-n} 2 \phi_1 \left( q^{-n}, \beta u^{-1} \left| \alpha^{-1} q^{-n+1} u, q \alpha^{-1} q u \right. \right) \]

\[ = (\beta u^{-1}; q)_n u^{-n} 2 \phi_1 \left( q^{-n}, \alpha u \left| \beta^{-1} q^{-n+1} u, q \beta^{-1} q u^{-1} \right. \right), \]

where \( x = \frac{u+u^{-1}}{2} \) or \( x = \cos \theta \) if \( u = e^{i \theta} \).

The Al-Salam-Chihara polynomials have the following generating function

\[ G(t, x) = \sum_{n=0}^{\infty} Q_n(x; \alpha, \beta | q) \frac{t^n}{(q; q)_n} = \frac{(\alpha t, \beta t; q)_\infty}{(te^{i \theta}, te^{-i \theta}; q)_\infty}. \]

They are orthogonal with respect to the linear functional \( \hat{\mathcal{L}}_q \):

\[ \hat{\mathcal{L}}_q(x^n) = \frac{1}{2\pi} \int_0^{\pi} (\cos \theta)^n \frac{(q, \alpha \beta, e^{2i \theta}, e^{-2i \theta}; q)_\infty}{(\alpha e^{i \theta}, \alpha e^{-i \theta}, \beta e^{i \theta}, \beta e^{-i \theta}; q)_\infty} d\theta, \tag{8} \]

where \( x = \cos \theta \). Note that

\[ \hat{\mathcal{L}}_q(Q_n(x)^2) = (q; q)_n (\alpha \beta; q)_n. \]

**Theorem 1.** We have

\[ Q_{n_1}(x) Q_{n_2}(x) = \sum_{n_3 \geq 0} C_{n_1, n_2}^{n_3} (\alpha, \beta; q) Q_{n_3}(x), \tag{9} \]

where

\[ C_{n_1, n_2}^{n_3} (\alpha, \beta; q) = (-1)^{N_2+n_3} \frac{(q; q)_{n_3}(q; q)_{n_2}}{(\alpha \beta; q)_{n_3}} \]

\[ \times \sum_{m_2, m_3} \frac{(\alpha \beta; q)_{n_1+n_3} \alpha^m \beta^{m_3+n_2-n_1-m_2-2m_3} q^{(m_3+2)} \sum (\alpha \beta; q)_{n_1+n_3-m_2-n_2-2m_3}(q; q)_{m_2}(q; q)_{n_3}(q; q)_{m_3+n_1-n_2}(q; q)_{m_3+n_1-n_2}}{(q; q)_{n_3+n_2-n_1-m_2-2m_3}(q; q)_{m_2}(q; q)_{m_3+n_1-n_2}(q; q)_{m_3+n_1-n_2}(q; q)_{m_3+n_1-n_2}). \]

**Proof.** Clearly \( C_{n_1, n_2}^{n_3} (\alpha, \beta; q) = \hat{\mathcal{L}}_q(Q_{n_1}(x) Q_{n_2}(x) Q_{n_3}(x)) / \hat{\mathcal{L}}_q(Q_{n_3}(x) Q_{n_3}(x)). \) Using the Askey-Wilson integral:

\[ \frac{(q; q)_\infty}{2\pi} \int_0^{\pi} \frac{\left( e^{2i \theta}, e^{-2i \theta}; q \right)_\infty}{\prod_{j=1}^{4} (t_j e^{i \theta}, t_j e^{-i \theta}; q)_\infty} d\theta = \frac{(t_1 t_2 t_3 t_4; q)_\infty}{\prod_{1 \leq j < k \leq 4} (t_j t_k; q)_\infty}, \]

one can prove [12, Theorem 3.5] that

\[ \hat{\mathcal{L}}_q(G(t_1, x) G(t_2, x) G(t_3, x)) \]

\[ = \frac{(at_1 t_2 t_3, \beta t_1 t_2 t_3, \alpha \beta; q)_\infty}{(t_1 t_2, t_1 t_3, t_2 t_3; q)_\infty} \phi_2 \left( (t_1 t_2, t_1 t_3, \beta t_1 t_2 t_3; q; \alpha \beta \right). \]
Therefore
\[ \sum_{n_1, n_2, n_3} \hat{N}_q(Q_{n_1}(x)Q_{n_2}(x)Q_{n_3}(x)) \frac{t_{1}^{n_1}}{(q; q)_{n_1}} \frac{t_{2}^{n_2}}{(q; q)_{n_2}} \frac{t_{3}^{n_3}}{(q; q)_{n_3}} = \sum_{k \geq 0} (\alpha_1 t_{1} t_{2} t_{3} q^k, \beta t_{1} t_{2} t_{3} q^k, \alpha_1 \beta; q)_\infty (\alpha \beta)_k (t_{1} t_{2} t_{3} q^k, t_{1} t_{3} q^k, t_{2} t_{3} q^k; q)_\infty \frac{t_{1}^{n_1}}{(q; q)_{n_1}} \frac{t_{2}^{n_2}}{(q; q)_{n_2}} \frac{t_{3}^{n_3}}{(q; q)_{n_3}} \] (10)

Using the Euler formulas:
\[ (t; q)_\infty = \sum_{n \geq 0} \frac{(-1)^n q^n}{(q; q)_n} t^n; \quad \frac{1}{(t; q)_\infty} = \sum_{n \geq 0} \frac{1}{(q; q)_n} t^n, \]
we can rewrite the sum in (10) as follows:
\[ \sum_{k \geq 0} (\alpha \beta)_k (q; q)_k \sum_{l_1, l_2 \geq 0} \frac{\alpha_1 \beta_2 q^{k(l_1 + l_2)} (-t_{1} t_{2} t_{3} l_1 t_2 q^{l_1/2} + l_2 q^{l_2/2})}{(q; q)_l_1 (q; q)_l_2} \times \sum_{m_1, m_2, m_3 \geq 0} q^{(m_1 + m_2 + m_3) k} t_{1}^{m_1} t_{2}^{m_2} t_{3}^{m_3} (q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3} \] (11)

Substituting
\[ \sum_{k \geq 0} (\alpha \beta q^{l_1 + l_2 + m_1 + m_2 + m_3}) (q; q)_k = \frac{1}{(\alpha \beta q^{l_1 + l_2 + m_1 + m_2 + m_3}; q)_\infty} \]
in (11), we get
\[ \sum_{l_1, l_2, m_1, m_2, m_3} \frac{t_{1}^{n_1} t_{2}^{n_2} t_{3}^{n_3}}{(q; q)_{n_1} (q; q)_{n_2} (q; q)_{n_3}} (\alpha \beta)_{n_1 + m_1 + m_2} \frac{q^{l_1} \beta_2 q^{l_2} q^{l_1/2} + l_2 q^{l_2/2}}{(q; q)_{l_1} (q; q)_{l_2}} (-1)^{l_1 + l_2} \] (12)

where \( l_1 + l_2 + m_1 + m_2 = n_1, l_1 + l_2 + m_1 + m_3 = n_2 \) and \( l_1 + l_2 + m_2 + m_3 = n_3 \).

Since \( l_1 + l_2 \equiv N_2 + N_3 \pmod{2} \), extracting the coefficient of \( t_{1}^{n_1} t_{2}^{n_2} t_{3}^{n_3} \) in (12) and dividing by \( (q, \alpha \beta; q)_{n_3} \), we obtain (13) where \( l_1 \) is replaced by \( m_2 \).

3. The new \( q \)-Laguerre polynomials

We define the new \( q \)-Laguerre polynomials \( L_n(x; q) \) by re-scaling Al-Salam-Chihara polynomials:
\[ L_n(x; q) = \left( \frac{\sqrt{y}}{q - 1} \right)^n Q_n \left( \frac{(q - 1)x + y + 1}{2\sqrt{y}}; \frac{1}{\sqrt{y}}, \sqrt{y}q/q \right). \] (13)

It follows from (13) that the polynomials \( L_n(x; q) \) satisfy the recurrence:
\[ L_{n+1}(x; q) = (x - y[n + 1]q - [n]q) L_n(x; q) - y[n]_q^2 L_{n-1}(x; q). \] (14)

We derive then the explicit formula for \( L_n(x; q) \):
\[ L_n(x; q) = \sum_{k=0}^{n} (-1)^{n-k} \frac{n^k q^{k-n}}{k! q^k} \sum_{j=0}^{k-1} (x - (1 - yq^{-1})j)_q \left[ \frac{n}{k} q^{k(n-k)} y^{n-k} \prod_{j=0}^{k-1} (x - (1 - yq^{-1})j)_q \right]. \] (15)
Thus
\[
L_1(x; q) = x - y,
\]
\[
L_2(x; q) = x^2 - (1 + 2y + qy)x + (1 + q)y^2,
\]
\[
L_3(x; q) = x^3 - (q^2y + 3y + q + 2 + 2qy)x^2
\]
\[
+ (q^3y^2 + yq^2 + q + 2qy + 3q^2y^2 + 1 + 4qy^2 + 2y + 3y^2)x
\]
\[
- (2q^2 + 2q + q^3 + 1)y^3.
\]

A combinatorial interpretation of these $q$-Laguerres polynomials can be derived from the Simion and Stanton’s combinatorial model for octabasic Laguerre polynomials [19]. For a subset $A$ of $[n]$, the functional digraph of an injection $f : A \to [n]$ consists of disjoint paths and cycles. Each path $P$ is of the form $a_0 \to a_1 \to \cdots \to a_l$, where $f(a_j) = a_{j+1}$ for $0 \leq j < l$, with $f^{-1}(a_0)$ empty, and $a_l \in [n] - A$. We put $\text{last}(P) = a_l$ and if $i = a_k \in P$ we write $\text{ind}(i, P) = k$ for the index of $i$ on the path $P$. For any path $P$ in the digraph and two integers $i < j$, we put
\[
n_P(i, j) = |\{a \in P : i < a < j\}|.
\]

For $p \in P$ and two integers $i < j$, we define
\[
m_P(p; i, j) = |\{a \in P : i < a < j, \text{ind}(p, P) < \text{ind}(a, P)\}|,
\]
that is, the number of points on the path “to the right” of $p$, whose values are strictly between $i$ and $j$. And finally, for $i \in A$, we denote by $F(i)$ the “first forward iterate” of $f$ which is smaller than $i$, i.e.,
\[
F(i) = \begin{cases} 
  f^p(i), & \text{where } p = \min\{m \geq 1, f^m(i) < i \text{ if such } m \text{ exists}\}; \\
  i, & \text{if } \{m \geq 1, f^m(i) < i \} \text{ is empty}.
\end{cases}
\]

For instance, suppose that the path $P = 2 \to 7 \to 1 \to 5 \to 3$ is a connected component of the functional digraph of $f$. Then $n_P(1, 4) = |\{2, 3\}| = 2$, $m_P(7; 1, 4) = |\{3\}| = 1$, and $F(2) = F(7) = 1$, $F(1) = 1$, and $F(5) = 3$.

For any $k \in [n]$, let $\alpha(k) = w(k) = 0$ if $k \notin A$, otherwise if $k$ is on a cycle or a path $P$ such that $k > \text{last}(P)$, then $\alpha(k) = 1$ and
\[
w(k) = F(k) - 1 - \sum_{\text{last}(Q) > k} n_Q(0, F(k));
\]
if $k$ is on a path $P$ such that $k < \text{last}(P)$, then $\alpha(k) = 0$ and
\[
w(k) = k - 1 - m_P(k; 0, k) - \sum_{\text{last}(Q) > \text{last}(P)} n_Q(0, k),
\]
where $Q$ ranges over all paths in the functional digraphs of $f$. Let
\[
w(A, f) = \sum_{k \in A} w(k) \quad \text{and} \quad \alpha(A, f) = \sum_{k \in A} \alpha(k).
\]

Example 1. Let $n = 9$, $A = \{2, 9\}$ and $\sigma = (6)(47)(3518)$ (in cycle notation with maximum at last). Then we have $\text{cyc}(\sigma) = 3$ and
\[
w(A, \sigma) = (3 - 1 - 1) + (5 - 1 - 1) + (1 - 1) + (4 - 1 - 2) = 5.
\]
Theorem 2. The $q$-Laguerre polynomials have the following interpretation:

$$L_n(x; q) = \sum_{A \subseteq [n], f: A \rightarrow [n]} (-1)^{|A|} x^{-|A|} y^{\alpha(A,f)} q^{w(A,f)},$$

where $f$ is injective.

Proof. This is the $a = 1, s = u = 1$ and $r = t = q$ special case of the quadrabasic Laguerre polynomials [19, p.313]. □

Remark 1. It is easy to see that the constant term $L_n(0)$ is equal to

$$L_n(0) = (-1)^n y^n n! q^n.$$

So the restriction of the statistic on permutations is a Mahonian statistic.

4. Moments of the $q$-Laguerre polynomials

Let $S_n$ be the set of permutations of $[n] := \{1, 2, \ldots, n\}$. For $\sigma \in S_n$ the crossing number of $\sigma$ is defined by

$$cr(\sigma) = \sum_{i=1}^{n} \# \{ j | j < i \leq \sigma(j) < \sigma(i) \} + \sum_{i=1}^{n} \# \{ j | j > i > \sigma(j) > \sigma(i) \},$$

while the number of weak excedances of $\sigma$ is defined by

$$wex(\sigma) = \# \{ i | 1 \leq i \leq n \text{ and } i \leq \sigma(i) \}.$$

We can depict these statistics by associating with each permutation $\sigma$ of $[n]$ a diagram by drawing an arc $i \rightarrow \sigma(i)$ above (resp. under) the segment $1 \rightarrow 2 \rightarrow \cdots \rightarrow n$ if $i \leq \sigma(i)$ (resp. $i > \sigma(i)$). For example, the permutation $\sigma = 9 \ 3 \ 7 \ 4 \ 6 \ 11 \ 5 \ 8 \ 1 \ 10 \ 2$ can be depicted as follows:

![Diagram](image)

Let $\mu_n^{(\ell)}(y, q)$ be the enumerating polynomial of permutations in $S_n$ with respect to weak excedances and crossing numbers:

$$\mu_n^{(\ell)}(y, q) := \sum_{\sigma \in S_n} y^{wex(\sigma)} q^{\ell^{cr(\sigma)}}.$$

Randrianarivony [17] and Corteel [3] have proved the following continued fraction expansion:

$$E(y, q, t) := \sum_{n \geq 0} \mu_n^{(\ell)}(y, q)t^n = \frac{1}{1 - b_0 t - \frac{\lambda_1 t^2}{1 - b_1 t - \frac{\lambda_2 t^2}{1 - b_2 t - \cdots}}},$$

where $b_n = y[n + 1]_q + [n]_q$ and $\lambda_n = y[n]_q^2$.

We derive then from the classical theory of orthogonal polynomials the following interpretation of the moments of the $q$-Laguerre polynomials.
Theorem 3. The n-th moment of the q-Laguerre polynomials is equal to $\mu_n^{(\ell)}(y, q)$. More precisely, let $L_q$ be the linear functional defined by $L_q(x^n) = \mu_n^{(\ell)}(y, q)$, then

$$L_q(L_{n_1}(x; q)L_{n_2}(x; q)) = y^{n_1}(n_1!_q)^2\delta_{n_1 n_2}. \tag{17}$$

The first values of the moment sequence are as follows:

$$\mu_1^{(\ell)}(y, q) = y,$$

$$\mu_2^{(\ell)}(y, q) = y + y^2,$$

$$\mu_3^{(\ell)}(y, q) = y + (3 + q)y^2 + y^3,$$

$$\mu_4^{(\ell)}(y, q) = y + (6 + 4q + q^2)y + (6 + 4q + q^2)y^3 + y^4.$$

Combining the results of Corteel [3], Williams [21, Proposition 4.11] and the classical theory of orthogonal polynomials, one can write the moments of the above q-Laguerre polynomials as a finite double sum (cf. (28)). Here we propose a direct proof of this result. Actually we shall give such a formula for the moments of Al-Salam-Chihara polynomials.

Definition 4. Define the $y$-versions of the q-Stirling numbers of the second kind by

$$X_n = \sum_{k=1}^{n} S_q(n, k, y) \prod_{j=0}^{k-1} (X - [j]_q(1 - yq^{-j})). \tag{18}$$

The y-versions of q-Stirling numbers of the first kind can be defined by the inverse matrix or equivalently

$$\prod_{j=0}^{n-1} (X - [j]_q(1 - yq^{-j})) = \sum_{k=1}^{n} s_q(n, k, y)X^k.$$

Remark 2. We have

$$S_q(n, k, y)|_{q=1} = S(n, k)(1 - y)^{n-k}, \quad S_q(n, k, 0) = S_q(n, k),$$

where $S(n, k)$ and $S_q(n, k)$ are, respectively, the Stirling numbers of the second kind and their well-known q-analogues, see [11].

Consider the rescaled Al-Salam-Chihara polynomials $P_n(x)$:

$$P_n(X) = Q_n(((q - 1)X + 1/\alpha^2 + 1)\alpha/2; \alpha, \beta|q)$$

$$= \alpha^{-n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} q^k(\alpha \beta q^k; q)_{n-k}(1 - q)^k q^{(k)}(\alpha^2)$$

$$\times \prod_{j=0}^{k-1} (X - [j]_q(1 - q^{-j}/\alpha^2)). \tag{19}$$

Theorem 1. The moments of the rescaled Al-Salam-Chihara polynomials $P_n(X)$ are

$$\mu_n(\alpha, \beta) = \sum_{k=1}^{n} S_q(n, k, 1/\alpha^2)(\alpha \beta; q)_k q^{-\binom{k}{2}}(1 - q)^{-k} \alpha^{-2k}.$$
Proof. Let \( L : X^n \to \mu_n(\alpha, \beta) \) be the linear functional. We check that these moments do satisfy \( L(P_n(X)) = 0 \) for \( n > 0 \). Let \( a_k \) be the coefficients in front of the product in (14), then we have, using \( y \)-Stirling orthogonality,

\[
L(P_n(X)) = \sum_{k=0}^{n} a_k \sum_{j=1}^{k} s_q(k, j, 1/\alpha^2) \sum_{t=1}^{j} S_q(j, t, 1/\alpha^2)(\alpha; q)_t q - (1/2)(1 - q)^{-t} \alpha^{-2t}
\]

\[
= \sum_{k=0}^{n} a_k(\alpha; q)_k q^{-k/2} (1 - q)^{-k} \alpha^{-2k}
\]

\[
= \alpha^{-n}(\alpha; q)_n \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} q^k = 0.
\]

Note that the last equality follows by applying the \( q \)-binomial formula. \( \square \)

**Theorem 2.** The generating function for the moments \( \mu_n(\alpha, \beta) \) is

\[
\sum_{n=0}^{\infty} \mu_n(\alpha, \beta)t^n = \sum_{k=0}^{\infty} \frac{(\alpha; q)_k q^{-k/2}(1 - q)^{-k} \alpha^{-2k} t^k}{\prod_{i=1}^{k}(1 - [i]_q t(1 - q^{-i}/\alpha^2))}.
\]  

(20)

**Proof.** By definition (18) we have

\[
S_q(n, k, y) = S_q(n - 1, k - 1, y) + [k]q(1 - yq^{-k})S_q(n - 1, k, y).
\]

It follows that (18) is equivalent to

\[
\sum_{n \geq k} S_q(n, k, y)t^n = \frac{t^k}{\prod_{i=1}^{k}(1 - [i]_q t(1 - q^{-i}y))},
\]  

(21)

which yields immediately (20) in view of Theorem 1. \( \square \)

The moment of \( q \)-Charlier polynomials corresponds to the \( \beta = 0, \alpha = -1/\sqrt{a(1-q)} \) case, while that of \( q \)-Laguerre polynomials corresponds to the \( \alpha = 1/\sqrt{y}, \alpha \beta = q \) case. Therefore,

\[
\sum_{n=0}^{\infty} \mu_n^{(c)}(a, q)t^n = \sum_{k=0}^{\infty} \frac{a(qt)^k}{\prod_{i=1}^{k}(q^i - q^i[i]_q t + a(1 - q)[i]_q t)},
\]

(22)

\[
\sum_{n=0}^{\infty} \mu_n^{(l)}(y, q)t^n = \sum_{k=0}^{\infty} \frac{k!_q(qty)^k}{\prod_{i=1}^{k}(q^i - q^i[i]_q t + [i]_q t y)}.
\]

(23)

**Theorem 3.** Let \( p = 1/q \). We have

\[
\sum_{k=0}^{\infty} \frac{(\alpha; q)_k q^{-k/2}(1 - q)^{-k} \alpha^{-2k} t^k}{\prod_{i=1}^{k}(1 - [i]_q t(1 - q^{-i}/\alpha^2))} = \sum_{i \geq 0} \frac{c_i(\alpha, \beta)}{1 - [i]_q t(1 - q^{-i}/\alpha^2)}.
\]

(24)

where

\[
c_i(\alpha, \beta) = \frac{(\alpha; q)_i}{(q; q)_i} \frac{q^{i-1} \alpha^{-2i}}{(\alpha^{-1} - 2i/\alpha^2; q)_i} \frac{(p^{1+i} \alpha \beta/\alpha^2; p)_\infty}{(p^{1+i+2i/\alpha^2}; p)_\infty}.
\]
Proof. Note the following partial fraction decomposition formula:

\[
\frac{t^k}{(1-a_1t)(1-a_2t)\ldots(1-a_k t)} = \frac{(-1)^k}{a_1 \cdots a_k} + \sum_{i=1}^k \frac{a_i^{-1} \prod_{j=1,j \neq i}^k (a_i - a_j)^{-1}}{1-a_i t}.
\]

Therefore

\[
\prod_{i=1}^k (1-[i]_q t(1-q^{-i}/\alpha^2)) = \sum_{i=0}^k \frac{\gamma_k(i)}{1-[i]_q t(1-q^{-i}/\alpha^2)}.
\]

where

\[
\gamma_k(i) = \frac{1}{k!_q} \left[ \begin{array}{c} k \\ i \end{array} \right]_q \frac{a^{2(k-i)} q_{(2)}^{k-i} q^{-2i}}{q^{(q^2-2i)/\alpha^2} q_i (q^{1+2i/\alpha^2}; q)_{k-i}} \quad (0 \leq i \leq k).
\]

Substituting this in (24) yields

\[
c_i(\alpha, \beta) = \sum_{k \geq i} \frac{(\alpha \beta; q)_k}{(q; q)_i} \left[ \begin{array}{c} k \\ i \end{array} \right]_q \frac{q^{k-i} \alpha^{-2i}}{q (q^{1-2i}/\alpha^2; q)_i (q^{1+2i/\alpha^2}; q)_{k-i}} \frac{(\alpha \beta q^i; q)_k}{(q; q)_k} \frac{q^k}{(q^{1+2i/\alpha^2}; q)_k}
\]

The theorem follows then by applying the \(1\Phi_1\) summation formula (see [10, II.5]).

By partial fraction decomposition (see [21, Theorem 4.12]), we get

\[
\sum_{n=0}^\infty \mu_n^{(c)}(a, q) t^n = \sum_{i \geq 0} \frac{a^i (1-a(1-q)p^{2i})/(a(1-q)p^i; p)^\infty}{q^i q^2 t^{i} (q^i - q^i [i]_q t + [i]_q t)(1-q)},
\]

\[
\sum_{n=0}^\infty \mu_n^{(c)}(y, q) t^n = \sum_{i \geq 0} \frac{y^i (q^{2i} - y)}{q^2 t^{i} (q^i - q^i [i]_q t + [i]_q t y)}.
\]

Note that (27) yields the following polynomial formula in \(y\) for \(\mu_n^{(c)}(y, q)\):

\[
\mu_n^{(c)}(y, q) = \sum_{k=1}^n \sum_{i=0}^{k-1} (-1)^i [k - i] q^i y^{k-i} \left( \binom{n}{i} q^{k-i} + \binom{n}{i} \right) y^k,
\]

while (20) does not yield such a polynomial formula in \(a\) for \(\mu_n^{(c)}(a, q)\).

On the other hand, it follows from (23) and (21) that

\[
S_q(n, k, y) = \frac{q^{-\frac{n}{2}}}{k!_q} \sum_{i=1}^k \left[ \begin{array}{c} k \\ i \end{array} \right]_q y^{-k} q^{-k^2/2} \frac{([i]_q (1-q^{-i} y)^n)}{(q^{1-2i} y; q)_i (q^{1+2i} y; q)_{k-i}}.
\]

Using Theorem 1 and the above explicit formula for q-Stirling numbers we can also write the moments \(\mu_n(\alpha, \beta)\) as a double sum.
5. LINEARIZATION COEFFICIENTS OF THE $q$-LAGUERRE POLYNOMIALS

The following is our main result of this section.

**Theorem 5.** The linearization coefficients of the $q$-Laguerre polynomials are

$$L_q(L_{n_1}(x; q)\ldots L_{n_k}(x; q)) = \sum_{\sigma \in D(n_1,\ldots,n_k)} y^{wex(\sigma)} y^{cr(\sigma)}. \quad (30)$$

A proof à la Viennot (cf. [12, 15]) of (30) would use the combinatorial interpretations for the moments and $q$-Laguerre polynomials to rewrite the left-hand side of (30) and then construct an adequate killing involution on the resulting set. For the time being we do not have such a proof to offer, instead we provide an inductive proof.

We first show that the above result is true for $(n_1, \ldots, n_k) = (1, \ldots, 1)$.

**Lemma 6.** Let $d_n(y, q) = \sum_{\sigma \in D_n} y^{wex(\sigma)} q^{cr(\sigma)}$. Then $L_q((x - y)^n) = d_n(y, q)$.

**Proof.** Note that

$$L_q((x - y)^n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} y^{n-k} \mu_n^{\ell}(y, q).$$

By binomial inversion, it suffices to prove that

$$\mu_n^{\ell}(y, q) = \sum_{k=0}^{n} \binom{n}{k} y^k d_{n-k}(y, q).$$

But the latter identity is obvious. \qed

The invariance of $\sum_{\sigma \in D(n_1,n_2,\ldots,n_k)} y^{wex(\sigma)} q^{cr(\sigma)}$ by permutating the $n_i$’s is a direct consequence of Theorem 5, but for our proof we need to first establish this property.

**Theorem 7.** For any permutation $\gamma \in S_k$ we have

$$\sum_{\sigma \in D(n_1,n_2,\ldots,n_k)} y^{wex(\sigma)} q^{cr(\sigma)} = \sum_{\sigma \in D(n_{\gamma(1)},n_{\gamma(2)},\ldots,n_{\gamma(k)})} y^{wex(\sigma)} q^{cr(\sigma)}.$$

Since the two cyclic permutations $(1, 2)$ and $(1, 2, 3, \ldots, k)$ generate the symmetric group $S_k$, Theorem 7 is a corollary of the following two lemmas (proved in the next two sections).

**Lemma 8.**

$$\sum_{\sigma \in D(n_1,n_2,\ldots,n_k)} y^{wex(\sigma)} q^{cr(\sigma)} = \sum_{\sigma \in D(n_2,n_3,\ldots,n_k,n_1)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

**Lemma 9.**

$$\sum_{\sigma \in D(n_1,n_2,\ldots,n_k)} y^{wex(\sigma)} q^{cr(\sigma)} = \sum_{\sigma \in D(n_2,n_1,n_3,\ldots,n_k)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

**Proof of Theorem 4.** Writing (14) as

$$(x - y)L_n(x) = L_{n+1}(x) + (yq + 1)[n]_q L_n(x) + y[n]_q^2 L_{n-1}(x),$$

we observe that
Theorem 10. 

When \(k\) is \(2\) we obtain

\[
\sum_{\pi \in D(1,n,n_2,\ldots,n_k)} w(\pi) = \sum_{\pi \in D(n+1,n_2,\ldots,n_k)} w(\pi) + (yq + 1)[n]_q \sum_{\pi \in D(n,n_2,\ldots,n_k)} w(\pi) 
+ y[n]_q^2 \sum_{\pi \in D(n-1,n_2,\ldots,n_k)} w(\pi),
\]

(31)

where \(w(\pi) = y^{\text{wex}(\pi)} q^{\text{cr}(\pi)}\). In view of Lemma 3 it suffices to prove (31).

We distinguish four cases for permutations \(\pi \in D(1,n,n_2,\ldots,n_k)\).

a) \(\pi(1), \pi^{-1}(1) \in \{2,\ldots,n+1\}\). Let \(\pi(1) = i\) and \(\pi(j) = 1\) with \(i, j \in \{2,\ldots,n+1\}\). Then we define the mapping \(\pi \rightarrow \pi' \in D(n-1,n_2,\ldots,n_k)\) by deleting \(1\) and \(j\) and adding the edge \(\pi^{-1}(j) \rightarrow i\) if \(i \neq j\). Clearly

\[
w(\pi) = yq^{(i-1)+(j-1)-2} w(\pi').
\]

Summing over all \(i,j \in \{2,\ldots,n+1\}\) yields the generating function:

\[
y[n]_q^2 \sum_{\pi \in D(n-1,n_2,\ldots,n_k)} y^{\text{wex}(\pi)} q^{\text{cr}(\pi)}.
\]

b) \(\pi(1) \in \{2,\ldots,n+1\}\) and \(\pi^{-1}(1) > n+1\). We define the mapping \(\pi \rightarrow \pi' \in D(n,n_2,\ldots,n_k)\) by deleting \(i := \pi(1)\) and replacing the two edges \(1 \rightarrow \pi(1) \rightarrow \pi^2(1)\) by \(1 \rightarrow \pi^2(1)\). Clearly

\[
w(\pi) = yq^{i-1} w(\pi').
\]

Summing over all \(i = 2,\ldots,n+1\) yields the generating function:

\[
qy[n]_q \sum_{\pi \in D(n,n_2,\ldots,n_k)} y^{\text{wex}(\pi)} q^{\text{cr}(\pi)}.
\]

c) \(\pi^{-1}(1) \in \{2,\ldots,n+1\}\) and \(\pi(1) > n+1\). We define the mapping \(\pi \rightarrow \pi' \in D(n,n_2,\ldots,n_k)\) by deleting \(i := \pi^{-1}(1)\) and replacing the two edges \(1 \leftarrow \pi^{-1}(1) \leftarrow \pi^{-2}(1)\) by \(1 \leftarrow \pi^{-2}(1)\). Clearly

\[
w(\pi) = q^{i-2} w(\pi').
\]

Summing over all \(i = 2,\ldots,n+1\) yields the generating function:

\[
[q]_q \sum_{\pi \in D(n,n_2,\ldots,n_k)} y^{\text{wex}(\pi)} q^{\text{cr}(\pi)}.
\]

d) \(\pi(1) > n+1\) and \(\pi^{-1}(1) > n+1\). Clearly we can consider \(\pi\) as a permutation in \(D(n+1,n_2,\ldots,n_k)\). The generating function is

\[
\sum_{\pi \in D(n+1,n_2,\ldots,n_k)} y^{\text{wex}(\pi)} q^{\text{cr}(\pi)}.
\]

Summing up we obtain \(\square\).

When \(k = 2\) Theorem 4 reduces to the orthogonality of the \(q\)-Laguerre polynomials \(\langle \rangle\).

When \(k = 3\) we can derive the following explicit formula from Theorem 1.

Theorem 10. We have

\[
L_q(L_{n_1}(x; q)L_{n_2}(x; q)L_{n_3}(x; q)) = \sum_s \frac{n_1^s q \cdot n_2^s q \cdot n_3^s q \cdot s_q y^s}{(n_1 + n_2 + n_3 - 2s)_q (s - n_3)_q (s - n_2)_q (s - n_1)_q} 
\times \sum_k \left[\frac{n_1 + n_2 + n_3 - 2s}{k}\right] y^k q^{\binom{k+1}{2} + \binom{n_1+n_2+n_3-2s-k}{2}}.
\]
Proof. By Theorem 1 with \( a = \sqrt[4]{y} \) and \( b = \sqrt{y}q \) we have

\[
L_q(\mathcal{L}_{1n_1}(x; q)\mathcal{L}_{2n_2}(x; q)\mathcal{L}_{3n_3}(x; q)) = L_q(\mathcal{L}_{n_3}(x; q)^2) \left( \frac{\sqrt[4]{y}}{q - 1} \right)^{n_1 + n_2 - n_3} \binom{n_3}{n_1, n_2}(a, b; q)
\]

\[
= \sum_{m_2, m_3} \frac{n_1!q n_2!q n_3!q (n_1 + m_3)!q y^{n_2+n_3-m_2-m_3} q^{(m_2)+\binom{n_3+n_2-n_1-m_2-2m_3+1}{2}}}{(n_3 + n_2 - n_1 - m_2 - 2m_3)!q} m_2!q (m_3 + n_1 - n_2)!q m_3!q.
\]

Substituting \( s = n_1 + m_3 \) and \( k = n_3 + n_2 - n_1 - m_2 - 2m_3 \) in the last sum yields the desired formula. \( \square \)

Remark 3. It would be interesting to give a combinatorial proof of the above result as in [12,15]. When \( q = 1 \) such a proof was given in [23].

We end this section with an example. If \( n = (2, 2, 1) \), by Theorem 8 we have

\[
L_q(L_2(x; q)L_2(x; q)L_1(x; q)) = \sum_s \frac{2!q^2!q!s!q y^s}{(5−2s)!q(s−1)!q(s−2)!q} \times \sum_{k \geq 0} \left[ \binom{5−2s}{k} y^{(k+1)+\binom{5−2s−k}{2}} \right] q^{(5−2s−k)}
\]

\[
= (1 + q)^3(1 + qy)y^2.
\] (32)

On the other hand, the sixteen derangements, depicted by their diagrams and the corresponding weights are tabulated as follows:
Summing up we get $\sum_{\sigma \in D(2,2,1)} y^{\text{wex}} q^{\sigma} \sigma = y^2(1 + qy)(1 + q)^3$, which coincides with (32).

6. PROOF OF LEMMA

For each fixed $k \in [n]$ define the two subsets of $S_n$:

$kS_n = \{ \sigma \in S_n \mid \sigma(i) > k \; \text{ for } 1 \leq i \leq k \}$,

$S_n^k = \{ \sigma \in S_n \mid \sigma(n + 1 - i) < n + 1 - k \; \text{ for } 1 \leq i \leq k \}$.

We first construct a simple bijection $\Phi_k : kS_n \rightarrow S_n^k$. Let $\sigma \in S_n^k$. For $1 \leq i \leq n$ we define $\sigma'(i) := \Phi_k(\sigma)(i)$ as follows:

$\sigma'(i) = \begin{cases} 
\sigma(i + k) - k, & \text{if } 1 \leq i \leq n - k \text{ and } \sigma(i + k) > k; \\
\sigma(i + k) + n - k, & \text{if } 1 \leq i \leq n - k \text{ and } \sigma(i + k) \leq k; \\
\sigma(i + k - n) - k, & \text{if } n - k + 1 \leq i \leq n.
\end{cases}$

We can illustrate the map by the diagrams of permutations.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\rightarrow$</th>
<th>$\sigma'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \quad k \quad k+1 \quad i \quad \sigma(i) \quad n$</td>
<td>$\rightarrow$</td>
<td>$1 \quad i-k \quad \sigma(i) - k \quad n-k \quad n$</td>
</tr>
<tr>
<td>$1 \quad k \quad \sigma(i) \quad i \quad n$</td>
<td>$\rightarrow$</td>
<td>$1 \quad \sigma(i) - k \quad i-k \quad n-k \quad n$</td>
</tr>
<tr>
<td>$1 \quad \sigma(i) \quad k \quad i \quad n$</td>
<td>$\rightarrow$</td>
<td>$1 \quad \sigma(i) - k \quad n-k \quad n-k+i \quad n-k+\sigma(i)$</td>
</tr>
<tr>
<td>$1 \quad \sigma(i) \quad k \quad i \quad n$</td>
<td>$\rightarrow$</td>
<td>$1 \quad i-k \quad n-k \quad n-k+i \quad n-k+\sigma(i)$</td>
</tr>
</tbody>
</table>

Table 1. The mapping $\Phi_k : \sigma \rightarrow \sigma'$.

For example, consider the permutation $\sigma \in 3S_{15}$, whose diagram is given below.
Then the diagram of $\Phi_3(\sigma)$ is given by

The main properties of $\Phi_k$ are summarized in the following result.

**Lemma 11.** For each positive integer $k \in [n]$, the map $\Phi_k : {}^kS_n \to {}^kS_n$ is a bijection such that for any $\sigma \in {}^kS_n$ there holds

$$(wex, cr)\Phi_k(\sigma) = (wex, cr)\sigma.$$  \hspace{1cm} (33)

Now, Lemma 8 is an immediate consequence of Lemma 11. Let $n = n_1 + n_2 + \cdots + n_k$. Then $D(n_1, n_2, \ldots, n_k) \subseteq n_1S_n$. By definition of $\Phi_{n_1}$, for any $\sigma \in n_1S_n$ and $i \in [n - n_1]$ satisfying $\sigma(i + n_1) > n_1$, we have $i - \Phi_{n_1}(\sigma)(i) = i + n_1 - \sigma(i + n_1)$, so $\Phi_{n_1}(D(n_1, n_2, \ldots, n_k)) \subseteq D(n_2, n_3, \ldots, n_k, n_1)$. Since the cardinality of $D(n_1, n_2, \ldots, n_k)$ is invariant by permutations of the $n_i$’s and $\Phi_{n_1}$ is bijective, we have $\Phi_{n_1}(D(n_1, n_2, \ldots, n_k)) = D(n_2, n_3, \ldots, n_k, n_1)$. The result follows then by applying (33).

<table>
<thead>
<tr>
<th>$i$</th>
<th>$L_i(\sigma)$</th>
<th>$R_i(\sigma')$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td>2</td>
<td><img src="image3" alt="Diagram" /></td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
<tr>
<td>3</td>
<td><img src="image5" alt="Diagram" /></td>
<td><img src="image6" alt="Diagram" /></td>
</tr>
</tbody>
</table>

**Table 2.** Forms of crossings in $L_i(\sigma)$ and $R_i(\sigma')$.

**Proof of Lemma 11** It is easy to see that $\Phi_k$ is a bijection. Let $\sigma \in {}^kS_n$ and $\sigma' = \Phi_k(\sigma)$. The equality $wex(\sigma') = wex(\sigma)$ follows directly from the definition of $\Phi_k$. It then remains to prove
that \( cr(\sigma') = cr(\sigma) \). We first decompose the crossings of \( \sigma \) and \( \sigma' \) into three subsets. Set

\[
\begin{align*}
L_1(\sigma) &= \{(i, j) \mid k < i < j \leq \sigma(i) < \sigma(j) \quad \text{or} \quad i > j > \sigma(i) > \sigma(j) > k\}, \\
L_2(\sigma) &= \{(i, j) \mid i < j \leq k < \sigma(i) < \sigma(j) \quad \text{or} \quad i > j > k \geq \sigma(i) > \sigma(j)\}, \\
L_3(\sigma) &= \{(i, j) \mid i \leq k < j \leq \sigma(i) < \sigma(j) \quad \text{or} \quad i > j > \sigma(i) > k \geq \sigma(j)\},
\end{align*}
\]

and

\[
\begin{align*}
R_1(\sigma') &= \{(i, j) \mid i < j \leq \sigma'(i) < \sigma'(j) \leq n - k \quad \text{or} \quad n - k \geq i > j > \sigma'(i) > \sigma'(j)\}, \\
R_2(\sigma') &= \{(i, j) \mid i < j \leq n - k < \sigma'(i) < \sigma'(j) \quad \text{or} \quad i > j > n - k \geq \sigma'(i) > \sigma'(j)\}, \\
R_3(\sigma') &= \{(i, j) \mid i < j \leq \sigma'(i) \leq n - k < \sigma'(j) \quad \text{or} \quad i > n - k \geq j > \sigma(i) > \sigma(j)\}.
\end{align*}
\]

The “forms” of the crossings in the \( L_i \)'s and \( R_i \)'s is given in Table 3. Clearly, we have \( cr(\sigma) = \sum_{i=1}^{3} |L_i(\sigma)| \) and \( cr(\sigma') = \sum_{i=1}^{3} |R_i(\sigma')| \) since \( \sigma \in S_n \) and \( \sigma' \in S_n' \).

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<th>\sigma</th>
<th>\longrightarrow</th>
<th>\sigma'</th>
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<tr>
<td>( k )</td>
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<td>( \sigma_i )</td>
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Table 3. Effects of the mapping \( \Phi_k \) on the crossings of \( \sigma \) and \( \sigma' \).
By the definition of $\Phi_k$, it is readily seen (see Row 1 in Table 3) that $(i, j) \in L_1(\sigma)$ if and only if $(i - k, j - k) \in R_1(\sigma')$, and thus $|L_1(\sigma)| = |R_1(\sigma')|$. Similarly, we have (see Row 2 in Table 3) that $|L_2(\sigma)| = |R_2(\sigma')|$. It then remains to prove that $|L_3(\sigma)| = |R_3(\sigma')|$. Let

$$L_4(\sigma) = \{(i, j) \mid \sigma(i) \leq k < j < i \leq \sigma(j) \quad \text{or} \quad i \leq k < \sigma(j) < \sigma(i) < j\}.$$  

Then it is not difficult to show (see Row 4 of Table 3) that $|R_3(\sigma')| = |L_4(\sigma)|$. It then suffices to prove that $|L_3(\sigma)| = |L_4(\sigma)|$.

Suppose $\sigma([1, k]) = \{i_1, i_2, \ldots, i_k\}$ and $\sigma^{-1}([1, k]) = \{j_1, j_2, \ldots, j_k\}$. Then by definition of $L_3(\sigma)$ and $L_4(\sigma)$ we have

$$|L_3(\sigma)| = \sum_{s=1}^{k} |\{\ell \mid k < \ell \leq i_s < \sigma(\ell)\}| + |\{\ell \mid \ell > j_s > \sigma(\ell) > k\}|,$$

$$|L_4(\sigma)| = \sum_{s=1}^{k} |\{\ell \mid \ell > i_s > \sigma(\ell) > k\}| + |\{\ell \mid k < \ell < j_s \leq \sigma(\ell)\}|.$$  

(34)

(35)

For any integer $i \in [n]$ set $A_i(\sigma) = \{j \mid j < i < \sigma(j)\}$. Note that it is easily seen that

$$|A_i(\sigma)| = |\{j \mid j < i < \sigma(j)\}| = |\{j \mid j > i > \sigma(j)\}| = |A_i(\sigma^{-1})|.$$  

(36)

Let $s \in [k]$. By elementary manipulations we get

$$|\{\ell \mid k < \ell \leq i_s < \sigma(\ell)\}| = |\{\ell \mid \ell \leq \ell < \sigma(\ell)\}| - |\{\ell \mid \ell \leq k < i_s < \sigma(\ell)\}|$$

$$= |A_{i_s}(\sigma)| + \chi(i_s < \sigma(i_s)) - |\{\ell \mid \ell \leq k < i_s < \sigma(\ell)\}|$$

$$= |A_{i_s}(\sigma)| + \chi(i_s < \sigma(i_s)) - |\{t \mid i_t > i_s\}|.$$  

(37)

By a similar reasoning, we obtain the following identities:

$$|\{\ell \mid \ell > j_s > \sigma(\ell) > k\}| = |A_{j_s}(\sigma^{-1})| - |\{t \mid j_t > j_s\}|.$$  

(38)

$$|\{\ell \mid \ell > i_s > \sigma(\ell) > k\}| = |A_{i_s}(\sigma^{-1})| - |\{t \mid j_t > i_s\}|$$

(39)

$$|\{\ell \mid k < \ell < j_s \leq \sigma(\ell)\}| = |A_{j_s}(\sigma)| + \chi(k < \sigma^{-1}(j_s) < j_s) - |\{t \mid i_t > j_s\}|.$$  

(40)

Inserting (37) and (38) in (34) and using (36), we get

$$|L_3(\sigma)| = \sum_{s=1}^{k} |A_{i_s}(\sigma)| + |A_{j_s}(\sigma)| + \chi(i_s < \sigma(i_s)) - |\{t \mid i_t > i_s\}| - |\{t \mid j_t > j_s\}|.$$  

(41)

Similarly, inserting (39) and (40) in (35) and using (36), we get

$$|L_4(\sigma)| = \sum_{s=1}^{k} |A_{i_s}(\sigma)| + |A_{j_s}(\sigma)| + \chi(k < \sigma^{-1}(j_s) < j_s) - |\{t \mid j_t > i_s\}| - |\{t \mid i_t > j_s\}|.$$  

(42)

Since the $i_t$’s and $j_t$’s are distinct we have $\sum_{s=1}^{k} |\{t \mid i_t > i_s\}| = \sum_{s=1}^{k} |\{t \mid j_t > j_s\}| = \binom{k}{2}$ and thus

$$\sum_{s=1}^{k} |\{t \mid i_t > i_s\}| + |\{t \mid j_t > j_s\}| = k(k - 1).$$  

(43)
On the other hand,
\[
\sum_{s=1}^{k} |\{(t \mid j_t > i_s)\}| + |\{(t \mid i_t > j_s)\}| = \sum_{s=1}^{k} |\{(t \mid i_t \neq j_s)\}|
\]
\[
= k^2 - \sum_{s=1}^{k} \chi(j_s \in \{i_1, i_2, \ldots, i_k\})
\]
\[
= k^2 - \sum_{s=1}^{k} \chi(\sigma^{-1}(j_s) \leq k).
\]
(44)

Also, it follows from the definition of the \(j_i\)’s that for any \(s \in [k]\), we have \(j_s > k\) and \(\sigma^{-1}(j_s) \neq j_s\), and thus
\[
\chi(k < \sigma^{-1}(j_s) < j_s) + \chi(\sigma^{-1}(j_s) > j_s) = \chi(\sigma^{-1}(j_s) > k).
\]
(45)

Inserting (53), (54) and (45) in (41) and (42) lead to \(|L_3(\sigma)| = |L_4(\sigma)|\) as desired.

7. PROOF OF LEMMA 3

For any two integers \(n_1, n_2\) satisfying \(N_2 := n_1 + n_2 \leq n\) we denote by \(S_n^{(n_1, n_2)}\) the set of permutations \(\sigma\) in \(S_n\) such that
\[
(i, \sigma(i)) \notin [1, n_1]^2 \cup [n_1 + 1, N_2]^2.
\]

In other words, any two integers in \([1, n_1]\) or \([n_1 + 1, N_2]\) are not connected by an arc in its graph.

We now construct a map \(\Gamma^{(n_1, n_2)} : \sigma \mapsto \sigma'\) from \(S_n^{(n_1, n_2)}\) to \(S_n^{(n_2, n_1)}\) as follows. For \(i = 1, \ldots, n\),
(1) If \(i > N_2\) and \(\sigma(i) > N_2\), set \(\sigma'(i) = \sigma(i)\).
(2) Suppose
\[
\{(i, \sigma(i)) \mid i < \sigma(i) \leq N_2\} = \{(i_1, N_2 + 1 - j_1), (i_2, N_2 + 1 - j_2), \ldots, (i_p, N_2 + 1 - j_p)\}
\]
\[
\{(\sigma(i), i) \mid \sigma(i) < i \leq N_2\} = \{(k_1, N_2 + 1 - \ell_1), (k_2, N_2 + 1 - \ell_2), \ldots, (k_q, N_2 + 1 - \ell_q)\}.
\]
Then set \(\sigma'(j_s) = N_2 + 1 - i_s\) and \(\sigma'(N_2 + 1 - k_t) = \ell_t\) for any \(s \in [p]\) and \(t \in [q]\).
(3) Let
\[
C = \{i \in [1, N_2] : \sigma(i) > N_2\} \quad \text{and} \quad D = \{i \in [1, N_2] : \sigma^{-1}(i) > N_2\}.
\]
It is easy to see that \(|C| = |D|\). Suppose \(C = \{c_1, c_2, \ldots, c_u\} < D = \{d_1, d_2, \ldots, d_u\} < \sigma(C) = \{r_1, r_2, \ldots, r_u\} < \sigma^{-1}(D) = \{s_1, s_2, \ldots, s_u\} < \). Let \(\alpha, \beta \in S_u\) be the (unique) permutations satisfying \(\sigma(c_i) = r_{\alpha(i)}\) and \(\sigma^{-1}(d_i) = s_{\beta(i)}\) for each \(1 \leq i \leq u\). Let
\[
E = [1, N_2] \setminus \{j_1, \ldots, j_p, N_2 + 1 - k_1, \ldots, N_2 + 1 - k_q\}
\]
\[
F = [1, N_2] \setminus \{N_2 + 1 - i_1, \ldots, N_2 + 1 - i_p, \ell_1, \ldots, \ell_q\}.
\]
Clearly, we have \(|E| = |C|\) and \(|F| = |D|\). Suppose \(E = \{e_1, \ldots, e_u\} < F = \{f_1, \ldots, f_u\} < \). Then set \(\sigma'(e_i) = r_{\alpha(i)}\) and \(\sigma'(s_i) = f_{\beta(i)}\) for each \(1 \leq i \leq u\).

We can illustrate the map through the diagrams of permutations. See Table 3.

For example, if we consider the permutation in \(S_{15}^{(3,4)}\) whose diagram is given by
\[ \sigma \longrightarrow \sigma' \]

<table>
<thead>
<tr>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( \sigma )</th>
<th>( \sigma' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( N_2 )</td>
<td>( \sigma(i) )</td>
<td>( N_2 )</td>
</tr>
<tr>
<td>1</td>
<td>( N_2 )</td>
<td>( \sigma(i) )</td>
<td>( N_2 )</td>
</tr>
</tbody>
</table>

\[
\begin{array}{c}
1 \quad n_i \quad N_2 + 1 - n_i \\
1 \quad k_i \quad N_2 + 1 - k_i \\
1 \quad e_j \quad N_2 \\
1 \quad e_j \quad N_2
\end{array}
\]

\[
\begin{array}{c}
1 \quad n_i \quad N_2 + 1 - j_i \\
1 \quad n_i \quad N_2 + 1 - f_i \\
1 \quad f_i \quad N_2 + 1 - k_i \\
1 \quad f_i \quad N_2
\end{array}
\]

Table 4. The mapping \( \Gamma^{(n_1, n_2)} : \sigma \mapsto \sigma' \)

then the diagram of \( \Gamma^{(n_1, n_2)}(\sigma) \) is given by

It is not hard to check that \( \Gamma^{(n_1, n_2)} \) is well defined from \( S_n^{(n_1, n_2)} \) to \( S_n^{(n_2, n_1)} \). Since each step of the construction of \( \Gamma^{(n_1, n_2)} \) is reversible, the map \( \Gamma^{(n_1, n_2)} \) is bijective. Actually we can prove, the details are left to the reader, that \( (\Gamma^{(n_1, n_2)})^{-1} = \Gamma^{(n_2, n_1)} \).

**Lemma 12.** For each positive integers \( n_1, n_2, n, \) with \( N_2 \leq n \), the map \( \Gamma^{(n_1, n_2)} \) is a bijection from \( S_n^{(n_1, n_2)} \) to \( S_n^{(n_2, n_1)} \) such that for each \( \sigma \in S_n^{(n_1, n_2)} \), we have

\[
(wex, cr)\Gamma^{(n_1, n_2)}(\sigma) = (wex, cr)\sigma.
\] (46)

As an immediate consequence, we obtain Lemma 9. Let \( n = n_1 + n_2 + \cdots + n_k \). Then \( D(n_1, n_2, \ldots, n_k) \subseteq S_n^{(n_1, n_2)} \). By definition of \( \Gamma^{(n_1, n_2)} \), for any \( \sigma \in S_n^{(n_1, n_2)} \) and \( i > N_2 \) satisfying
\(\sigma(i) > N_2\), we have

\[ i - \Gamma^{(n_1,n_2)}(\sigma)(i) = i - \sigma(i), \]

so \(\Gamma^{(n_1,n_2)}(\mathcal{D}(n_1,n_2,\ldots,n_k)) \subseteq \mathcal{D}(n_2,n_3,\ldots,n_k,n_1)\). Since the cardinality of \(\mathcal{D}(n_1,n_2,\ldots,n_k)\) doesn’t depend of the order of the \(n_i\)’s and \(\Gamma^{(n_1,n_2)}\) is a bijection, we have

\[ \Gamma^{(n_1,n_2)}(\mathcal{D}(n_1,n_2,\ldots,n_k)) = \mathcal{D}(n_2,n_3,\ldots,n_k,n_1). \]

Lemma 9 then follows from (46).

<table>
<thead>
<tr>
<th>(i)</th>
<th>(G_i^{(n_1,n_2)}(\gamma))</th>
<th>(G_i^{(n_2,n_1)}(\gamma))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td>2</td>
<td><img src="image3" alt="Diagram" /></td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
<tr>
<td>3</td>
<td><img src="image5" alt="Diagram" /></td>
<td><img src="image6" alt="Diagram" /></td>
</tr>
<tr>
<td>4</td>
<td><img src="image7" alt="Diagram" /></td>
<td><img src="image8" alt="Diagram" /></td>
</tr>
<tr>
<td>5</td>
<td><img src="image9" alt="Diagram" /></td>
<td><img src="image10" alt="Diagram" /></td>
</tr>
</tbody>
</table>

**Table 5.** Forms of the crossings in \(G_i^{(n_1,n_2)}(\gamma)\) and \(G_i^{(n_2,n_1)}(\gamma)\).

**Proof of Lemma** It was shown above that \(\Gamma^{(n_1,n_2)}\) is bijective. Let \(\sigma \in S_{n_1,n_2}^{(n_1,n_2)}\) and \(\sigma' := \Gamma^{(n_1,n_2)}(\sigma)\). The equality \(\text{we}x(\sigma') = \text{we}x(\sigma)\) is an immediate consequence of the definition
Lemma 13. It then remains to prove that \( cr(\sigma') = cr(\sigma) \). The idea is the same than for the proof of Lemma \( \text{[13]} \). We first decompose the number of crossings of \( \sigma \) and \( \sigma' \). For each permutation \( \gamma \in S_n \), set

\[
G_{1}^{(n_1,n_2)}(\gamma) = \{(i,j) \mid N_2 < i < j \leq \gamma(i) < \gamma(j) \} \quad \text{or} \quad \gamma > \gamma(i) > \gamma(j) > N_2, \]
\[
G_{2}^{(n_1,n_2)}(\gamma) = \{(i,j) \mid i < j < \gamma(i) < \gamma(j) \leq N_2 \} \quad \text{or} \quad N_2 \geq i > j > \gamma(i) > \gamma(j), \]
\[
G_{3}^{(n_1,n_2)}(\gamma) = \{(i,j) \mid i < \gamma(i) < \gamma(j) \leq N_2 \} \quad \text{or} \quad i > j > \gamma(i) > \gamma(j), \]
\[
G_{4}^{(n_1,n_2)}(\gamma) = \{(i,j) \mid \gamma(i) < \gamma(j) \leq N_2 \} \quad \text{or} \quad i > \gamma(i) > N_2 \geq \gamma(j), \]
\[
G_{5}^{(n_1,n_2)}(\gamma) = \{(i,j) \mid i < \gamma(i) \leq N_2 < \gamma(j) \} \quad \text{or} \quad i > N_2 \geq j > \gamma(i) > \gamma(j). \]

Clearly, for any \( \gamma \in S_n^{(n_1,n_2)} \), we have \( cr(\gamma) = \sum_{i=1}^{5} |G_{i}^{(n_1,n_2)}(\gamma)| \). In particular,

\[
|G_{i}^{(n_1,n_2)}(\sigma)| = |G_{i}^{(n_2,n_1)}(\sigma')| \quad \text{and} \quad cr(\sigma) = \sum_{i=1}^{5} |G_{i}^{(n_1,n_2)}(\sigma)| \quad \text{and} \quad cr(\sigma') = \sum_{i=1}^{5} |G_{i}^{(n_2,n_1)}(\sigma')|. \tag{47} \]

The “forms” of the crossings in the \( G_{i}^{(n_1,n_2)} \)’s and \( G_{i}^{(n_2,n_1)} \)’s are given in Table \( \text{[13]} \). By the definition of \( (n_1,n_2) \), it is readily seen (see Row 1 in Table \( \text{[13]} \) that \( G_{i}^{(n_1,n_2)}(\sigma) = G_{i}^{(n_2,n_1)}(\sigma') \) and thus \( |G_{i}^{(n_1,n_2)}(\sigma)| = |G_{i}^{(n_2,n_1)}(\sigma')| \). By similar considerations we can prove (see Table \( \text{[13]} \) that \( |G_{i}^{(n_1,n_2)}(\sigma)| = |G_{i}^{(n_2,n_1)}(\sigma')| \) for \( i = 2, 3, 4 \). It then suffices to prove that \( |G_{5}^{(n_1,n_2)}(\sigma)| = |G_{5}^{(n_2,n_1)}(\sigma')| \) which will follow from the following lemma.

Lemma 13. Let \( n_1, n_2, n \) be positive integers with \( N_2 \leq n \) and \( \gamma \in S_n^{(n_1,n_2)} \). Suppose that

\[
B(\gamma) := \{(i, \gamma(i)) \mid i < \gamma(i) \leq N_2\} = \{(i_1, \gamma_1), (i_2, \gamma_2), \ldots, (i_p, \gamma_p)\}
\[
B(\gamma^{-1}) := \{(\gamma(i), i) \mid \gamma(i) < i \leq N_2\} = \{(k_1, \ell_1), (k_2, \ell_2), \ldots, (k_q, \ell_q)\},
\]

with \( i_1 < i_2 < \cdots < i_p \) and \( k_1 < k_2 < \cdots < k_q \). Then we have

\[
|G_5^{(n_1,n_2)}(\gamma)| = \sum_{r=1}^{p} (j_r - i_r) + \sum_{r=1}^{q} (\ell_r - k_r - 1) - \binom{p + q}{2}. \tag{48} \]

Suppose

\[
B(\sigma) = \{(i_1, N_2 + 1 - j_1), \ldots, (i_p, N_2 + 1 - j_p)\}
\]
\[
B(\sigma^{-1}) = \{(k_1, N_2 + 1 - \ell_1), \ldots, (k_q, N_2 + 1 - \ell_q)\},
\]

then, by construction of \( \sigma' \), we have

\[
B(\sigma') = \{(j_1, N_2 + 1 - i_1), \ldots, (j_p, N_2 + 1 - i_p)\}
\]
\[
B(\sigma^{-1}) = \{(\ell_1, N_2 + 1 - k_1), \ldots, (\ell_q, N_2 + 1 - k_q)\}. \]

By symmetry, the identity \( \text{[18]} \) is also valid on \( S_n^{(n_2,n_1)} \). Applying \( \text{[18]} \) to \( \sigma' \) and \( \sigma \) lead to

\[
|G_5^{(n_1,n_2)}(\sigma)| = |G_5^{(n_1,n_2)}(\sigma')|. \]

This conclude the proof of Lemma \( \text{[12]} \). It then remains to prove Lemma \( \text{[13]} \).
Proof of Lemma 13 By definition of $G_{5}^{(n_{1},n_{2})}(\gamma)$, we have

$$|G_{5}^{(n_{1},n_{2})}(\gamma)| = |\{(i,j) \mid i < j < \gamma(i) \leq N_{2} < \gamma(j)\}| + |\{(i,j) \mid \gamma(j) < \gamma(i) < j \leq N_{2} < i\}|$$
$$+ |\{i \mid i < \gamma(i) \leq N_{2} < \gamma^{2}(i)\}|.$$  

(49)

Now, by elementary manipulations and the definition of $B(\gamma)$ we get

$$|\{(i,j) \mid i < j < \gamma(i) \leq N_{2} < \gamma(j)\}| = \sum_{r=1}^{p} |\{x \mid i_{r} < x < j_{r} \leq N_{2} < \gamma(x)\}|$$
$$= \sum_{r=1}^{p} |\{x \mid i_{r} < x < j_{r}\}| - |\{x \mid i_{r} < x < j_{r}, \gamma(x) \leq N_{2}\}|.$$
But for any \( r \in [1, p] \), we have \(|\{ x | i_r < x < j_r \}| = j_r - i_r - 1\) and 
\[ |\{ x | i_r < x < j_r, \gamma(x) \leq N_2 \}| \]

\[ = |\{ x | i_r < x < j_r, x < \gamma(x) \leq N_2 \}| + |\{ x | i_r < x < j_r, \gamma(x) < x \leq N_2 \}| \]

\[ = |\{ t | \gamma(t) < j_r \}| + |\{ t | i_r < \ell_t < j_r \}| \]

by definition of \( B(\gamma) \) and \( B(\gamma^{-1}) \).

| (i, j) | \( i < j < \gamma(i) \leq N_2 < \gamma(j) \) | \( = \sum_{r=1}^{p} (j_r - i_r - 1) - \sum_{r=1}^{p} |\{ t | i_r < i_t \}| - \sum_{r=1}^{p} |\{ t | \ell_t < j_r \}| \) \]  \( (50) \)

Since \(|\{ (i, j) | \gamma(j) < \gamma(i) < j \leq N_2 < i \}| = |\{ (i, j) | i < j < \gamma^{-1}(i) \leq N_2 < \gamma^{-1}(j) \}| \), it follows from (50) that 
\[ |\{ (i, j) | \gamma(j) < \gamma(i) < j \leq N_2 < i \}| = \sum_{r=1}^{q} (\ell_r - k_r - 1) - \sum_{r=1}^{q} |\{ t | k_r < k_t \}| - \sum_{r=1}^{q} |\{ t | j_t < \ell_r \}| \]

\[ (51) \]

Remarking that \(|\{ i | i < \gamma(i) \leq N_2 < \gamma^2(i) \}| = |\{ t | \gamma(j_t) > N_2 \}| \) and inserting (50) and (51) in (19) lead to 
\[ |G_{n_r}^{(n_1, n_2)}(\gamma)| = \sum_{r=1}^{p} (j_r - i_r - 1) + \sum_{r=1}^{q} (\ell_r - k_r - 1) + |\{ t | \gamma(j_t) > N_2 \}| - \sum_{r=1}^{p} |\{ t | i_r < i_t \}| 
- \sum_{r=1}^{p} |\{ t | \ell_t < j_r \}| - \sum_{r=1}^{q} |\{ t | k_r < k_t \}| - \sum_{r=1}^{q} |\{ t | j_t < \ell_r \}| \]

\[ (52) \]

Since the \( i_r \)'s and \( k_r \)'s are distinct we have 
\[ \sum_{r=1}^{p} |\{ t | i_r < i_t \}| = \left( \frac{p}{2} \right) \quad \text{and} \quad \sum_{r=1}^{q} |\{ t | k_r < k_t \}| = \left( \frac{q}{2} \right). \]

\[ (53) \]

On the other hand, 
\[ |\{ t | \ell_t < j_r \}| + \sum_{r=1}^{q} |\{ t | j_t < \ell_r \}| = \sum_{r=1}^{p} |\{ t | \ell_t \neq j_r \}| \]

\[ = pq - |\{ t | j_t \in \{ \ell_1, \ell_2, \ldots, \ell_q \}| \]

\[ = pq - \sum_{s=1}^{k} |\{ t | \gamma(j_t) \leq N_2 \}|, \]

\[ (54) \]

where the last identity follows from the definitions of \( B(\gamma) \) and \( B(\gamma^{-1}) \). Inserting (53) and (54) in (12) lead to (18). This concludes the proof of Lemma 13. \( \square \)
References
