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A NON-HOMOGENEOUS ORBIT CLOSURE OF A DIAGONAL SUBGROUP

FRANÇOIS MAUCOURANT

Abstract. Let $G = \text{SL}(n, \mathbb{R})$ with $n \geq 6$. We construct examples of lattices $\Gamma \subset G$, subgroups $A$ of the diagonal group $D$ and points $x \in G/\Gamma$ such that the closure of the orbit $Ax$ is not homogeneous and such that the action of $A$ does not factor through the action of a one-parameter non-unipotent group. This contradicts a conjecture of Margulis.

1. Introduction

1.1. Topological rigidity and related questions. Let $G$ be a real Lie group, $\Gamma$ a lattice in $G$, meaning a discrete subgroup of finite covolume, and $A$ a closed connected subgroup. We are interested in the action of $A$ on $G/\Gamma$ by left multiplication; we will restrict ourselves to the topological properties of these actions, referring the reader to [4] and [5] for references and recent developments on related measure theoretical problems.

Two linked questions arise when one studies continuous actions of topological groups: what are the closed invariant sets, and what are the orbit closures?

In the homogeneous action setting we are considering, there is a class of closed sets that admit a simple description: a closed subset $X \subset G$ is said to be homogeneous if there exists a closed connected subgroup $H \subset G$ such that $X = Hx$ for some (and hence every) $x \in X$. Let us say that the action of $A$ on $G/\Gamma$ is topologically rigid if for any $x \in G/\Gamma$, the closure $\overline{Ax}$ of the orbit $Ax$ is homogeneous.

The most basic example of a topologically rigid action is when $G = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$, $A$ any vector subspace of $G$. It turns out that the behavior of elements of $A$ for the adjoint action on the Lie algebra $\mathfrak{g}$ of $G$ plays a important role for our problem. Recall that an element $g \in G$ is said to be $\text{Ad}$-unipotent if $\text{Ad}(g)$ is unipotent, and $\text{Ad}$-split over $\mathbb{R}$ if $\text{Ad}(g)$ is diagonalizable over $\mathbb{R}$. If the closed, connected subgroup $A$ of $G$ is generated by $\text{Ad}$-unipotent elements, a celebrated theorem of Ratner [13] asserts that the action of $A$ is always topologically rigid, settling a conjecture due to Raghunathan.

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When $A$ is generated by elements which are $\text{Ad}$-split over $\mathbb{R}$, much less is known. Consider the model case of $G = \text{SL}(n, \mathbb{R})$ and $A$ the group of diagonal matrices with nonnegative entries. If $n = 2$, it is easy to produce nonhomogeneous orbit closures (see e.g. [7]); more generally, a similar phenomenon can be observed when $A$ is a one-parameter subgroup of the diagonal group (see [3], 4.1). However, for $A$ the full diagonal group, if $n \geq 3$, to the best of our knowledge, the only nontrivial example of a nonhomogeneous $A$-orbit closure is due to Rees, later generalized in [7]. In an unpublished preprint, Rees exhibited a lattice $\Gamma$ of $G = \text{SL}(3, \mathbb{R})$ and a point $x \in G/\Gamma$ such that for the full diagonal group $A$, the orbit closure $Ax$ is not homogeneous. Her construction was based on the following property of the lattice: there exists a $\gamma \in \Gamma \cap A$ such that the centralizer $C_G(\gamma)$ of $\gamma$ is isomorphic to $\text{SL}(2, \mathbb{R}) \times \mathbb{R}^*$, and such that $C_G(\gamma) \cap \Gamma$ is, in this product decomposition and up to finite index, $\Gamma_0 \times \langle \gamma \rangle$, where $\Gamma_0$ is a lattice in $\text{SL}(2, \mathbb{R})$ (see [4], [7]). Thus in this case the action of $A$ on $C_G(\gamma) / C_G(\gamma) \cap \Gamma$ factors to the action of a 1-parameter non-unipotent subgroup on $\text{SL}(2, \mathbb{R}) / \Gamma_0$, which, as we saw, has many non-homogeneous orbits.

Rees’ example shows that factor actions of 1-parameter non-$\text{Ad}$-unipotent groups are obstructions to the topological rigidity of the action of diagonal subgroups. The following conjecture of Margulis [8, conjecture 1.1] (see also [6, 4.4.11]) essentially states that these are the only ones:

**Conjecture 1.** Let $G$ be a connected Lie group, $\Gamma$ a lattice in $G$, and $A$ a closed, connected subgroup of $G$ generated by $\text{Ad}$-split over $\mathbb{R}$ elements. Then for any $x \in G/\Gamma$, one of the following holds:

(a) $\overline{Ax}$ is homogeneous, or

(b) There exists a closed connected subgroup $F$ of $G$ and a continuous epimorphism $\phi$ of $F$ onto a Lie group $L$ such that
   - $A \subset F$,
   - $Fx$ is closed in $G/\Gamma$,
   - $\phi(F_x)$ is closed in $L$, where $F_x$ denotes the stabilizer $\{g \in F | gx = x\}$,
   - $\phi(A)$ is a one-parameter subgroup of $L$ containing no nontrivial $\text{Ad}_L$-unipotent elements.

A first step toward this conjecture has been done by Lindenstrauss and Weiss [7], who proved that in the case $G = \text{SL}(n, \mathbb{R})$ and $A$ the full diagonal group, if the closure of a $A$-orbit contains a compact $A$-orbit that satisfy some irrationality conditions, then this closure is homogeneous. See also [8]. Recently, using an approach based on measure theory, Einsiedler, Katok and Lindenstrauss proved that if moreover $\Gamma = \text{SL}(n, \mathbb{Z})$, then the set of bounded $A$-orbits has Hausdorff dimension $n - 1$ [7, Theorem 10.2].

1.2. **Statement of the results.** In this article we exhibit some counterexamples to the above conjecture when $G = \text{SL}(n, \mathbb{R})$ for $n \geq 6$ and $A$ is some strict subgroup of the diagonal group of matrices with nonnegative entries. Let $D$ be
the diagonal subgroup of $G$; note that $D$ has dimension $n - 1$. Our main result is:

**Theorem 1.** Assume $n \geq 6$.

1. There exists a $(n - 3)$ dimensional closed and connected subgroup $A$ of $D$, and a point $x \in \text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})$ such that the closure of the $A$-orbit of $x$ satisfies neither condition (a) nor condition (b) of the conjecture.

2. There exists a lattice $\Gamma$ of $\text{SL}(n, \mathbb{R})$, a $(n - 2)$ dimensional closed and connected subgroup $A$ of $D$ and a point $x \in \text{SL}(n, \mathbb{R})/\Gamma$ such that the closure of the $A$-orbit of $x$ satisfies neither condition (a) nor condition (b) of the conjecture.

It will be clear from the proofs that these examples however satisfy a third condition:

3. There exists a closed connected subgroup $F$ of $G$ and two continuous epimorphisms $\phi_1, \phi_2$ of $F$ onto Lie groups $L_1, L_2$ such that
   - $A \subset F$,
   - $F \cdot x$ is closed in $G/\Gamma$,
   - For $i = 1, 2$, $\phi_i(F \cdot x)$ is closed in $L_i$,
   - $(\phi_1, \phi_2) : F \to L_1 \times L_2$ is surjective
   - $(\phi_1, \phi_2) : A \to \phi_1(A) \times \phi_2(A)$ is not surjective.

Construction of these examples is the subject of Section 2, whereas the proof that they satisfy the required properties is postponed to Section 3.

1.3. **Toral endomorphisms.** To conclude this introduction, we would like to mention that the idea behind this construction can be also used to yield examples of 'non-homogeneous' orbits for diagonal toral endomorphisms.

Let $1 < p_1 < \cdots < p_q$, with $q \geq 2$, be integers generating a multiplicative non-lacunary semigroup of $\mathbb{Z}$ (that is, the $\mathbb{Q}$-subspace $\oplus_{1 \leq i \leq q} \mathbb{Q} \log(p_i)$ has dimension at least 2). We consider the abelian semigroup $\Omega$ of endomorphisms of the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ generated by the maps $z \mapsto p_i z \mod \mathbb{Z}^n$, $1 \leq i \leq q$.

In the one-dimensional situation, described by Furstenberg [5], every $\Omega$-orbit is finite or dense. If $n \geq 2$, Berend [1] showed that minimal sets are the finite orbits of rational points, but there are others obvious closed $\Omega$-invariant sets, namely the orbits of rational affine subspaces. Meiri and Peres [10] showed that closed invariant sets have integer Hausdorff dimension.

Note that the study of the orbit of a point lying in a proper rational affine subspace reduces to the study of finitely many orbits in lower dimensional tori, although some care must be taken about the pre-periodic part of the rational affine subspace (for example, if $q = n = 2$, and if $\alpha \in T^1$ is irrational with non-dense $p_1$-orbit, the orbit closure of the point $(\alpha, 1/p_2) \in T^2$ is the union of a horizontal circle and a finite number of strict closed infinite subsets of some horizontal circles).
With this last example in mind, Question 5.2 of [10] can be re-formulated: is a proper closed invariant set necessarily a subset of a finite union of rational affine tori? Or, equivalently, if a point is outside any rational affine subspace, does it necessarily have a dense orbit? It turns out that this is not the case at least for \( n \geq 2q \), as the following example shows.

**Theorem 2.** Let \( N \) be an integer greater than \( q^{\frac{\log p_2}{\log p_1}} \), and let \( z \) be the point in the \( 2q \)-dimensional torus \( T^{2q} \) defined by the coordinates modulo 1:

\[
z = (z_1, \ldots, z_{2q}) = \left( \sum_{k \geq 1} p_1^{-N^{2k}}, \ldots, \sum_{k \geq 1} p_q^{-N^{2k}}, \sum_{k \geq 1} p_1^{-N^{2k+1}}, \ldots, \sum_{k \geq 1} p_q^{-N^{2k+1}} \right).
\]

Then the point \( z \in T^{2q} \) is not contained in any rational affine subspace, but its orbit \( \Omega z \) is not dense.

The proof of Theorem 2 will be the subject of Section 4.

2. Sketch of proof of Theorem 1

2.1. The direct product setup. We now describe how these examples are built. Choose two integers \( n_1 \geq 3, n_2 \geq 3 \), such that \( n_1 + n_2 = n \). For \( i = 1, 2 \), let \( \Gamma_i \) be a lattice in \( G_i = \text{SL}(n_i, \mathbb{R}) \).

Let \( g_i \) be an element of \( G_i \) such that \( g_i \Gamma_i g_i^{-1} \) intersects the diagonal subgroup \( D_i \) of \( \text{SL}(n_i, \mathbb{R}) \) in a lattice, in other words \( g_i \Gamma_i \) has a compact \( D_i \)-orbit; such elements exist, see [11]. In fact, we will need an additional assumption on \( g_i \), namely that the tori \( g_i^{-1}D_i g_i \) are irreducible over \( \mathbb{Q} \). The precise definition of this property and the proof of the existence of such a \( g_i \), a consequence of a theorem of Prasad and Rapinchuk [12, Theorem 1], will be the subject of Section 3.1.

Let \( \pi_i : G_i \to G_i/\Gamma_i \) be the canonical quotient map. Define for \( i = 1, 2 \):

\[
y_i = \pi_i \left( \begin{array}{c|c|c|c|c}
1 & 0 & \cdots & 0 & 1 \\
0 & 1 & \cdots & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 1 & 0 & \vdots \\
0 & \cdots & \cdots & 0 & 1 \\
\end{array} \right) g_i.
\]

The \( D_i \)-orbit of \( y_i \) is dense, by the following argument. It is easily seen that the closure of \( D_i y_i \) contains the compact \( D_i \)-orbit \( T_i = \pi_i(D_i g_i) \). The \( \mathbb{Q} \)-irreducibility of \( T_i \) is sufficient to show that the assumptions of the theorem of Lindenstrauss and Weiss [4, Theorem 1.1] are satisfied (Lemma 3.1); thus, by this theorem, we obtain that there exists a group \( H_i < G_i \) such that \( H_i y_i = D_i y_i \). Again because
of $\mathbb{Q}$-irreducibility, the group $H_i$ is necessarily the full group, i.e. $H_i = G_i$ (proof of Lemma 3.2).

Let $A_1$ be the $(n-3)$ dimensional subgroup of $G_1 \times G_2$ given by:

$$A_1 = \left\{ (\text{diag}(a_1, \ldots, a_{n_1}), \text{diag}(b_1, \ldots, b_{n_2})) : \prod_{i=1}^{n_1} a_i = \prod_{j=1}^{n_2} b_j = \frac{a_1b_1}{a_{n_1}b_{n_2}} = 1, a_i > 0, b_j > 0 \right\}.$$ (1)

Then the $A_1$-orbit of $(y_1, y_2)$ is not dense in $G_1 \times G_2 / \Gamma_1 \times \Gamma_2$ (Lemma 3.3), but $G_1 \times G_2$ is the smallest closed connected subgroup $F$ of $G_1 \times G_2$ such that $A_1(y_1, y_2) \subset F(y_1, y_2)$ (Lemma 3.7).

This yields a counterexample to Conjecture 4 which can be summarized as follows:

**Proposition 1.** For $i = 1, 2$, let $n_i \geq 3$ and $\Gamma_i$ be a lattice in $G_i = \text{SL}(n_i, \mathbb{R})$. For $A_1$, $y_1, y_2$ depicted as above, the $A_1$-orbit of $(y_1, y_2)$ in $G_1 \times G_2 / \Gamma_1 \times \Gamma_2$ satisfies neither condition (a) nor condition (b) of Conjecture 4.

2.2. **Theorem 3.** part (1). In order to obtain the first part of Theorem 3, choose $\Gamma_i = \text{SL}(n_i, \mathbb{Z})$, $\Gamma = \text{SL}(n, \mathbb{Z})$ and consider the embedding of $G_1 \times G_2$ in $G$, where matrices are written in blocks:

$$\Psi : (M_{n_1,n_1}, N_{n_2,n_2}) \mapsto \begin{bmatrix} M_{n_1,n_1} & 0_{n_1,n_2} \\ 0_{n_2,n_1} & N_{n_2,n_2} \end{bmatrix}.$$ (2)

This embedding gives rise to an embedding $\overline{\Psi}$ of $G_1 \times G_2 / \Gamma_1 \times \Gamma_2$ into $G / \Gamma$. Let $y_1, y_2$ be two points as above, let $x = \overline{\Psi}(y_1, y_2)$ and take $A = \Psi(A_1)$. We claim that this point $x$ and this group $A$ satisfy Theorem 3, part (1). In fact, since the image of $\overline{\Psi}$ is a closed connected $A$-invariant subset of $\text{SL}(n, \mathbb{R}) / \text{SL}(n, \mathbb{Z})$, everything takes place in this direct product.

2.3. **Theorem 3.** part (2). The second part of Theorem 3 is obtained as follows. Let $\sigma$ be the nontrivial field automorphism of the quadratic extension $\mathbb{Q}(\sqrt{2}) / \mathbb{Q}(\sqrt{2})$. Consider for any $m \geq 1$:

$$\text{SU}(m, \mathbb{Z}[\sqrt{2}], \sigma) = \left\{ M \in \text{SL}(m, \mathbb{Z}[\sqrt{2}]) : (\sigma^m)M = I_m \right\}.$$ Then $\text{SU}(m, \mathbb{Z}[\sqrt{2}], \sigma)$ is a lattice in $\text{SL}(m, \mathbb{R})$, as will be proved in Section 3.3 (see [4]. Appendix) for $m = 3$). Define for $i = 1, 2$, $\Gamma_i = \text{SU}(n_i, \mathbb{Z}[\sqrt{2}], \sigma)$, and $\Gamma = \text{SU}(n, \mathbb{Z}[\sqrt{2}], \sigma)$. Now consider the map:

$$\varphi : G_1 \times G_2 \times \mathbb{R} \to G,$$
Define $M$ to be the image of $\varphi$. This time, $\varphi$ factors into a finite covering $\overline{\varphi}$ of homogeneous spaces:

$$\overline{\varphi} : G_1 \times G_2 \times \mathbb{R}/\Gamma_1 \times \Gamma_2 \times \log(\alpha) \mathbb{Z} \to M/M \cap \Gamma \subset G/\Gamma,$$

where $\alpha = (3 + 2\sqrt{2}) + \sqrt{2}(2 + 2\sqrt{2})$ satisfies $\alpha^{-1} = \sigma(\alpha)$. Consider the points $y_i$ constructed above, and let $x = \overline{\varphi}(y_1, y_2, 0)$. Choose:

$$A = \left\{ \text{diag}(a_1, \ldots, a_n) \mid \prod_{i=1}^{n} a_i = \frac{a_1a_{n+1}}{a_{n+1}a_n} = 1, \ a_i > 0 \right\} \subset \text{SL}(n, \mathbb{R}).$$

We claim that this lattice $\Gamma$, this point $x$ and this group $A$ satisfy Theorem 1, part (2). What happens here is that the $A$-orbit of $x$ is a circle bundle over an $A_1$-orbit (up to the finite cover $\overline{\varphi}$), like in Rees’ example.

### 3. Proof of Theorem 1

#### 3.1. Q-irreducible tori

Fix $i \in \{1, 2\}$. Recall that $\Gamma_i$ is a lattice in $G_i = \text{SL}(n_i, \mathbb{R})$. Since $n_i \geq 3$, by Margulis’s arithmeticity Theorem [H, Theorem 6.1.2], there exists a semisimple algebraic Q-group $H_i$ and a surjective homomorphism $\theta$ from the connected component of identity of the real points of this group $H_i^0(\mathbb{R})$ to $\text{SL}(n_i, \mathbb{R})$, with compact kernel, such that $\theta(H_i(\mathbb{Z}) \cap H_i^0(\mathbb{R}))$ is commensurable with $\Gamma_i$.

Following Prasad and Rapinchuk, we say that a Q-torus $T \subset H_i$ is Q-irreducible if it does not contain any proper subtorus defined over $\mathbb{Q}$. By [12, Theorem 1,(ii)], there exists a maximal Q-anisotropic Q-torus $T_i \subset H_i$, which is Q-irreducible. Because any two maximal R-tori of $\text{SL}(n_i, \mathbb{R})$ are R-conjugate, there exists $g_i \in G_i$ such that $\theta(T_i^0(\mathbb{R})) = g_i^{-1}D_i g_i$. The subgroup $T_i(\mathbb{Z})$ is a cocompact lattice in $T_i(\mathbb{R})$ since $T_i$ is Q-anisotropic [2 Theorem 8.4 and Definition 10.5].

Because $\theta(H_i(\mathbb{Z}) \cap H_i^0(\mathbb{R}))$ and $\Gamma_i$ are commensurable and $\theta$ has compact kernel, it follows that both $\Gamma_i \cap g_i^{-1}D_i g_i$ and $\theta(T_i^0(\mathbb{Z})) \cap \Gamma_i \cap g_i^{-1}D_i g_i$ are also cocompact lattices in $g_i^{-1}D_i g_i$. The resulting topological torus $\pi_i(D_i g_i) \subset G_i/\Gamma_i$ will be denoted $T_i$. Write $z_i = \pi_i(g_i)$, so that $T_i = D_i z_i$.

For every $1 \leq k < l \leq n_i$, define as in [O]:

$$N_{k,l}^{(i)} = \left\{ \text{diag}(a_1, \ldots, a_n) : \prod_{s=1}^{n_i} a_s = 1, \ a_k = a_l, \ a_s > 0 \right\} \subset D_i,$$

Of interest to us amongst the consequences of Q-irreducibility is the fact that an element of $\Gamma_i \cap g_i^{-1}D_i g_i$ lying in a wall of a Weyl chamber is necessarily trivial. This is expressed in the following form:

**Lemma 3.1.** For every $1 \leq k < l \leq n_i$, and any closed connected subgroup $L$ of positive dimension of $N_{k,l}^{(i)}$, the $L$-orbit of $z_i$ is not compact.
Proof. Assume the contrary, that is $Lz_i$ is compact. This implies that $g_i^{-1}Lg_i \cap \Gamma_i$ is a uniform lattice in $g_i^{-1}Lg_i$, so $g_i^{-1}Lg_i \cap \theta(H_i(Z))$ is also a uniform lattice. Since $L$ is nontrivial, there exists an element $\gamma \in H_i(Z) \cap H_i^0(R)$ of infinite order, such that $g_i\theta(\gamma)g_i^{-1}$ is in $L$. Note that since $\theta$ has compact kernel, $T_i(Z)$ is a lattice in $\theta^{-1}(\theta(T_i^0(R)))$ and is then a subgroup of finite index in $H_i(Z) \cap H_i^0(R) \cap \theta^{-1}(\theta(T_i^0(R)))$, so there exists $n > 0$ such that $\gamma^n$ belongs to $T_i(Z)$.

Consider the representation:

$$\rho : H_i^0(R) \to GL(sl(n_i, R)),$$

$$x \mapsto Ad(g_i\theta(x)g_i^{-1}).$$

Recall that $\chi(diag(a_1, \ldots, a_{n_i})) = a_k/a_l$ is a weight of $Ad$ with respect to $D_i$, so $\chi$ is a weight of $\rho$ with respect to $T_i$. By [12, Proposition 1, (iii)], the $Q$-irreducibility of $T_i$ implies that $\chi(\gamma^n) \neq 1$, but this contradicts the fact that $\theta(\gamma^n) \in g_i^{-1}N_{k,l}^{(i)}g_i$.

3.2. Contraction and expansion. For real $s$, denote by $a_i(s)$ the following $n_i \times n_i$-matrix:

$$(a_i(s) = diag(e^{s/2}, 1, \ldots, 1, e^{-s/2}), n_i-2 \text{ times})$$

and write simply $N_i$ for $N_i^{(i)}$. Write also:

$$h_i(t) = \begin{bmatrix}
1 & 0 & \ldots & 0 & t \\
0 & 1 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 1 & 0 \\
0 & \ldots & \ldots & 0 & 1
\end{bmatrix}$$

Then the following commutation relation holds:

$$a_i(s)h_i(t) = h_i(e^s t)a_i(s),$$

that is the direction $h_i$ is expanded for positive $s$; note that both $h_i$ and $a_i$ commute with elements of $N_i$. It is easy to check from Equation (1) that

$$A_1 = \{(a_1(s)d_1, a_2(-s)d_2) : s \in R, d_i \in N_i, i = 1, 2\}.$$

Recall that $y_i = h_i(1)z_i$.

Lemma 3.2.  
(1) If $s \leq 0$, for any $d \in N_i$ the point $a_i(s)dy_i$ lies in the compact set $K_i = h_i([0, 1])[T_i]$.

(2) The $D_i$-orbit of $y_i$ is dense in $G_i/\Gamma_i$.

(3) The set $\{a_i(s)dy_i : s \geq 0, d \in N_i\}$ is dense in $G_i/\Gamma_i$.

Proof. The first statement is clear from the commutation relation. It also implies that $D_iy_i$ contains the compact torus $T_i$ in its closure.
To prove the second point, we rely heavily on the paper of Lindenstrauss and Weiss. [4, Theorem 1.1] applies here, since the hypotheses of their Theorem is precisely the conclusion of Lemma 3.1 for \( L = N^{i(k)}_{k,l} \). So the following holds: there exists a reductive subgroup \( H_i \), containing \( D_i \), such that \( D_i y_i = H_i y_i \), and \( H_i \cap \Gamma_i \) is a lattice in \( H_i \). Write \( L = D_i \cap C_G(H_i) \).

Since \( D_i y_i \) is not closed, \( H_i \neq D_i \), so there exists a nontrivial root relatively to \( D_i \) for the Adjoint representation of \( H_i \) on its Lie algebra, which is a subalgebra of \( \mathfrak{s}(n, \mathbb{R}) \). Thus there exist \( k, l \) such that \( L \subset N^{i(k)}_{k,l} \). By [4, step 4.1 of Lemma 4.2], \( L z_i \) is compact, so by Lemma 3.1, \( L \) is trivial. By [4, Proposition 3.1], \( H_i \) is the connected component of the identity of \( C_G(L) \), so \( H_i = G_i \), as desired.

The third claim follows from the first and second claim together with the fact that \( K_i \) has empty interior. \( \square \)

3.3. Topological properties of the \( A_1 \)-orbit.

Lemma 3.3. The \( A_1 \)-orbit of \( (y_1, y_2) \) is not dense in \( G_1 \times G_2 / \Gamma_1 \times \Gamma_2 \).

Proof. Consider the open set \( U = K_1^+ \times K_2^+ \). We claim that the \( A_1 \)-orbit of \( (y_1, y_2) \) does not intersect \( U \). Indeed, if \( (a_1(s) d_1, a_2(-s) d_2) \in A_1 \) with \( s \in \mathbb{R} \) and \( d_i \in N_i \), the previous Lemma implies that if \( s \geq 0 \), \( a_2(-s) d_2 y_2 \in K_2 \), and if \( s \leq 0 \), \( a_1(s) d_1 y_1 \in K_1 \).

The following elementary result will be useful:

Lemma 3.4. Let \( p_i : G_1 \times G_2 \to G_i \) be the first (resp. second) coordinate morphism. If \( F \subset G_1 \times G_2 \) is a subgroup such that \( p_i(F) = G_i \) for \( i = 1, 2 \), and \( A_1 \subset F \), then \( F = G_1 \times G_2 \).

Proof. Let \( F_1 = \text{Ker}(p_1) \cap F \). Since \( F_1 \) is normal in \( F \), \( p_2(F_1) \) is normal in \( p_2(F) = G_2 \). Note that \( N_2 \subset p_2(A_1 \cap \text{Ker}(p_1)) \subset p_2(F_1) \) is not finite, and that \( G_2 \) is almost simple, consequently the normal subgroup \( p_2(F_1) \) of \( G_2 \) is equal to \( G_2 \). Let \( (a, b) \in G_1 \times G_2 \), by assumption there exists \( f \in F \) such that \( p_1(f) = a \). Let \( f_1 \in F_1 \) be such that \( p_2(f_1) = bp_2(f)^{-1} \), then \( (a, b) = f_1 f \in F \). \( \square \)

We will have to apply several times the two following well-known Lemmas:

Lemma 3.5. Let \( L \) be a Lie group, \( \Lambda \subset L \) a lattice, \( M, N \) two closed, connected subgroups of \( L \), such that for some \( w \in L / \Lambda \), \( M w \) and \( N w \) are closed. Then \( (M \cap N) w \) is closed.

Proof. This is a weaker form of [4, Lemma 2.2]. \( \square \)

Lemma 3.6. Let \( L \) be a connected Lie group, \( \Lambda \subset L \) a discrete subgroup, \( M, N \) two subgroups of \( L \), such that \( M \) is closed and connected, and \( N \) is a countable union of closed sets. For any \( w \in L / \Lambda \), if \( M w \subset N w \), then \( M \subset N \).

Proof. Up to changing \( \Lambda \) by one of its conjugate in \( L \), one can assume that \( w = \Lambda \in L / \Lambda \). By assumption, \( M \Lambda \subset N \Lambda \) so \( M \subset N \Lambda \subset L \). Recall that \( M \)
is closed, that $\Lambda$ is countable, and that $N$ is a countable union of closed sets, so Baire’s category Theorem applies, and there exists $\lambda \in \Lambda$ and an open set $U$ of $M$ such that $U \subset N\lambda$, so $UU^{-1} \subset N$. Since $M$ is a connected subgroup, $UU^{-1}$ generates $M$, so $M \subset N$.

The following lemma will be useful both for proving that the closure of $A_1(y_1, y_2)$ is not homogeneous, and for proving it does not fiber over a 1-parameter group orbit.

**Lemma 3.7.** Let $F$ be a closed connected subgroup of $G_1 \times G_2$ such that $F(y_1, y_2)$ contains the closure of $A_1(y_1, y_2)$. Then $F = G_1 \times G_2$.

**Proof.** By Lemma 3.2, the set of first coordinates of the set

$$\{(a(s)d_1y_1, a(-s)d_2y_2) : s \geq 0, d_i \in N_i\},$$

is dense in $G_1/\Gamma_1$ and the second coordinates lies in the compact set $K_2$, so the closure of $A_1(y_1, y_2)$ contains points of arbitrary first coordinate with their second coordinate in $K_2$. Consequently, the set of first coordinates of $F(y_1, y_2)$ is the whole $G_1/\Gamma_1$, and similarly for the set of second coordinates. For $i = 1, 2$, Lemma 3.6 now applies to $L = M = G_i$, $\Lambda = \Gamma_i$, $N = p_i(F)$, which is a countable union of closed sets because $G_1 \times G_2$ is $\sigma$-compact, and $w = y_i$, and so $p_i(F) = G_i$.

In order to apply Lemma 3.4 and finish the proof, we have to show that $A_1 \subset F$. Again, this follows from a direct application of Lemma 3.6 to $L = G_1 \times G_2$, $\Lambda = \Gamma_1 \times \Gamma_2$, $M = A_1$, $N = F$, $w = (y_1, y_2)$.

3.4. **Proof of Theorem 4, part (1).** We now proceed to proving Theorem 4, part (1). The proof of Proposition 4 is similar and is omitted.

Recall that in this case, we fixed $A = \Psi(A_1)$ and $x = \overline{\Psi}(y_1, y_2)$.

Assume $Ax$ is homogeneous, that is $Ax = Fx$ for a closed connected subgroup $F$ of $G$. Since $Ax \subset \overline{\Psi}(G_1 \times G_2/\Gamma_1 \times \Gamma_2)$, which is closed in $G/\Gamma$, Lemma 3.6 imply that $F \subset \Psi(G_1 \times G_2)$. By Lemma 3.7, $F = \Psi(G_1 \times G_2)$, so $Fx = G/\Gamma$ and $Ax$ is dense in $\overline{\Psi}(G_1 \times G_2)$, which is a contradiction.

Now assume $Ax$ fibers over the orbit of a one-parameter subgroup. Let $F$ be a closed connected subgroup, $L$ a Lie group and $\phi : F \to L$ a continuous epimorphism satisfying the (b) of the conjecture. Let $F' = F \cap \Psi(G_1 \times G_2)$, we have $A \subset \overline{F'}$. By Lemma 3.3, $F'x$ is closed in $Fx \cap \overline{\Psi}(G_1 \times G_2)$, so is closed in $G/\Gamma$. By Lemma 3.7, $F' = \Psi(G_1 \times G_2)$ necessarily. Let $H = \ker(\phi \circ \Psi) \subset G_1 \times G_2$, so $A_1/(A_1 \cap H)$ is a one-parameter group by assumption (b).

The subgroup $H$ is a normal subgroup of the semisimple group $G_1 \times G_2$, which has only four kind of normal subgroups : finite, $G_1 \times G_2$, $G_1 \times \text{finite}$ and $\text{finite} \times G_2$. None of these possible normal subgroups have the property that they intersect $A_1$ in a codimension 1 subgroup, so this is a contradiction.
3.5. The arithmetic lattice. Here we prove that \( \text{SU}(n, \mathbb{Z}[\sqrt{2}], \sigma) \) is a lattice in \( \text{SL}(n, \mathbb{R}) \). Let \( P, Q \) be the polynomials with coefficients in \( \mathbb{Q}(\sqrt{2}) \) such that for any \( X, Y \in M_n(\mathbb{C}) \)
\[
det(X + \sqrt{2}Y) = P(X, Y) + \sqrt{2}Q(X, Y).
\]
For an integral domain \( A \subset \mathbb{C} \), consider the set of pairs of matrices:
\[
G(A) = \{(X, Y) \in M_n(A)^2 : \left. {}^tX X - \sqrt{2} Y Y = I_n, \quad {}^tX Y - {}^tY X = 0, \right. \quad P(X, Y) = 1, \ Q(X, Y) = 0\},
\]
which implies that \((X - \sqrt{2}Y)(X + \sqrt{2}Y) = I_n \) and \( \det(X + \sqrt{2}) = 1 \) for all \( (X, Y) \in G(A) \). Endow \( G(A) \) with the multiplication given by
\[
(X, Y)(X', Y') = (XX' + \sqrt{2}YY', XY' + YX'),
\]
which is such that the map \( \phi : G(A) \to \text{SL}(n, \mathbb{C}) \), \( (X, Y) \mapsto X + \sqrt{2}Y \) is a morphism. With this structure, \( G \) is an algebraic group, which is clearly defined over \( \mathbb{Q}(\sqrt{2}) \). Let \( \tau \) be the nontrivial field automorphism of \( \mathbb{Q}(\sqrt{2})/\mathbb{Q} \), it can be checked that the map \( \phi \) is an isomorphism between \( G(\mathbb{R}) \) and \( \text{SL}(n, \mathbb{R}) \), and that moreover \( \phi' : G'(\mathbb{R}) \to \text{SL}(n, \mathbb{C}) \), \( (X, Y) \mapsto X + i\sqrt{2}Y \) is an isomorphism onto \( \text{SU}(n) \). Let \( H = \text{Res}_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}} G = G \times G' \). Then \( H \) is defined over \( \mathbb{Q} \) (see for example [4, 6.1.3], for definition and properties of the restriction of scalars functor). It follows from a Theorem of Borel and Harish-Chandra [4, Theorem 3.1.7] that \( H(\mathbb{Z}) \) is a lattice in \( H(\mathbb{R}) \). Since \( \text{SU}(n) \) is compact, it follows that the projection of \( H(\mathbb{Z}) \) onto the first factor of \( G(\mathbb{R}) \times G'(\mathbb{R}) \) is again a lattice. Using the isomorphism between \( G(\mathbb{R}) \) and \( \text{SL}(n, \mathbb{R}) \), this projection can be identified with
\[
G(\mathbb{Z}[\sqrt{2}]) = \text{SU}(n, \mathbb{Z}[\sqrt{2}] + \sqrt{2}\mathbb{Z}[\sqrt{2}], \sigma) = \text{SU}(n, \mathbb{Z}[\sqrt{2}], \sigma).
\]

3.6. Proof of Theorem [1, part (2)]. Note that, as stated implicitly in Section 2.3,
\[
\varphi(\Gamma_1 \times \Gamma_2 \times (\log \alpha)\mathbb{Z}) \subset \Gamma \cap M,
\]
so \( \Gamma \cap M \) is a lattice in \( M \), and \( M/(\Gamma \cap M) \) is a closed, \( A \)-invariant subset of \( G/T \). Notice also that the map \( \Psi \) defined by Equation (2) defines an embedding \( \Psi : G_1 \times G_2/\Gamma_1 \times \Gamma_2 \to G/T \).

Assume \( \overline{Ax} \) is homogeneous, that is \( \overline{Ax} = Fx \) for a closed connected subgroup \( F \) of \( G \). Since \( Ax \subset M/(\Gamma \cap M) \), which is closed in \( G/T \), Lemma 3.3 again by Lemma 3.3, \( F'x \) is a closed subset of \( \overline{Im(\Psi)} \). Since \( A_1 \subset F' \), \( \Psi(A_1)x \subset F'x \) and Lemma 3.7 implies that \( F' = \overline{\Psi(G_1 \times G_2)} \). Since \( A \) contains \( \varphi(e, e, t) \) for all \( t \in \mathbb{R} \), we have \( M = AF' \subset F \) so \( F = M \) necessarily.

By Lemma 3.3, the \( A_1 \)-orbit of \((y_1, y_2)\) is not dense; the topological transitivity of the action of \( A_1 \) on \( G_1 \times G_2/\Gamma_1 \times \Gamma_2 \) implies that moreover the closure of this orbit has empty interior. Thus, the \( A_1 \times \mathbb{R} \)-orbit of \((y_1, y_2, 0)\) is also nowhere
dense in $G_1 \times G_2 \times \mathbb{R}/\Gamma_1 \times \Gamma_2 \times (\log \alpha)\mathbb{Z}$. The map $\psi$ being a finite covering, the $A$-orbit of $x$ is nowhere dense. This is a contradiction with $F = M$.

Now assume $Ax$ fibers over the orbit of a one-parameter non-$\text{Ad}$-unipotent subgroup. Let $F$ be a closed connected subgroup, $L$ a Lie group and $\phi : F \to L$ a continuous epimorphism satisfying the (b) of the conjecture. Let $F' = F \cap \Psi(G_1 \times G_2)$ and $F'' = F \cap M$, we have $A_1 \subset F'$ and $A \subset F''$. Similarly, $F'x$ and $F''x$ are closed in $G/\Gamma$. Again, by Lemma 3.7, $F' = \Psi(G_1 \times G_2)$ necessarily, and like before, $AF' \subset F'' \subset M$ so $F'' = M$.

Let $H = \text{Ker}(\phi \circ \varphi) \subset G_1 \times G_2 \times \mathbb{R}$, so $A_1 \times \mathbb{R}/(A_1 \times \mathbb{R} \cap H)$ is a one-parameter group. This time, possibilities for the closed normal subgroup $H$ are: finite $\times \Lambda$, $G_1 \times G_2 \times \Lambda$, $G_1 \times \text{finite} \times \Lambda$ and finite $\times G_2 \times \Lambda$, where $\Lambda$ is a closed subgroup of $\mathbb{R}$. Of all these possibilities, only $G_1 \times G_2 \times \Lambda$, where $\Lambda$ is discrete, has the required property that $A_1 \times \mathbb{R}/(A_1 \times \mathbb{R} \cap H)$ is a one-parameter group. This proves that $\Psi(G_1 \times G_2) \subset \text{Ker}(\phi)$, so $F \subset N_G(\Psi(G_1 \times G_2))$. However, the normalizer of $\Psi(G_1 \times G_2)$ in $G$ is the group of block matrices having for connected component of the identity the group $M$. So by connectedness of $F$, $F \subset M$, and since $M = F'' \subset F$, we have $F = M$. Thus $L = F/\text{Ker}(\phi) = \mathbb{R}/\Lambda$ is abelian, and a fortiori every element of $L$ is unipotent; this contradicts (b).

4. Proof of Theorem 2

The proof of Theorem 2 is divided in two independent lemmas.

Lemma 4.1. The family $(z_1, \ldots, z_{2q}, 1)$ is linearly independent over $\mathbb{Q}$.

Proof. Consider a linear combination:

$$\sum_{i=1}^{q} a_i z_i + b_i z_{i+q} = c.$$  

We can assume that $a_i, b_i$ and $c$ are integers. Let $k_0 \geq 1$, write

$$\left( \prod_{i=1}^{q} p_i \right)^{N^{2k_0+1}} \left( \sum_{i=1}^{q} \sum_{k=1}^{k_0} a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} - c \right) =$$  

$$- \left( \prod_{i=1}^{q} p_i \right)^{N^{2k_0+1}} \left( \sum_{i=1}^{q} \sum_{k \geq k_0+1} a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} \right).$$
It is clear the left hand side is an integer. Since $1 < p_1 < \cdots < p_q$, the right hand side is less in absolute value than

$$
p_q^{N^{2k_0+1}} 2q \sup_i (|a_i|, |b_i|) \sum_{k \geq 0} \left( p_1^{-N^{2k_0+2}} \right)^{N^{2k}} \leq 4q \sup_i (|a_i|, |b_i|) p_q^{N^{2k_0+1}} \frac{p_1^{-N^{2k_0+2}}}{p_1^-} \leq 4q \sup_i (|a_i|, |b_i|) \exp(N^{2k_0+1}(q \log p_q - N \log p_1)).
$$

Since $N > q^{\log(p_0)}$, the last expression tends to zero. This proves the right-hand side of (3) is zero for large enough $k_0$, so for all large $k$,

$$
\sum_{i=1}^q a_i p_i^{-N^{2k}} + b_i p_i^{-N^{2k+1}} = 0.
$$

The $p_i$ being distincts, this implies that for $i \in \{1, \ldots, q\}$, $a_i = b_i = 0$. \hfill \Box

The following Lemma implies easily that the orbit of $z$ under $\Omega$ cannot be dense.

**Lemma 4.2.** For all $\epsilon > 0$, there exists $L > 0$, such that for all $n_1, \ldots, n_q \geq 0$ with $\sum_{i=1}^q n_i \geq L$, there exists $j \in \{1, \ldots, 2q\}$ such that $p_1^{n_1} \cdots p_q^{n_q} z_j$ lies in the interval $[0, \epsilon]$ modulo 1.

**Proof.** Let $s \in \{1, \ldots, q\}$ such that for all $r \in \{1, \ldots, q\}$, $p_s^{n_s} \geq p_r^{n_r}$. Let $k_0$ be the integer part of $\log(n_s)/2 \log(N)$, then either $N^{2k_0} \leq n_s \leq N^{2k_0+1}$, or $N^{2k_0+1} \leq n_s \leq N^{2k_0+2}$. In the first case, take $j = s$, then:

$$
p_1^{n_1} \cdots p_q^{n_q} z_j = p_1^{n_1} \cdots p_q^{n_q} \sum_{k \geq 1} p_s^{-N^{2k}} = p_1^{n_1} \cdots p_q^{n_q} \sum_{k \geq k_0+1} p_s^{-N^{2k}} \mod 1.
$$

We have

$$
\sum_{k \geq k_0+1} p_s^{-N^{2k}} \leq 2p_s^{-N^{2k_0+2}},
$$

so, using the fact that for all $r \in \{1, \ldots, q\}$, $p_r^{n_r} \leq p_s^{n_s} \leq p_s^{N^{2k_0+1}}$, we obtain:

$$
p_1^{n_1} \cdots p_q^{n_q} \sum_{k \geq k_0+1} p_s^{-N^{2k}} \leq 2p_s^{N^{2k_0+1}-N^{2k_0+2}} \leq 2p_s^{N^{2k_0+1}(q-N)},
$$

but by hypothesis we have $N > q^{\log(p_0)} > q$, so the preceding bound is small whenever $k_0$ is large. Because of the definition of $k_0$, we have

$$
k_0 \geq \frac{\log \frac{\sum_{i=1}^q n_i \log p_i}{q \log p_q}}{2 \log N} \geq \frac{\log \frac{L \log p_1}{q \log p_q}}{2 \log N},
$$

so $k_0$ is arbitrary large when $L$ is large.

In the second case $N^{2k_0+1} \leq n_s \leq N^{2k_0+2}$, one can proceed similarly with $j = s + q$. 

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References


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