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To cite this version:
Philippe Carmona. Directed polymer in random environment and last passage percolation. ESAIM: Probability and Statistics, EDP Sciences, 2010, 14, pp.263–270. <hal-00315297>
Directed polymer in random environment and last passage percolation

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August 28, 2008

Abstract

The sequence of random probability measures $\nu_n$ that gives a path of length $n$, $\frac{1}{n}$ times the sum of the random weights collected along the paths, is shown to satisfy a large deviations principle with good rate function the Legendre transform of the free energy of the associated directed polymer in a random environment.

Consequences on the asymptotics of the typical number of paths whose collected weight is above a fixed proportion are then drawn.

Keywords: directed polymer, random environment, partition function, last passage percolation.

Mathematic Classification : 60K37
1 Introduction

Last passage percolation

To each site \((k, x)\) of \(\mathbb{N} \times \mathbb{Z}^d\) is assigned a random weight \(\eta(k, x)\). The \((\eta(k, x))_{k \geq 1, x \in \mathbb{Z}^d}\) are taken IID under the probability measure \(Q\).

The set of oriented paths of length \(n\) starting from the origin is

\[ \Omega_n = \{ \omega = (\omega_0, \ldots, \omega_n) : \omega_i \in \mathbb{Z}^d, \omega_0 = 0, |\omega_i - \omega_{i-1}| = 1 \} \].

The weight (energy, reward) of a path is the sum of weights of visited sites:

\[ H_n = H_n(\omega, \eta) = \sum_{k=1}^{n} \eta(k, \omega_k) \quad (n \geq 1, \omega \in \Omega_n). \]

Observe that when \(\eta(k, x)\) are Bernoulli(p) distributed

\[ Q(\eta(k, x) = 1) = 1 - Q(\eta(k, x) = 0) = p \in (0, 1), \]

the quantity \(\frac{H_n}{n}(\omega, \eta)\) is the proportion of open sites visited by \(\omega\), and it is natural to consider for \(0 < \rho < 1\),

\[ N_n(\rho) = \text{number of paths of length } n \text{ such that } H_n(\omega, \eta) \geq n\rho. \]

The problem of \(\rho\)-percolation, as we learnt it from Comets, Popov and Vachkovskaia [8] and Kesten and Sidoravicius [12], is to study the behaviour of \(N_n(\rho)\) for large \(n\) and different values of \(\rho\).

Directed polymer in a random environment

We are going to consider fairly general environment distributions, by requiring first that they have exponential moments of any order:

\[ \lambda(\beta) = \log Q(e^{\beta \eta(k, x)}) < +\infty \quad (\beta \in \mathbb{R}), \]

and second that they satisfy a logarithmic Sobolev inequality (see e.g. [2]): in particular we can apply our result to bounded support and Gaussian environments.

The polymer measure is the random probability measure defined on the set of oriented paths of length \(n\) by:

\[ \mu_n(\omega) = \left(2d\right)^{-n} \frac{e^{\beta H_n(\omega, \eta)}}{Z_n(\beta)} \quad (\omega \in \Omega_n), \]

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with $Z_n(\beta)$ the partition function

$$Z_n(\beta) = Z_n(\beta, \eta) = (2d)^n \sum_{\omega \in \Omega_n} e^{\beta H_n(\omega, \eta)} = \mathbf{P} \left( e^{\beta H_n(\omega, \eta)} \right),$$

where $\mathbf{P}$ is the law of simple random walk on $\mathbb{Z}^d$ starting from the origin. Bolthausen [3] proved the existence of a deterministic limiting free energy

$$p(\beta) = \lim_{n \to +\infty} \frac{1}{n} \mathbb{Q}(\log Z_n(\beta)) = \mathbb{Q} \text{ a.s.} \lim_{n \to +\infty} \frac{1}{n} \log Z_n(\beta).$$

Thanks to Jensen’s inequality, we have the upper bound $p(\beta) \leq \lambda(\beta)$ and it is conjectured (and partially proved, see [7, 6]) that the behaviour of a typical path under the polymer measure is diffusive iff $\beta \in \mathbb{C}_\eta$ the critical region

$$\mathbb{C}_\eta = \{ \beta \in \mathbb{R} : p(\beta) = \lambda(\beta) \}.$$

In dimension $d = 1$, $\mathbb{C}_\eta = \{0\}$ and in dimensions $d \geq 3$, $\mathbb{C}_\eta$ contains a neighborhood of the origin (see [3, 9]).

The main theorem

The connection between Last passage percolation and Directed polymer in random environment is made by the family $(\nu_n)_{n \in \mathbb{N}}$ of random probability measures on the real line:

$$\nu_n(A) = \frac{1}{|\Omega_n|} \sum_{\omega \in \Omega_n} 1_{\left( \frac{H_n(\omega, \eta)}{n} \in A \right)} = \mathbf{P} \left( \frac{H_n(\omega, \eta)}{n} \in A \right).$$

Indeed,

$$N_n(\rho) = \sum_{\omega \in \Omega_n} 1_{\left( H_n(\omega, \eta) \geq n \rho \right)} = (2d)^n \nu_n([\rho, +\infty)) .$$

The main result of the paper is

**Theorem 1.** $\mathbb{Q}$ almost surely, the family $(\nu_n)_{n \in \mathbb{N}}$ satisfies a large deviations principle with good rate function $I = p^*$ the Legendre transform of the free energy of the directed polymer.

Let $m = \mathbb{Q}(\eta(k, x))$ be the average weight of a path $m = \mathbb{Q}(\frac{H_n(\omega, \eta)}{n})$. It is natural to consider the quantities:

$$N_n(\rho) = \begin{cases} \sum_{\omega \in \Omega_n} 1_{\left( H_n(\omega, \eta) \geq n \rho \right)} & \text{if } \rho \geq m , \\ \sum_{\omega \in \Omega_n} 1_{\left( H_n(\omega, \eta) \leq n \rho \right)} & \text{if } \rho < m . \end{cases}$$
A simple exchange of limits $\beta \to \pm \infty$, and $n \to +\infty$, yields the following
\[
\rho^\pm = Q \text{ a.s. } \lim_{n \to +\infty} \max_{\omega \in \Omega_n} \pm \frac{H_n(\omega, \eta)}{n} = \lim_{\beta \to +\infty} \frac{p(\pm \beta)}{\beta} \in [0, +\infty].
\]
Repeating the proof of Theorem 1.1 of [8] gives

**Corollary 2.** For $-\rho^- < \rho < \rho^+$, we have $Q$ almost surely,
\[
\lim_{n \to +\infty} (N_n(\rho))^\frac{1}{n} = (2d)e^{-I(\rho)}.
\]

We can then translate our knowledge of the critical region $C_\eta$, into the following remark. Let
\[
\mathcal{V}_\eta = \{ \rho \in \mathbb{R} : I(\rho) = \lambda^*(\rho) \}.
\]
In dimension $d = 1$, $\mathcal{V}_\eta = \{m\}$ and in dimensions $d \geq 3$, $\mathcal{V}_\eta$ contains a neighbourhood of $m$.
This means that in dimensions $d \geq 3$, the typical large deviation of $\frac{H_n}{n}(\omega, \eta)$ close to its mean is the same as the large deviation of $\frac{1}{n}(\eta_1 + \cdots + \eta_n)$ close to its mean, with $\eta_i$ IID. There is no influence of the path $\omega$: this gives another justification to the name weak-disorder region given to the critical set $C_\eta$.

## 2 Proof of the main theorem

Observe that for any $\beta \in \mathbb{R}$ we have:
\[
\int e^{\beta n x} d\nu_n(x) = P\left(e^{\beta H_n(\omega, \eta)}\right) = Z_n(\beta) \quad Q \text{ a.s..}
\]
Consequently, since $e^u + e^{-u} \geq e^{\|u\|}$, we obtain for any $\beta > 0$,
\[
\limsup_{n \to +\infty} \frac{1}{n} \log \left( \int e^{\beta n |x|} d\nu_n(x) \right) \leq p(\beta) + p(-\beta) < +\infty,
\]
and the family $(\nu_n)_{n \geq 0}$ is exponentially tight (see Dembo and Zeitouni [10], or Feng and Kurtz [11]). We only need to show now that for a lower semi-continuous function $I$, and for $x \in \mathbb{R}$
\[
\lim \inf_{\delta \to 0} \frac{1}{n} \log \nu_n((x - \delta, x + \delta)) = I(x),
\]
\[
\lim \inf_{\delta \to 0} \frac{1}{n} \log \nu_n([x - \delta, x + \delta]) = I(x).
\]
From these, we shall infer that \((\nu_n)_{n \in \mathbb{N}}\) follows a large deviations principle with good rate function \(I\). Eventually, equation (1) and

\[
\lim_{n \to \infty} \frac{1}{n} \log Z_n(\beta) = p(\beta)
\]

will imply, by Varadhan’s lemma that \(I\) and \(p\) are Legendre conjuguate:

\[
I(x) = p^*(x) = \sup_{\beta \in \mathbb{R}} (x\beta - p(\beta)).
\]

The strategy of proof finds its origin in Varadhan’s seminal paper [13], and has already successfully been applied in [5]. Let us define for \(\lambda > 0, x \in \mathbb{Z}, a \in \mathbb{R}\)

\[
V_n^{(\lambda)}(x, a; \eta) = \log P^x(e^{-\lambda |H_n(\omega_n) - a|}) = V^{(\lambda)}(0, a; \tau_{0,x} \circ \eta),
\]

with \(\tau_{k,x}\) the translation operator on the environment defined by:

\[
\tau_{k,x} \circ \eta(i, y) = \eta(k + i, x + y),
\]

and \(P^x\) the law of simple random walk starting from \(x\).

**Step 1** The functions \(v_n^{(\lambda)}(a) = Q(V^{(\lambda)}(0, a; \eta))\) satisfy the inequality

\[
v_{n+m}^{(\lambda)}(a+b) \geq v_{n}^{(\lambda)}(a) + v_{m}^{(\lambda)}(b) \quad (n, m \in \mathbb{N}; a, b \in \mathbb{R}). \quad (4)
\]

**Proof.** Since \(|H_{n+m} - (a+b)| \leq |H_n - b| + |(H_{n+m} - H_n) - a|\) we have

\[
V_{n+m}^{(\lambda)}(x, a; \eta) \geq \log P^x(e^{-\lambda |H_n - b|} e^{-\lambda |(H_{n+m} - H_n) - a|})
\]

\[
= \log P^x(e^{-\lambda |H_n - b|} e^{V_m^{(\lambda)}(0, a; \tau_{0,x} \circ \eta)})
\]

\[
= \log \sum_y P^x(e^{-\lambda |H_n - b|} 1_{(S_n = y)}) e^{V_m^{(\lambda)}(0, a; \tau_{0,y} \circ \eta)}
\]

\[
= V_n^{(\lambda)}(x, b; \eta) + \log \left( \sum_y \sigma_n(y) e^{V_m^{(\lambda)}(0, a; \tau_{0,y} \circ \eta)} \right)
\]

\[
\geq V_n^{(\lambda)}(x, b; \eta) + \sum_y \sigma_n(y) V_m^{(\lambda)}(0, a; \tau_{0,y} \circ \eta) \quad \text{(Jensen’s inequality)},
\]

with \(\sigma_n\) the probability measure on \(\mathbb{Z}^d:\)

\[
\sigma_n(y) = \frac{1}{V_n^{(\lambda)}(x, b; \eta)} P^x(e^{-\lambda |H_n - b|} 1_{(S_n = y)}) \quad (y \in \mathbb{Z}^d).
\]
Observe that the random variables $\sigma_n(y)$ are measurable with respect to the sigma field $G_n = \sigma(\eta(i, x) : i \leq n, x \in \mathbb{Z}^d)$, whereas the random variables $V_{m}^{(\lambda)}(0, a; \tau_{n,y} \circ \eta)$ are independent from $G_n$. Hence, by stationarity,

$$v_{n+m}(x, a; \eta) = Q\left(V_{n+m}^{(\lambda)}(x, a; \eta)\right)$$

$$\geq v_n^{(\lambda)}(b) + \sum_y Q(\sigma_n(y))Q\left(V_m^{(\lambda)}(0, a; \tau_{n,y} \circ \eta)\right)$$

$$= v_n^{(\lambda)}(b) + \sum_y Q(\sigma_n(y))v_m^{(\lambda)}(a)$$

$$= v_n^{(\lambda)}(b) + v_m^{(\lambda)}(a)\left(\sum \sigma_n(y)\right)$$

$$= v_n^{(\lambda)}(b) + v_m^{(\lambda)}(a).$$

\[\square\]

**Step 2** There exists a function $I^{(\lambda)} : \mathbb{R} \to \mathbb{R}^+$ convex, non negative, Lipschitz with constant $\lambda$, such that

$$-\lim_{n \to \infty} \frac{1}{n} v_n^{(\lambda)}(a_n) = I^{(\lambda)}(\xi) \quad (\text{if } \frac{a_n}{n} \to \xi \in \mathbb{R}). \quad (5)$$

**Proof.** This is a standard subadditivity argument (see e.g. Varadhan [13] or Alexander [1]) combined with the Lipschitz property of $V^{(\lambda)}$: from $|H_n - a| \leq |H_n - b| + |a - b|$ we infer that

$$V_n^{(\lambda)}(0, a; \eta) \geq V_n^{(\lambda)}(0, a; \eta) + \lambda|a - b|.$$

\[\square\]

**Step 3** $Q$ almost surely, for any $\xi \in \mathbb{R}$,

$$\lim_{n \to \infty} -\frac{1}{n} \log P\left(e^{-\lambda|H_n-a_n|}\right) = I^{(\lambda)}(\xi). \quad (6)$$

**Proof.** Since the functions are Lipschitz, it is enough to prove that for any fixed $\xi \in \mathbb{Q}$, (6) holds a.s. This is where we use the restrictive assumptions made on the distribution of the environment. If the distribution of $\eta$ is with bounded support, or Gaussian, or more generally satisfies a logarithmic Sobolev inequality, then it has the gaussian concentration of measure
property (see [2]): for any 1-Lipschitz function $F$ of independent random variables distributed as $\eta$, 

$$
P(|F - P(F)| \geq r) \leq 2e^{-r^2/2} \quad (r > 0).
$$

It is easy to prove, as in Proposition 1.4 of [4], that the function 

$$(\eta(k,x), k \leq n, |x| \leq n) \rightarrow \log P\left( e^{-\lambda|H_n(\omega,\eta) - a|} \right)$$

is Lipschitz, with respect to the euclidean norm, with Lipschitz constant at most $\lambda \sqrt{n}$. Therefore, the Gaussian concentration of measure yields 

$$Q\left( \left| V_n^{(\lambda)}(0,a;\eta) - v_n^{(\lambda)}(a) \right| \geq u \right) \leq 2e^{-\frac{u^2}{2n}}.$$ 

We conclude by a Borel Cantelli argument combined with [5] 

$$\square$$

Observe that for fixed $\xi \in \mathbb{R}$, the function $\lambda \rightarrow I^{(\lambda)}(\xi)$ is increasing ; we shall consider the limit: 

$$I(\xi) = \lim_{\lambda \to +\infty} I^{(\lambda)}(\xi)$$

which is by construction non negative, convex and lower semi continuous.

**Step 4** The function $I$ satisfy [2] and [3].

**Proof.** Given, $\xi \in \mathbb{R}$ and $\lambda > 0, \delta > 0$, we have 

$$P\left( \left| \frac{H_n}{n}(\omega, \eta) - \xi \right| \leq \delta \right) = P\left( e^{-\lambda n} \frac{H_n}{n}(\omega, \eta) - \xi \right) \leq e^{\lambda n \delta} P\left( e^{-\lambda |H_n - n\xi|} \right).$$

Therefore, 

$$\limsup \frac{1}{n} \log \nu_n([\xi - \delta, \xi + \delta]) \leq \lambda \delta - I^{(\lambda)}(\xi)$$

$$\limsup \limsup_{\delta \to 0} \frac{1}{n} \log \nu_n([\xi - \delta, \xi + \delta]) \leq -I^{(\lambda)}(\xi)$$

and we obtain by letting $\lambda \to +\infty$, 

$$\limsup \limsup_{\delta \to 0} \frac{1}{n} \log \nu_n([\xi - \delta, \xi + \delta]) \leq -I(\xi).$$
Given \( \xi \in \mathbb{R} \) such that \( I(\xi) < +\infty \), and \( \delta > 0 \), we have for \( \lambda > 0 \),
\[
\mathbb{P}\left(\left|\frac{H_n}{n} - \xi\right| < \delta\right) \geq \mathbb{P}\left(e^{-\lambda|H_n - n\xi|}\right) - e^{-\lambda\delta n}.
\]
Hence, if we choose \( \lambda > 0 \) large enough such that \( \lambda\delta > I(\xi) \geq I^{(\lambda)}(\xi) \), we obtain
\[
\liminf_{n \to +\infty} \frac{1}{n} \log \nu_n((\xi - \delta, \xi + \delta)) \geq -I^{(\lambda)}(\xi) \geq -I(\xi)
\]
and therefore
\[
\liminf_{\delta \to 0} \liminf_{n \to +\infty} \frac{1}{n} \log \nu_n((\xi - \delta, \xi + \delta)) \geq -I(\xi).
\]

\( \square \)

References


