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CRITICAL POTENTIALS OF THE EIGENVALUES AND EIGENVALUE GAPS OF SCHRÖDINGER OPERATORS

AHMAD EL SOUFI AND NAZIH MOUKADEM

Abstract. Let $M$ be a compact Riemannian manifold with or without boundary, and let $-\Delta$ be its Laplace-Beltrami operator. For any bounded scalar potential $q$, we denote by $\lambda_i(q)$ the $i$-th eigenvalue of the Schrödinger type operator $-\Delta + q$ acting on functions with Dirichlet or Neumann boundary conditions in case $\partial M \neq \emptyset$. We investigate critical potentials of the eigenvalues $\lambda_i$ and the eigenvalue gaps $G_{ij} = \lambda_j - \lambda_i$, considered as functionals on the set of bounded potentials having a given mean value on $M$. We give necessary and sufficient conditions for a potential $q$ to be critical or to be a local minimizer or a local maximizer of these functionals. For instance, we prove that a potential $q \in L^\infty(M)$ is critical for the functional $\lambda_2$ if and only if, $q$ is smooth, $\lambda_2(q) = \lambda_3(q)$ and there exist second eigenfunctions $f_1, \ldots, f_k$ of $-\Delta + q$ such that $\sum_j f_j^2 = 1$. In particular, $\lambda_2$ (as well as any $\lambda_i$) admits no critical potentials under Dirichlet Boundary conditions. Moreover, the functional $\lambda_2$ never admits locally minimizing potentials.

1. Introduction and Statement of main Results

Let $M$ be a compact connected Riemannian manifold of dimension $d$, possibly with nonempty boundary $\partial M$, and let $-\Delta$ be its Laplace-Beltrami operator acting on functions with, in the case where $\partial M \neq \emptyset$, Dirichlet or Neumann boundary conditions. In all the sequel, as soon as the Neumann Laplacian will be considered, the boundary of $M$ will be assumed to be sufficiently regular (e.g. $C^1$, but weaker regularity assumptions may suffice, see [3]) in order to guarantee the compactness of the embedding $H^1(M) \hookrightarrow L^2(M)$ and, hence, the compactness of the resolvent of the Neumann Laplacian (note that it is well known, using standard arguments like in [13, p.89], that compactness results for Sobolev spaces on Euclidean domains remain valid in the Riemannian setting).

For any bounded real valued potential $q$ on $M$, the Schrödinger type operator $-\Delta + q$ has compact resolvent (see [10, Theorem IV.3.17] and observe that a bounded $q$ leads to a relatively compact operator with respect to $-\Delta$). Therefore, its spectrum consists of a nondecreasing and unbounded sequence of eigenvalues with finite multiplicities:

$$\text{Spec}(-\Delta + q) = \{\lambda_1(q) < \lambda_2(q) \leq \lambda_3(q) \leq \cdots \leq \lambda_i(q) \leq \cdots \}.$$  

Each eigenvalue $\lambda_i(q)$ can be considered as a (continuous) function of the potential $q \in L^\infty(M)$ and there are both physical and mathematical motivations to study existence and properties of extremal potentials of the

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functionals \( \lambda_i \) as well as of the differences, called gaps, between them. A very rich literature is devoted to the existence and the determination of maximizing or minimizing potentials for the eigenvalues (especially the fundamental one, \( \lambda_1 \)) and the eigenvalue gaps (especially the first one, \( \lambda_2 - \lambda_1 \)) under various constraints often motivated by physical considerations (see, for instance, [1, 2, 4, 6, 7, 10, 11, 12, 13, 17, 19] and the references therein).

Note that, since the function \( \lambda_i \) commutes with constant translations, that is, \( \lambda_i(q + c) = \lambda_i(q) + c \), such constraints are necessary.

Our aim in this paper is to investigate critical points, including ”local minimizers” and ”local maximizers”, of the eigenvalue functionals \( q \mapsto \lambda_i(q) \) and the eigenvalue gap functionals \( q \mapsto \lambda_j(q) - \lambda_i(q) \), the potentials \( q \) being subjected to the constraint that their mean value (or, equivalently, their integral) over \( M \) is fixed. All along this paper, the mean value of an integrable function \( q \) will be denoted \( \bar{q} \), that is,

\[ \bar{q} = \frac{1}{V(M)} \int_M q \, dv, \]

\( V(M) \) and \( dv \) being respectively the Riemannian volume and the Riemannian volume element of \( M \).

Actually, most of the results below can be extended, modulo some slight changes, to the case where this constraint is replaced by the more general one

\[ \int_M F(q) \, dv = \text{constant}, \]

where \( F : \mathbb{R} \to \mathbb{R} \) is a continuous function such that \( F'(x) \neq 0 \) if \( x \neq 0 \), like \( F(x) = |x|^\alpha \) or \( F(x) = x|x|^{\alpha-1} \) with \( \alpha \geq 1 \). However, for simplicity and clarity reasons, we preferred to focus only on the mean value constraint. Therefore, we fix a constant \( c \in \mathbb{R} \) and consider the functionals

\[ \lambda_i : q \in L^\infty_c(M) \mapsto \lambda_i(q) \in \mathbb{R}, \]

where \( L^\infty_c(M) = \{ q \in L^\infty(M) \mid \bar{q} = c \} \). The tangent space to \( L^\infty_c(M) \) at any point \( q \) is given by

\[ L^\infty_c(M) := \left\{ u \in L^\infty(M) \mid \int_M u \, dv = 0 \right\}. \]

1.1. Critical potentials of the eigenvalue functionals.

Since it is always nondegenerate, the first eigenvalue gives rise to a differentiable functional in the sense that, for any \( q \in L^\infty_c(M) \) and any \( u \in L^\infty_c(M) \), the function \( t \mapsto \lambda_1(q + tu) \) is differentiable in \( t \). A potential \( q \in L^\infty_c(M) \) will be termed critical for this functional if \( \frac{d}{dt} \lambda_1(q + tu) \big|_{t=0} = 0 \) for any \( u \in L^\infty_c(M) \).

In the case of empty boundary or of Neumann boundary conditions, the constant function \( 1 \) belongs to the domain of the operator \( -\Delta + q \) and one obtains, as a consequence of the min-max principle, that the constant potential \( c \) is a global maximizer of \( \lambda_1 \) over \( L^\infty_c(M) \) (see also [3] and [13]). Constant potential \( c \) is actually the only critical one for \( \lambda_1 \). On the other hand, under Dirichlet boundary conditions, the functional \( \lambda_1 \) admits no critical potentials in \( L^\infty_c(M) \). Indeed, we have the following
Theorem 1.1. (1) Assume that either \( \partial M = \emptyset \) or \( \partial M \neq \emptyset \) and Neumann boundary conditions are imposed. Then, for any potential \( q \) in \( L^\infty_c(M) \), we have

\[ \lambda_1(q) \leq \lambda_1(c) = c, \]

where the equality holds if and only if \( q = c \). Moreover, the constant potential \( c \) is the only critical one of the functional \( \lambda_1 \) over \( L^\infty_c(M) \).

(2) Assume that \( \partial M \neq \emptyset \) and that Zero Dirichlet boundary conditions are imposed. Then the functional \( \lambda_1 \) does not admit any critical potential in \( L^\infty_c(M) \).

Higher eigenvalues are continuous but not differentiable in general. Nevertheless, perturbation theory enables us to prove that, for any function \( u \in L^\infty(M) \), the function \( t \mapsto \lambda_i(q + tu) \) admits left and right derivatives at \( t = 0 \) (see section 2.2). A generalized notion of criticality can be naturally defined as follows:

Definition 1.1. A potential \( q \) is said to be critical for the functional \( \lambda_i \) if, for any \( u \in L^\infty_c(M) \), the left and right derivatives of \( t \mapsto \lambda_i(q + tu) \) at \( t = 0 \) have opposite signs, that is

\[ \left. \frac{d}{dt} \lambda_i(q + tu) \right|_{t=0^+} \times \left. \frac{d}{dt} \lambda_i(q + tu) \right|_{t=0^-} \leq 0. \]

It is immediate to check that \( q \) is critical for \( \lambda_i \) if and only if, for any \( u \in L^\infty_c(M) \), one of the two following inequalities holds:

\[ \lambda_i(q + tu) \leq \lambda_i(q) + o(t) \quad \text{as} \quad t \to 0 \]

or

\[ \lambda_i(q + tu) \geq \lambda_i(q) + o(t) \quad \text{as} \quad t \to 0. \]

In all the sequel, we will denote by \( E_i(q) \) the eigenspace corresponding to the \( i \)-th eigenvalue \( \lambda_i(q) \) whose dimension coincides with the number of indices \( j \in \mathbb{N} \) such that \( \lambda_j(q) = \lambda_i(q) \).

As for the first eigenvalue, the functionals \( \lambda_i, \ i \geq 2, \) admit no critical potentials under Dirichlet boundary conditions.

Theorem 1.2. Assume that \( \partial M \neq \emptyset \) and that Zero Dirichlet boundary conditions are imposed. Then, \( \forall i \in \mathbb{N}^* \), the functional \( \lambda_i \) does not admit any critical potential in \( L^\infty_c(M) \).

Under the two remaining boundary conditions, the following theorem gives a necessary condition for a potential \( q \) to be critical for the functional \( \lambda_i \). This condition is also sufficient for the indices \( i \) such that \( \lambda_i(q) > \lambda_{i-1}(q) \) or \( \lambda_i(q) < \lambda_{i+1}(q) \), which means that \( \lambda_i(q) \) is the first one or the last one in a cluster of equal eigenvalues.

Theorem 1.3. Assume that either \( \partial M = \emptyset \) or \( \partial M \neq \emptyset \) and Neumann boundary conditions are imposed. Let \( i \) be a positive integer.

If \( q \in L^\infty_c(M) \) is a critical potential of the functional \( \lambda_i \), then \( q \) is smooth and there exists a finite family of eigenfunctions \( f_1, \ldots, f_k \) in \( E_i(q) \) such that \( \sum_{1 \leq j \leq k} f_j^2 = 1. \)
Reciprocally, if \( \lambda_i(q) > \lambda_{i-1}(q) \) or \( \lambda_i(q) < \lambda_{i+1}(q) \), and if there exists a family of eigenfunctions \( f_1, \ldots, f_k \in E_i(q) \) such that \( \sum_{1 \leq j \leq k} f_j^2 = 1 \), then \( q \) is a critical potential of the functional \( \lambda_i \).

Note that the identity \( \sum_{1 \leq j \leq k} f_j^2 = 1 \), with \( f_1, \ldots, f_k \in E_i(q) \), immediately implies another one (that we obtain from \( \Delta \sum_{1 \leq j \leq k} f_j^2 = 0 \)):

\[
q = \lambda_i(q) - \sum_{1 \leq j \leq k} |\nabla f_j|^2,
\]

from which we can deduce the smoothness of \( q \).

**Remark 1.1.** 1. The identity \( \sum_{1 \leq j \leq k} f_j^2 = 1 \) with \(-\Delta f_j + q f_j = \lambda_i(q) f_j\), means that the map \( f = (f_1, \ldots, f_k) \) from \( M \) to the Euclidean sphere \( S^{k-1} \) is harmonic with energy density \( |\nabla f|^2 = \lambda_i(q) - q \) (see [2]). Hence, a necessary (and sometime sufficient) condition for a potential \( q \) to be critical for the functional \( \lambda_i \) is that the function \( \lambda_i(q) - q \) is the energy density of a harmonic map from \( M \) to a Euclidean sphere.

2. If one replaces the constraint on the mean value \( \frac{1}{|M|} \int_M q dv = c \) by the general constraint \( \int_M F(q) dv = c \), then the necessary and sufficient condition \( \sum_{1 \leq j \leq k} f_j^2 = 1 \) of Theorem 1.3 becomes (even under Dirichlet boundary conditions) \( \sum_{1 \leq j \leq k} f_j^2 = F'(q) \). In particular, \( q \) is a critical potential of the functional \( \lambda_i \) if and only if \( F'(q) \geq 0 \) and \( F'(q)^{\frac{1}{2}} \) is a first eigenfunction of \(-\Delta + q\), see [3] for a discussion of the case \( F(q) = |q|^\alpha \).

Under each one of the boundary conditions we consider, a constant function can never be an eigenfunction associated to an eigenvalue \( \lambda_i(q) \) with \( i \geq 2 \). Hence, an immediate consequence of Theorem 1.3 is the following

**Corollary 1.1.** If \( q \in L_c^{\infty}(M) \) is a critical potential of the functional \( \lambda_i \) with \( i \geq 2 \), then the eigenvalue \( \lambda_i(q) \) is degenerate, that is \( \lambda_i(q) = \lambda_{i-1}(q) \) or \( \lambda_i(q) = \lambda_{i+1}(q) \).

If \( \{f_1, \ldots, f_k\} \) is an \( L^2 \)-orthonormal basis of \( E_i(-\Delta) \), then the function \( \sum_{1 \leq j \leq k} f_j^2 \) is invariant under the isometry group of \( M \). Indeed, for any isometry \( \rho \) of \( M \), \( \{f_1 \circ \rho, \ldots, f_k \circ \rho\} \) is also an orthonormal basis of \( E_i(-\Delta) \) and then, there exists a matrix \( A \in O(d) \) such that \( (f_1 \circ \rho, \ldots, f_k \circ \rho) = A.(f_1, \ldots, f_k) \). In particular, if \( M \) is homogeneous, that is, the isometry group acts transitively on \( M \), then \( \sum_{1 \leq j \leq k} f_j^2 \) would be constant. Another consequence of Theorem 1.3 is then the following

**Corollary 1.2.** If \( M \) is homogeneous, then constant potentials are critical for all the functionals \( \lambda_i \) such that \( \lambda_i(-\Delta) < \lambda_{i+1}(-\Delta) \) or \( \lambda_i(-\Delta) > \lambda_{i-1}(-\Delta) \).

Recall that Euclidean spheres, projective spaces and flat tori are examples of homogeneous Riemannian spaces.

A potential \( q \in L_c^{\infty}(M) \) is said to be a local minimizer (resp. local maximizer) of the functional \( \lambda_i \) (in a weak sense) if, for any \( u \in L_c^{\infty}(M) \), the function \( t \mapsto \lambda_i(q + tu) \) admits a local minimum (resp. maximum) at \( t = 0 \). The result of Corollary 1.1 takes the following more precise form in the case of a local minimizer or maximizer.
Theorem 1.4. Let \( q \in L^\infty_c(M) \) and \( i \geq 2 \).

1. If \( q \) is a local minimizer of the functional \( \lambda_i \), then \( \lambda_i(q) = \lambda_{i-1}(q) \).
2. If \( q \) is a local maximizer of the functional \( \lambda_i \), then \( \lambda_i(q) = \lambda_{i+1}(q) \).

Since the first eigenvalue is simple, we always have \( \lambda_2(q) > \lambda_1(q) \). The previous results, applied to the functional \( \lambda_2 \) can be summarized as follows.

Corollary 1.3. Assume that either \( \partial M = \emptyset \) or \( \partial M \neq \emptyset \) and Neumann boundary conditions are imposed. A potential \( q \in L^\infty_c(M) \) is critical for the functional \( \lambda_2 \) if and only if, \( q \) is smooth, \( \lambda_2(q) = \lambda_3(q) \) and there exist eigenfunctions \( f_1, \ldots, f_k \) in \( E_2(q) \) such that \( \sum_{1 \leq j \leq k} f_j^2 = 1 \).

Moreover, the functional \( \lambda_2 \) admits no local minimizers in \( L^\infty_c(M) \).

In [8], Ilias and the first author have proved that, under some hypotheses on \( M \), satisfied in particular by compact rank-one symmetric spaces, irreducible homogeneous Riemannian spaces and some flat tori, the constant potential \( c \) is a global maximizer of \( \lambda_2 \) over \( L^\infty_c(M) \). In [3, 8], they studied the critical points of \( \lambda_1 \) considered as a functional on the set of Riemannian metrics of fixed volume on \( M \).

1.2. Critical potentials of the eigenvalue gaps functionals.

We consider now the eigenvalue gaps functionals \( q \mapsto G_{ij}(q) = \lambda_j(q) - \lambda_i(q) \), where \( i \) and \( j \) are two distinct positive integers, and define their critical potentials as in Definition 1.1. These functionals are invariant under translations, that is, \( G_{ij}(q + c) = G_{ij}(q) \). Therefore, critical potentials of \( G_{ij} \) with respect to fixed mean value deformations are also critical with respect to arbitrary deformations.

Theorem 1.5. If \( q \in L^\infty_c(M) \) is a critical potential of the gap functional \( G_{ij} = \lambda_j - \lambda_i \), then there exist a finite family of eigenfunctions \( f_1, \ldots, f_k \) in \( E_i(q) \) and a finite family of eigenfunctions \( g_1, \ldots, g_l \) in \( E_j(q) \), such that \( \sum_{1 \leq p \leq k} f_p^2 = \sum_{1 \leq p \leq l} g_p^2 \).

Reciprocally, if \( \lambda_1(q) < \lambda_{i+1}(q) \) and \( \lambda_j(q) > \lambda_{j-1}(q) \), and if there exist \( f_1, \ldots, f_k \) in \( E_i(q) \) and \( g_1, \ldots, g_l \) in \( E_j(q) \) such that \( \sum_{1 \leq p \leq k} f_p^2 = \sum_{1 \leq p \leq l} g_p^2 \), then \( q \) is a critical potential of \( G_{ij} \).

In the particular case of the gap between two consecutive eigenvalues, we have the following

Corollary 1.4. A potential \( q \in L^\infty_c(M) \) is critical for the gap functional \( G_{i,i+1} = \lambda_{i+1} - \lambda_i \) if and only if, either \( \lambda_{i+1}(q) = \lambda_i(q) \), or there exist a family of eigenfunctions \( f_1, \ldots, f_k \) in \( E_i(q) \) and a family of eigenfunctions \( g_1, \ldots, g_l \) in \( E_{i+1}(q) \), such that \( \sum_{1 \leq p \leq k} f_p^2 = \sum_{1 \leq p \leq l} g_p^2 \).

Remark 1.2. The characterization of critical potentials of \( G_{ij} \) given in Theorem 1.5 remains valid under the constraint \( \int_M F(q) dv = c \).

An immediate consequence of Theorem 1.5 is the following

Corollary 1.5. Let \( q \in L^\infty_c(M) \) be a critical potential of the gap functional \( G_{ij} = \lambda_j - \lambda_i \). If \( \lambda_i(q) \) (resp. \( \lambda_j(q) \)) is nondegenerate, then \( \lambda_j(q) \) (resp. \( \lambda_i(q) \)) is degenerate.
The following is an immediate consequence of the discussion above concerning homogeneous Riemannian manifolds.

**Corollary 1.6.** If $M$ is a homogeneous Riemannian manifold, then, for any positive integer $i$, constant potentials are critical points of the gap functional $G_{i,i+1} = \lambda_{i+1} - \lambda_i$.

Potentials $q$ such that $\lambda_{i+1}(q) = \lambda_i(q)$ are of course global minimizers of the gap functional $G_{i,i+1}$. These potentials are also the only local minimizers of $G_{i,i+1}$. Indeed, we have the following

**Theorem 1.6.** If $q \in L^\infty_c(M)$ is a local minimizer of the gap functional $G_{ij} = \lambda_j - \lambda_i$, then, either $\lambda_i(q) = \lambda_{i+1}(q)$, or $\lambda_j(q) = \lambda_{j+1}(q)$.

In particular, $q$ is a local minimizer of the gap functional $G_{i,i+1} = \lambda_{i+1} - \lambda_i$ if and only if $G_{i,i+1}(q) = 0$.

Finally, let us apply the results of this section to the first gap $G_{1,2}$.

**Corollary 1.7.** A potential $q \in L^\infty_c(M)$ is critical for the gap functional $G_{1,2} = \lambda_2 - \lambda_1$ if and only if $\lambda_2(q)$ is degenerate and there exists a family of eigenfunctions $g_1, \ldots, g_l$ in $E_2(q)$ such that $\sum_{1 \leq j \leq l} g_j = f$, where $f$ is a basis of $E_1(q)$.

The functional $G_{1,2}$ does not admit any local minimizer in $L^\infty_c(M)$.

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2. PROOF OF RESULTS

2.1. Variation Formula and proof of Theorem 1.1.

Given on $M$ a potential $q$ and a function $u \in L^\infty(M)$, we consider the family of operators $-\Delta + q + tu$. Suppose that $\Lambda(t)$ is a differentiable family of eigenvalues of $-\Delta + q + tu$ and that $f_t$ is a differentiable family of corresponding normalized eigenfunctions, that is, $\forall t$,

$$(-\Delta + q + tu)f_t = \Lambda(t)f_t,$$

and

$$\int_M f_t^2 dv = 1,$$

with $f_t|_{\partial M} = 0$ or $\frac{\partial f_t}{\partial v}|_{\partial M} = 0$ if $\partial M \neq \emptyset$. The following formula, giving the derivative of $\Lambda$, is already known at least in the case of Euclidean domains with Dirichlet boundary conditions.

**Proposition 2.1.**

$$\Lambda'(0) = \int_M u f_0^2 dv.$$

**Proof.** First, we have, for all $t$,

$$\Lambda(t) = \Lambda(t) \int_M (f_t)^2 dv = \int_M f_t(-\Delta + q + tu) f_t dv.$$

Differentiating at $t = 0$, we get

$$\Lambda'(0) = \frac{d}{dt} \left( \int_M f_t(-\Delta + q) f_t dv + t \int_M u(f_t)^2 dv \right) \bigg|_{t=0}.$$
Now, noticing that the function $\frac{d}{dt}f_t|_{t=0}$ satisfies the same boundary conditions as $f_0$ in case $\partial M \neq \emptyset$, and using integration by parts, we obtain
\[
\frac{d}{dt} \int_M f_t(-\Delta + q) f_t \, dv \bigg|_{t=0} = 2 \int_M (-\Delta + q) f_0 \frac{d}{dt} f_t \bigg|_{t=0} \, dv
\]
\[
= 2 \Lambda(0) \int_M f_0 \frac{d}{dt} f_t \bigg|_{t=0} \, dv
\]
\[
= \Lambda(0) \frac{d}{dt} \int_M f_t^2 \, dv \bigg|_{t=0} = 0.
\]
On the other hand, we have
\[
\frac{d}{dt} (t \int_M u f_t^2 \, dv) \bigg|_{t=0} = \int_M u f_0^2 \, dv + \left( t \int_M u \frac{d}{dt} f_t^2 \, dv \right) \bigg|_{t=0}
\]
\[
= \int_M u f_0^2 \, dv.
\]
Finally, $\Lambda'(0) = \int_M u f_0^2 \, dv$. □

Proof. (of Theorem 1.1) (i) First, let us show that constant potentials are maximizing for $\lambda_1$. Indeed, let $c$ be a constant potential and let $q$ be an arbitrary one in $L^\infty_0(M)$. From the variational characterization of $\lambda_1(-\Delta + q)$ in the case $\partial M = \emptyset$ as well as in the case of Neumann boundary conditions, we get
\[
\lambda_1(-\Delta + q) = \inf_{f \in H^1(M)} \frac{\int_M (|\nabla f|^2 + q f^2) \, dv}{\|f\|^2_{L^2(M)}}
\]
\[
\leq \frac{\int_M (|\nabla 1|^2 + q 1^2) \, dv}{\|1\|^2_{L^2(M)}} = \frac{\int_M q \, dv}{V(M)} = c.
\]
Hence, $\lambda_1(q) \leq \lambda_1(c)$ and the constant potential $c$ maximizes the functional $\lambda_1$ on $L^\infty_0(M)$. In particular, constant potentials are critical for this functional.

Now, suppose that $q \in L^\infty_0(M)$ is a critical potential for $\lambda_1$. For any $u \in L^\infty_0(M)$, we consider a differentiable family $f_t$ of normalized eigenfunctions corresponding to the first eigenvalue of $(-\Delta + q + tu)$ and apply the variation formula above to obtain:
\[
\frac{d}{dt} \lambda_1(q + tu) \bigg|_{t=0} = \int_M u f_0^2 \, dv.
\]
Hence, $\int_M u f_0^2 \, dv = 0$ for any $u \in L^\infty_0(M)$, which implies that $f_0$ is constant on $M$. Since $(-\Delta + q)f_0 = qf_0 = \lambda_1(q)f_0$, the potential $q$ must be constant on $M$.

(ii) Let $f_0$ be the first nonnegative Dirichlet eigenfunction of $-\Delta + q$ satisfying $\int_M f_0^2 \, dv = 1$. The function $u = V(M)f_0^2 - 1$ belongs to $L^\infty_0(M)$ and we have
\[
\frac{d}{dt} \lambda_1(q + tu) \bigg|_{t=0} = \int_M u f_0^2 \, dv = V(M) \int_M f_0^2 \, dv - 1 > 0,
\]
where the last inequality comes from Cauchy-Schwarz inequality and the fact that $f_0$ is not constant (recall that $f_0|_{\partial M} = 0$). Therefore, the potential $q$ is not critical for $\lambda_1$. □
2.2. Characterization of critical potentials. Let $i$ be a positive integer and let $m \geq 1$ be the dimension of the eigenspace $E_i(q)$ associated to the eigenvalue $\lambda_i(q)$. For any function $u \in L^\infty(M)$, perturbation theory of unbounded self-adjoint operators (see for instance Kato’s book [10]) that we apply to the one parameter family of operators $-\Delta + q + tu$, tells us that, there exists a family of $m$ eigenfunctions $f_{1,t}, \ldots, f_{m,t}$ associated with a family of $m$ (non ordered) eigenvalues $\Lambda_1(t), \ldots, \Lambda_m(t)$ of $-\Delta + q + tu$, all depending analytically in $t$ in some interval $(-\varepsilon, \varepsilon)$, and satisfying

- $\Lambda_1(0) = \cdots = \Lambda_m(0) = \lambda_i(q)$, 
- $\forall t \in (-\varepsilon, \varepsilon)$, the $m$ functions $f_{1,t}, \ldots, f_{m,t}$ are orthonormal in $L^2(M)$.

From this, one can easily deduce the existence of two integers $k \leq m$ and $l \leq m$, and a small $\delta > 0$ such that

\[ \lambda_i(q + tu) = \begin{cases} \Lambda_k(t) & \text{if } t \in (-\delta, 0) \\ \Lambda_l(t) & \text{if } t \in (0, \delta). \end{cases} \]

Hence, the function $t \mapsto \lambda_i(q + tu)$ admits a left sided and a right sided derivatives at $t = 0$ with

\[ \frac{d}{dt} \lambda_i(q + tu) \bigg|_{t=0^-} = \Lambda_k'(0) = \int_M u f_{k,0}^2 dv \]
and

\[ \frac{d}{dt} \lambda_i(q + tu) \bigg|_{t=0^+} = \Lambda_l'(0) = \int_M u f_{l,0}^2 dv. \]

To any function $u \in L^\infty(M)$ and any integer $i \in \mathbb{N}$, we associate the quadratic form $Q^i_u$ on $E_i(q)$ defined by

\[ Q^i_u(f) = \int_M u f^2 dv. \]

The corresponding symmetric linear transformation $L^i_u : E_i(q) \rightarrow E_i(q)$ is given by

\[ L^i_u(f) = P_i(uf), \]

where $P_i : L^2(M) \rightarrow E_i(q)$ is the orthogonal projection of $L^2(M)$ onto $E_i(q)$.

It follows immediately that

**Proposition 2.2.** If the potential $q$ is critical for the functional $\lambda_i$, then, $\forall u \in L^\infty(M)$, the quadratic form $Q^i_u(f) = \int_M u f^2 dv$ is indefinite on the eigenspace $E_i(q)$.

The following lemma enables us to establish a converse to this proposition.

**Lemma 2.1.** $\forall k, l \leq m$, we have

\[ \int_M u f_{k,0} f_{l,0} dv = \begin{cases} 0 & \text{if } k \neq l \\ \Lambda'_k(0) & \text{if } k = l. \end{cases} \]

In other words, $\Lambda'_1(0), \ldots, \Lambda'_m(0)$ are the eigenvalues of the symmetric linear transformation $L^i_u : E_i(q) \rightarrow E_i(q)$ and the functions $f_{1,0}, \ldots, f_{m,0}$ constitute an orthonormal eigenbasis of $L^i_u$. 
Proof. Differentiating at \( t = 0 \) the equality \((-\Delta + q + tu)f_{k,t} = \Lambda_k(t)f_{k,t}\), we obtain

\[
uf_{k,0} + (-\Delta + q) \frac{d}{dt}f_{k,t} \bigg|_{t=0} = \Lambda'_k(0)f_{k,0} + \Lambda_k(0)\frac{d}{dt}f_{k,t} \bigg|_{t=0},
\]

and then,

\[
\int_M uf_{k,0}f_{1,0} \, dv = \Lambda'_k(0) \int_M f_{k,0}f_{1,0} \, dv + \Lambda_k(0) \int_M f_{1,0} \frac{d}{dt}f_{k,t} \bigg|_{t=0} \, dv - \int_M f_{1,0}(-\Delta + q) \frac{d}{dt}f_{k,t} \bigg|_{t=0} \, dv.
\]

Integration by parts gives, after noticing that \( \Lambda_k(0) = \Lambda_l(0) = \lambda_i(q) \) and that the functions \( \frac{d}{dt}f_{k,t} \big|_{t=0} \) satisfy the considered boundary conditions,

\[
\int_M f_{1,0}(-\Delta + q) \frac{d}{dt}f_{k,t} \bigg|_{t=0} \, dv = \int_M \frac{d}{dt}f_{k,t} \bigg|_{t=0} \, dv(-\Delta + q)f_{1,0} \, dv = \Lambda_k(0) \int_M f_{1,0} \frac{d}{dt}f_{k,t} \bigg|_{t=0} \, dv,
\]

and finally,

\[
\int_M uf_{k,0}f_{1,0} \, dv = \Lambda'_k(0) \int_M f_{k,0}f_{1,0} \, dv = \Lambda'_k(0)\delta_{kl}.
\]

\(\square\)

**Proposition 2.3.** Assume that \( \lambda_i(q) > \lambda_{i-1}(q) \) or \( \lambda_i(q) < \lambda_{i+1}(q) \). Then the following conditions are equivalent:

i) the potential \( q \) is critical for \( \lambda_i \)

ii) \( \forall u \in L^\infty_+(M), the quadratic form \( Q^l_u(f) = \int_M uf^2 dv \) is indefinite on the eigenspace \( E_i(q) \).

iii) \( \forall u \in L^\infty_+(M), the linear transformation \( L^l_u \) admits eigenvalues of both signs.

**Proof.** Conditions (ii) and (iii) are clearly equivalent and the fact that (i) implies (ii) was established in Proposition 2.2. Let us show that (iii) implies (i). Assume that \( \lambda_i(q) > \lambda_{i-1}(q) \) and let \( u \in L^\infty_+(M) \) and \( \Lambda_1(t), \ldots, \Lambda_m(t) \) be as above. For small \( t \), we will have, for continuity reasons, \( \forall k \leq m, \Lambda_k(t) > \lambda_{i-1}(q + tu) \) and then, \( \lambda_i(q + tu) \leq \Lambda_k(t) \). Since \( \lambda_i(q + tu) \in \{\Lambda_1(t), \ldots, \Lambda_m(t)\} \), we get

\[
\lambda_i(q + tu) = \min_{k \leq m} \Lambda_k(t).
\]

It follows that

\[
\frac{d}{dt} \lambda_i(q + tu) \bigg|_{t=0^-} = \max_{k \leq m} \Lambda'_k(0)
\]

and

\[
\frac{d}{dt} \lambda_i(q + tu) \bigg|_{t=0^+} = \min_{k \leq m} \Lambda'_k(0).
\]

Thanks to Lemma 2.1, Condition (iii) implies that \( \min_{k \leq m} \Lambda'_k(0) \leq 0 \leq \max_{k \leq m} \Lambda'_k(0) \) which implies the criticality of \( q \).

The case \( \lambda_i(q) < \lambda_{i+1}(q) \) can be treated in a similar manner. \(\square\)
2.3. Proof of Theorems 1.2 and 1.3. Let $q$ be a potential in $L^\infty_c(M)$. To prove Theorem 1.2 we first notice that, since $f|_{\partial M} = 0$ for any $f \in E_i(q)$, the constant function 1 does not belong to the vector space $F$ generated in $L^2(M)$ by $\{f^2; f \in E_i(q)\}$. Hence, there exists a function $u$ orthogonal to $F$ and such that $\langle u, 1 \rangle_{L^2(M)} < 0$. The function $u_0 = u - \bar{u}$ belongs to $L^\infty_c(M)$ and the quadratic form $Q_{u_0}^i (f) = \int_M u_0 f^2 dv = -\bar{u} ||f||^2_{L^2(M)}$ is positive definite on $E_i(q)$. Hence, the potential $q$ is not critical for $\lambda_i$ (see Proposition 2.2).

The proof of Theorem 1.3 follows directly from the two propositions above and the following lemma.

**Lemma 2.2.** Let $i$ be a positive integer. The two following conditions are equivalent:

i) $\forall u \in L^\infty_c(M)$, the quadratic form $Q_u^i(f) = \int_M u f^2 dv$ is indefinite on the eigenspace $E_i(q)$.

ii) there exists a family of eigenfunctions $f_1, \ldots, f_k$ in $E_i(q)$ such that $\sum_{1 \leq j \leq k} f_j^2 = 1$.

**Proof.** To see that (i) implies (ii) we introduce the convex cone $C$ generated in $L^2(M)$ by the set $\{f^2; f \in E_i(q)\}$, that is $C = \{\sum_{j \in J} f_j^2; f_j \in E_i(q), J \subset \mathbb{N}, J \text{ is finite}\}$. Condition (ii) is then equivalent to the fact that the constant function 1 belongs to $C$. Let us suppose, for a contradiction, that $1 \notin C$. Then, applying classical separation theorems (in the finite dimensional vector subspace of $L^2(M)$ generated by $\{f^2; f \in E_i(q)\}$ and 1, see [18]), we prove the existence of a function $u \in L^2(M)$ such that $\bar{u} = \frac{1}{\int_M u \cdot 1 dv} \int_M u \cdot 1 dv < 0$ and $\int_M u f^2 dv \geq 0$ for any $f \in C$. Hence, the function $u_0 = u - \bar{u}$ belongs to $L^\infty_c(M)$ and satisfies, $\forall f \in E_i(q)$,

$$Q_{u_0}^i(f) = \int_M u f^2 dv - \frac{1}{V(M)} \int_M u dv \int_M f^2 dv \geq -\bar{u} ||f||^2_{L^2(M)}.$$

The quadratic form $Q_{u_0}^i$ is then positive definite which contradicts (i) (see Proposition 2.2).

Reciprocally, the existence of $f_1, \ldots, f_k$ in $E_i(q)$ satisfying $\sum_{1 \leq j \leq k} f_j^2 = 1$ implies that, $\forall u \in L^\infty_c(M)$,

$$\sum_{j \leq k} Q_u^i(f_j) = \sum_{j \leq k} \int_M u f_j^2 dv = \int_M u = 0,$$

which implies that the quadratic form $Q_u^i$ is indefinite on $E_i(q)$. \qed

Finally, let us check that the condition $\sum_{1 \leq j \leq k} f_j^2 = 1$, with $f_j \in E_i(q)$, implies that $q$ is smooth. Indeed, since $q \in L^\infty(M)$, we have, for any eigenfunction $f \in E_i(q)$, $\Delta f \in L^2(M)$ and then, $f \in H^{2,2}(M)$. Using standard regularity theory and Sobolev embeddings (see, for instance, [15]), we obtain by an elementary iteration, that $f \in H^{2,2}(M)$ for some $p > n$, and, then, $f \in C^1(M)$. From $\sum_{1 \leq j \leq k} f_j^2 = 1$ and $\Delta \sum_{1 \leq j \leq k} f_j^2 = 0$, we get

$$q = \lambda_i(q) - \sum_{1 \leq j \leq k} |\nabla f_j|^2,$$
which implies that $q$ is continuous. Again, elliptic regularity theory tells us that the eigenfunctions of $-\Delta + q$ are actually smooth, and, hence, $q$ is smooth.

2.4. Proof of Theorem 1.4. Assume that the potential $q$ is a local minimizer of the functional $\lambda_i$ on $L_c^\infty(M)$ and let us suppose for a contradiction that $\lambda_i(q) > \lambda_{i-1}(q)$. Let $u$ be a function in $L_c^\infty(M)$ and let $\Lambda_1(t), \ldots, \Lambda_m(t)$ be a family of $m$ eigenvalues of $-\Delta + q + tu$, where $m$ is the multiplicity of $\lambda_i(q)$, depending analytically in $t$ and such that $\Lambda_1(0) = \cdots = \Lambda_m(0) = \lambda_i(q)$. For continuity reasons, we have, for sufficiently small $t$ and any $k \leq m$, $\Lambda_k(t) > \lambda_{i-1}(q + tu)$. Hence, $\forall k \leq m$ and $\forall t$ sufficiently small,

$$\Lambda_k(t) \geq \lambda_i(q + tu) \geq \lambda_i(q) = \Lambda_k(0).$$

Consequently, $\forall k \leq m, \Lambda_k(0) = 0$. Applying Lemma 2.1 above we deduce that the symmetric linear transformation $L_i^t$ and then the quadratic form $Q_i^t$ is identically zero on the eigenspace $E_i(q)$. Therefore, $\forall u \in L_c^\infty(M)$ and $\forall f \in E_i(q)$, we have $\int_M u^2 v_g = 0$. In conclusion, $\forall f \in E_i(q)$, $f$ is constant on $M$ which is impossible for $i \geq 2$. The same arguments work to prove Assertion (ii).

2.5. Proof of Theorem 1.5. Let $q$ be a potential and let $i$ and $j$ be two distinct positive integers such that $\lambda_i(q) \neq \lambda_j(q)$. We denote by $m$ (resp. $n$) the dimension of the eigenspace $E_i(q)$ (resp. $E_j(q)$). Given a function $u$ in $L_c^\infty(M)$, we consider, as above, $m$ (resp. $n$) $L^2(M)$-orthonormal families of eigenfunctions $f_{1,t}, \ldots, f_{m,t}$ (resp. $g_{1,t}, \ldots, g_{n,t}$) associated with $m$ (resp. $n$) families of eigenvalues $\Lambda_1(t), \ldots, \Lambda_m(t)$ (resp. $\Gamma_1(t), \ldots, \Gamma_n(t)$) of $-\Delta + q + tu$, all depending analytically in $t \in (-\varepsilon, \varepsilon)$, such that $\Lambda_1(0) = \cdots = \Lambda_m(0) = \lambda_i(q)$ (resp. $\Gamma_1(0) = \cdots = \Gamma_n(0) = \lambda_j(q)$). Hence, there exist four integers $k \leq m, k' \leq m, l \leq n$ and $l' \leq n$, such that

$$\frac{d}{dt}(\lambda_j - \lambda_i)(q + tu)\bigg|_{t=0^-} = \Gamma_j'(0) - \Lambda_k'(0)$$

and

$$\frac{d}{dt}(\lambda_j - \lambda_i)(q + tu)\bigg|_{t=0^+} = \Gamma_j'(0) - \Lambda_{k'}'(0).$$

Recall that (Lemma 2.1) the eigenfunctions $f_{1,0}, \ldots, f_{m,0}$ (resp. $g_{1,0}, \ldots, g_{n,0}$) constitutes an $L^2(M)$-orthonormal basis of $E_i(q)$ (resp. $E_j(q)$) which diagonalizes the quadratic form $Q_u(i)$ (resp. $Q_u(j)$). Therefore, the family $(f_{k,0} \otimes g_{l,0})_{k \leq m, l \leq n}$ constitutes a basis of the space $E_i(q) \otimes E_j(q)$ which diagonalizes the quadratic form $S_u^{ij}(f \otimes g)$ given by

$$S_u^{ij}(f \otimes g) = \|f\|^2_{L^2(M)}Q_u^i(g) - \|g\|^2_{L^2(M)}Q_u^j(f)$$

$$= \int_M u(\|f\|^2_{L^2(M)}g^2 - \|g\|^2_{L^2(M)}f^2)dv.$$
The corresponding eigenvalues are \((\Gamma'_i(0) - \Lambda'_k(0))_{k \leq m, l \leq n}\). The criticality of \(q\) for \(\lambda_j - \lambda_i\) then implies that this quadratic form admits eigenvalues of both signs, which means that it is indefinite.

On the other hand, in the case where \(\lambda_i(q) < \lambda_{i+1}(q)\) and \(\lambda_j(q) > \lambda_{j-1}(q)\), we have, as in the proof of Lemma 2.2, separation theorems enable us to prove that, for sufficiently small \(t\), \(\lambda_i(q+tu) = \min_{l \leq m} \Lambda_k(t)\) and \(\lambda_j(q+tu) = \min_{l \leq n} \Gamma_l(t)\), which yields

\[
\frac{d}{dt}(\lambda_j - \lambda_i)(q + tu)\big|_{t=0^-} = \max_{l \leq n} \Gamma'_l(0) - \min_{k \leq m} \Lambda'_k(0)
\]

and

\[
\frac{d}{dt}(\lambda_j - \lambda_i)(q + tu)\big|_{t=0^+} = \min_{l \leq n} \Gamma'_l(0) - \max_{k \leq m} \Lambda'_k(0)
\]

One deduces the following

**Proposition 2.4.** If the potential \(q \in L^\infty_c(M)\) is critical for the functional \(G_{ij} = \lambda_j - \lambda_i\), then, \(\forall u \in L^\infty_c(M)\), the quadratic form \(S_u^{i,j}\) is indefinite on \(E_i(q) \otimes E_j(q)\).

Reciprocally, if \(\lambda_i(q) < \lambda_{i+1}(q)\) and \(\lambda_j(q) > \lambda_{j-1}(q)\), and if, \(\forall u \in L^\infty_c(M)\), the quadratic form \(S_u^{i,j}(q)\) is indefinite on \(E_i(q) \otimes E_j(q)\), then \(q\) is a critical potential of the functional \(G_{ij}\).

The following lemma will completes the proof of Theorem 1.5

**Lemma 2.3.** The two following conditions are equivalent:

1. \(\forall u \in L^\infty_c(M)\), the quadratic form \(S_u^{i,j}\) is indefinite on \(E_i(q) \otimes E_j(q)\).
2. There exist a finite family of eigenfunctions \(f_1, \ldots, f_k\) in \(E_i(q)\) and a finite family of eigenfunctions \(g_1, \ldots, g_l\) in \(E_j(q)\), such that \(\sum_{1 \leq p \leq k} f_p^2 = \sum_{1 \leq p \leq l} g_p^2\).

The proof of this lemma is similar to that of Lemma 2.2. Here, we consider the two convex cones \(C_i\) and \(C_j\) in \(L^2(M)\) generated respectively by \(\{f^2 : f \in E_i(q), f \neq 0\}\) and \(\{g^2 : g \in E_j(q), g \neq 0\}\). Condition (ii) is then equivalent to the fact that these two cones admit a nontrivial intersection. In the proof of Lemma 2.2, separation theorems enable us to prove that, if \(C_i \cap C_j = \emptyset\), then there exists a function \(u\) such that \(\int_M u f^2 dv < 0\) for any \(f \in E_i(q)\), and \(\int_M u g^2 dv \geq 0\) for any \(g \in E_j(q)\), which implies that \(S_u^{i,j}\) is positive definite on \(E_i(q) \otimes E_j(q)\). Since \(S_u^{i,j} = 0\), we have, \(S_u^{i,j} = S_u^{i,j}\) with \(w_0 = u - \bar{u} \in L^\infty_c(M)\). Proposition 2.4 enables us to conclude.

Reciprocally, assume the existence of \(f_1, \ldots, f_k \in E_i(q)\) and \(g_1, \ldots, g_l \in E_j(q)\) satisfying \(\sum_{1 \leq p \leq k} f_p^2 = \sum_{1 \leq p \leq l} g_p^2\). Then, \(\forall u \in L^\infty_c(M)\),

\[
\sum_{1 \leq p \leq k} \sum_{1 \leq p' \leq l} S_u^{i,j}(f_p \otimes g_{p'}) = \cdots = 0,
\]

which implies that \(S_u^{i,j}\) is indefinite on \(E_i(q) \otimes E_j(q)\).
2.6. **Proof of Theorem 1.6.** Let \( q \) be a local minimizer of \( G_{ij} = \lambda_j - \lambda_i \) and let us suppose for a contradiction that \( \lambda_i(q) < \lambda_{i+1}(q) \) and \( \lambda_j(q) > \lambda_{j-1}(q) \). Given a function \( u \) in \( L^2(\mathcal{M}) \), we consider, as above, \( m \) (resp. \( n \)) families of eigenvalues \( \Lambda_1(t), \ldots, \Lambda_m(t) \) (resp. \( \Gamma_1(t), \ldots, \Gamma_n(t) \)) of \(-\Delta + q + tu\), with \( m = \dim E_1(q) \) and \( n = \dim E_j(q) \), such that \( \Lambda_1(0) = \cdots = \Lambda_m(0) = \lambda_i(q) \) and \( \Gamma_1(0) = \cdots = \Gamma_n(0) = \lambda_j(q) \). As in the proof of Theorem 1.4, we will have for sufficiently small \( t \), \( \lambda_i(q + tu) = \max_{k \leq m} \Lambda_k(t) \) and \( \lambda_j(q + tu) = \min_{l \leq n} \Gamma_l(t) \). Hence, \( \forall k \leq m \) and \( l \leq n \),

\[
\Gamma_l(t) - \Lambda_k(t) \geq \lambda_j(q + tu) - \lambda_i(q + tu) = G_{ij}(q + tu) \geq G_{ij}(q) = \Gamma_l(0) - \Lambda_k(0).
\]

It follows that, \( \forall k \leq m \) and \( l \leq n \), \( \Gamma_l(0) - \Lambda_k(0) = 0 \) and, then, the quadratic form \( S_u^{ij} \) is identically zero on \( E_i(q) \otimes E_j(q) \) (recall that \( \Gamma_l(0) - \Lambda_k(0) \) are the eigenvalues of \( S_u^{ij} \)). This implies that, \( \forall f \in E_i(q) \) and \( \forall g \in E_j(q) \), the function \(|f|^2_{L^2(\mathcal{M})} g^2 - |g|^2_{L^2(\mathcal{M})} f^2 \) is constant equal to zero (since its integral vanishes) which is clearly impossible unless \( i = j \).

**References**


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