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ABSTRACT. We prove the uniqueness of the renormalized solution to the elliptic equation $-\mathrm{div}(\mathbf{A}(x,u)Du) = f + \mathrm{div}(g)$. The data $f + \mathrm{div}(g)$ belongs to $L^1 + H^{-1}$ and we assume a local condition on the diffusion matrix $\mathbf{A}(x,s)$ with respect to s.

1. INTRODUCTION

The present paper is concerned with the uniqueness of the solution to the quasilinear elliptic boundary–value problem on Ω

$$\begin{cases} -\mathrm{div}(\mathbf{A}(x,u)Du) = f + \mathrm{div}(g) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $f \in L^1(\Omega)$, $g \in (L^2(\Omega))^N$ and $\mathbf{A}(x,s)$ is a Carathéodory function with matrix values.

When f belongs to $L^2(\Omega)$ (i.e. the right-hand side of (1.1) lies in $H^{-1}(\Omega)$) the variational solution of (1.1) is unique under a global Lipschitz condition on the function $\mathbf{A}(x,s)$ with respect to the variable s (or a global and strong control of the modulus of continuity), see Artola [1986], Carrillo and Chipot [1985] and for more general and nonlinear operators Boccardo et al. [1992], Chipot and Michaille [1989]. Moreover in Carrillo and Chipot [1985], Chipot and Michaille [1989] the authors show that if $\mathbf{A}(x,s)$ is Hölder continuous in s with a Hölder exponent greater of equal to $1/2$ and if $\mathbf{A}(x,s)$ is Lipschitz continuous in x then the solution is unique. For this last result the quasilinear character of the equation and the regularity of $\mathbf{A}(x,s)$ in x are crucial.

In the case where f lies in $L^1(\Omega)$ and if $\mathbf{A}$ is uniformly coercive we cannot expect to have a solution of (1.1) in the sense of distributions without any growth condition on $\mathbf{A}(x,s)$ with respect to s. Moreover it is well know that, in the simple case where $\mathbf{A}$ does not depend on s, a solution in the sense of distribution exists (see e.g. Boccardo and Gallouët [1989]) but it is not unique in general (see the counter example in Serrin [1964]). In the present paper we use the framework of renormalized solution (see Dal Maso et al. [1999], Murat [1993, 1994]) which insures the existence of such a solution when f belongs to $L^1(\Omega)$, $\mathbf{A}$ is uniformly coercive and $\mathbf{A} \in L^\infty(\Omega \times]-K,K[)^{N\times N}$ for any $K > 0$.

Uniqueness results have been recently obtained in Blanchard et al. [2005] in the framework of renormalized solutions and in Porretta [2004] in the very close framework of entropy solutions for equations (1.1) with f belonging to L^1 with very general

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and global conditions on the matrix field **A**. Roughly speaking the modulus of continuity of **A** with respect to s has to be controlled by $\exp(c|s|)$ $(c>0)$ in Porretta [2004] and by a function which satisfies an appropriate differential inequality in Blanchard et al. [2005].

In the present paper we state in Theorem 3.2 that the renormalized solution of (1.1) is unique if **A** is locally Hölder continuous in s with a Hölder exponent greater or equal to 1/2 and under a global control of the modulus of continuity of **A** with respect to the space variable x. The main novelty between our and known uniqueness results is the very local condition on **A**, i.e. we do not assume any control on the growth of the modulus of continuity of A in s as in the above cited papers. The price to pay to get rid of this global behavior is to assume a regularity with respect to x. The results obtained in the present paper rely on the mixing of the assumptions and the techniques developed in Carrillo and Chipot [1985] (see also Chipot and Michaille [1989]) with those used to study L^1–problems with the help of renormalized solutions (see Lemma 3.3 and Remark 3.4).

At last the question of the uniqueness under a local condition is s remains still open in general.

The paper is organized as follows. In Section 2 we give the assumptions on the data and we recall the definition of a renormalized solution of (1.1). Section 3 is devoted to a comparison result stated in Theorem 3.1 which implies the uniqueness of the solution given in Theorem 3.2.

2. Assumptions and definitions

In the whole paper we assume that $\mathbf{A}:\Omega\times\mathbb{R}\mapsto\mathbb{R}^{N\times N}$ is a Carathéodory function with $\mathbf{A}(x,s)=(a_{ij}(x,s))_{1\le i,j\le N}$ and such that

$$\exists\alpha>0,\quad \mathbf{A}(x,s)\xi\cdot\xi\ge\alpha|\xi|^2,\quad \forall\xi\in\mathbb{R}^N,\ \forall s\in\mathbb{R},\text{ a.e. in }\Omega; \tag{2.1}$$

$$\forall K>0,\quad \exists C_K>0\quad |\mathbf{A}(x,s)|\le C_K,\quad \forall s\in[-K,K],\text{ a.e. in }\Omega; \tag{2.2}$$

for any r in $\mathbb{R}$ and any $1\le i,j\le N$, the function $a_{ij}(r,\cdot)$ belongs to $W^{1,\infty}(\Omega)$ and there exists $M>0$ such that

$$\Big|\frac{\partial a_{ij}}{\partial x_k}(x,r)\Big|\le M\sum_{1\le i,j\le N}a_{i,j}(x,r),\quad \forall r\in\mathbb{R},\ \forall 1\le i,j\le N,\text{ a.e. in }\Omega. \tag{2.3}$$

Moreover we assume that for any $K>0$ there exists a nonnegative, non decreasing continuous function ω_K such that

$$|\mathbf{A}(x,s)-\mathbf{A}(x,r)|\le\omega_K(|s-r|)\quad \forall r,s\in\mathbb{R}\text{ with }|s|\le K,\ |r|\le K,\text{ a.e. in }\Omega; \tag{2.4}$$

$$\int_{0^+}\frac{ds}{\omega_K^2(s)}=+\infty. \tag{2.5}$$

The data f and g are such that

$$f\in L^1(\Omega); \tag{2.6}$$

$$g\in(L^2(\Omega))^N. \tag{2.7}$$

Remark 2.1. Assumptions (2.1) and (2.2) are classical in the framework of renormalized solutions and allow to obtain the existence of such a solution with a data belonging to $L^1 + H^{-1}$. Conditions (2.4) and (2.5) concern a local condition on the modulus of continuity of the matrix field $\mathbf{A}(x,s)$ in s. If the matrix field $\mathbf{A}(x,s)$ does not depend on x and is locally Hölder continuous with an exponent greater or equal to $1/2$ then assumptions (2.3), (2.4) and (2.5) are satisfied. Assumption (2.3) is crucial when $\mathbf{A}(x,s)$ depends on x. As an example, if b is an element of $W^{1,\infty}(\Omega)$ and h is a non negative locally Hölder continuous function with an exponent greater or equal to $1/2$, then $\mathbf{A}(x,s) = (\exp(s^2) + b(x)h(s))I$ verifies (2.1)–(2.5).

Remark 2.2. Since ω_K is a nonnegative, non decreasing continuous function satisfying (2.5) we can assume without loss of generality that there exists C_K such that

$$\forall 0 < r < 1, \quad \frac{r}{\omega_K^2(r)} \le C_K. \tag{2.8}$$

Indeed it is sufficient to take $\omega_K(r) + \sqrt{r}$ in place of ω_K in (2.4) which also verifies condition (2.5).

For any $K > 0$ we denote by T_K the truncation function at height $\pm K$, $T_K(s) = \max(-K, \min(K,s))$ for any $s \in \mathbb{R}$ and we define the continuous function h_n by

$$h_n(s) = 1 - \left|\frac{T_{2n}(s) - T_n(s)}{n}\right|. \tag{2.9}$$

We now recall the definition of the gradient of functions whose truncates belong to $H_0^1(\Omega)$ (see Bénilan et al. [1995]).

Definition 2.3. Let $u : \Omega \mapsto \mathbb{R}$ be a measurable function, finite almost everywhere in Ω, such that $T_K(u) \in H_0^1(\Omega)$ for any $K > 0$. Then there exists a unique measurable vector field $v : \Omega \mapsto \mathbb{R}^N$ such that

$$DT_K(u) = \mathbb{1}_{\{|u|<K\}} v \quad \text{a.e. in } \Omega.$$

This function v is called the gradient of u and is denoted by Du.

Following Dal Maso et al. [1999] (see also Murat [1993, 1994]) we now recall the definition of a renormalized solution to (1.1).

Definition 2.4. A measurable function u defined from Ω into $\mathbb{R}$ is called a renormalized solution of (1.1) if

$$\forall K > 0, \quad T_K(u) \in H_0^1(\Omega); \tag{2.10}$$

if for any function $h \in W^{1,\infty}(\mathbb{R})$ such that $\operatorname{supp} h$ is compact, u satisfies the equation

$$\begin{aligned} -\operatorname{div}\big[h(u)\mathbf{A}(x,u)Du\big] &+ h'(u)\mathbf{A}(x,u)Du\cdot Du \\ &= fh(u) + \operatorname{div}(gh(u)) - h'(u)g\cdot Du \quad \text{in } \mathscr{D}'(\Omega), \end{aligned} \tag{2.11}$$

$$\lim_{n\to+\infty} \frac{1}{n}\int_{n<|u|<2n} \mathbf{A}(x,u)Du\cdot Du\,dx = 0 \tag{2.12}$$

Remark 2.5. Condition (2.10) and Definition 2.3 allow to define Du almost everywhere in Ω. In (2.11) which is formally obtained by the point-wise multiplication of (1.1) by $h(u)$ every terms are well defined. Indeed since $\text{supp}(h)$ is compact, we have $\text{supp}(h) \subset [-K,K]$ for $K>0$ sufficiently large. It follows that $h(u)\mathbf{A}(x,u)Du = h(u)\mathbf{A}(x,T_K(u))DT_K(u)$ almost everywhere in Ω and then it belongs to $(L^2(\Omega))^N$. Similarly $h'(u)\mathbf{A}(x,u)Du\cdot Du$ is identified to $h'(u)\mathbf{A}(x,T_K(u))DT_K(u)\cdot DT_K(u)$ which belongs to $L^1(\Omega)$. The same arguments imply that the right hand side of (2.11) lies in $L^1(\Omega)+H^{-1}(\Omega)$. Condition (2.12) is classical in the framework of renormalized solutions and gives additional information on Du for large value of $|u|$.

It is well know that under assumptions (2.1), (2.2), (2.6) and (2.7) there exists at least one renormalized solution to equation (1.1), see e.g. Blanchard et al. [2005], Lions and Murat, Murat [1993, 1994].

3. Main result

In Theorem 3.1 below we give a comparison result from which it follows a uniqueness result.

Theorem 3.1. *Assume that* (2.1)–(2.5) *hold true. Let* f_1 *and* f_2 *belong to* $L^1(\Omega)$ *and let* g_1 *and* g_2 *belong to* $(L^2(\Omega))^N$ *such that*

$$f_1 + \text{div}(g_1) \le f_2 + \text{div}(g_2) \quad \textit{in } \mathscr{D}'(\Omega) \tag{3.1}$$

Let u_1 *be a renormalized solution of* (1.1) *with* (f_1,g_1) *in place of* (f,g) *and let* u_2 *be a renormalized solution of* (1.1) *with* (f_2,g_2) *in place of* (f,g). *Then* $u_1 \le u_2$ *almost everywhere in* Ω.

An immediate consequence is the uniqueness of the solution for a fixed data $f + \text{div}(g) \in L^1(\Omega)+H^{-1}(\Omega)$.

Theorem 3.2. *Assume that* (2.1)–(2.7) *hold true. Then the renormalized solution of* (1.1) *is unique.*

To prove Theorem 3.1 we mix the methods developed by Chipot and Carillo in Carrillo and Chipot [1985] (see also Chipot and Michaille [1989]) together with the techniques of renormalized solutions. The main tool is the following lemma which is a truncated version to Theorem 4 in Carrillo and Chipot [1985].

Lemma 3.3. *For any* φ *belonging to* $\mathscr{C}^1(\overline{\Omega})$

$$\lim_{n\to+\infty}\int_{\{u_1-u_2>0\}} \big(h_n(u_1)\mathbf{A}(x,u_1)Du_1 - h_n(u_2)\mathbf{A}(x,u_2)Du_2\big)\cdot D\varphi\, dx = 0. \tag{3.2}$$

Remark 3.4. In Carrillo and Chipot [1985] (see also Chipot and Michaille [1989]) when $f+\text{div}(g)$ belongs to $H^{-1}(\Omega)$ and under more restrictive conditions on the matrix field $\mathbf{A}(x,s)$ (roughly speaking $\mathbf{A}(x,s)$ is bounded) the authors state that

$$\int_{\{u_1-u_2>0\}} \big(\mathbf{A}(x,u_1)Du_1 - \mathbf{A}(x,u_2)Du_2\big)\cdot D\varphi\, dx = 0.$$

In the $L^1(\Omega)+H^{-1}(\Omega)$ case, since we do not have any growth assumption on $\mathbf{A}(x,s)$ with respect to s, we cannot expect to have $\mathbf{A}(x,u_1)Du_1$ in $L^1_{\text{loc}}(\Omega)$. It follows that the above equality does not have any sense or equivalently that the limit in (3.2) cannot be written in terms of u_1 and u_2.

Proof of Lemma 3.3. Let φ belong to $\mathscr{C}^1(\overline{\Omega})$ with $\varphi \geq 0$ on Ω and let n be a positive integer. Assume without loss of generality that $\omega_{2n+1}(r) > 0\ \forall r > 0$. Following Chipot and Michaille [1989] let us define for any $0 < \varepsilon < 1$

$$I_n(\varepsilon) = \int_\varepsilon^1 \frac{1}{\omega_{2n+1}^2(s)}\,ds, \tag{3.3}$$

$$F_n^\varepsilon(r) = \begin{cases} 1 & \text{if } r \geq 1, \\ \dfrac{1}{I_n(\varepsilon)} \displaystyle\int_\varepsilon^r \frac{1}{\omega_{2n+1}^2(s)}\,ds & \text{if } 1 > r > \varepsilon, \\ 0 & \text{if } r \leq \varepsilon. \end{cases} \tag{3.4}$$

From the definition of F_n^ε and the regularity of ω_{2n+1} it follows that F_n^ε is a nonnegative Lipschitz continuous function such that $F_n^\varepsilon(s) = 0\ \forall s \leq \varepsilon$.

Let us consider the test function $W_n^\varepsilon = F_n^\varepsilon(T_1(T_{2n+1}(u_1) - T_{2n+1}(u_2)))\varphi$ which belongs to $L^\infty(\Omega) \cap H_0^1(\Omega)$ due to (2.10) and the regularities of F_n^ε and φ. Moreover we have

$$\begin{aligned} DW_n^\varepsilon = {} & F_n^\varepsilon(T_1(T_{2n+1}(u_1) - T_{2n+1}(u_2)))D\varphi + \varphi T_1'(T_{2n+1}(u_1) - T_{2n+1}(u_2)) \\ & \times (F_n^\varepsilon)'(T_1(T_{2n+1}(u_1) - T_{2n+1}(u_2)))(DT_{2n+1}(u_1) - DT_{2n+1}(u_2)) \end{aligned}$$

almost everywhere in Ω.

Choosing $h = h_n$ in (2.11) written in u_1 yields

$$\begin{aligned} & \int_\Omega h_n(u_1)\mathbf{A}(x,u_1)Du_1 \cdot DW_n^\varepsilon dx + \int_\Omega h_n'(u_1)\mathbf{A}(x,u_1)Du_1 \cdot Du_1 W_n^\varepsilon dx \\ & \quad = \int_\Omega h_n(u_1) f_1 W_n^\varepsilon dx - \int_\Omega h_n(u_1) g_1 \cdot DW_n^\varepsilon dx - \int_\Omega h_n'(u_1) g_1 \cdot Du_1 W_n^\varepsilon dx. \end{aligned} \tag{3.5}$$

Since $\mathrm{supp}(h_n) = [-2n, 2n]$ we have $h_n(u_1)F_n^\varepsilon(T_1(T_{2n+1}(u_1) - T_{2n+1}(u_2))) = h_n(u_1)F_n^\varepsilon(T_1(u_1 - u_2))$ almost everywhere in Ω and

$$h_n(u_1)\mathbb{1}_{\{|T_{2n+1}(u_1) - T_{2n+1}(u_2)|<1\}}(DT_{2n+1}(u_1) - DT_{2n+1}(u_2)) = h_n(u_1)\mathbb{1}_{\{|u_1-u_2|<1\}}(Du_1 - Du_2)$$

almost everywhere in Ω. It follows that (3.5) can be rewritten as

$$\begin{aligned} & \int_{\{|u_1-u_2|<1\}} h_n(u_1)\mathbf{A}(x,u_1)Du_1 \cdot (Du_1 - Du_2)(F_n^\varepsilon)'(T_1(u_1 - u_2))\varphi dx \\ & + \int_\Omega h_n(u_1)\mathbf{A}(x,u_1)Du_1 \cdot D\varphi F_n^\varepsilon(T_1(u_1 - u_2))dx + \int_\Omega h_n'(u_1)\mathbf{A}(x,u_1)Du_1 \cdot Du_1 F_n^\varepsilon(T_1(u_1 - u_2))\varphi dx \\ & \quad = \int_\Omega h_n(u_1) f_1 W_n^\varepsilon dx - \int_\Omega h_n(u_1) g_1 \cdot DW_n^\varepsilon dx - \int_\Omega h_n'(u_1) g_1 \cdot Du_1 F_n^\varepsilon(T_1(u_1 - u_2))\varphi dx. \end{aligned}$$

Subtracting the equivalent equality written in u_2 gives

$$\begin{aligned}
&\int_{\{|u_1-u_2|<1\}} (h_n(u_1)\mathbf{A}(x,u_1)Du_1 - h_n(u_2)\mathbf{A}(x,u_2)Du_2)\cdot(Du_1-Du_2)(F_n^\varepsilon)'(T_1(u_1-u_2))\varphi\,dx\\
&\qquad\quad +\int_\Omega (h_n(u_1)\mathbf{A}(x,u_1)Du_1 - h_n(u_2)\mathbf{A}(x,u_2)Du_2)\cdot D\varphi F_n^\varepsilon(T_1(u_1-u_2))\,dx\\
&\quad +\int_\Omega (h_n'(u_1)\mathbf{A}(x,u_1)Du_1\cdot Du_1 - h_n'(u_2)\mathbf{A}(x,u_2)wDu_2\cdot Du_2)F_n^\varepsilon(T_1(u_1-u_2))\varphi\,dx\\
&\qquad\quad =\int_\Omega (h_n(u_1)f_1-h_n(u_2)f_2)W_n^\varepsilon\,dx - \int_\Omega (h_n(u_1)g_1-h_n(u_2)g_2)\cdot DW_n^\varepsilon\,dx\\
&\qquad\qquad -\int_\Omega (h_n'(u_1)g_1\cdot Du_1 - h_n'(u_2)g_2\cdot Du_2)F_n^\varepsilon(T_1(u_1-u_2))\varphi\,dx,
\end{aligned}$$

which reads as

$$A_{n,\varepsilon}+B_{n,\varepsilon}+C_{n,\varepsilon}=D_{n,\varepsilon}+E_{n,\varepsilon}+F_{n,\varepsilon}. \tag{3.6}$$

In the following we pass to the limit in (3.6) as ε tends to 0, and then as n tends to $+\infty$. We claim that

$$\liminf_{\varepsilon\to 0} A_{n,\varepsilon}\geq 0, \tag{3.7}$$

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}|C_{n,\varepsilon}|=0, \tag{3.8}$$

$$\limsup_{n\to+\infty}\limsup_{\varepsilon\to 0}(D_{n,\varepsilon}+E_{n,\varepsilon})\leq 0, \tag{3.9}$$

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}|F_{n,\varepsilon}|=0. \tag{3.10}$$

Proof of (3.7). We split $A_{n,\varepsilon}$ into

$$A_{n,\varepsilon}=A^1_{n,\varepsilon}+A^2_{n,\varepsilon}+A^3_{n,\varepsilon} \tag{3.11}$$

with

$$\begin{aligned}
A^1_{n,\varepsilon}&=\int_{\{|u_1-u_2|<1\}} h_n(u_1)\mathbf{A}(x,u_1)(Du_1-Du_2)\cdot(Du_1-Du_2)(F_n^\varepsilon)'(T_1(u_1-u_2))\varphi\,dx,\\
A^2_{n,\varepsilon}&=\int_{\{|u_1-u_2|<1\}} h_n(u_1)(\mathbf{A}(x,u_1)-\mathbf{A}(x,u_2))Du_2\cdot(Du_1-Du_2)(F_n^\varepsilon)'(T_1(u_1-u_2))\varphi\,dx,\\
A^3_{n,\varepsilon}&=\int_{\{|u_1-u_2|<1\}} (h_n(u_1)-h_n(u_2))\mathbf{A}(x,u_2)Du_2\cdot(Du_1-Du_2)(F_n^\varepsilon)'(T_1(u_1-u_2))\varphi\,dx.
\end{aligned}$$

Since h_n, φ and $(F_n^\varepsilon)'$ are nonnegative functions the coercivity of the matrix field $\mathbf{A}(x,s)$ yields that

$$\liminf_{\varepsilon\to 0} A^1_{n,\varepsilon}\geq \liminf_{\varepsilon\to 0}\alpha\int_{\{|u_1-u_2|<1\}} h_n(u_1)|Du_1-Du_2|^2\,dx\geq 0. \tag{3.12}$$

Recalling that $\operatorname{supp}(h_n)=[-2n,2n]$, assumption (2.4) implies that

$$\begin{aligned}
\mathbb{1}_{\{|u_1-u_2|<1\}}h_n(u_1)|\mathbf{A}(x,u_1)-\mathbf{A}(x,u_2)| &\leq \mathbb{1}_{\{|u_1-u_2|<1\}}h_n(u_1)\mathbb{1}_{\{|u_1|<2n+1\}}\\
&\qquad\times\mathbb{1}_{\{|u_2|<2n+1\}}|\mathbf{A}(x,u_1)-\mathbf{A}(x,u_2)|\\
&\leq \mathbb{1}_{\{|u_1-u_2|<1\}}h_n(u_1)\mathbb{1}_{\{|u_1|<2n+1\}}\\
&\qquad\times\mathbb{1}_{\{|u_2|<2n+1\}}\omega_{2n+1}(|u_1-u_2|),
\end{aligned} \tag{3.13}$$

almost everywhere in Ω. Young's inequality and (3.13) then lead to

$$|A^2_{n,\varepsilon}| \le \frac{\alpha}{2}\int_{\{|u_1-u_2|<1\}} h_n(u_1)|Du_1-Du_2|^2 (F^\varepsilon_n)'(T_1(u_1-u_2))\varphi dx$$
$$+\frac{1}{2\alpha}\int_{\substack{\{|u_1-u_2|<1\}\cap\\ \{|u_1|<2n+1\}\cap\{|u_2|<2n+1\}}} |Du_2|^2\omega^2_{2n+1}(|u_1-u_2|)(F^\varepsilon_n)'(T_1(u_1-u_2))\varphi dx.$$

Assumption (2.1), the definition (3.4) of F^ε_n and (3.12) give that

$$|A^2_{n,\varepsilon}| \le \frac{1}{2}A^1_{n,\varepsilon} + \frac{\|\varphi\|_{L^\infty(\Omega)}}{2\alpha I_n(\varepsilon)}\int_\Omega |DT_{2n+1}(u_2)|^2 dx.$$

Since $\lim_{\varepsilon\to 0} I_n(\varepsilon) = +\infty$, from (3.12) and the above inequality we get

$$\liminf_{\varepsilon\to 0}(A^1_{n,\varepsilon} + A^2_{n,\varepsilon}) \ge \frac{1}{2}\liminf_{\varepsilon\to 0} A^1_{n,\varepsilon} \ge 0. \tag{3.14}$$

As far as $A^3_{n,\varepsilon}$ is concerned, due to Remark 2.2 we can choose ω_{2n+1} such that

$$\forall 0<r<1 \quad \frac{r}{\omega_{2n+1}(r)} \le C, \quad \text{with } C>0.$$

Because the function h_n is Lipschitz continuous we deduce that

$$\| A^3_{n,\varepsilon}\| \le \frac{C}{nI_n(\varepsilon)}\int_{\{|u_1|<2n+1\}\cap\{|u_2|<2n+1\}} |\mathbf{A}(x,u_2)Du_2|(|Du_1|+|Du_2|)\varphi dx$$

from which it follows that

$$\lim_{\varepsilon\to 0}|A^3_{n,\varepsilon}| = 0. \tag{3.15}$$

From (3.11), (3.12), (3.14) and (3.15) we conclude that (3.7) holds.

Proof of (3.8). Due to the definitions of $I_n(\varepsilon)$ and F^ε_n we have $0 \le F^\varepsilon_n(T_1(u_1-u_2)) \le 1$ almost everywhere in Ω. Therefore we have

$$\left|\int_\Omega h'_n(u_1)\mathbf{A}(x,u_1)Du_1\cdot Du_1 F^\varepsilon_n(T_1(u_1-u_2))\varphi dx\right| \le \frac{\|\varphi\|_{L^\infty(\Omega)}}{n}\int_{\{n<|u_1|<2n\}} \mathbf{A}(x,u_1)Du_1\cdot Du_1 dx$$

and condition (2.12) allows to obtain (3.8).

Proof of (3.9). Since W^ε_n belongs to $H^1_0(\Omega)\cap L^\infty(\Omega)$ and since $h_n(u_1)$ belongs to $H^1(\Omega)\cap L^\infty(\Omega)$ we have

$$\int_\Omega (h_n(u_1)f_1 - h_n(u_2)f_2)W^\varepsilon_n dx - \int_\Omega (g_1 h_n(u_1) - g_2 h_n(u_2))\cdot DW^\varepsilon_n dx \tag{3.16}$$
$$= \int_\Omega (f_1-f_2)h_n(u_1)W^\varepsilon_n dx - \int_\Omega (g_1-g_2)\cdot D(h_n(u_1)W^\varepsilon_n)dx$$
$$+\int_\Omega f_2(h_n(u_1)-h_n(u_2))W^\varepsilon_n dx - \int_\Omega (h_n(u_1)-h_n(u_2))g_2\cdot DW^\varepsilon_n dx + \int_\Omega h'_n(u_1)W^\varepsilon_n(g_1-g_2)\cdot Du_1 dx.$$

Due to the definitions of h_n and W^ε_n, the field $h_n(u_1)W^\varepsilon_n$ is a non negative element of $H^1_0(\Omega)\cap L^\infty(\Omega)$. Assumption (3.1) on $f_1+\mathrm{div}(g_1)$ and $f_2+\mathrm{div}(g_2)$ leads to

$$\int_\Omega (f_1-f_2)h_n(u_1)W^\varepsilon_n dx - \int_\Omega (g_1-g_2)\cdot D(h_n(u_1)W^\varepsilon_n)dx \le 0. \tag{3.17}$$

We now prove that the third, forth and fifth terms of (3.16) tend to zero as ε goes to zero and then as n goes to infinity.

Recalling the definition of W_n^ε we have

$$\left|\int_\Omega f_2(h_n(u_1)-h_n(u_2))W_n^\varepsilon dx\right| \le \|\varphi\|_{L^\infty(\Omega)}\|F_n^\varepsilon\|_{L^\infty(\Omega)}\int_\Omega |f_2||h_n(u_1)-h_n(u_2)|dx. \tag{3.18}$$

Since $h_n \to 1$ in L^∞ weak-$*$ and almost everywhere in Ω as n goes to infinity and since $f_2 \in L^1(\Omega)$ the Lebesgue convergence theorem and the fact that $|F_n^\varepsilon| \le 1$ uniformly with respect to ε and n imply that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\left|\int_\Omega f_2(h_n(u_1)-h_n(u_2))W_n^\varepsilon dx\right| = 0. \tag{3.19}$$

We now turn to the forth term of (3.16). Since h_n is a Lipschitz continuous function we obtain (recalling the definition of W_n^ε)

$$\begin{aligned}\left|\int_\Omega (h_n(u_1)-h_n(u_2))g_2\cdot DW_n^\varepsilon dx\right| \\ \le \frac{1}{n}\int_{\{|u_1|<2n+1\}\cap\{|u_2|<2n+1\}} |u_1-u_2||g_2|(F_n^\varepsilon)'(T_1(u_1-u_2))|Du_1-Du_2|\varphi dx \\ + \int_\Omega |h_n(u_1)-h_n(u_2)||g_2||D\varphi||F_n^\varepsilon(T_1(u_1-u_2))|dx.\end{aligned} \tag{3.20}$$

On the one hand, due to arguments already used we know that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\int_\Omega |h_n(u_1)-h_n(u_2)||g_2||D\varphi||F_n^\varepsilon(T_1(u_1-u_2))|dx = 0. \tag{3.21}$$

On the other hand, Remark 2.2 yields that

$$\begin{aligned}\frac{1}{n}\int_{\substack{\{|u_1-u_2|<1\}\cap\\ \{|u_1|<2n+1\}\cap\{|u_2|<2n+1\}}} |u_1-u_2||g_2|\frac{|Du_1-Du_2||\varphi|}{I_n(\varepsilon)\omega_{2n+1}(|u_1-u_2|)}dx \\ \le \frac{C}{nI_n(\varepsilon)}\|g_2\|_{(L^2(\Omega))^N}\left[\|DT_{2n+1}(u_1)\|_{(L^2(\Omega))^N} + \|DT_{2n+1}(u_2)\|_{(L^2(\Omega))^N}\right].\end{aligned}$$

Because $I_n(\varepsilon) \to +\infty$ as ε goes to zero, from (3.20), (3.21) and the above inequality it follows that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\left|\int_\Omega (h_n(u_1)-h_n(u_2))g_2\cdot DW_n^\varepsilon dx\right| = 0. \tag{3.22}$$

For the last term in the right-hand side of (3.16), Hölder's inequality gives

$$\left|\int_\Omega h_n'(u_1)W_n^\varepsilon(g_1-g_2)\cdot Du_1 dx\right| \le \|W_n^\varepsilon\|_{L^\infty(\Omega)}\|g_1-g_2\|_{(L^2(\Omega))^N}\left(\frac{1}{n^2}\int_{\{|u_1|<2n\}}|Du_1|^2dx\right)^{1/2}$$

and condition (2.12) then implies

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\left|\int_\Omega h_n'(u_1)W_n^\varepsilon(g_1-g_2)\cdot Du_1 dx\right| = 0. \tag{3.23}$$

Gathering (3.16), (3.17), (3.19), (3.22) and (3.23) we conclude that (3.9) holds true.

Proof of (3.10). We have

$$\left|\int_\Omega (h_n'(u_1)g_1\cdot Du_1 - h_n'(u_2)g_2\cdot Du_2)W_n^\varepsilon dx\right)$$
$$\leq \|W_n^\varepsilon\|_{L^\infty(\Omega)}\left(\|g_1\|_{(L^2(\Omega))^N}\frac{1}{2n}\|DT_{2n}(u_1)\|_{(L^2(\Omega))^N} + \|g_2\|_{(L^2(\Omega))^N}\frac{1}{2n}\|DT_{2n}(u_2)\|_{(L^2(\Omega))^N}\right).$$

Recalling that W_n^ε is bounded in $L^\infty(\Omega)$ uniformly with respect to n and ε, condition (2.12) implies (3.10).

We are now in a position to prove Lemma 3.3. With arguments already used we know that

$$\lim_{\varepsilon\to 0} B_{n,\varepsilon} = \int_{\{u_1-u_2>0\}} (h_n(u_1)\mathbf{A}(x,u_1)Du_1 - h_n(u_2)\mathbf{A}(x,u_2)Du_2)\cdot D\varphi dx. \tag{3.24}$$

From equality (3.6) together with (3.7)–(3.10) and (3.24) it follows that

$$\limsup_{n\to+\infty}\int_{\{u_1-u_2>0\}} (h_n(u_1)\mathbf{A}(x,u_1)Du_1 - h_n(u_2)\mathbf{A}(x,u_2)Du_2)\cdot D\varphi dx \leq 0. \tag{3.25}$$

Taking $M-\varphi$ in place of φ in (3.25), with M sufficiently large so that $M-\varphi\geq 0$, gives

$$\liminf_{n\to+\infty}\int_{\{u_1-u_2>0\}} (h_n(u_1)\mathbf{A}(x,u_1)Du_1 - h_n(u_2)\mathbf{A}(x,u_2)Du_2)\cdot D\varphi dx \geq 0. \tag{3.26}$$

At last (3.25) and (3.26) allow to conclude that (3.2) holds true. The proof of Lemma 3.3 is complete. □

With the help of Lemma 3.3 we now turn to Theorem 3.1.

Proof of Theorem 3.1. We use Lemma 3.3 with $\varphi(x) = \exp(c\sum_{i=1}^N x_i)$, where $c>0$.

Since $h_n(s) = 0$, $\forall |s|\geq 2n$, we have

$$h_n(u_1)\mathbf{A}(x,u_1)Du_1\mathbb{1}_{\{u_1-u_2>0\}} = h_n(T_{2n}(u_1))\mathbf{A}(x,T_{2n}(u_1))DT_{2n}(u_1)\mathbb{1}_{\{T_{2n}(u_1)-T_{2n}(u_2)>0\}}$$

almost everywhere in Ω. To shorten the notations we denote by u_1^{2n} the field $T_{2n}(u_1)$ and by u_2^{2n} the field $T_{2n}(u_2)$. It follows that (3.2) can be rewritten as

$$\lim_{n\to+\infty}\int_{\{u_1^{2n}-u_2^{2n}>0\}} \left(h_n(u_1^{2n})\mathbf{A}(x,u_1^{2n})Du_1^{2n} - h_n(u_2^{2n})\mathbf{A}(x,u_2^{2n})Du_2^{2n}\right)\cdot D\varphi dx = 0. \tag{3.27}$$

Let us define

$$\tilde{a}_{i,j}^n(x,r) = \int_0^r a_{i,j}(x,s)h_n(s)ds.$$

Due to the regularity (2.10) of $T_K(u_1)$ and $T_K(u_2)$, assumption (2.3) implies that both $\tilde{a}_{i,j}^n(u_1^{2n})$ and $\tilde{a}_{i,j}^n(u_2^{2n})$ belong to $H_0^1(\Omega)$ and for $l=1,2$

$$\frac{\partial \tilde{a}_{i,j}^n(u_l^{2n})}{\partial x_k} = h_n(u_l^{2n})a_{i,j}(u_l^{2n})\frac{\partial u_l^{2n}}{\partial x_k} + \int_0^{u_l^{2n}} h_n(s)\frac{\partial a_{i,j}}{\partial x_k}(x,s)ds. \tag{3.28}$$

Since $\frac{\partial\varphi}{\partial x_k} = c\varphi$, using (3.28) we have

$$\begin{aligned}
&\int_{\{u_1^{2n}-u_2^{2n}>0\}} \big(h_n(u_1^{2n})\mathbf{A}(x,u_1^{2n})Du_1^{2n} - h_n(u_2^{2n})\mathbf{A}(x,u_2^{2n})Du_2^{2n}\big)\cdot D\varphi dx \\
&\qquad = c\int_{\{u_1^{2n}-u_2^{2n}>0\}} \sum_{1\le i,j\le N}\Big(\frac{\partial \tilde{a}_{i,j}^n(u_1^{2n})}{\partial x_j} - \frac{\partial \tilde{a}_{i,j}^n(u_2^{2n})}{\partial x_j}\Big)\varphi dx \\
&\qquad\qquad + c\int_{\{u_1^{2n}-u_2^{2n}>0\}} \sum_{1\le i,j\le N}\int_{u_1^{2n}}^{u_2^{2n}} h_n(s)\frac{\partial a_{i,j}}{\partial x_j}(x,s)ds\varphi dx.
\end{aligned}$$

Let us define $w_{2n} = (u_1^{2n}-u_2^{2n})^+$ which belongs to $L^\infty(\Omega)\cap H_0^1(\Omega)$ and is such that $u_1^{2n} = w_{2n}+u_2^{2n}$ almost everywhere on $\{u_1^{2n}-u_2^{2n}>0\}$. Since $\tilde{a}_{i,j}^n(u_2^{2n}+w_{2n}) - \tilde{a}_{i,j}^n(u_2^{2n})$ lies in $L^\infty(\Omega)\cap H_0^1(\Omega)$, a few computations and the integration by part formula give

$$\begin{aligned}
&\int_{\{u_1^{2n}-u_2^{2n}>0\}} \big(h_n(u_1^{2n})\mathbf{A}(x,u_1^{2n})Du_1^{2n} - h_n(u_2^{2n})\mathbf{A}(x,u_2^{2n})Du_2^{2n}\big)\cdot D\varphi dx \\
&\qquad = c\int_\Omega \sum_{1\le i,j\le N}\Big(\frac{\partial \tilde{a}_{i,j}^n(u_2^{2n}+w_{2n})}{\partial x_j} - \frac{\partial \tilde{a}_{i,j}^n(u_2^{2n})}{\partial x_j}\Big)\varphi dx \\
&\qquad\qquad + c\int_\Omega \sum_{1\le i,j\le N}\int_{u_2^{2n}+w_{2n}}^{u_2^{2n}} h_n(s)\frac{\partial a_{i,j}}{\partial x_j}(x,s)\varphi dx \\
&\qquad = -c^2\int_\Omega \sum_{1\le i,j\le N}\Big(\tilde{a}_{i,j}^n(u_2^{2n}+w_{2n}) - \tilde{a}_{i,j}^n(u_2^{2n})\Big)\varphi dx \\
&\qquad\qquad + c\int_\Omega \sum_{1\le i,j\le N}\int_{u_2^{2n}+w_{2n}}^{u_2^{2n}} h_n(s)\frac{\partial a_{i,j}}{\partial x_j}(x,s)\varphi dx \\
&\qquad = -c\int_\Omega\int_{u_2^{2n}}^{u_2^{2n}+w_{2n}} h_n(s)\Big(c\sum_{1\le i,j\le N} a_{i,j}(x,s) + \sum_{1\le i,j\le N}\frac{\partial a_{i,j}}{\partial x_j}(x,s)\Big)ds\varphi dx.
\end{aligned}$$

Because $\varphi \ge 0$ in Ω, from assumption (2.3) and the coercivity (2.1) of the matrix field $\mathbf{A}$ we obtain for c sufficiently large independently of n ($c > 2N^2M$ for example) that

$$\begin{aligned}
\text{(3.29)}\quad &\int_{\{u_1^{2n}-u_2^{2n}>0\}} \big(h_n(u_1^{2n})\mathbf{A}(x,u_1^{2n})Du_1^{2n} - h_n(u_2^{2n})\mathbf{A}(x,u_2^{2n})Du_2^{2n}\big)\cdot D\varphi dx \\
&\qquad \le -\frac{\alpha c}{2}\int_\Omega\int_{u_2^{2n}}^{u_2^{2n}+w_{2n}} h_n(s)ds\varphi dx.
\end{aligned}$$

Since u_1 and u_2 are finite almost everywhere in Ω while h_n converges to 1 almost everywhere in $\mathbb{R}$ and is bounded by 1 we obtain

$$\text{(3.30)}\qquad \lim_{n\to+\infty}\int_{u_2^{2n}}^{u_2^{2n}+w_{2n}} h_n(s)ds = \int_{u_2}^{u_2+w} ds = w \quad \text{almost everywhere in } \Omega,$$

where $w = (u_1-u_2)^+$.

Finally from (3.27), (3.29), (3.30) and Fatou lemma it follows that

$$\int_\Omega w dx \le 0,$$

which leads to $u_1 \le u_2$ almost everywhere in Ω.

The proof of Theorem 3.1 is complete. $\square$

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