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► **To cite this version:**

Olivier Guibé. Uniqueness of the solution to quasilinear elliptic equations under a local condition on the diffusion matrix. *Advances in Mathematical Sciences and Applications*, 2007, 17 (2), pp.357–368. <hal-00311535>

HAL Id: hal-00311535

<https://hal.archives-ouvertes.fr/hal-00311535>

Submitted on 18 Aug 2008

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UNIQUENESS OF THE SOLUTION TO QUASILINEAR ELLIPTIC EQUATIONS UNDER A LOCAL CONDITION ON THE DIFFUSION MATRIX

OLIVIER GUIBÉ

ABSTRACT. We prove the uniqueness of the renormalized solution to the elliptic equation $-\operatorname{div}(\mathbf{A}(x, u)Du) = f + \operatorname{div}(g)$. The data $f + \operatorname{div}(g)$ belongs to $L^1 + H^{-1}$ and we assume a local condition on the diffusion matrix $\mathbf{A}(x, s)$ with respect to s .

1. INTRODUCTION

The present paper is concerned with the uniqueness of the solution to the quasilinear elliptic boundary-value problem on Ω

$$(1.1) \quad \begin{cases} -\operatorname{div}(\mathbf{A}(x, u)Du) = f + \operatorname{div}(g) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of \mathbb{R}^N , $f \in L^1(\Omega)$, $g \in (L^2(\Omega))^N$ and $\mathbf{A}(x, s)$ is a Carathéodory function with matrix values.

When f belongs to $L^2(\Omega)$ (i.e. the right-hand side of (1.1) lies in $H^{-1}(\Omega)$) the variational solution of (1.1) is unique under a global Lipschitz condition on the function $\mathbf{A}(x, s)$ with respect to the variable s (or a global and strong control of the modulus of continuity), see Artola [1986], Carrillo and Chipot [1985] and for more general and nonlinear operators Boccardo et al. [1992], Chipot and Michaille [1989]. Moreover in Carrillo and Chipot [1985], Chipot and Michaille [1989] the authors show that if $\mathbf{A}(x, s)$ is Hölder continuous in s with a Hölder exponent greater or equal to $1/2$ and if $\mathbf{A}(x, s)$ is Lipschitz continuous in x then the solution is unique. For this last result the quasilinear character of the equation and the regularity of $\mathbf{A}(x, s)$ in x are crucial.

In the case where f lies in $L^1(\Omega)$ and if \mathbf{A} is uniformly coercive we cannot expect to have a solution of (1.1) in the sense of distributions without any growth condition on $\mathbf{A}(x, s)$ with respect to s . Moreover it is well known that, in the simple case where \mathbf{A} does not depend on s , a solution in the sense of distribution exists (see e.g. Boccardo and Gallouët [1989]) but it is not unique in general (see the counter example in Serin [1964]). In the present paper we use the framework of renormalized solution (see Dal Maso et al. [1999], Murat [1993, 1994]) which insures the existence of such a solution when f belongs to $L^1(\Omega)$, \mathbf{A} is uniformly coercive and $\mathbf{A} \in L^\infty(\Omega \times]-K, K])^{N \times N}$ for any $K > 0$.

Uniqueness results have been recently obtained in Blanchard et al. [2005] in the framework of renormalized solutions and in Porretta [2004] in the very close framework of entropy solutions for equations (1.1) with f belonging to L^1 with very general

Date: August 18, 2008.

Key words and phrases. uniqueness, quasilinear elliptic equations, renormalized solutions.

and global conditions on the matrix field \mathbf{A} . Roughly speaking the modulus of continuity of \mathbf{A} with respect to s has to be controlled by $\exp(c|s|)$ ($c > 0$) in Porretta [2004] and by a function which satisfies an appropriate differential inequality in Blanchard et al. [2005].

In the present paper we state in Theorem 3.2 that the renormalized solution of (1.1) is unique if \mathbf{A} is locally Hölder continuous in s with a Hölder exponent greater or equal to $1/2$ and under a global control of the modulus of continuity of \mathbf{A} with respect to the space variable x . The main novelty between our and known uniqueness results is the very local condition on \mathbf{A} , i.e. we do not assume any control on the growth of the modulus of continuity of \mathbf{A} in s as in the above cited papers. The price to pay to get rid of this global behavior is to assume a regularity with respect to x . The results obtained in the present paper rely on the mixing of the assumptions and the techniques developed in Carrillo and Chipot [1985] (see also Chipot and Michaille [1989]) with those used to study L^1 -problems with the help of renormalized solutions (see Lemma 3.3 and Remark 3.4).

At last the question of the uniqueness under a local condition in s remains still open in general.

The paper is organized as follows. In Section 2 we give the assumptions on the data and we recall the definition of a renormalized solution of (1.1). Section 3 is devoted to a comparison result stated in Theorem 3.1 which implies the uniqueness of the solution given in Theorem 3.2.

2. ASSUMPTIONS AND DEFINITIONS

In the whole paper we assume that $\mathbf{A} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^{N \times N}$ is a Carathéodory function with $\mathbf{A}(x, s) = (a_{ij}(x, s))_{1 \leq i, j \leq N}$ and such that

$$(2.1) \quad \exists \alpha > 0, \quad \mathbf{A}(x, s) \xi \cdot \xi \geq \alpha |\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \forall s \in \mathbb{R}, \text{ a.e. in } \Omega;$$

$$(2.2) \quad \forall K > 0, \quad \exists C_K > 0 \quad |\mathbf{A}(x, s)| \leq C_K, \quad \forall s \in [-K, K], \text{ a.e. in } \Omega;$$

for any r in \mathbb{R} and any $1 \leq i, j \leq N$, the function $a_{ij}(r, \cdot)$ belongs to $W^{1, \infty}(\Omega)$ and there exists $M > 0$ such that

$$(2.3) \quad \left| \frac{\partial a_{ij}}{\partial x_k}(x, r) \right| \leq M \sum_{1 \leq i, j \leq N} a_{i, j}(x, r), \quad \forall r \in \mathbb{R}, \forall 1 \leq i, j \leq N, \text{ a.e. in } \Omega.$$

Moreover we assume that for any $K > 0$ there exists a nonnegative, non decreasing continuous function ω_K such that

$$(2.4) \quad |\mathbf{A}(x, s) - \mathbf{A}(x, r)| \leq \omega_K(|s - r|) \quad \forall r, s \in \mathbb{R} \text{ with } |s| \leq K, |r| \leq K, \text{ a.e. in } \Omega;$$

$$(2.5) \quad \int_{0^+} \frac{ds}{\omega_K^2(s)} = +\infty.$$

The data f and g are such that

$$(2.6) \quad f \in L^1(\Omega);$$

$$(2.7) \quad g \in (L^2(\Omega))^N.$$

Remark 2.1. Assumptions (2.1) and (2.2) are classical in the framework of renormalized solutions and allow to obtain the existence of such a solution with a data belonging to $L^1 + H^{-1}$. Conditions (2.4) and (2.5) concern a local condition on the modulus of continuity of the matrix field $\mathbf{A}(x, s)$ in s . If the matrix field $\mathbf{A}(x, s)$ does not depend on x and is locally Hölder continuous with an exponent greater or equal to $1/2$ then assumptions (2.3), (2.4) and (2.5) are satisfied. Assumption (2.3) is crucial when $\mathbf{A}(x, s)$ depends on x . As an example, if b is an element of $W^{1,\infty}(\Omega)$ and h is a non negative locally Hölder continuous function with an exponent greater or equal to $1/2$, then $\mathbf{A}(x, s) = (\exp(s^2) + b(x)h(s))I$ verifies (2.1)–(2.5).

Remark 2.2. Since ω_K is a nonnegative, non decreasing continuous function satisfying (2.5) we can assume without loss of generality that there exists C_K such that

$$(2.8) \quad \forall 0 < r < 1, \quad \frac{r}{\omega_K^2(r)} \leq C_K.$$

Indeed it is sufficient to take $\omega_K(r) + \sqrt{r}$ in place of ω_K in (2.4) which also verifies condition (2.5).

For any $K > 0$ we denote by T_K the truncation function at height $\pm K$, $T_K(s) = \max(-K, \min(K, s))$ for any $s \in \mathbb{R}$ and we define the continuous function h_n by

$$(2.9) \quad h_n(s) = 1 - \left| \frac{T_{2n}(s) - T_n(s)}{n} \right|.$$

We now recall the definition of the gradient of functions whose truncates belong to $H_0^1(\Omega)$ (see Bénénil et al. [1995]).

Definition 2.3. Let $u : \Omega \rightarrow \mathbb{R}$ be a measurable function, finite almost everywhere in Ω , such that $T_K(u) \in H_0^1(\Omega)$ for any $K > 0$. Then there exists a unique measurable vector field $v : \Omega \rightarrow \mathbb{R}^N$ such that

$$DT_K(u) = \mathbb{1}_{\{|u| < K\}} v \quad \text{a.e. in } \Omega.$$

This function v is called the gradient of u and is denoted by Du .

Following Dal Maso et al. [1999] (see also Murat [1993, 1994]) we now recall the definition of a renormalized solution to (1.1).

Definition 2.4. A measurable function u defined from Ω into \mathbb{R} is called a renormalized solution of (1.1) if

$$(2.10) \quad \forall K > 0, \quad T_K(u) \in H_0^1(\Omega);$$

if for any function $h \in W^{1,\infty}(\mathbb{R})$ such that $\text{supp } h$ is compact, u satisfies the equation

$$(2.11) \quad -\text{div}[h(u)\mathbf{A}(x, u)Du] + h'(u)\mathbf{A}(x, u)Du \cdot Du \\ = fh(u) + \text{div}(gh(u)) - h'(u)g \cdot Du \quad \text{in } \mathcal{D}'(\Omega),$$

$$(2.12) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{n < |u| < 2n} \mathbf{A}(x, u)Du \cdot Dudx = 0$$

Remark 2.5. Condition (2.10) and Definition 2.3 allow to define Du almost everywhere in Ω . In (2.11) which is formally obtained by the point-wise multiplication of (1.1) by $h(u)$ every terms are well defined. Indeed since $\text{supp}(h)$ is compact, we have $\text{supp}(h) \subset [-K, K]$ for $K > 0$ sufficiently large. It follows that $h(u)\mathbf{A}(x, u)Du = h(u)\mathbf{A}(x, T_K(u))DT_K(u)$ almost everywhere in Ω and then it belongs to $(L^2(\Omega))^N$. Similarly $h'(u)\mathbf{A}(x, u)Du \cdot Du$ is identified to $h'(u)\mathbf{A}(x, T_K(u))DT_K(u) \cdot DT_K(u)$ which belongs to $L^1(\Omega)$. The same arguments imply that the right hand side of (2.11) lies in $L^1(\Omega) + H^{-1}(\Omega)$. Condition (2.12) is classical in the framework of renormalized solutions and gives additional information on Du for large value of $|u|$.

It is well know that under assumptions (2.1), (2.2), (2.6) and (2.7) there exists at least one renormalized solution to equation (1.1), see e.g. Blanchard et al. [2005], Lions and Murat, Murat [1993, 1994].

3. MAIN RESULT

In Theorem 3.1 below we give a comparison result from which it follows a uniqueness result.

Theorem 3.1. *Assume that (2.1)–(2.5) hold true. Let f_1 and f_2 belong to $L^1(\Omega)$ and let g_1 and g_2 belong to $(L^2(\Omega))^N$ such that*

$$(3.1) \quad f_1 + \text{div}(g_1) \leq f_2 + \text{div}(g_2) \quad \text{in } \mathcal{D}'(\Omega)$$

Let u_1 be a renormalized solution of (1.1) with (f_1, g_1) in place of (f, g) and let u_2 be a renormalized solution of (1.1) with (f_2, g_2) in place of (f, g) . Then $u_1 \leq u_2$ almost everywhere in Ω .

An immediate consequence is the uniqueness of the solution for a fixed data $f + \text{div}(g) \in L^1(\Omega) + H^{-1}(\Omega)$.

Theorem 3.2. *Assume that (2.1)–(2.7) hold true. Then the renormalized solution of (1.1) is unique.*

To prove Theorem 3.1 we mix the methods developed by Chipot and Carrillo in Carrillo and Chipot [1985] (see also Chipot and Michaille [1989]) together with the techniques of renormalized solutions. The main tool is the following lemma which is a truncated version to Theorem 4 in Carrillo and Chipot [1985].

Lemma 3.3. *For any φ belonging to $\mathcal{C}^1(\bar{\Omega})$*

$$(3.2) \quad \lim_{n \rightarrow +\infty} \int_{\{u_1 - u_2 > 0\}} (h_n(u_1)\mathbf{A}(x, u_1)Du_1 - h_n(u_2)\mathbf{A}(x, u_2)Du_2) \cdot D\varphi dx = 0.$$

Remark 3.4. In Carrillo and Chipot [1985] (see also Chipot and Michaille [1989]) when $f + \text{div}(g)$ belongs to $H^{-1}(\Omega)$ and under more restrictive conditions on the matrix field $\mathbf{A}(x, s)$ (roughly speaking $\mathbf{A}(x, s)$ is bounded) the authors state that

$$\int_{\{u_1 - u_2 > 0\}} (\mathbf{A}(x, u_1)Du_1 - \mathbf{A}(x, u_2)Du_2) \cdot D\varphi dx = 0.$$

In the $L^1(\Omega) + H^{-1}(\Omega)$ case, since we do not have any growth assumption on $\mathbf{A}(x, s)$ with respect to s , we cannot expect to have $\mathbf{A}(x, u_1)Du_1$ in $L^1_{\text{loc}}(\Omega)$. It follows that the above equality does not have any sense or equivalently that the limit in (3.2) cannot be written in terms of u_1 and u_2 .

Proof of Lemma 3.3. Let φ belong to $\mathcal{C}^1(\overline{\Omega})$ with $\varphi \geq 0$ on Ω and let n be a positive integer. Assume without loss of generality that $\omega_{2n+1}(r) > 0 \forall r > 0$. Following Chipot and Michaille [1989] let us define for any $0 < \varepsilon < 1$

$$(3.3) \quad I_n(\varepsilon) = \int_{\varepsilon}^1 \frac{1}{\omega_{2n+1}^2(s)} ds,$$

$$(3.4) \quad F_n^\varepsilon(r) = \begin{cases} 1 & \text{if } r \geq 1, \\ \frac{1}{I_n(\varepsilon)} \int_{\varepsilon}^r \frac{1}{\omega_{2n+1}^2(s)} ds & \text{if } 1 > r > \varepsilon, \\ 0 & \text{if } r \leq \varepsilon. \end{cases}$$

From the definition of F_n^ε and the regularity of ω_{2n+1} it follows that F_n^ε is a nonnegative Lipschitz continuous function such that $F_n^\varepsilon(s) = 0 \forall s \leq \varepsilon$.

Let us consider the test function $W_n^\varepsilon = F_n^\varepsilon(T_1(T_{2n+1}(u_1) - T_{2n+1}(u_2)))\varphi$ which belongs to $L^\infty(\Omega) \cap H_0^1(\Omega)$ due to (2.10) and the regularities of F_n^ε and φ . Moreover we have

$$\begin{aligned} DW_n^\varepsilon &= F_n^\varepsilon(T_1(T_{2n+1}(u_1) - T_{2n+1}(u_2)))D\varphi + \varphi T_1'(T_{2n+1}(u_1) - T_{2n+1}(u_2)) \\ &\quad \times (F_n^\varepsilon)'(T_1(T_{2n+1}(u_1) - T_{2n+1}(u_2)))(DT_{2n+1}(u_1) - DT_{2n+1}(u_2)) \end{aligned}$$

almost everywhere in Ω .

Choosing $h = h_n$ in (2.11) written in u_1 yields

$$(3.5) \quad \begin{aligned} &\int_{\Omega} h_n(u_1)\mathbf{A}(x, u_1)Du_1 \cdot DW_n^\varepsilon dx + \int_{\Omega} h_n'(u_1)\mathbf{A}(x, u_1)Du_1 \cdot Du_1 W_n^\varepsilon dx \\ &= \int_{\Omega} h_n(u_1)f_1 W_n^\varepsilon dx - \int_{\Omega} h_n(u_1)g_1 \cdot DW_n^\varepsilon dx - \int_{\Omega} h_n'(u_1)g_1 \cdot Du_1 W_n^\varepsilon dx. \end{aligned}$$

Since $\text{supp}(h_n) = [-2n, 2n]$ we have $h_n(u_1)F_n^\varepsilon(T_1(T_{2n+1}(u_1) - T_{2n+1}(u_2))) = h_n(u_1)F_n^\varepsilon(T_1(u_1 - u_2))$ almost everywhere in Ω and

$$h_n(u_1)\mathbb{1}_{\{|T_{2n+1}(u_1) - T_{2n+1}(u_2)| < 1\}}(DT_{2n+1}(u_1) - DT_{2n+1}(u_2)) = h_n(u_1)\mathbb{1}_{\{|u_1 - u_2| < 1\}}(Du_1 - Du_2)$$

almost everywhere in Ω . It follows that (3.5) can be rewritten as

$$\begin{aligned} &\int_{\{|u_1 - u_2| < 1\}} h_n(u_1)\mathbf{A}(x, u_1)Du_1 \cdot (Du_1 - Du_2)(F_n^\varepsilon)'(T_1(u_1 - u_2))\varphi dx \\ &+ \int_{\Omega} h_n(u_1)\mathbf{A}(x, u_1)Du_1 \cdot D\varphi F_n^\varepsilon(T_1(u_1 - u_2))dx + \int_{\Omega} h_n'(u_1)\mathbf{A}(x, u_1)Du_1 \cdot Du_1 F_n^\varepsilon(T_1(u_1 - u_2))\varphi dx \\ &= \int_{\Omega} h_n(u_1)f_1 W_n^\varepsilon dx - \int_{\Omega} h_n(u_1)g_1 \cdot DW_n^\varepsilon dx - \int_{\Omega} h_n'(u_1)g_1 \cdot Du_1 F_n^\varepsilon(T_1(u_1 - u_2))\varphi dx. \end{aligned}$$

Subtracting the equivalent equality written in u_2 gives

$$\begin{aligned}
& \int_{\{|u_1-u_2|<1\}} (h_n(u_1)\mathbf{A}(x, u_1)Du_1 - h_n(u_2)\mathbf{A}(x, u_2)Du_2) \cdot (Du_1 - Du_2)(F_n^\varepsilon)'(T_1(u_1 - u_2))\varphi dx \\
& \quad + \int_{\Omega} (h_n(u_1)\mathbf{A}(x, u_1)Du_1 - h_n(u_2)\mathbf{A}(x, u_2)Du_2) \cdot D\varphi F_n^\varepsilon(T_1(u_1 - u_2)) dx \\
& \quad + \int_{\Omega} (h'_n(u_1)\mathbf{A}(x, u_1)Du_1 \cdot Du_1 - h'_n(u_2)\mathbf{A}(x, u_2)Du_2 \cdot Du_2) F_n^\varepsilon(T_1(u_1 - u_2))\varphi dx \\
& \quad = \int_{\Omega} (h_n(u_1)f_1 - h_n(u_2)f_2)W_n^\varepsilon dx - \int_{\Omega} (h_n(u_1)g_1 - h_n(u_2)g_2) \cdot DW_n^\varepsilon dx \\
& \quad \quad - \int_{\Omega} (h'_n(u_1)g_1 \cdot Du_1 - h'_n(u_2)g_2 \cdot Du_2) F_n^\varepsilon(T_1(u_1 - u_2))\varphi dx,
\end{aligned}$$

which reads as

$$(3.6) \quad A_{n,\varepsilon} + B_{n,\varepsilon} + C_{n,\varepsilon} = D_{n,\varepsilon} + E_{n,\varepsilon} + F_{n,\varepsilon}.$$

In the following we pass to the limit in (3.6) as ε tends to 0, and then as n tends to $+\infty$. We claim that

$$(3.7) \quad \liminf_{\varepsilon \rightarrow 0} A_{n,\varepsilon} \geq 0,$$

$$(3.8) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} |C_{n,\varepsilon}| = 0,$$

$$(3.9) \quad \limsup_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} (D_{n,\varepsilon} + E_{n,\varepsilon}) \leq 0,$$

$$(3.10) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} |F_{n,\varepsilon}| = 0.$$

Proof of (3.7). We split $A_{n,\varepsilon}$ into

$$(3.11) \quad A_{n,\varepsilon} = A_{n,\varepsilon}^1 + A_{n,\varepsilon}^2 + A_{n,\varepsilon}^3$$

with

$$\begin{aligned}
A_{n,\varepsilon}^1 &= \int_{\{|u_1-u_2|<1\}} h_n(u_1)\mathbf{A}(x, u_1)(Du_1 - Du_2) \cdot (Du_1 - Du_2)(F_n^\varepsilon)'(T_1(u_1 - u_2))\varphi dx, \\
A_{n,\varepsilon}^2 &= \int_{\{|u_1-u_2|<1\}} h_n(u_1)(\mathbf{A}(x, u_1) - \mathbf{A}(x, u_2))Du_2 \cdot (Du_1 - Du_2)(F_n^\varepsilon)'(T_1(u_1 - u_2))\varphi dx, \\
A_{n,\varepsilon}^3 &= \int_{\{|u_1-u_2|<1\}} (h_n(u_1) - h_n(u_2))\mathbf{A}(x, u_2)Du_2 \cdot (Du_1 - Du_2)(F_n^\varepsilon)'(T_1(u_1 - u_2))\varphi dx.
\end{aligned}$$

Since h_n , φ and $(F_n^\varepsilon)'$ are nonnegative functions the coercivity of the matrix field $\mathbf{A}(x, s)$ yields that

$$(3.12) \quad \liminf_{\varepsilon \rightarrow 0} A_{n,\varepsilon}^1 \geq \liminf_{\varepsilon \rightarrow 0} \alpha \int_{\{|u_1-u_2|<1\}} h_n(u_1)|Du_1 - Du_2|^2 dx \geq 0.$$

Recalling that $\text{supp}(h_n) = [-2n, 2n]$, assumption (2.4) implies that

$$\begin{aligned}
(3.13) \quad \mathbb{1}_{\{|u_1-u_2|<1\}} h_n(u_1)|\mathbf{A}(x, u_1) - \mathbf{A}(x, u_2)| &\leq \mathbb{1}_{\{|u_1-u_2|<1\}} h_n(u_1)\mathbb{1}_{\{|u_1|<2n+1\}} \\
&\quad \times \mathbb{1}_{\{|u_2|<2n+1\}} |\mathbf{A}(x, u_1) - \mathbf{A}(x, u_2)| \\
&\leq \mathbb{1}_{\{|u_1-u_2|<1\}} h_n(u_1)\mathbb{1}_{\{|u_1|<2n+1\}} \\
&\quad \times \mathbb{1}_{\{|u_2|<2n+1\}} \omega_{2n+1}(|u_1 - u_2|),
\end{aligned}$$

almost everywhere in Ω . Young's inequality and (3.13) then lead to

$$\begin{aligned} |A_{n,\varepsilon}^2| &\leq \frac{\alpha}{2} \int_{\{|u_1-u_2|<1\}} h_n(u_1) |Du_1 - Du_2|^2 (F_n^\varepsilon)'(T_1(u_1 - u_2)) \varphi dx \\ &\quad + \frac{1}{2\alpha} \int_{\substack{\{|u_1-u_2|<1\} \cap \\ \{|u_1|<2n+1\} \cap \{|u_2|<2n+1\}}} |Du_2|^2 \omega_{2n+1}^2(|u_1 - u_2|) (F_n^\varepsilon)'(T_1(u_1 - u_2)) \varphi dx. \end{aligned}$$

Assumption (2.1), the definition (3.4) of F_n^ε and (3.12) give that

$$|A_{n,\varepsilon}^2| \leq \frac{1}{2} A_{n,\varepsilon}^1 + \frac{\|\varphi\|_{L^\infty(\Omega)}}{2\alpha I_n(\varepsilon)} \int_{\Omega} |DT_{2n+1}(u_2)|^2 dx.$$

Since $\lim_{\varepsilon \rightarrow 0} I_n(\varepsilon) = +\infty$, from (3.12) and the above inequality we get

$$(3.14) \quad \liminf_{\varepsilon \rightarrow 0} (A_{n,\varepsilon}^1 + A_{n,\varepsilon}^2) \geq \frac{1}{2} \liminf_{\varepsilon \rightarrow 0} A_{n,\varepsilon}^1 \geq 0.$$

As far as $A_{n,\varepsilon}^3$ is concerned, due to Remark 2.2 we can choose ω_{2n+1} such that

$$\forall 0 < r < 1 \quad \frac{r}{\omega_{2n+1}(r)} \leq C, \quad \text{with } C > 0.$$

Because the function h_n is Lipschitz continuous we deduce that

$$\|A_{n,\varepsilon}^3\| \leq \frac{C}{n I_n(\varepsilon)} \int_{\substack{\{|u_1|<2n+1\} \cap \{|u_2|<2n+1\}}} |\mathbf{A}(x, u_2) Du_2| (|Du_1| + |Du_2|) \varphi dx$$

from which it follows that

$$(3.15) \quad \lim_{\varepsilon \rightarrow 0} |A_{n,\varepsilon}^3| = 0.$$

From (3.11), (3.12), (3.14) and (3.15) we conclude that (3.7) holds.

Proof of (3.8). Due to the definitions of $I_n(\varepsilon)$ and F_n^ε we have $0 \leq F_n^\varepsilon(T_1(u_1 - u_2)) \leq 1$ almost everywhere in Ω . Therefore we have

$$\left| \int_{\Omega} h_n'(u_1) \mathbf{A}(x, u_1) Du_1 \cdot Du_1 F_n^\varepsilon(T_1(u_1 - u_2)) \varphi dx \right| \leq \frac{\|\varphi\|_{L^\infty(\Omega)}}{n} \int_{\{|n < |u_1| < 2n\}} \mathbf{A}(x, u_1) Du_1 \cdot Du_1 dx$$

and condition (2.12) allows to obtain (3.8).

Proof of (3.9). Since W_n^ε belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ and since $h_n(u_1)$ belongs to $H^1(\Omega) \cap L^\infty(\Omega)$ we have

$$\begin{aligned} (3.16) \quad &\int_{\Omega} (h_n(u_1) f_1 - h_n(u_2) f_2) W_n^\varepsilon dx - \int_{\Omega} (g_1 h_n(u_1) - g_2 h_n(u_2)) \cdot DW_n^\varepsilon dx \\ &= \int_{\Omega} (f_1 - f_2) h_n(u_1) W_n^\varepsilon dx - \int_{\Omega} (g_1 - g_2) \cdot D(h_n(u_1) W_n^\varepsilon) dx \\ &+ \int_{\Omega} f_2 (h_n(u_1) - h_n(u_2)) W_n^\varepsilon dx - \int_{\Omega} (h_n(u_1) - h_n(u_2)) g_2 \cdot DW_n^\varepsilon dx + \int_{\Omega} h_n'(u_1) W_n^\varepsilon (g_1 - g_2) \cdot Du_1 dx. \end{aligned}$$

Due to the definitions of h_n and W_n^ε , the field $h_n(u_1) W_n^\varepsilon$ is a non negative element of $H_0^1(\Omega) \cap L^\infty(\Omega)$. Assumption (3.1) on $f_1 + \operatorname{div}(g_1)$ and $f_2 + \operatorname{div}(g_2)$ leads to

$$(3.17) \quad \int_{\Omega} (f_1 - f_2) h_n(u_1) W_n^\varepsilon dx - \int_{\Omega} (g_1 - g_2) \cdot D(h_n(u_1) W_n^\varepsilon) dx \leq 0.$$

We now prove that the third, fourth and fifth terms of (3.16) tend to zero as ε goes to zero and then as n goes to infinity.

Recalling the definition of W_n^ε we have

$$(3.18) \quad \left| \int_{\Omega} f_2(h_n(u_1) - h_n(u_2)) W_n^\varepsilon dx \right| \leq \|\varphi\|_{L^\infty(\Omega)} \|F_n^\varepsilon\|_{L^\infty(\Omega)} \int_{\Omega} |f_2| |h_n(u_1) - h_n(u_2)| dx.$$

Since $h_n \rightarrow 1$ in L^∞ weak-* and almost everywhere in Ω as n goes to infinity and since $f_2 \in L^1(\Omega)$ the Lebesgue convergence theorem and the fact that $|F_n^\varepsilon| \leq 1$ uniformly with respect to ε and n imply that

$$(3.19) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \left| \int_{\Omega} f_2(h_n(u_1) - h_n(u_2)) W_n^\varepsilon dx \right| = 0.$$

We now turn to the fourth term of (3.16). Since h_n is a Lipschitz continuous function we obtain (recalling the definition of W_n^ε)

$$(3.20) \quad \left| \int_{\Omega} (h_n(u_1) - h_n(u_2)) g_2 \cdot DW_n^\varepsilon dx \right| \\ \leq \frac{1}{n} \int_{\{|u_1| < 2n+1\} \cap \{|u_2| < 2n+1\}} |u_1 - u_2| |g_2| |(F_n^\varepsilon)'(T_1(u_1 - u_2))| |Du_1 - Du_2| \varphi dx \\ + \int_{\Omega} |h_n(u_1) - h_n(u_2)| |g_2| |D\varphi| |F_n^\varepsilon(T_1(u_1 - u_2))| dx.$$

On the one hand, due to arguments already used we know that

$$(3.21) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} |h_n(u_1) - h_n(u_2)| |g_2| |D\varphi| |F_n^\varepsilon(T_1(u_1 - u_2))| dx = 0.$$

On the other hand, Remark 2.2 yields that

$$\frac{1}{n} \int_{\substack{\{|u_1 - u_2| < 1\} \cap \\ \{|u_1| < 2n+1\} \cap \{|u_2| < 2n+1\}}} |u_1 - u_2| |g_2| \frac{|Du_1 - Du_2| |\varphi|}{I_n(\varepsilon) \omega_{2n+1}(|u_1 - u_2|)} dx \\ \leq \frac{C}{n I_n(\varepsilon)} \|g_2\|_{(L^2(\Omega))^N} [\|DT_{2n+1}(u_1)\|_{(L^2(\Omega))^N} + \|DT_{2n+1}(u_2)\|_{(L^2(\Omega))^N}].$$

Because $I_n(\varepsilon) \rightarrow +\infty$ as ε goes to zero, from (3.20), (3.21) and the above inequality it follows that

$$(3.22) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \left| \int_{\Omega} (h_n(u_1) - h_n(u_2)) g_2 \cdot DW_n^\varepsilon dx \right| = 0.$$

For the last term in the right-hand side of (3.16), Hölder's inequality gives

$$\left| \int_{\Omega} h_n'(u_1) W_n^\varepsilon (g_1 - g_2) \cdot Du_1 dx \right| \leq \|W_n^\varepsilon\|_{L^\infty(\Omega)} \|g_1 - g_2\|_{(L^2(\Omega))^N} \left(\frac{1}{n^2} \int_{\{|u_1| < 2n\}} |Du_1|^2 dx \right)^{1/2}$$

and condition (2.12) then implies

$$(3.23) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \left| \int_{\Omega} h_n'(u_1) W_n^\varepsilon (g_1 - g_2) \cdot Du_1 dx \right| = 0.$$

Gathering (3.16), (3.17), (3.19), (3.22) and (3.23) we conclude that (3.9) holds true.

Proof of (3.10). We have

$$\begin{aligned} & \left| \int_{\Omega} (h'_n(u_1)g_1 \cdot Du_1 - h'_n(u_2)g_2 \cdot Du_2) W_n^\varepsilon dx \right| \\ & \leq \|W_n^\varepsilon\|_{L^\infty(\Omega)} \left(\|g_1\|_{(L^2(\Omega))^N} \frac{1}{2n} \|DT_{2n}(u_1)\|_{(L^2(\Omega))^N} + \|g_2\|_{(L^2(\Omega))^N} \frac{1}{2n} \|DT_{2n}(u_2)\|_{(L^2(\Omega))^N} \right). \end{aligned}$$

Recalling that W_n^ε is bounded in $L^\infty(\Omega)$ uniformly with respect to n and ε , condition (2.12) implies (3.10).

We are now in a position to prove Lemma 3.3. With arguments already used we know that

$$(3.24) \quad \lim_{\varepsilon \rightarrow 0} B_{n,\varepsilon} = \int_{\{u_1 - u_2 > 0\}} (h_n(u_1)\mathbf{A}(x, u_1)Du_1 - h_n(u_2)\mathbf{A}(x, u_2)Du_2) \cdot D\varphi dx.$$

From equality (3.6) together with (3.7)–(3.10) and (3.24) it follows that

$$(3.25) \quad \limsup_{n \rightarrow +\infty} \int_{\{u_1 - u_2 > 0\}} (h_n(u_1)\mathbf{A}(x, u_1)Du_1 - h_n(u_2)\mathbf{A}(x, u_2)Du_2) \cdot D\varphi dx \leq 0.$$

Taking $M - \varphi$ in place of φ in (3.25), with M sufficiently large so that $M - \varphi \geq 0$, gives

$$(3.26) \quad \liminf_{n \rightarrow +\infty} \int_{\{u_1 - u_2 > 0\}} (h_n(u_1)\mathbf{A}(x, u_1)Du_1 - h_n(u_2)\mathbf{A}(x, u_2)Du_2) \cdot D\varphi dx \geq 0.$$

At last (3.25) and (3.26) allow to conclude that (3.2) holds true. The proof of Lemma 3.3 is complete. \square

With the help of Lemma 3.3 we now turn to Theorem 3.1.

Proof of Theorem 3.1. We use Lemma 3.3 with $\varphi(x) = \exp(c \sum_{i=1}^N x_i)$, where $c > 0$.

Since $h_n(s) = 0$, $\forall |s| \geq 2n$, we have

$$h_n(u_1)\mathbf{A}(x, u_1)Du_1 \mathbb{1}_{\{u_1 - u_2 > 0\}} = h_n(T_{2n}(u_1))\mathbf{A}(x, T_{2n}(u_1))DT_{2n}(u_1) \mathbb{1}_{\{T_{2n}(u_1) - T_{2n}(u_2) > 0\}}$$

almost everywhere in Ω . To shorten the notations we denote by u_1^{2n} the field $T_{2n}(u_1)$ and by u_2^{2n} the field $T_{2n}(u_2)$. It follows that (3.2) can be rewritten as

$$(3.27) \quad \lim_{n \rightarrow +\infty} \int_{\{u_1^{2n} - u_2^{2n} > 0\}} (h_n(u_1^{2n})\mathbf{A}(x, u_1^{2n})Du_1^{2n} - h_n(u_2^{2n})\mathbf{A}(x, u_2^{2n})Du_2^{2n}) \cdot D\varphi dx = 0.$$

Let us define

$$\tilde{a}_{i,j}^n(x, r) = \int_0^r a_{i,j}(x, s)h_n(s)ds.$$

Due to the regularity (2.10) of $T_K(u_1)$ and $T_K(u_2)$, assumption (2.3) implies that both $\tilde{a}_{i,j}^n(u_1^{2n})$ and $\tilde{a}_{i,j}^n(u_2^{2n})$ belong to $H_0^1(\Omega)$ and for $l = 1, 2$

$$(3.28) \quad \frac{\partial \tilde{a}_{i,j}^n(u_l^{2n})}{\partial x_k} = h_n(u_l^{2n})a_{i,j}(u_l^{2n}) \frac{\partial u_l^{2n}}{\partial x_k} + \int_0^{u_l^{2n}} h_n(s) \frac{\partial a_{i,j}}{\partial x_k}(x, s)ds.$$

Since $\frac{\partial \varphi}{\partial x_k} = c\varphi$, using (3.28) we have

$$\begin{aligned} & \int_{\{u_1^{2n} - u_2^{2n} > 0\}} (h_n(u_1^{2n})\mathbf{A}(x, u_1^{2n})Du_1^{2n} - h_n(u_2^{2n})\mathbf{A}(x, u_2^{2n})Du_2^{2n}) \cdot D\varphi dx \\ &= c \int_{\{u_1^{2n} - u_2^{2n} > 0\}} \sum_{1 \leq i, j \leq N} \left(\frac{\partial \bar{a}_{i,j}^n(u_1^{2n})}{\partial x_j} - \frac{\partial \bar{a}_{i,j}^n(u_2^{2n})}{\partial x_j} \right) \varphi dx \\ & \quad + c \int_{\{u_1^{2n} - u_2^{2n} > 0\}} \sum_{1 \leq i, j \leq N} \int_{u_1^{2n}}^{u_2^{2n}} h_n(s) \frac{\partial a_{i,j}}{\partial x_j}(x, s) ds \varphi dx. \end{aligned}$$

Let us define $w_{2n} = (u_1^{2n} - u_2^{2n})^+$ which belongs to $L^\infty(\Omega) \cap H_0^1(\Omega)$ and is such that $u_1^{2n} = w_{2n} + u_2^{2n}$ almost everywhere on $\{u_1^{2n} - u_2^{2n} > 0\}$. Since $\bar{a}_{i,j}^n(u_2^{2n} + w_{2n}) - \bar{a}_{i,j}^n(u_2^{2n})$ lies in $L^\infty(\Omega) \cap H_0^1(\Omega)$, a few computations and the integration by part formula give

$$\begin{aligned} & \int_{\{u_1^{2n} - u_2^{2n} > 0\}} (h_n(u_1^{2n})\mathbf{A}(x, u_1^{2n})Du_1^{2n} - h_n(u_2^{2n})\mathbf{A}(x, u_2^{2n})Du_2^{2n}) \cdot D\varphi dx \\ &= c \int_{\Omega} \sum_{1 \leq i, j \leq N} \left(\frac{\partial \bar{a}_{i,j}^n(u_2^{2n} + w_{2n})}{\partial x_j} - \frac{\partial \bar{a}_{i,j}^n(u_2^{2n})}{\partial x_j} \right) \varphi dx \\ & \quad + c \int_{\Omega} \sum_{1 \leq i, j \leq N} \int_{u_2^{2n}}^{u_2^{2n} + w_{2n}} h_n(s) \frac{\partial a_{i,j}}{\partial x_j}(x, s) \varphi dx \\ &= -c^2 \int_{\Omega} \sum_{1 \leq i, j \leq N} \left(\bar{a}_{i,j}^n(u_2^{2n} + w_{2n}) - \bar{a}_{i,j}^n(u_2^{2n}) \right) \varphi dx \\ & \quad + c \int_{\Omega} \sum_{1 \leq i, j \leq N} \int_{u_2^{2n}}^{u_2^{2n} + w_{2n}} h_n(s) \frac{\partial a_{i,j}}{\partial x_j}(x, s) \varphi dx \\ &= -c \int_{\Omega} \int_{u_2^{2n}}^{u_2^{2n} + w_{2n}} h_n(s) \left(c \sum_{1 \leq i, j \leq N} a_{i,j}(x, s) + \sum_{1 \leq i, j \leq N} \frac{\partial a_{i,j}}{\partial x_j}(x, s) \right) ds \varphi dx. \end{aligned}$$

Because $\varphi \geq 0$ in Ω , from assumption (2.3) and the coercivity (2.1) of the matrix field \mathbf{A} we obtain for c sufficiently large independently of n ($c > 2N^2M$ for example) that

$$(3.29) \quad \begin{aligned} & \int_{\{u_1^{2n} - u_2^{2n} > 0\}} (h_n(u_1^{2n})\mathbf{A}(x, u_1^{2n})Du_1^{2n} - h_n(u_2^{2n})\mathbf{A}(x, u_2^{2n})Du_2^{2n}) \cdot D\varphi dx \\ & \leq -\frac{\alpha c}{2} \int_{\Omega} \int_{u_2^{2n}}^{u_2^{2n} + w_{2n}} h_n(s) ds \varphi dx. \end{aligned}$$

Since u_1 and u_2 are finite almost everywhere in Ω while h_n converges to 1 almost everywhere in \mathbb{R} and is bounded by 1 we obtain

$$(3.30) \quad \lim_{n \rightarrow +\infty} \int_{u_2^{2n}}^{u_2^{2n} + w_{2n}} h_n(s) ds = \int_{u_2}^{u_2 + w} ds = w \quad \text{almost everywhere in } \Omega,$$

where $w = (u_1 - u_2)^+$.

Finally from (3.27), (3.29), (3.30) and Fatou lemma it follows that

$$\int_{\Omega} w dx \leq 0,$$

which leads to $u_1 \leq u_2$ almost everywhere in Ω .

The proof of Theorem 3.1 is complete. \square

Acknowledgments. The author thanks the referee for his comments.

REFERENCES

- M. Artola. Sur une classe de problèmes paraboliques quasi-linéaires. *Boll. Un. Mat. Ital. B (6)*, 5(1):51–70, 1986.
- P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J.L. Vazquez. An L^1 -theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa*, 22:241–273, 1995.
- D. Blanchard, F. Désir, and O. Guibé. Quasi-linear degenerate elliptic problems with L^1 data. *Nonlinear Anal.*, 60(3):557–587, 2005. ISSN 0362-546X.
- L. Boccardo and T. Gallouët. On some nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, 87:149–169, 1989.
- L. Boccardo, T. Gallouët, and F. Murat. Unicité de la solution de certaines équations elliptiques non linéaires. *C. R. Acad. Sci. Paris Sér. I Math.*, 315(11):1159–1164, 1992. ISSN 0764-4442.
- J. Carrillo and M. Chipot. On some nonlinear elliptic equations involving derivatives of the nonlinearity. *Proc. Roy. Soc. Edinburgh Sect. A*, 100(3-4):281–294, 1985. ISSN 0308-2105.
- M. Chipot and G. Michaille. Uniqueness results and monotonicity properties for strongly nonlinear elliptic variational inequalities. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 16(1):137–166, 1989.
- G. Dal Maso, F. Murat, L. Orsina, and A. Prignet. Renormalized solutions of elliptic equations with general measure data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(4):741–808, 1999.
- P.-L. Lions and F. Murat. Solutions renormalisées d'équations elliptiques. (in preparation).
- F. Murat. Soluciones renormalizadas de EDP elipticas non lineales. Technical Report R93023, Laboratoire d'Analyse Numérique, Paris VI, 1993. Cours à l'Université de Séville.
- F. Murat. Equations elliptiques non linéaires avec second membre L^1 ou mesure. In *Compte Rendus du 26ème Congrès d'Analyse Numérique*, les Karellis, 1994.
- A. Porretta. Uniqueness of solutions for nonlinear elliptic Dirichlet problems. *NoDEA Nonlinear Differential Equations Appl.*, 11(4):407–430, 2004.
- J. Serrin. Pathological solution of elliptic differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 18:385–387, 1964.

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