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On uniqueness of large solutions of nonlinear parabolic equations in nonsmooth domains

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Abstract We study the existence and uniqueness of the positive solutions of the problem (P):
\[ \partial_t u - \Delta u + u^q = 0 \] in \( \Omega \times (0, \infty) \), \( u = \infty \) on \( \partial \Omega \times (0, \infty) \), and \( u(., 0) \in L^1(\Omega) \), when \( \Omega \) is a bounded domain in \( \mathbb{R}^N \). We construct a maximal solution, prove that this maximal solution is a large solution whenever \( q < N/(N-2) \) and it is unique if \( \partial \Omega = \partial \Omega^c \). If \( \partial \Omega \) has the local graph property, we prove that there exists at most one solution to problem (P).

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1 Introduction
Let \( q > 1 \) and let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with boundary \( \partial \Omega := \Gamma \). It has been proved by Keller and Osserman that there exists a maximal solution \( \overline{u} \) to the stationary equation
\[ -\Delta u + |u|^{q-1}u = 0 \text{ in } \Omega. \] (1.1)
When \( 1 < q < N/(N-2) \) this maximal solution is a large solution in the sense that
\[ \lim_{\rho(x) \to 0} \overline{u}(x) = \infty \] (1.2)
where \( \rho(x) = \text{dist}(x, \partial \Omega) \). Furthermore Véron proves in that \( \overline{u} \) is the unique large solution whenever \( \partial \Omega = \partial \Omega^c \). When \( q \geq N/(N-2) \) his proof of uniqueness does not apply. Marcus and Véron prove in that, there exists at most one large solution, provided \( \partial \Omega \) is locally the graph of a continuous function. The aim of this article is to extend these questions to the parabolic equation
\[ \partial_t u - \Delta u + |u|^{q-1}u = 0 \text{ in } \Omega \times (0, \infty). \] (1.3)
We are interested into positive solutions which satisfy
\[ \lim_{t \to 0} u(., t) = f \text{ in } L^1_{loc}(\Omega). \] (1.4)
where \( f \in L_{1}^{1}(\Omega) \) and
\[
\lim_{(x,t) \to (y,s)} u(x, t) = \infty \quad \forall (y, s) \in \Gamma \times (0, \infty).
\] (1.5)

Notice that if the initial and boundary conditions are exchanged, i.e. \( u(., t) \) blows-up when \( t \to 0 \) and coincides with a locally integrable function on \( \Gamma \times (0, \infty) \), this problem is associated with the study of the initial trace, and much work has been done by Marcus and Véron \[4\] in the case of a smooth domain. In particular they obtain the existence and uniqueness when \( q \) is subcritical, i.e. \( 1 < q < 1 + 2/N \).

In this article we prove two series of results:

**Theorem A** Assume \( q > 1 \) and \( \Omega \) is a bounded domain. Then for any \( f \in L_{1}^{1}(\Omega) \) there exists a maximal solution \( \overline{u}_f \) to problem (2.5) satisfying (1.4). If \( 1 < q < N/(N-2) \), \( \overline{u}_f \) satisfies (1.5). At end, if \( 1 < q < N/(N-2) \) and \( \partial \Omega = \partial \Omega^c \), \( \overline{u}_f \) is the unique solution of the problem which satisfies (1.5).

The proof of uniqueness is based upon the construction of self-similar solutions of (2.5) in \( \mathbb{R}^N \setminus \{0\} \times (0, \infty) \), with a persistent strong singularity on the axis \( \{0\} \times (0, \infty) \) and a zero initial trace on \( \mathbb{R}^N \setminus \{0\} \). This solution, which is studied in Appendix, is reminiscent of the very singular solution of Brezis, Peletier and Terman \[2\], although the method of construction is far different. The uniqueness is a delicate adaptation to the parabolic framework of the proof by contradiction of \[12\].

**Theorem B** Assume \( q > 1 \), \( \Omega \) is a bounded domain and \( \partial \Omega \), is locally a continuous graph. Then for any \( f \in L_{1}^{1}(\Omega) \) there exists at most one solution to problem (2.5) satisfying (1.4) and (1.5).

For proving this result, we adapt the idea which was introduced in \[6\] of constructing local super and subsolutions by small translations of the domain, but the non-uniformity of the boundary blow-up creates an extra-difficulty. In an appendix we study a self-similar equation which plays a key-role in our construction,
\[
\begin{aligned}
H'' + \left( \frac{N-1}{r} + \frac{r}{2} \right) H' + \frac{1}{q-1} H - |H|^{q-1} &= 0 \\
\lim_{r \to 0} H(r) &= \infty \\
\lim_{r \to \infty} r^{2/(q-1)} H(r) &= 0.
\end{aligned}
\] (1.6)

We prove the existence and the uniqueness of the positive solution of (1.6) when \( 1 < q < N/(N-2) \) and we give precise asymptotics when \( r \to 0 \) and \( r \to \infty \).

This article is organised as follows: 1- Introduction. 2- The maximal solution 3- The case \( 1 < q < N/(N-2) \). 4- The local continuous graph property. 5- Appendix.

2 The maximal solution

In this section \( \Omega \) is an open domain of \( \mathbb{R}^N \), with a compact boundary \( \Gamma := \partial \Omega \). If \( G \) is any open subset of \( \mathbb{R}^N \) and \( 0 < T \leq \infty \), we denote \( Q^G_T := G \times (0, T) \). If \( f \in L_{1}^{1}(\Omega) \), we
consider the problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u + |u|^{q-1}u &= 0 \quad \text{in } Q_T^\infty \\
\lim_{t \to 0} u(., t) &= f(.) \quad \text{in } L^1_{loc}(\Omega) \\
\lim_{(x,t) \to (y,s)} u(x,t) &= \infty \quad \forall (y,s) \in \Gamma \times (0, \infty).
\end{aligned}
\]  

(2.1)

By the next result, we reduce the lateral blow-up condition by a locally uniform one in which we set \(\rho(x) = \text{dist}(x, \Gamma)\).

Lemma 2.1 The following two conditions are equivalent

\[
\lim_{(x,t) \to (y,s)} u(x,t) = \infty \quad \forall (y,s) \in \Gamma \times (0, \infty) \tag{2.2}
\]

and

\[
\lim_{\rho(x) \to 0} u(x,t) = \infty \quad \text{uniformly on } [\tau, T], \tag{2.3}
\]

for any \(0 < \tau < T < \infty\).

Proof. It is clear that (2.3) is equivalent to the fact that (2.2) holds uniformly on \(\Gamma \times [\tau, T]\). By contradiction, we assume that (2.2) does not hold uniformly for some \(T > \tau > 0\). Then there exists \(\beta > 0\) such that for any \(\delta > 0\), there exist two couples \((y_\delta, s_\delta) \in \Gamma \times [\tau, T]\) and \((x_\delta, t_\delta) \in \Omega \times [\tau, T]\) such that

\[
|x_\delta - y_\delta| + |t_\delta - s_\delta| \leq \delta \quad \text{and} \quad u(x_\delta, t_\delta) \leq \beta. \tag{2.4}
\]

Taking \(\delta = 1/n, n \in \mathbb{N}^+\), we can assume that \(\{\delta\}\) is discrete and that \(y_\delta \to y \in \Gamma\) and \(s_\delta \to s \in [\tau, T]\). Thus \(x_\delta \to y\) and \(t_\delta \to s\). Therefore (2.4) contradicts (2.2). \(\square\)

Theorem 2.2 For any \(q > 1\) and \(f \in L^1_{loc+}(\Omega)\), there exists a maximal solution \(u := \mathfrak{u}_f\) of

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u + |u|^{q-1}u &= 0 \quad \text{in } Q_T^\infty \\
\lim_{t \to 0} u(., t) &= f(.) \quad \text{in } L^1_{loc}(\Omega).
\end{aligned}
\]  

(2.5)

which satisfies

\[
\lim_{t \to 0} u(., t) = f(.) \quad \text{in } L^1_{loc}(\Omega). \tag{2.6}
\]

Proof. Let \(\Omega_n\) be an increasing sequence of smooth bounded domains such that \(\overline{\Omega}_n \subset \Omega_{n+1} \subset \Omega\) and \(\cup \Omega_n = \Omega\). For each \(n\) let \(u_{n,f}\) be the increasing limit when \(k \to \infty\) of the \(u_{n,k,f}\) solution of

\[
\begin{aligned}
\frac{\partial u_{n,k,f}}{\partial t} - \Delta u_{n,k,f} + u_{n,k,f}^q &= 0 \quad \text{in } Q_T^\infty \\
u_{n,k,f}(x,t) &= k \quad \text{in } \partial \Omega_n \times (0, \infty) \\
u_{n,k,f}(x,0) &= f\chi_{\Omega_n} \quad \text{in } \Omega_n.
\end{aligned}
\]  

(2.7)

By the maximum principle and a standard approximation argument \(n \mapsto u_{n,k,f}\) is decreasing thus \(n \mapsto u_{n,f}\) too. The limit \(\mathfrak{u}_f\) of the \(u_{n,f}\) satisfies (2.5) and (2.6). It is independent of the exhaustion \(\{\Omega_n\}\) of \(\Omega\). Let \(u\) be a positive solution of (2.5) in \(Q_T^\infty\) which satisfies (2.6). Since the initial trace of \(u\) is a locally integrable function, \(u^\dagger \in L^1_{loc}(\Omega \times [0, \infty))\). By
Fubini we can assume that, for any \( n, u \in L^1_{loc}(\partial \Omega_n \times [0, \infty)) \). Because \((u - u_{n,k,f})_+ \leq u\) and tends to 0 when \( k \to \infty \), it follows by Lebesgue’s theorem that
\[
\lim_{k \to \infty} \|(u - u_{n,k,f})_+\|_{L^1(\partial \Omega_n \times (0,T))} = 0 \quad \forall T > 0.
\]
Applying the maximum principle in \( \Omega_n \times (0, \infty) \) yields to
\[
u(n,k,f) \leq \lim_{k \to \infty} u_{n,k,f} = u_{n,f} = \lim_{n \to \infty} u_{n,f} = u_f.
\]

**Theorem 2.3** For any \( q > 1 \) and \( f \in L^1_{loc}(\Omega) \), there exists a minimal nonnegative solution \( u_f \) of (2.5) in \( Q^{\infty}_{\Omega} \) which satisfies (2.6).

Proof. The scheme of the construction is similar to the one of \( \pi_f \): with the same exhaustion \( \{\Omega_n\} \) of \( \Omega \), we consider the solution \( u_{n,0,f} \) solution of
\[
\begin{aligned}
\partial_t u_{n,0,f} - \Delta u_{n,0,f} + u_{n,0,f}^q &= 0 \quad \text{in } Q^{\infty}_{\Omega_n} \\
u_{n,0,f}(x,t) &= 0 \quad \text{in } \partial \Omega_n \times (0, \infty) \\
u_{n,0,f}(x,0) &= f \chi_{\Omega_n} \quad \text{in } \Omega_n.
\end{aligned}
\]

By the maximum principle, \( n \mapsto u_{n,0,f} \) is increasing and dominated by \( u_f \). Therefore it converges to some solution \( u_f \) of (2.5), which satisfies (2.6) as \( u_{n,0,f} \) and \( u_f \) do it. Using the same argument as in the proof of Theorem 2.2, there holds \( u_{n,0,f} \leq u \) in \( Q^{\infty}_{\Omega_n} \) for a suitable exhaustion. Thus \( u_f \leq u \). \( \square \)

Remark. Because of the lack of regularity of \( \partial \Omega \), there is no reason for \( \pi_f \) (resp. \( u_f \)) to tend to infinity (resp. zero) on \( \partial \Omega \times (0, \infty) \).

The next statement will be very usefull for proving uniqueness results.

**Theorem 2.4** Assume \( q > 1 \), \( f \in L^1_{loc}(\Omega) \) and \( u_f \) is a nonnegative solution of (2.5) satisfying (2.6). Then there exists a nonnegative solution \( u_0 \) of (2.5) satisfying (2.9) such that
\[
0 \leq u_f - u_0 \leq u_f,
\]
and
\[
0 \leq \pi_f - u_f \leq \pi_f - u_0.
\]

Proof. Step 1: construction of \( u_0 \). The function \( w = u_f - u_0 \) is a nonnegative subsolution of (2.5) which satisfies (2.11) such that
\[
\lim_{t \to 0} w(.,t) = 0 \quad \text{in } L^1_{loc}(\Omega).
\]
Using the above considered exhaustion of \( \Omega \), we denote by \( v_n \) the solution of
\[
\begin{aligned}
\partial_t v_n - \Delta v_n + v_n^q &= 0 \quad \text{in } Q^{\infty}_{\Omega_n} \\
v_n(x,t) &= u_f - u_0 \quad \text{in } \partial \Omega_n \times (0, \infty) \\
v_n(x,0) &= 0 \quad \text{in } \Omega_n.
\end{aligned}
\]
By the maximum principle
\[ u_f - u_f \leq v_n \leq u_f \quad \text{in} \quad Q_{\Omega_{\infty}}. \]
Therefore \( v_{n+1} \geq v_n \) on \( \partial \Omega_n \times (0, \infty) \); this implies that the same inequality holds in \( Q_{\Omega_{\infty}} \).

If we denote by \( u_0 \) the limit of the \( \{v_n\} \), it is a solution of (2.5) in \( Q_{\Omega_{\infty}} \). For any compact \( K \in \Omega \), there exists \( n_K \) and \( \alpha > 0 \) such that \( \text{dist} (K, \Omega_n^c) > \alpha \) for \( n \geq n_K \) therefore \( v_n \) remains uniformly bounded on \( K \) by Brezis-Friedman estimate [3]. Thus the local equicontinuity of the \( v_n \) (consequence of the regularity theory for parabolic equations) implies that \( u_0 \) satisfies (2.9).

**Step 2: proof of (2.11).** We follow a method introduced in [8] in a different context. For \( n \in \mathbb{N} \) and \( k > 0 \) fixed, we set
\[ Z_{f,n} = u_{f,n} - u_f \quad \text{and} \quad Z_{0,n} = u_{0,n} - u_0, \]
where we assume that the \( n \) are chosen such that \( u_f, u_0 \in L^1_{\text{loc}}(\partial \Omega_n \times [0, \infty)) \), and
\[ \phi(r,s) = \begin{cases} \frac{r^q - s^q}{r - s} & \text{if } r \neq s \\ 0 & \text{if } r = s. \end{cases} \]

By convexity,
\[ \begin{cases} r_0 \geq s_0, \quad r_1 \geq s_1 \\ r_1 \geq r_0, \quad s_1 \geq s_0 \end{cases} \implies \phi(r_1, s_1) \geq \phi(r_0, s_0). \]
Therefore
\[ \phi(u_{f,n}, u_f) \geq \phi(u_{0,n}, u_0) \quad \text{in} \quad Q_{\Omega_{T}}, \]
and
\[ 0 = \partial_t(Z_{f,n} - Z_{0,n}) - \Delta(Z_{f,n} - Z_{0,n}) + u_{f,n} - u_f - u_{0,n} + u_0 \]
\[ = \partial_t(Z_{f,n} - Z_{0,n}) - \Delta(Z_{f,n} - Z_{0,n}) + \phi(u_{f,n}, u_f)(Z_{f,n} - Z_{f,n} - \phi(u_{0,n}, u_0)Z_{0,n}), \]
which implies
\[ \partial_t(Z_{f,n} - Z_{0,n}) - \Delta(Z_{f,n} - Z_{0,n}) + \phi(u_{f,n}, u_f)(Z_{f,n} - Z_{0,n}) \leq 0. \]
But \( Z_{f,n} - Z_{0,n} = 0 \) in \( \Omega_n \times \{0\} \) and
\[ \int_0^\infty \int_{\partial \Omega_n} |Z_{f,n} - Z_{0,n}| \, dS \, dt = 0 \]
by approximations. By the maximum principle \( Z_{f,n,k} - Z_{0,n,k} \leq 0 \). Letting \( n \to \infty \) yields to
\[ \overline{u}_f - u_f \leq \overline{u}_0 - u_0, \]
which ends the proof. \( \square \)
Theorem 3.1 Assume $1 < q < N/(N-2)$ and $f \in L^1_{\text{loc}}(\Omega)$. Then the maximal solution $\overline{u}_f$ of (2.5) in $Q_1^\Omega$ which satisfies (2.7) satisfies also (2.3).

Proof. In Appendix we construct the self-similar solution $V := V_N$ of (2.5) in $Q_\infty^N \setminus \{0\}$ which has initial trace zero in $\mathbb{R}^N \setminus \{0\}$ and satisfies

$$\lim_{|x| \to 0} V_N(x,t) = \infty,$$

locally uniformly on $[\tau, \infty)$, for any $\tau > 0$. Furthermore, $V_N(x,t) = t^{-1/(q-1)} H_N(|x|/\sqrt{t})$. If $a \in \partial \Omega$, the restriction to $\Omega_n$ of the function $V_N(x-a,t)$ is bounded from above by $u_{n,f}$. Letting $n \to \infty$ yields to

$$V_N(x-a,t) \leq \overline{u}_f(x,t) \quad \forall (x,t) \in Q_\infty^\Omega. \tag{3.1}$$

If we consider $x \in \Omega$ and denote by $a_x$ a projection of $x$ onto $\partial \Omega$, there holds

$$t^{-1/(q-1)} H_N(|x-a_x|/\sqrt{t}) = V_N(x-a_x,t) \leq \overline{u}_f(x,t). \tag{3.2}$$

Using (5.2), we derive that $\overline{u}_f$ satisfies (2.3).

Theorem 3.2 Assume $1 < q < N/(N-2)$, $f \in L^1_{\text{loc}}(\Omega)$ and $\partial \Omega = \partial \Omega^\tau$. Then $\overline{u}_f$ is the unique solution of (2.5) in $Q_1^\Omega$ which satisfies (2.6) and (2.3).

Proof. Assume that $u_f$ is a solution of (2.5) in $Q_1^\Omega$ such that (2.6) and (2.3) hold. By Theorem 2.4 there exists a positive solution $u_0$ with zero initial trace such that

$$0 \leq u_f - u_0 \leq \overline{u}_f \tag{3.3}$$

and (2.11) are satisfied. Since $u_f(x,t) \leq (t-1)^{-1/(q-1)}$ (notice that this last expression is the maximal solution of (2.5) in $Q_\infty^N$), the function $u_0$ satisfies also (2.3). Therefore, it is sufficient to prove that $\overline{u}_f = u_0 := u$.

Step 1: bilateral estimates. Since $\partial \Omega = \partial \Omega^\tau$, for any $a \in \partial \Omega$, there exists a sequence $\{a_n\} \subset \Omega^\tau$ converging to $a$. If $u$ is any solution of (2.5) in $Q_1^\Omega$ which satisfies (2.3) and (2.9), there holds

$$V_N(x-a_n,t) \leq u(x,t) \implies V_N(x-a,t) \leq u(x,t).$$

In particular, if $a = a_x$, we see that $u$ satisfies (3.2). In order to obtain an estimate from above we consider for $r < \rho(x)$ the solution $(y,t) \mapsto u_{x,r}(y,t)$ of

$$\begin{cases}
\partial_t u_{x,r} - \Delta u_{x,r} + u_{x,r}^q = 0 & \text{in } Q_\infty^{B_r(x)} \\
\lim_{(y,t) \to (z,0)} u_{x,r}(y,t) = 0 & \forall z \in B_r(x) \\
\lim_{|x|/r} u_{x,r}(x,t) = \infty & \text{locally uniformly on } [\tau, \infty), \text{for any } \tau > 0
\end{cases} \tag{3.4}$$
Then
\[ \mathfrak{m}_0(y, t) \leq u_{x,t}(y, t) \implies \mathfrak{m}_0(y, t) \leq u_{x,\rho(x)}(y, t) \quad \forall (y, t) \in Q^{B_{\rho(x)}(x)}._\infty. \]
In particular, with \( u_{0,t} = u_t \),
\[ \mathfrak{m}_0(x, t) \leq u_{\rho(x)}(0, t) = (\rho(x))^{-2/(q-1)} u_1(0, t/(\rho(x))^2). \]
Therefore
\[ t^{-1/(q-1)} H_N(\rho(x)/\sqrt{t}) \leq u(x, t) \leq \mathfrak{m}_0(x, t) \leq (\rho(x))^{-2/(q-1)} u_1(0, t/(\rho(x))^2). \] (3.5)
The function \( s \mapsto u_1(0, s) \) is increasing by the same argument as the one of Corollary 4.3 and bounded from above by the unique solution \( P \) of
\[
\begin{align*}
-\Delta P + P^q &= 0 \quad \text{in } B_1, \\
\lim_{|x| \to 1} P(x) &= \infty.
\end{align*}
\] (3.6)
Therefore it converges to \( P \) locally uniformly in \( B_1 \) and \( \lim_{s \to \infty} u_1(0, s) = P(0) \). Thus
\[ t/(\rho(x))^2 \to \infty \implies (\rho(x))^{-2/(q-1)} u_1(0, t/(\rho(x))^2) \approx P(0)(\rho(x))^{-2/(q-1)}. \] (3.7)
On the other hand, if \( t/(\rho(x))^2 \to \infty \), equivalently \( \rho(x)/\sqrt{t} \to 0 \),
\[ t^{-1/(q-1)} H_N(\rho(x)/\sqrt{t}) \approx \lambda_N q^{-1/(q-1)} (\rho(x)/\sqrt{t})^{-2/(q-1)} = \lambda_N q (\rho(x))^{-2/(q-1)}, \] (3.8)
by (3.4).

Next, in order to obtain an estimate from above of \( u_1(0, s) \) when \( s \to 0 \), we compare \( u_1 \) to a solution \( u_\Theta \) of (2.5) in \( Q^\infty_{\infty} \), where \( \Theta \) is a polyhedra inscribed in \( B_1 \); this polyhedra is a finite intersection of half spaces \( \Gamma_i \) containing \( \Pi \). In each of the half space \( \Gamma_i \), with boundary \( \gamma_i \), we can consider the solution \( W_i \) of (2.5) in \( Q^\infty_{\infty} \) which tends to infinity on \( \gamma_i \times (0, \infty) \) and has value 0 on \( \Gamma_i \times \{0\} \). This solution depends only on the distance to \( \gamma_i \) and \( t \). Thus it is expressed by the function \( V_i \) defined in Proposition 5.3 when \( N = 1 \). Moreover, since a sum of solutions is a super solution,
\[ u_1 \leq u_\Theta \leq \sum_i W_i \implies u_1(0, s) \leq \sum_i H_i(\text{dist}(0, \gamma_i)/\sqrt{s}). \] (3.9)
We can choose the hyperplanes \( \gamma_i \) such that for any \( \delta \in (0, 1) \), there exists \( C_\delta \in \mathbb{N}_+ \) such that
\[ u_1(0, s) \leq C_\delta H_1((1 - \delta)/\sqrt{s}). \] (3.10)
Using (3.3) we derive
\[ u(x, t) \geq c_{N,q} (\rho(x))^{2/(q-1)-N_i} t^{N/2-1/(q-1)} e^{-(\rho(x))^2/4t}, \]
when \( \rho(x)/\sqrt{t} \to \infty \), and
\[ \mathfrak{m}_0(x, t) \leq C H_1((1-\delta)\rho(x)/\sqrt{t}) \leq C (1-\delta)^{2/(q-1)-1}(\rho(x))^{2/(q-1)-1} t^{1/2-1/(q-1)} e^{-(1-\delta)\rho(x)^2/4t}. \]
Therefore, there exists $\theta > 1$ such that
\[ \underline{w}_0(x, t) \leq C(\rho(x))^{2/(q-1)-N/2} e^{-(\rho(x))^2/4\theta t} \leq C u(x, \theta t), \] (3.11)
when $\rho(x)/\sqrt{t} \rightarrow \infty$. Finally, when $m^{-1} \leq \rho(x)/\sqrt{t} \leq m$ for some $m > 1$, (3.5) shows that $(\rho(x))^{-2/(q-1)} u_1(0, t/(\rho(x))^2)$ and $t^{-1/(q-1)} H_N(\rho(x)/\sqrt{t})$ are comparable. In conclusion, there exist constants $C > P(0)/\lambda_{N,q} > 1$ and $\theta > 1$ such that
\[ u(x, t) \leq \underline{w}_0(x, t) \leq C u(x, \theta t) \quad \forall (x, t) \in Q^\Omega_\infty. \] (3.12)

**Step 2: End of the proof.** Let $\tau > 0$ and $C' > C$ be fixed. The function
\[ t \mapsto u_\tau(x, t) := C' u(x, t + \theta \tau) \]
is a supersolution of (2.5) in $\Omega \times (0, \infty)$ which satisfies $u_\tau(x, 0) = C' u(x, \theta \tau) > \underline{w}_0(x, \tau)$ by (3.12). Furthermore,
\[ C' u(x, t + \theta \tau) \geq C' (t + \theta \tau)^{-1/(q-1)} H_N(\rho(x)/\sqrt{t + \theta \tau}) = C' \lambda_{N,q} (1 + o(1))(\rho(x))^{-2/(q-1)}, \]
as $\rho(x) \rightarrow 0$, locally uniformly for $t \in [0, \infty)$. Similarly,
\[ \bar{w}_0(x, t + \tau) \leq (\rho(x))^{-2/(q-1)} u_1(0, (t + \tau)/(\rho(x))^2) = P(0)(1 + o(1))(\rho(x))^{-2/(q-1)}, \]
as $\rho(x) \rightarrow 0$, and also locally uniformly for $t \in [0, \infty)$. Therefore $(\bar{w}_0(x, t) - u_\tau(x, t))_+$ vanishes in a neighborhood of $\partial \Omega \times [0, T]$ for any $T > 0$. By the maximum principle
\[ u_\tau(x, t) \geq \bar{w}_0(x, t) \quad \forall (x, t) \in \Omega \times (0, \infty). \]

Letting $\tau \rightarrow 0$ and $C' \rightarrow C$ yields to
\[ u(x, t) \leq \underline{w}_0(x, t) \leq C u(x, t) \quad \forall (x, t) \in Q^\Omega_\infty. \] (3.13)

The conclusion of the proof is contradiction, following an idea introduced in [8] and developed by [12] in the elliptic case. We assume $u \neq \underline{w}_0$, thus $u < \underline{w}_0$. By convexity the function
\[ w = u - \frac{1}{2C} (\underline{w}_0 - u) \]
is a supersolution and $w < u$. Moreover $w > w' := ((1 + C)/2C) u$ and $w'$ is a subsolution. Consequently, there exists a solution $u_1$ of (2.5) which satisfies
\[ w' < u_1 \leq w \implies \underline{w}_0 - u_1 \geq (1 + K^{-1}) (\underline{w}_0 - u) \quad \text{in} \quad Q^\Omega_\infty. \] (3.14)

Notice that $u_1$ satisfies (2.9) and (2.3), therefore it satisfies (3.13) as $u$ does it. Replacing $u$ by $u_1$ and introducing the supersolution
\[ w_1 = u_1 - \frac{1}{2C} (\underline{w}_0 - u_1) \]
and the subsolution $w'_1 := ((1 + C)/2C) u_1$ we see that there exists a solution $u_2$ of (2.5) such that
\[ w'_1 < u_2 \leq w_1 \implies \underline{w}_0 - u_2 \geq (1 + K^{-1})^2 (\underline{w}_0 - u) \quad \text{in} \quad Q^\Omega_\infty. \] (3.15)
By induction, we construct a sequence of positive solutions $u_k$ of (2.5), subject to (2.9) and (2.3) such that
\[ u_0 - u_k \geq (1 + K^{-1})^k (u_0 - u) \quad \text{in} \quad Q_\infty^\Omega. \] (3.16)
This is clearly a contradiction since $(1 + K^{-1})^k \to \infty$ as $k \to \infty$ and $u_0$ is locally bounded in $Q_\infty^\Omega$. □

4 The local continuous graph property

In this section, we assume that $\partial \Omega$ is compact and is locally the graph of a continuous function, which means that there exists a finite number of open sets $\Omega_j$ ($j = 1, \ldots, k$) such that $\Gamma \cap \Omega_j$ is the graph of a continuous function. Our main result is the following

**Theorem 4.1** Assume $q > 1$ and $f \in L^1_{loc} (\Omega)$. Then there exists at most one positive solution of (2.5) in $Q_\infty^\Omega$ satisfying (2.6) and (2.3).

Suppose $u_f$ satisfies (2.5) in $Q_\infty^\Omega$ satisfying (2.6) and (2.3), then clearly the maximal solution $u_f$ endows the same properties. In order to prove that $u_f = u_f$, we can assume that $f = 0$ by Theorem 2.4. We denote by $u$ this large solution with zero initial trace. We consider some $j \in \{ 1, \ldots, k \}$, perform a rotation, denote by $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ the coordinates in $\mathbb{R}^N$ and represent $\Gamma \cap \Omega_j$ as the graph of a continuous positive function $\phi$ defined in $C = \{ x' \in \mathbb{R}^{N-1} : |x'| \leq R \}$. We identify $C$ with $\{ x = (x', 0) : |x'| \leq R \}$ and set
\[ \Gamma_1 = \{ x = (x', \phi(x')) : x' \in C \}, \]
\[ \Gamma_2 = \{ x = (x', x_N) : x' \in \partial C, 0 \leq x_N < \phi(x') \}, \]
and
\[ G_R = \{ x \in \mathbb{R}^N : |x'| < R, 0 < x_N < \phi(x') \}. \]
We can assume that $\overline{G}_R \subset \Omega \cup \Gamma_1$, \[ \inf \{ \phi(x') : x' \in C \} = R_0 > 0 \quad \text{and} \quad \sup \{ \phi(x') : x' \in C \} = R_1 > R_0. \]
For $\sigma > 0$, small enough, we consider $\phi_\sigma \in C^\infty (C)$ satisfying
\[ \phi(x') - \sigma/2 \leq \phi_\sigma(x') \leq \phi(x') + \sigma/2 \quad \forall x' \in C, \]
and set
\[ G_{\sigma, R} = \{ x \in \mathbb{R}^N : |x'| < R, 0 < x_N < \phi_\sigma(x') - \sigma \} \]
and
\[ G'_{\sigma, R} = \{ x \in \mathbb{R}^N : |x'| < R, 0 < x_N < \phi_\sigma(x') + \sigma \}. \]
The upper boundaries of $G_{\sigma}$ and $G'_{\sigma}$ are defined by
\[ \Gamma_{1, \sigma} = \{ x = (x', \phi_\sigma(x') - \sigma) : x' \in C \}, \]
\[ \Gamma'_{1, \sigma} = \{ x = (x', \phi_\sigma(x') + \sigma) : x' \in C \}. \]
and the remaining boundaries are
\[ \Gamma_{2, \sigma} = \{ x = (x', x_N) : x' \in \partial C, 0 \leq x_N \leq \phi_\sigma(x') - \sigma \}, \]
\[ \Gamma'_{2, \sigma} = \{ x = (x', x_N) : x' \in \partial C, 0 \leq x_N \leq \phi_\sigma(x') + \sigma \}. \]
In order to have the monotonicity of the domains, we can also assume
\[ \phi_\sigma(x') - \sigma < \phi_{\sigma'}(x') - \sigma' < \phi_\sigma(x') + \sigma' < \phi_{\sigma'}(x') + \sigma \quad \forall 0 < \sigma' < \sigma \quad \forall x' \in C, \] (4.1)
thus, under the condition \( 0 < \sigma' < \sigma \),
\[ G_{\sigma, R} \subset G_{\sigma', R} \subset G_R \subset G'_{\sigma', R} \subset G'_{\sigma, R}. \] (4.2)
The localization procedure is to consider the restriction of \( u \) to \( Q^{\infty}_{G_R} := G_R \times (0, \infty) \), thus \( u \) is regular in \( G_R \cup \Gamma_2 \times [0, \infty) \) and satisfies
\[ \lim_{x_N \to \phi(x')} u(x', x_N, t) = \infty, \] (4.3)
uniformly with respect to \((x', t) \in C \times [\tau, T]\), for any \( 0 < \tau < T \). We construct \( v_\sigma \) as solution of
\[ \partial_t v_\sigma - \Delta v_\sigma + v_\sigma q_\sigma = 0 \quad \text{in} \quad Q^{\infty}_{G_R} := G_{\sigma, R} \times (0, \infty), \] (4.4)
subject to the initial condition
\[ \lim_{t \to 0} v_\sigma(x, t) = 0 \quad \text{locally uniformly in} \quad G_{\sigma, R}, \] (4.5)
and the boundary conditions
\[ \lim_{x_N \to \phi(x') - \sigma} v_\sigma(x', x_N, t) = \infty \quad \forall (x', t) \in C \times (0, \infty], \] (4.6)
uniformly on any set \( K \times [\tau, T] \), where \( T > \tau > 0 \) and \( K \) is a compact subset of \( C \) and
\[ v_\sigma(x, t) = 0 \quad \forall (x, t) \in \Gamma_2 \times [0, \infty). \] (4.7)
We also construct \( w_\sigma \) as solution of
\[ \partial_t w_\sigma - \Delta w_\sigma + w_\sigma q_\sigma = 0 \quad \text{in} \quad Q^{G_R}_{T} := G'_{\sigma, R} \times (0, \infty), \] (4.8)
subject to the initial condition
\[ \lim_{t \to 0} w_\sigma(x, t) = 0 \quad \text{locally uniformly in} \quad G'_{\sigma, R}, \] (4.9)
and the boundary conditions
\[
\begin{align*}
(i) \quad w_\sigma(x, t) &= 0 \quad \forall (x, t) \in \Gamma'_1 \times [0, T], \\
(i') \quad \lim_{(x, s) \to (y, t)} w_\sigma(x, t) &= \infty \quad \forall (y, s) \in \Gamma'_2 \times [0, T].
\end{align*}
\] (4.10)
The functions \( v_\sigma \) and \( w_\sigma \) inherit the following properties in which the local graph property plays a fundamental role, allowing translations of the truncated domains in the \( x_N \)-direction.
Lemma 4.2 For $\sigma > \sigma' > 0$ there holds
\begin{equation}
 v_{\sigma'} \leq v_{\sigma} \quad \text{in } Q^G_{\infty, R},
\end{equation}
\begin{equation}
 w_{\sigma'} \leq w_{\sigma} \quad \text{in } Q^G_{\infty, R},
\end{equation}
\begin{align}
 (i) & \quad v_{\sigma}(x', x_N - 2\sigma, t) \leq u(x', x_N, t) \quad \text{in } Q^G_{\infty, R} \\
 (ii) & \quad u(x', x_N, t) \leq v_{\sigma}(x, t) + w_{\sigma}(x, t) \quad \text{in } Q^G_{\infty, R}.
\end{align}

Proof. The inequalities (4.11) and (4.12) are the direct consequence of the fact that the domains $G_{\sigma, R}$ and $G_{\sigma', R}$ are Lipschitz and the functions $v_{\sigma}$ and $w_{\sigma}$ are constructed by approximations of solutions of (2.5) with bounded boundary data. For proving (4.13)-(i), we compare, for $\tau > 0$, $u(x, t - \tau)$ and $v_{\sigma}(x', x_N - 2\sigma, t)$ in $Q^G_{\infty, R}$. Because $u$ satisfies (2.5), and $v_{\sigma}(x', x_N - 2\sigma, 0) = 0$ in $G_R$, (4.13)-(i) follows by the maximum principle. The proof of (4.13)-(ii) needs no translation, but the fact that the sum of two solutions is a supersolution.

\[\square\]

Corollary 4.3 There exist $v_0 = \lim_{\sigma \to 0} v_{\sigma}$ and $w_0 = \lim_{\sigma \to 0} w_{\sigma}$ and there holds
\begin{equation}
 v_0 \leq u \leq v_0 + w_0 \quad \text{in } Q^G_{\infty, R}.
\end{equation}

Moreover, the functions $t \mapsto v_0(x, t)$ and $t \mapsto w_0(x, t)$ are increasing on $(0, \infty)$, $\forall x \in G_R$.

Proof. The first assertion follows from (4.11)-(4.12), and (4.14) from (4.13). Since $v_0$ is the limit, when $\sigma \to 0$ of $v_{\sigma}$ which satisfy equation (4.14) in $Q^G_{\infty, R}$, initial condition (4.5) and boundary conditions (4.6), (4.7), it is sufficient to prove the monotonicity of $t \mapsto v_{\sigma}(\cdot, t)$. Moreover $v_{\sigma}$ is the limit, when $k \to 0$ of the $v_{k, \sigma}$ solutions of (2.5) in $Q^G_{\infty, R}$, which satisfy the same boundary conditions as $v_{\sigma}$ on $\Gamma_{2, \sigma} \times [0, T]$, the same zero initial condition and
\[\lim_{x_N \to \phi(x') - \sigma} v_{k, \sigma}(x', x_N, t) = k.\]

For $\tau > 0$, we define $V_{\tau}$ by $V_{\tau}(x, t) = (v_{k, \sigma}(x, t) - v_{k, \sigma}(x, t + \tau))_{+}$. Because $\partial G_{\sigma, R}$ is Lipschitz and $V_{\tau}$ is a subsolution of (2.5) which vanishes on $\partial G_{\sigma, R} \times [0, T]$ and at $t = 0$, it is identically zero. This implies $v_{k, \sigma}(x, t) \leq v_{k, \sigma}(x, t + \tau)$, and the monotonicity property of $v_0$, by strict maximum principle and letting $\sigma \to 0$. The proof of the monotonicity of $w_0$ is similar. \[\square\]

The key step of the proof is the following result.

Proposition 4.4 Let $\epsilon, \tau > 0$. Then there exists $\delta_0 > 0$ such that, if we denote
\[G_{\delta, R'} = \{ x = (x', x_N) : |x'| < R', \phi(x') - \delta \leq x_N < \phi(x') \},\]
there holds, for $R' < R/\sqrt{N - 1}$,
\begin{equation}
 w_0(x, t) \leq \epsilon v_0(x, t + \tau) \quad \forall (x, t) \in Q^G_{\infty, R'}. 
\end{equation}
Proof. Using the result in Appendix, we recall that $V := V_1$ is the unique positive and self-similar solution of the problem

$$\begin{cases}
\partial_t V - \partial_{zz} V + V^q = 0 \quad \text{in} \; \mathbb{R}_+ \times \mathbb{R}_+ \\
\lim_{t \to 0} V(z, t) = 0 \quad \forall z > 0 \\
\lim_{z \to 0} V(z, t) = \infty \quad \forall t > 0,
\end{cases} \quad (4.16)$$

and it is expressed by $V_1(z, t) = t^{-1/(q-1)}H_1(x/\sqrt{t})$, where $H_1$ satisfies (5.2)-(5.3) with $N = 1$. We set $R_N = R/\sqrt{N-1}$ so that

$$C_\infty := \{x' = (x_1, ..., x_{N-1}) : \sup_{j \leq N-1} |x_j| < R_N\} \subset \{x' : |x'| \leq R\}$$

and we define

$$\tilde{w}(x, t) = W(x_N, t) + \sum_{j=1}^{N-1} (W(x_j - R, t) + W(R - x_j, t)).$$

The function $\tilde{w}$ a super solution in $\Theta \times \mathbb{R}^+$ where $\Theta := \{(x', x_N) : x' \in C_\infty, x_N > 0\}$ which blows up on

$$\{x : x_N = 0, \sup_{j \leq N-1} |x_j| \leq R\} \bigcup \{x : x_N \geq 0, x_j = \pm R\}.$$ 

Therefore $w_0 \leq \tilde{w}$ in $Q_G^{G_{R_N}}$. Moreover $\tilde{w}(x, t) \to 0$ when $t \to 0$, uniformly on

$$G_{\alpha, R} := \{x = (x_1, x_2) : |x_1| \leq R', \alpha \leq x_2 \leq \phi(x_1)\},$$

for any $\alpha \in (0, R_0]$ and $R' \in (0, R_N)$. Since for any $\tau > 0$, $v_0(x, t + \tau) \to \infty$ when $\rho(x) \to 0$, locally uniformly on $[0, \infty)$, and $\tilde{w}(x, t)$ remains uniformly bounded on $Q_\infty^{G_{R, R'}}$, for any $\delta > R_0$, it follows that for any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$w_0(x, t) \leq \tilde{w}(x, t) \leq \epsilon v_0(x, t + \tau) \quad \forall (x, t) \in Q_\infty^{G_{R, R'}}.$$ 

□

Proof of Theorem 4.1. Assume $u$ is a solution of (2.5) satisfying (2.6) and (2.3). Then there holds in $Q_\infty^{G_{R, R'}}$,

$$v_0(., t) \leq u(., t) \leq v_0(., t) + \epsilon v_0(., t + \tau). \quad (4.17)$$

Therefore

$$v_0(., t + \tau) \leq u(., t + \tau) \leq v_0(., t + \tau) + \epsilon v_0(., t + 2\tau),$$

from which follows

$$(1 + \epsilon)u(., t + \tau) \geq (1 + \epsilon)v_0(., t + \tau) \geq v_0(., t) + \epsilon v_0(., t + \tau)$$

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since $t \mapsto v_0(., t)$ is increasing by Corollary 4.13. The maximal solution $\overline{v}_0$ satisfies (4.17) too; consequently the following inequality is verified in $Q^{G_j}_{\infty, \omega}$,

$$(1 + \epsilon)u(., t + \tau) \geq \overline{v}_0(., t).$$

(4.18)

Since $\partial \Omega$ is compact, there exists $\delta^* > 0$ such that (4.18) holds whenever $t \in [0, T]$ ($T > 0$ arbitrary) and $\rho(x) \leq \delta^*$. Furthermore

$$\lim_{t \to 0} \max_{\overline{v}_0(x, t) - (1 + \epsilon)u(x, t + \tau)} : \rho(x) \geq \delta^* = 0$$

because of (2.6). Since $(\overline{v}_0(x, t) - (1 + \epsilon)u(x, t + \tau))_+$ is a subsolution, which vanishes at $t = 0$ and near $\partial \Omega \times [0, T]$, it follows that (4.18) holds in $Q^{G_j}_f$. Letting $\epsilon \to 0$ and $\tau \to 0$ yields to $u \geq \overline{v}_0$.

**Remark.** The existence of large solutions when $q \geq N/(N - 2)$ is a difficult problem as it is already in the elliptic case. We conjecture that the necessary and sufficient conditions, obtained by Dbersin-Le Gall when $q = 2$ [4] and Labutin [1] in the general case $q > 1$, and expressed by mean of a Wiener type criterion involving the $C^{N, q}_\delta$-Bessel capacity, are still valid. As in [1], it is clear that if $\partial \Omega$ satisfies the exterior segment property and $1 < q < (N - 1)/(N - 3)$, then $\overline{v}_0$ is a large solution.

5 Appendix

The proof of this result is based upon the existence of solution of (2.5) in $Q^{\mathbb{R}^N \setminus \{0\}}$ with a persistent singularity on $\{0\} \times [0, \infty)$.

**Proposition 5.1** For any $q > 1$, there exists a unique positive function $V := V_N$ defined in $\mathbb{R}^N \setminus \{0\}$ satisfying, for any $\tau > 0$,

$$\begin{cases}
\partial_t V - \Delta V + V^q = 0 \quad \text{in } Q^{\mathbb{R}^N \setminus \{0\}} \\
\lim_{(x,t) \to (y,0)} V(x, t) = 0 \quad \forall y \in \mathbb{R}^N \setminus \{0\} \\
\lim_{|x| \to 0} V(x, t) = \infty \quad \text{locally uniformly on } [\tau, \infty), \text{ for any } \tau > 0
\end{cases}$$

Then $V_N(x, t) = t^{-1/(q-1)}H_N(|x|/\sqrt{t})$, where $H := H_N$ is the unique positive function satisfying

$$\begin{cases}
H'' + \left(\frac{N - 1}{r} + \frac{r}{2}\right)H' + \frac{1}{q - 1}H - H^q = 0 \quad \text{in } \mathbb{R}_+ \\
\lim_{r \to 0} H(r) = \infty \\
\lim_{r \to \infty} r^{2/(q-1)}H(r) = 0.
\end{cases}$$

(5.2)

Furthermore there holds

$$H_N(r) = c_{N, q}r^{2/(q-1)} - N e^{-r^2/4}(1 + O(r^{-2})) \quad \text{as } r \to \infty,$$

(5.3)

and

$$H_N(r) = \lambda_{N, q}r^{-2/(q-1)}(1 + O(r)) \quad \text{as } r \to 0,$$

(5.4)
Proof. If we assume $1 < q < N/(N-2)$, the $C_{2.1,q'}$ parabolic capacity of the axis $\{0\} \times \mathbb{R} \subset \mathbb{R}^{N+1}$ is positive, therefore there exists a unique solution $u := u_\mu$ to the problem

$$\partial_t u - \Delta u + |u|^{q-1}u = \mu \quad \in \mathbb{R}^N \times \mathbb{R},$$

(5.5)

(see [1]) where $\mu$ is the uniform measure on $\{0\} \times \mathbb{R}_+$ defined by

$$\int \zeta d\mu = \int_0^\infty \zeta(0,t) dt \quad \forall \zeta \in C^\infty_0(\mathbb{R}^{N+1}).$$

If we denote $T_\ell[u](x,t) = \ell^{2/(q-1)}u(\ell x, \ell^2 t)$ for $\ell > 0$, then $T_\ell$ leaves the equation (2.5) invariant, and $T_\ell[u_\mu] = u_{\ell^{2/(q-1)}-N\mu}$. If we replace $\mu$ by $k\mu$ ($k > 0$), we obtain

$$T_\ell[u_{k\mu}] = u_{\ell^{2/(q-1)}-Nk\mu}.$$ 

Moreover, any solution of (2.5) in $\mathbb{R}^N \setminus \{0\} \times \mathbb{R}_+$ which vanishes on $\mathbb{R}^N \setminus \{0\} \times \{0\}$ is bounded from above by the maximum solution $u := U$ of

$$-\Delta u + u^q = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$ 

(5.7)

This is obtained by considering the solution $U_\varepsilon$ of

$$
\begin{aligned}
-\Delta u + u^q &= 0 \quad \text{in } \mathbb{R}^N \setminus B_\varepsilon \\
\lim_{|x| \to \varepsilon} u(x) &= \infty.
\end{aligned}
$$

(5.8)

Actually,

$$U(x) := \lim_{\varepsilon \to 0} U_\varepsilon(x) = \lambda_{N,q}|x|^{-2/(q-1)}$$

with $\lambda_{N,q} := \left[\frac{2}{q-1}\left(\frac{2q}{q-1} - N\right)\right]^{1/(q-1)}$, an expression which exists since $1 < q < N/(N-2)$. If we let $k \to \infty$ in (5.6), using the monotonicity of $\mu \mapsto u_\mu$, we obtain that $u_{k\mu} \to u_{\infty\mu}$, $u_{\infty\mu} \leq U$ and

$$T_\ell[u_{\infty\mu}] = u_{\ell^{2/(q-1)}-N\infty\mu} = u_{\infty\mu} \quad \forall \ell > 0.$$ 

(5.10)

This implies that $u_{\infty\mu}$ is self-similar, that is

$$u_{\infty\mu}(x,t) = t^{-1/(q-1)}h(x/\sqrt{t}).$$

Furthermore, $h(.)$ is positive and radial as $x \mapsto u_\mu(x,t)$ is, and it solves

$$h'' + \left(\frac{N-1}{r} + \frac{r}{2}\right)h' + \frac{1}{q-1}h - h^q = 0 \quad \text{in } \mathbb{R}_+.$$ 

(5.11)

Since $u_\mu(x,0) = 0$ for $x \neq 0$, the a priori bounds $u_{k\mu} \leq U$, the equicontinuity of the $\{u_{k\mu}\}_{k>0}$ implies that $u_{\infty\mu}(x,0) = 0$ for $x \neq 0$; therefore

$$\lim_{r \to \infty} r^{2/(q-1)}h(r) = 0.$$ 

(5.12)
The same argument as the one used in the proof of Corollary 4.3 implies that \( t \mapsto u_\mu(x,t) \) is increasing, therefore \( \lim_{x \to 0} u_\mu(x,t) = \infty \) for \( t > 0 \). This implies \( \lim_{r \to 0} h(r) = \infty \). Then the proof of (5.3) follows from [10, Appendix]. When \( r \to 0 \), \( h \) could have two possible behaviours:

(i) either

\[
h(r) = \lambda N,q r^{2/(q-1)} (1 + O(r)),
\]

(ii) or there exists \( c \geq 0 \) such that

\[
h(r) = cm_N(r)(1 + O(r)),
\]

where \( m_N(r) \) is the Newtonian kernel if \( N \geq 2 \) and \( m_1(r) = 1 + o(1) \).

If (ii) were true with \( c > 0 \) (the case \( c = 0 \) implying that \( h = 0 \) because of the behavior at \( \infty \) and maximum principle), it would lead to

\[
u_{\infty \mu}(x) = c|x|^{2-N}t^{N-2-1/(q-1)}(1 + o(1)) \quad \text{as} \quad x \to 0,
\]

for all \( t > 0 \). Therefore

\[
\int_T \int_{B_1} u_{k\mu}^q \, dx \, dt < C(\epsilon),
\]

for any \( \epsilon > 0 \) and \( k \in (0, \infty) \). We write (5.5) under the form

\[
\partial_t u_{k\mu} - \Delta u_{k\mu} = g_k + k\mu
\]

where \( g_k = -u_{k\mu}^q \), then \( u_{k\mu} = u_{k\mu}' + u_{k\mu}'' \), where

\[
\partial_t u_{k\mu}' - \Delta u_{k\mu}' = k\mu
\]

and

\[
\partial_t u_{k\mu}'' - \Delta u_{k\mu}'' = g_k.
\]

By linearity \( u_{k\mu}' = ku_{\mu}' \). Because of (5.16) \( u_{k\mu}' \) remains uniformly bounded in \( L^1(B_1 \times (\epsilon, T)) \). This clearly contradicts \( \lim_{k \to \infty} u_{k\mu}' = \infty \). Thus (5.4) holds. The proof of uniqueness is an easy adaptation of [8, Lemma 1.1]: the fact that the domain is not bounded being compensated by the strong decay estimate (5.3). This unique solution is denoted by \( V_N \) and \( h = H_N \). \( \square \)

References


