On Non-Rank Facets in Stable Set Polytopes of Webs with Clique Number Four
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Abstract

Graphs with circular symmetry, called webs, are relevant for describing the stable set polytopes of two larger graph classes, quasi-line graphs [6,10] and claw-free graphs [5,6]. Providing a decent linear description of the stable set polytopes of claw-free graphs is a longstanding problem [7]. However, even the problem of finding all facets of stable set polytopes of webs is open. So far, it is only known that stable set polytopes of webs with clique number \( \leq 3 \) have rank facets only [3,13] while there are examples with clique number \( > 4 \) having non-rank facets [8,10,9]. The aim of the present paper is to treat the remaining case with clique number \( = 4 \): we provide an infinite sequence of such webs whose stable set polytopes admit non-rank facets.

Key words: web, rank-perfect graph, stable set polytope, (non-)rank facet

1 Introduction

A natural generalization of odd holes and odd antiholes are graphs with circular symmetry of their maximum cliques and stable sets, called webs: a web \( W^k_n \) is a graph with nodes \( 1, \ldots, n \) where \( ij \) is an edge iff \( i \) and \( j \) differ by at most \( k \) (modulo

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n) and \( i \neq j \). These graphs belong to the classes of quasi-line graphs and claw-free graphs and are, besides line graphs, relevant for describing the stable set polytopes of those larger graph classes [5,6,10]. (The line graph of a graph \( H \) is obtained by taking the edges of \( H \) as nodes and connecting two nodes iff the corresponding edges of \( H \) are incident. A graph is quasi-line (resp. claw-free) if the neighborhood of any node can be partitioned into two cliques (resp. does not contain any stable set of size 3.).) All facets of the stable set polytope of line graphs are known from matching theory [4]. In contrary, providing all facets of the stable set polytopes of claw-free graphs is a long-standing problem [7] but we are even still far from having a complete description for the stable set polytopes of webs (and, therefore, of quasi-line and claw-free graphs, too).

In particular, as shown by Giles & Trotter [6], the stable set polytopes of claw-free graphs contain facets with a much more complex structure than those defining the matching polytope. Oriolo [10] discussed which of them occur in quasi-linegraphs. In particular, these non-rank facets rely on certain combinations of joined webs.

Several further authors studied the stable set polytopes of webs. Obviously, webs with clique number 2 are either even or odd holes (their stable set polytopes are known due to [1,11]). Dahl [3] studied webs with clique number 3 and showed that their stable set polytopes admit rank facets only. On the other hand, Kind [8] found (by means of the PORTA software \(^3\)) examples of webs with clique number > 4 whose stable set polytopes have non-rank facets. Oriolo [10] and Liebling et al. [9] presented further examples of such webs. It is natural to ask whether the stable set polytopes of webs with clique number = 4 admit rank facets only.

The aim of the present paper is to answer that question by providing an infinite sequence of webs with clique number = 4 whose stable set polytopes have non-rank facets.

2 Results on Stable Set Polytopes

The \textit{stable set polytope} \( \text{STAB}(G) \) of \( G \) is defined as the convex hull of the incidence vectors of all stable sets of the graph \( G = (V,E) \) (a set \( V' \subseteq V \) is a stable set if the nodes in \( V' \) are mutually non-adjacent). A linear inequality \( a^T x \leq b \) is said to be valid for \( \text{STAB}(G) \) if it holds for all \( x \in \text{STAB}(G) \). We call a stable set \( S \) of \( G \) a root of \( a^T x \leq b \) if its incidence vector \( \chi^S \) satisfies \( a^T \chi^S = b \). A valid inequality for \( \text{STAB}(G) \) is a facet if and only if it has \( |V| \) roots with affinely independent incidence vectors. (Note that the incidence vectors of the roots of \( a^T x \leq b \) have to be linearly independent if \( b > 0 \).

\(^3\) By PORTA it is possible to generate all facets of the convex hull of a given set of integer points, see http://www.zib.de
The aim is to find a system $Ax \leq b$ of valid inequalities s.t. $\text{STAB}(G) = \{ x \in \mathbb{R}^{[G]}_+ : Ax \leq b \}$ holds. Such a system is unknown for the most graphs and it is, therefore, of interest to study certain linear relaxations of $\text{STAB}(G)$ and to investigate for which graphs $G$ these relaxations coincide with $\text{STAB}(G)$.

One relaxation of $\text{STAB}(G)$ is the \textit{fractional stable set polytope} $\text{QSTAB}(G)$ given by all “trivial” facets, the nonnegativity constraints

$$x_i \geq 0$$

for all nodes $i$ of $G$ and by the clique constraints

$$\sum_{i \in Q} x_i \leq 1$$

for all cliques $Q \subseteq G$ (a set $V' \subseteq V$ is a clique if the nodes in $V'$ are mutually adjacent). Obviously, a clique and a stable set have at most one node in common. Therefore, $\text{QSTAB}(G)$ contains all incidence vectors of stable sets of $G$ and $\text{STAB}(G) \subseteq \text{QSTAB}(G)$ holds for all graphs $G$. The two polytopes coincide precisely for perfect graphs $[1,11]$.

A graph $G$ is called \textit{perfect} if, for each (node-induced) subgraph $G' \subseteq G$, the chromatic number $\chi(G')$ equals the clique number $\omega(G')$. That is, for all $G' \subseteq G$, as many stable sets cover all nodes of $G'$ as a maximum clique of $G'$ has nodes (maximum cliques resp. maximum stable sets contain a maximal number of nodes).

In particular, for all imperfect graphs $G$ follows $\text{STAB}(G) \subset \text{QSTAB}(G)$ and, therefore, further constraints are needed to describe their stable set polytopes. A natural way to generalize clique constraints is to investigate rank constraints

$$\sum_{i \in G'} x_i \leq \alpha(G')$$

associated with arbitrary (node-)induced subgraphs $G' \subseteq G$ where $\alpha(G')$ denotes the stability number of $G'$, i.e., the cardinality of a maximum stable set in $G'$ (note that $\alpha(G') = 1$ holds iff $G'$ is a clique). For convenience, we often write (2) in the form $x(G') \leq \alpha(G')$.

Let $\text{RSTAB}(G)$ denote the \textit{rank polytope} of $G$ given by all nonnegativity constraints (0) and all rank constraints (2). A graph $G$ is called \textit{rank-perfect} $[14]$ if $\text{STAB}(G)$ coincides with $\text{RSTAB}(G)$.

By construction, every perfect graph is rank-perfect. Some further graphs are rank-perfect by definition: \textit{near-perfect} $[12]$ (resp. \textit{t}-perfect $[1]$, \textit{h}-perfect $[7]$) graphs, where rank constraints associated with cliques and the graph itself (resp. edges and odd cycles, cliques and odd cycles) are allowed. Moreover, the result of Edmonds and Pulleyblank $[4]$ implies that line graphs are rank-perfect as well (see $[15]$ for a list with more examples).
Recall that a web $W^k_n$ is a graph with nodes 1, \ldots, $n$ where $ij$ is an edge if $i$ and $j$ differ by at most $k$ (i.e., if $|i - j| \leq k \mod n$ and $i \neq j$). We assume $k \geq 1$ and $n \geq 2(k + 1)$ in the sequel in order to exclude the degenerated cases when $W^k_n$ is a stable set or a clique. $W^1_n$ is a hole and $W^{k-1}_{2k+1}$ an odd antihole for $k \geq 2$. All webs $W^k_9$ on nine nodes are depicted in Figure 1. It is easy to see that $\omega(W^k_n) = k + 1$ and $\alpha(W^k_n) = \lfloor \frac{n}{k+1} \rfloor$ holds. Note that webs are also called circulant graphs $C^k_n$ [2]. Furthermore, similar graphs $W(n, k)$ were introduced in [13].

So far, the following is known about stable set polytopes of webs. The webs $W^1_n$ are holes, hence they are perfect if $n$ is even and near-perfect if $n$ is odd (recall that we suppose $n \geq 2(k + 1)$). Dahl [3] showed that all webs $W^2_n$ with clique number 3 are rank-perfect. But there are several webs with clique number $\geq 4$ known to be not rank-perfect [8, 10, 9], e.g., $W^4_{31}, W^5_{25}, W^6_{29}, W^7_{33}, W^8_{28}, W^9_{31}$; these results are summarized in Table 1.

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<th>$\omega = 3$</th>
<th>$\omega = 4$</th>
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<td>All webs rank-perfect?</td>
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<td>Yes</td>
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<td>No</td>
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<td>Infinitely many not rank-perfect webs?</td>
<td>No</td>
<td>No</td>
<td>?</td>
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A conjecture due to Ben Rebea (see [10]) claims that the stable set polytopes of quasi-line graphs admit only one type of facets besides nonnegativity constraints (0) and clique constraints (1), so-called clique family inequalities: Let $G = (V, E)$ be a graph, $\mathcal{F}$ be a family of (at least three inclusion-wise) maximal cliques of $G$, $p \leq |\mathcal{F}|$ be an integer, and define two sets as follows:

$$I(\mathcal{F}, p) = \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| \geq p\}$$

$$O(\mathcal{F}, p) = \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| = p - 1\}$$

The clique family inequality $(\mathcal{F}, p)$

$$\sum_{i \in I(\mathcal{F}, p)} x_i + (p - r - 1) \sum_{i \in O(\mathcal{F}, p)} x_i \leq (p - r) \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor$$

with $r = |\mathcal{F}| \mod p$ and $r > 0$ is valid for the stable set polytope of every graph by
Oriolo [10]. Since webs are quasi-line graphs in particular, the stable set polytopes of webs should admit, according to Ben Rebea’s conjecture, facets coming from cliques and clique family inequalities only.

In order to answer the question whether the webs with clique number \( n = 4 \) are rank-perfect or not, we introduce clique family inequalities associated with certain subwebs and prove the following: the clique family inequality associated with \( W_{2l}^2 \subset W_{3l}^3 \) induces a non-rank facet of \( \text{STAB}(W_{3l}^3) \) if \( l \geq 11 \) and \( 2 = l \mod 3 \) (Theorem 6).

3 Non-Rank Facets of \( \text{STAB}(W_{n}^{3}) \)

Consider a web \( W_{n}^{k} \). We say that a clique family inequality \((\mathcal{F}, p)\) of \( \text{STAB}(W_{n}^{k}) \) is associated with a proper subweb \( W_{n'}^{k'} \subset W_{n}^{k} \) if \( \mathcal{F} = \{Q_{i} : i \in W_{n'}^{k'} \} \) is chosen as clique family, \( p = k' + 1 \), and \( Q_{i} = \{i, \ldots, i + k\} \) denotes the maximum clique of \( W_{n}^{k} \) starting in node \( i \). In order to explore the special structure of such inequalities, we need the following fact from Trotter [13].

**Observation 1** [13] \( W_{n'}^{k'} \) is an induced subweb of \( W_{n}^{k} \) if and only if there is a subset \( V' = \{i_{1}, \ldots, i_{n'}\} \subset V(W_{n}^{k}) \) s.t. \( |V' \cap Q_{i_{j}}| = k' + 1 \) for every \( 1 \leq j \leq n' \).

We now prove the following.

**Lemma 2** Let \( W_{n'}^{k'} \subset W_{n}^{k} \) be any proper induced subweb. The clique family inequality \((\mathcal{F}, p)\) of \( \text{STAB}(W_{n}^{k}) \) associated with \( W_{n'}^{k'} \) is

\[
(k' + 1 - r) \sum_{i \in I(\mathcal{F}, p)} x_{i} + (k' - r) \sum_{i \in O(\mathcal{F}, p)} x_{i} \leq (k' + 1 - r) \alpha(W_{n'}^{k'}) \tag{4}
\]

with \( p = k' + 1 \), \( r = n' \mod (k' + 1) \), \( r > 0 \); we have \( W_{n'}^{k'} \subseteq I(\mathcal{F}, p) \) and the union of \( I(\mathcal{F}, p) \) and \( O(\mathcal{F}, p) \) covers all nodes of \( W_{n}^{k} \).

**Proof.** Let \( W_{n'}^{k'} \) be a proper subweb of \( W_{n}^{k} \) and choose \( \mathcal{F} = \{Q_{i} : i \in W_{n'}^{k'}\} \), \( p = k' + 1 \). Obviously \( |\mathcal{F}| = |W_{n'}^{k'}| = n' \) follows. Let \( V' = \{i_{1}, \ldots, i_{n'}\} \) be the node set of \( W_{n'}^{k'} \) in \( W_{n}^{k} \). Observation 1 implies that \( Q_{i_{j}} = \{i_{j}, \ldots, i_{j} + k\} \) contains the nodes \( i_{j}, \ldots, i_{j+k} \) from \( V' \). Obviously, the node \( i_{j+k} \) belongs exactly to the \((k'+1)\) cliques \( Q_{i_{j}}, \ldots, Q_{i_{j+k}} \) from \( \mathcal{F} \). Since all indices are taken modulo \( n \), every node in \( W_{n'}^{k'} \) is covered precisely \((k'+1)\) times by \( \mathcal{F} \) and \( p = k' + 1 \) yields, therefore, \( W_{n'}^{k'} \subseteq I(\mathcal{F}, p) \). Furthermore, \( |\mathcal{F}| = n' \) and \( p = \omega(W_{n'}^{k'}) \) implies \( \frac{|\mathcal{F}|}{p} = \alpha(W_{n'}^{k'}) \). Hence the clique family inequality given by \((\mathcal{F}, p)\) is (4) which finishes the proof. \( \square \)
Let us turn to the clique family inequality associated with $W_{2l}^2 \subset W_{3l}^3$, i.e. $n$ is divisible by 3 (for some $l \geq 3$ by $n \geq 2(k+1)$). Observation 1 easily yields that every third node of $W_{3l}^3$ does not belong to the subweb $W_{2l}^2$ and that $W_{2l}^2 = I(\mathcal{F}, 3)$ holds if we choose $\mathcal{F} = \{Q_i : i \in W_{2l}^2\}$, see Figure 2.

Fig. 2. The subweb $W_{2l}^2 \subset W_{3l}^3$

Furthermore, the nodes in $W_{3l}^3 - W_{2l}^2 = O(\mathcal{F}, 3)$ induce the hole $W_{1l}^1$. Thus, the clique family inequality $(\mathcal{F}, 3)$

$$(3-r) x(W_{2l}^2) + (2-r) x(W_{1l}^1) \leq (3-r) \alpha(W_{2l}^2)$$

associated with $W_{2l}^2 \subset W_{3l}^3$ is a non-rank constraint if $r = 1$ holds. The aim of this section is to prove that $(\mathcal{F}, 3)$ is a non-rank facet of $\text{STAB}(W_{3l}^3)$ whenever $l \geq 11$ and $2 = l \mod 3$ (note: $2 = l \mod 3$ implies $r = 1 = 2l \mod 3$).

For that, we have to present $3l$ roots of $(\mathcal{F}, 3)$ whose incidence vectors are linearly independent. (Recall that a root of $(\mathcal{F}, 3)$ is a stable set of $W_{3l}^3$ satisfying $(\mathcal{F}, 3)$ at equality.)

It follows from [13] that a web $W_n^k$ produces the full rank facet $x(W_n^k) \leq \alpha(W_n^k)$ iff $(k+1) \nmid n$. Thus $W_{2l}^2$ is facet-producing if $2 = l \mod 3$ and the maximum stable sets of $W_{2l}^2$ yield already $2l$ roots of $(\mathcal{F}, 3)$ whose incidence vectors are linearly independent.

Let $V = V(W_{3l}^3)$ and $V' = V(W_{2l}^2)$. We need a set $S$ of further $l$ roots of $(\mathcal{F}, 3)$ which have a non-empty intersection with $V - V'$, called mixed roots, and are independent, in order to prove that $(\mathcal{F}, 3)$ is a facet of $\text{STAB}(W_{3l}^3)$.

We show that there exists a set $S$ of $l$ mixed roots of $(\mathcal{F}, 3)$ whenever $l \geq 11$. Due to $2 = l \mod 3$, we set $l = 2 + 3l'$ and obtain $|V| = 3l = 6 + 9l'$. Thus, $V$ can be partitioned into 2 blocks $D_1, D_2$ with 3 nodes each and $l'$ blocks $B_1, \ldots, B_{l'}$ with 9 nodes each s.t. every block ends with a node in $V - V'$ (this is possible since every third node of $V$ belongs to $V - V'$ say $i \in V'$ if $3 \nmid i$ and $i \in V - V'$ if $3|i$). Figure 3 shows a block $D_i$ and a block $B_j$ (where circles represent nodes in $V'$ and squares represent nodes in $V - V'$). For the studied mixed roots of $(\mathcal{F}, 3)$ we choose the black filled nodes in Figure 3:
The set $\{i \mid i \leq D\}$ of every block $B_j$ to discuss what happens between two consecutive blocks. Since the first 3 nodes $2V$ other nodes are in $S$, consider a set $B$ due to $\alpha$ $S$ is a stable set provided the two blocks $S$ before $B$ the nodes in $V$ nodes, hence we choose orderings $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$ of the blocks s.t. $D_1, D_2$ are not neighbored.

**Lemma 3** The set $S$ containing the 3rd node of the blocks $D_1, D_2$ as well as the 4th and 8th node of any block $B_j$ is a root of $(\mathcal{F}, 3)$ with $|S \cap V'| = 2l'$ and $|S \cap (V - V')| = 2$ for every ordering $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$ of the blocks s.t. $D_1, D_2$ are not neighbored.

**PROOF.** Consider a set $S$ constructed that way. Since every block ends with a node in $V - V'$ by definition and every third node of $V$ is in $V - V'$, we have that the last node of $D_1$ and the 3rd, 6th, and 9th node of $B_j$ belong to $V - V'$ while all other nodes are in $V'$. Thus, the two last nodes in $D_1$ and $D_2$ are the two studied nodes in $S \cap (V - V')$ and the 4th and 8th node in $B_j$ for $1 \leq j \leq l'$ are the studied 2$l'$ nodes in $S \cap V'$ (see Figure 3).

$S$ is a stable set provided the two blocks $D_1$ and $D_2$ are not neighbored: Obviously, there is no edge between the 4th and 8th node of any block $B_j$. Thus, we only have to discuss what happens between two consecutive blocks. Since the first 3 nodes of every block $B_j$ do not belong to $S$, there is no problem with having any block before $B_j$, i.e., $B_k B_j$ or $D_i B_j$. For the remaining case $B_j D_i$, notice that the last node of $B_j$ and the first two nodes of $D_i$ do not belong to $S$ and there cannot be an edge between two nodes of $S$ in that case, too.

This shows that $S$ is a stable set satisfying $|S \cap V'| = 2l'$ and $|S \cap (V - V')| = 2$. Due to $\alpha(W'_{2l'}) = \left\lceil \frac{2(2+3l')}{3} \right\rceil = 2l' + 1$, the set $S$ is finally a root of $(\mathcal{F}, 3)$. □

Lemma 3 implies that there exist mixed roots $S$ of $(\mathcal{F}, 3)$ with $|S| = 2 + 2l'$ if $l' \geq 2$. The next step is to show that there are $l$ such roots if $l' \geq 3$ (resp. $l \geq 11$).

In the sequel, we denote by $S_{i,m}$ the stable set constructed as in Lemma 3 when $D_1 = \{i - 2, i - 1, i\}$ and $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$. If there are more than $\left\lceil \frac{l'}{2} \right\rceil$ blocks between $D_1$ and $D_2$, there are less than $\left\lfloor \frac{l'}{2} \right\rfloor$ blocks between $D_2$ and $D_1$. Hence it suffices to consider $m \leq \left\lfloor \frac{l'}{2} \right\rfloor$.

By construction, $S_{i,m}$ contains a second node from $V - V'$, namely, the third node $i + 9m + 3$ of block $D_2$. If $2l'$ and $m = \frac{l'}{2}$, then $(i + 9m + 3 + 9m + 3) = i + 9l' + 6 = i(m \bmod n)$ and, therefore, $S_{i,m} = S_{i + 9m + 3, m}$ follows.

We are supposed to construct distinct mixed roots $S_{i,m}$ of $(\mathcal{F}, 3)$ with $2 + 2l'$ nodes, hence we choose orderings $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$ with $1 \leq m < \frac{l'}{2}$ and obtain easily:
Lemma 4 If \( l' \geq 3 \), then the stable sets \( S_{i,m} \) for each \( i \in V - V' \) obtained from any ordering \( V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'} \) with \( 1 \leq m < \frac{l'}{2} \) yield \(|V - V'| = l\) roots of \((\mathcal{F}, 3)\) with \( 2 + 2l' \) nodes each.

Consequently, we can always choose a set of \( 3l \) roots of \((\mathcal{F}, 3)\) if \( l' \geq 3 \) resp. \( l \geq 11 \).

If \( S \) is a set of \( l \) distinct mixed roots, denote by \( A_S \) the square matrix containing the incidence vectors of the \( 2l \) maximum stable sets of \( W_{2l}^2 \) and the \( l \) mixed roots in \( S \). \( A_S \) can be arranged s.t. the first \( 2l \) and the last \( l \) columns correspond to the nodes in \( W_{2l}^2 \) and \( W_{l}^1 \), respectively, and the first \( 2l \) rows contain the incidence vectors of the maximum stable sets of \( W_{2l}^2 \) where the last rows contain the incidence vectors of the \( l \) mixed roots in \( S \). (Note that the nodes corresponding to the last \( l \) columns of \( A_S \) are \( 3, 6, \ldots, 3l \).) Then \( A_S \) has the block structure

\[
A_S = \begin{pmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{pmatrix}
\]

where the \( 2l \times 2l \)-matrix \( A_{11} \) is invertible (recall: \( W_{2l}^2 \) is facet-producing by [13] in the considered case with \( 1 = 2l \mod 3 \) resp. \( 2 = l \mod 3 \)).

It is left to find a set \( S \) of \( l \) distinct mixed roots s.t. \( A_{22} \) is an invertible \( l \times l \)-matrix (then \( A_S \) is invertible due to its block structure).

Lemma 5 For every \( l \geq 11 \), there is a set \( S \) of \( l \) mixed roots of \((\mathcal{F}, 3)\) containing \( 2 \) nodes from \( V - V' \) s.t. the \( l \times l \)-submatrix \( A_{22} \) of \( A_S \) is invertible.

PROOF. Every root \( S_{i,m} \) of \((\mathcal{F}, 3)\) corresponds to a row in \((A_{21}|A_{22})\) of \( A_S \) having precisely two 1-entries in the columns belonging to \( A_{22} \) (by \(|S_{i,m} \cap (V - V')| = 2 \) for all \( i \in V - V' \)). Lemma 4 ensures that no such roots coincide if \( 1 \leq m < \frac{l'}{2} \) for all \( i \in V - V' \).

The idea of finding cases when \( A_{22} \) is invertible goes as follows: Let \( S_{3j,1} \) for \( 1 \leq j \leq l - 4 \) be the first \( l - 4 \) roots in \( S \) with \( S_{3j,1} \cap (V - V') = \{3j, 3(j+4)\} \). Choose as the remaining 4 roots in \( S \) the stable sets \( S_{3j,2} \) for \( l - 10 \leq j \leq l - 7 \) with \( S_{3j,2} \cap (V - V') = \{3j, 3(j+7)\} \). Then take their incidence vectors \( \chi^{S_{3j,1}} \) for \( 1 \leq j \leq l - 4 \) as the first \( l - 4 \) rows and \( \chi^{S_{3j,2}} \) for \( l - 10 \leq j \leq l - 7 \) as the last 4 rows of \((A_{21}|A_{22})\). By construction, \( A_{22} \) is the \( l \times l \)-matrix in Figure 4 (1-entries are shown only, the column \( i \) corresponds to the node \( 3i \)).

\( A_{22} \) has only 1-entries on the main diagonal (coming from the first nodes in \( V - V' \) of \( S_{3j,1} \) for \( 1 \leq j \leq l - 4 \) and from the second nodes in \( V - V' \) of \( S_{3j,2} \) for \( l - 10 \leq j \leq l - 7 \)). The only non-zero entries of \( A_{22} \) below the main diagonal come from the first nodes in \( V - V' \) of \( S_{3j,2} \) for \( l - 10 \leq j \leq l - 7 \). Hence, \( A_{22} \)
Fig. 4. The $l \times l$-matrix $A_{22}$

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</tbody>
</table>


The $l \times l$-matrix $A_{22}$ has the form

$$A_{22} = \begin{pmatrix} A'_{22} & 0 \\ 0 & A''_{22} \end{pmatrix}$$

where both $A'_{22}$ and $A''_{22}$ are invertible due to the following reasons:

$A'_{22}$ is an $(l - 11) \times (l - 11)$-matrix having 1-entries on the main diagonal and 0-entries below the main diagonal by construction. Hence $A'_{22}$ is clearly invertible.

$A''_{22}$ is an $11 \times 11$-matrix which has obviously the circular 1’s property. In other words, $A''_{22}$ is equivalent to the matrix $A(C_{11})$ containing the incidence vectors of the maximum stable sets of the odd antihole $C_{11}$ as rows. Since $A(C_{11})$ is invertible due to Padberg [11], the matrix $A''_{22}$ is invertible, too. (Note that $l = 11$ implies $A_{22} = A''_{22}$.)

This completes the proof that $A_{22}$ is invertible for every $l \geq 11$ if we choose the set $S$ of $l$ roots of $(F, 3)$ as constructed above. □

Finally, we have shown that, for every $l \geq 11$ with $2 = l \mod 3$, there are $3l$ roots of $(F, 3)$ whose incidence vectors are linearly independent:
Theorem 6 For any $W^2_{2l} \subset W^3_{3l}$ with $2 = l \mod 3$ and $l \geq 11$, the clique family inequality

$$2x(W^2_{2l}) + 1x(W^1_{4l}) \leq 2\alpha(W^2_{2l})$$

associated with $W^2_{2l}$ is a non-rank facet of $STAB(W^3_{3l})$.

This gives us an infinite sequence of not rank-perfect webs $W^3_{3l}$ with clique number 4, namely $W^3_{33}, W^3_{42}, W^3_{51}, ..., W^3_{60}$, ... and answers the question whether the webs $W^3_n$ with clique number 4 are rank-perfect negatively. Thus, we can update Table 1 as follows:

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$\omega = 2$</th>
<th>$\omega = 3$</th>
<th>$\omega = 4$</th>
<th>$\omega \geq 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>All webs rank-perfect?</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Infinitely many not rank-perfect webs?</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>?</td>
</tr>
</tbody>
</table>

4 Concluding Remarks

It is open whether there exist, for each $\omega \geq 5$, infinitely many not rank-perfect webs, see Table 2. We believe that this is the case.

Assuming Ben Rebea’s Conjecture as true, we conjecture further that all non-rank facets of $STAB(W^k_n)$ are clique family inequalities $(\mathcal{F}, p)$

$$(k' + 1 - r) \sum_{i \in I(\mathcal{F}, p)} x_i + (k' - r) \sum_{i \in O(\mathcal{F}, p)} x_i \leq (k' + 1 - r) \alpha(W^k_{n'})$$

associated with certain subwebs $W^k_{n'} \subset W^k_n$. All non-rank facets would have, therefore, coefficients at most $k - 1$ and $k - 2$ (since $k' < k$ follows by $W^k_{n'} \subset W^k_n$ and $(k' + 1 - r) \leq k'$ by $r > 0$). This would imply that the stable set polytopes of webs $W^3_n$ could have non-rank facets with coefficients 2 and 1 only.

References


