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ENNUMERATION OF CONNECTED UNIFORM HYPERGRAPHS

TSIRY ANDRIAMAMPIANENA AND VLADY RAVELOMANANA

Abstract. In this paper, we are concerned in counting exactly and asymptotically connected labeled $b$-uniform hypergraphs ($b \geq 3$). Enumerative results on connected graphs are generalized here to connected uniform hypergraphs. For this purpose, these structures are counted according to the number of vertices and hyperedges. First, we show how to compute step by step the associated exponential generating functions (EGFs) by means of differential equations and provide combinatorial interpretations of the obtained results. Next, we turn on asymptotic enumeration. We establish Wright-like inequalities for hypergraphs and by means of complex analysis, we obtain the asymptotic number of connected $b$-uniform hypergraphs with $n$ vertices and $(n + \ell)/(b-1)$ hyperedges whenever $\ell = o(n^{1/3}/b^{1/3})$. This latter result confirms a conjecture made by Karoński and Luczak in [20] about the validity of their formula for excesses in the ‘Wright’s range’.

1. Introduction

In this paper we are concerned with counting exactly and asymptotically members of families of labeled connected $b$-uniform hypergraphs with a given number of vertices and hyperedges and without multiple hyperedges. A labeled hypergraph $\mathcal{H} = (V, E)$ is given by a set $V$ of $n$ vertices with a family $E$ of subsets of $V$ of cardinal $\geq 2$ (see Berge [4]). A member of $E$ is called hyperedge and $\mathcal{H} = (V, E)$ is said $b$-uniform ($b \geq 2$) iff each member of $E$ contains exactly $b$ vertices. Therefore, 2-uniform hypergraphs are simply graphs. Let $\mathcal{H} = (V, E)$ be a hypergraph, uniform or not, then its excess is defined as (see [21]):

\[
\text{excess}(\mathcal{H}) = \sum_{e \in E} (|e| - 1) - |V|.
\]

Therefore, if $\mathcal{H} = (V, E)$ is a $b$-uniform hypergraph then its excess is given by the expression

\[
\text{excess}(\mathcal{H}) = |E| \times (b - 1) - |V|.
\]

The notion of excess was first used in [22] where the author obtained substantial enumerative results in the study of connected graphs according to their number of vertices and edges. Wright’s results appeared to be very important in the study of random graphs [1, 15, 16]. Later, Bender, Canfield and McKay [2] but also Pittel and Wormald [24] generalized Wright’s results and obtained the asymptotic number of connected graphs of any given number of vertices and edges.

Key words and phrases. $b$-uniform hypergraphs; enumerative and analytic combinatorics; saddle-point method; generalized Wright’s coefficients; random hypergraphs.
In contrary, much less is known about the number of hypergraphs of a given size. As far as we know, the most important results in these directions are those of Karoński and Łuczak in [20, 21]. In [24], the two authors used ‘purely combinatorial arguments’ to obtain their results. In this paper, our aim is to obtain analogous results to that of Wright [25, 29]. To do so, our approach is based on generating functions. Following the previously cited works, namely [18], connected hypergraphs with excess \(-1\) are called \textit{hypertrees}, connected hypergraphs with excess \(0\) are called \textit{unicyclic components} or \textit{unicycles}. Since these structures are labeled, we will use exponential generating functions (EGFs, for short) [15] to encode their number. Then, denote by \(H_\ell\) the EGF of labeled connected \(b\)-uniform hypergraphs with excess \(\ell\). The purpose of this work is to compute the sequence of EGFs \((H_\ell)_{\ell \geq -1}\).

The outline of this paper is as follows. In the second section, we establish the differential equation satisfied by the EGFs \(H_\ell\) \((\ell \geq -1)\). We show how these EGFs can be computed exactly from this combinatorial equation and we retrieve some results that appeared in [25, 20]. In the third section, we give the forms of the expression of \(H_\ell\). We show that for every \(\ell \geq -1\), \(H_\ell\) can be expressed in terms of the EGF of rooted hypertrees and we give combinatorial interpretations of the forms of these EGFs. The next section is devoted to the asymptotic enumeration of uniform hypergraphs. First, we establish Wright-like inequalities for hypergraphs. Next, these inequalities are combined with methods from complex analysis and lead us to the asymptotic number of connected hypergraphs with \(n\) vertices and \((n + o(n^{1/3}))/((b - 1))\) hyperedges.

2. Combinatorial equations satisfied by \(H_\ell\)

2.1. Hypergraphs surgery. Let us start with a figure that illustrates, with 4-uniform hypergraph, the main idea from which we deduce the enumeration.

The figure on the left side is a connected 4-uniform hypergraph with 14 vertices and 5 hyperedges one of which is distinguished, namely the dashed hyperedge \(\{2, 4, 8, 12\}\). The figure on the right side is a 4-uniform hypergraph with also 14 vertices but with only 4 hyperedges. This latter hypergraph is not connected but contains 3 components in which one or more vertices are distinguished (resp. \(\{2, 8\}\), \(\{4\}\) and \(\{12\}\)). The above figures reflect combinatorial relations between families of connected hypergraphs with on first hand a distinguished hyperedge and on the other hand marked vertices. For instance, we refer the reader to Bergeron, Labelle and Leroux [3] for the use of distinguishing/marking and pointing in combinatorial species. The following lemma describes the relationships between number of edges and excesses in \(b\)-uniform connected components with distinguished hyperedge and marked vertices:
Lemma 2.1. Consider a set $M$ of connected $b$-uniform hypergraphs with one or more distinguished vertices. For any couple $(j, k)$, denote by $m_{jk}$ the number of connected components in $M$ of excess $j$ and with $k$ marked vertices. Then, the hypergraph obtained when creating a (new) hyperedge connecting all the distinguished vertices of all the components in $M$ is (i) connected, (ii) $b$-uniform and (iii) has excess $\ell$ if and only if

$$\sum_{j,k} k m_{jk} = b \quad \text{and} \quad \sum_{j,k} (j + k) m_{jk} = \ell + 1.$$  

Proof. It is clear that the created hypergraph is connected and is $b$-uniform if and only if the total number $\left( \sum_{j,k} k m_{jk} \right)$ of distinguished vertices in the set is equal to $b$. Let $N$ be the number of (connected) hypergraphs in the considered set $M$ and let $n$ be the total number of vertices in this set. Let us assign an arbitrary order to the members of the set and let $n_1$, $s_1$, and $k_1$ be respectively the number of vertices, hyperedges and distinguished vertices in the $i$-th hypergraph. The excess of the newly created hypergraph is then equals to $\ell$ if and only if

$$\sum_{i=1}^{N} s_i (b - 1) + (b - 1) - n = \ell.$$  

We get $\sum_{i=1}^{N} s_i (b - 1) + \sum_{i=1}^{N} k_i - \sum_{i=1}^{N} n_i = \ell + 1$ and

$$\sum_{i=1}^{N} \left( \{ s_i (b - 1) - n_i \} + k_i \right) = \ell + 1.$$  

Therefore, $\sum_{j,k} m_{jk} (j + k) = \ell + 1$. $\square$

2.2. Combinatorial equations. In this paragraph, the previous correspondences are expressed in terms of EGFs. Let us consider the bivariate EGF $H_\ell$. We have

$$H_\ell(w, z) = \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} h_\ell(s, n) \frac{w^s}{s!} \frac{n^\ell}{n!},$$  

where $w$ (resp. $z$) is the variable related to the number of hyperedges (resp. labeled vertices). In (3), $h_\ell(s, n)$ denotes the number of connected $b$-uniform hypergraphs with excess $\ell$ with $s$ hyperedges and $n$ vertices. Using (2), we note that $h_\ell(s, n) \neq 0$ iff $(n + \ell)/(b - 1) \in \mathbb{N}$. The following theorem is inspired by the observations of paragraph 2.1 and gives recursive relation between the EGFs $H_\ell$.

Theorem 2.2. The bivariate EGFs $(H_\ell)_{\ell \geq -1}$ of labeled connected $b$-uniform hypergraphs satisfy

$$w \frac{\partial}{\partial w} H_\ell(w, z) = w \sum_{(m_{jk}) \in S_\ell} \left\{ \prod_{j,k} \frac{1}{m_{jk}!} \left( \frac{z^k}{k!} \frac{\partial^k}{\partial z^k} H_j(w, z) \right)^{m_{jk}} \right\} - w \frac{\partial}{\partial w} H_{\ell-b+1}(w, z)$$

where $S_\ell$ is the following set of matrix:

$$S_\ell = \left\{ (m_{jk})_{-1 \leq j \leq \ell} \quad \text{with} \quad m_{jk} \in \mathbb{N} \quad \text{such that} \quad \sum_{j,k} m_{jk} (j + k) = \ell + 1 \quad \text{and} \quad \sum_{j,k} km_{jk} = b \right\}$$

$H_{-2} \equiv 0$ if $j \leq -2$.

Proof. This equation relates, in terms of generating functions, the bijection between two sets of objects described by a) and b) as follows. a) In the left-hand side of (5), we have the EGF of the set of connected hypergraphs with excess $\ell$ and with a marked hyperedge. b) In the right-hand side, there are union of sets of components with one or more distinguished vertices that can be obtained from the removal, in a connected hypergraph of excess $\ell$, of a hyperedge. After such removal, in each newly created component, the vertices which belonged to the removed hyperedge are marked. If there is $k$ such distinguished vertices, in terms of EGFs, we then have $\sum_{k=0}^{b} \frac{\partial^k}{\partial z^k} H_j(w, z)$. The second member of our equation can be interpreted as the creation of a (future) hyperedge with a total of $b$ distinguished vertices in order to reconnect a set of hypergraphs. In the case where there is only one component, necessarily its excess is $\ell - b - 1$ and there are $b$ of its vertices that are distinguished. These $b$ vertices must not form an already existing hyperedge because we consider here hypergraphs without multiple hyperedges. It is the reason why we have to subtract the term $w \frac{\partial}{\partial w} H_{\ell-b+1}(w, z)$.
in the RHS of (1). Observe that by the previous lemma, the definition of the set $S_\ell$, viz. (3), ensures that the hypergraph obtained by the creation of a hyperedge connecting the marked vertices in the RHS is with excess $\ell$ and that the hyperedge which is created is formed with $b$ vertices.

**Remark 2.3.** We note that it is sufficient to determine the univariate EGFs since the corresponding bivariate EGFs can be deduced from the univariate ones simply using the relation

$$H_\ell(w,z) = w^{\ell/(b-1)} H_\ell(w^{1/(b-1)} z).$$

Saving the justification of its use for later, let us denote by $T(z)$ the (univariate) EGF corresponding to rooted hypertrees. Since a rooted hypertree is either a root or a root with a non-empty set of rooted (sub)hypertrees, borrowing methods from symbolic combinatorics (cf. [11]), we get

$$T(z) = z \exp \left( \frac{T(z)(b-1)}{(b-1)!} \right).$$

**Remark 2.4.** Throughout this paper, we use the notation $H_\ell$ followed by the couple of variables $(w, z)$ to express the bivariate EGF, the notation $H_\ell$ followed by the variable $(z)$ to express the univariate EGF. Whenever the variable are intentionally omitted, the EGF in used is $H_\ell \equiv H_\ell(T(z))$ where $T(z)$ is the EGF of rooted $(b$-uniform) hypertrees implicitly given by (3).

The EGFs $H_\ell \equiv H_\ell(T(z))$ satisfies the following.

**Corollary 2.5.** For excess $\ell = -1$

$$H_{-1} = T - \frac{(b-1)T^b}{b!}, \quad T \equiv T(z)$$

and for $\ell \geq 0$

$$H_\ell(w,z) = \frac{1}{b-1} \left( \ell H_\ell + T \frac{d}{dT} H_\ell \right) = \sum_{(m_{jk}) \in S_\ell^*} \left\{ \prod_{j,k} \frac{1}{m_{jk}!} \left( z^k \frac{d^k}{dz^k} H_j(z) \right)^{m_{jk}} \right\}$$

$$- \frac{1}{b-1} \left( (\ell-1) H_{\ell-1}(z) + z \frac{d}{dz} H_{\ell-1}(z) \right)$$

where $S_\ell^*$ is the same as $S_\ell$ (see (3)) without the matrix where all coefficients equal zero except for the coefficients $m_{1,1} = b-1$ and $m_{\ell,1} = 1$.

**Sketch of proof.** Use the fact that

$$w \frac{\partial}{\partial w} H_j(w,z) = \frac{1}{b-1} \left( j H_j(w,z) + z \frac{\partial}{\partial z} H_j(w,z) \right),$$

with (3) and (3) and set $w = 1$. For $\ell = -1$, we have $S_{-1} = \{(m_{-11}, m_{-12}, \ldots, m_{-1b}) = (b, 0, 0, \ldots, 0)\}$. Therefore, we obtain

$$\frac{1}{b-1} \left( -H_{-1} + z \frac{d}{dz} H_{-1}(z) \right) = \left( z \frac{d}{dz} H_{-1}(z) \right)^{b-1}$$

and using the fact that $z \frac{d}{dz} H_{-1}(z) = T(z)$, it yields (3). To prove (4), first we note that for $\ell \geq 0$ the range of the matrix in $S_\ell^*$ can be rearranged so that the line index ranges from $-1$ to $\ell - 1$ and the column index ranges from $1$ to $b$. After some algebra, we get

$$\frac{1}{b-1} \left( \ell H_\ell + z \frac{d}{dz} H_\ell(z) \right) = J_\ell + \left( z \frac{d}{dz} H_{-1}(z) \right)^{b-1} \left( z \frac{d}{dz} H_\ell(z) \right)$$

where $J_\ell$ is the RHS of (3). Again using $z \frac{d}{dz} H_{-1}(z) = T$, we obtain

$$\frac{1}{b-1} \left( \ell H_\ell + \left( z \frac{d}{dz} H_\ell(z) \right) \left( 1 - \frac{T^{b-1}}{(b-2)!} \right) \right) = J_\ell.$$
From (8), we have
\[ dT \quad T \quad z \quad (1 - \frac{T^{b-1}}{(b-2)!}) \]
and by the chain rule for differentiation we get the desired result. Note also that \( \frac{d^k}{dz^k} H_j(z) \) can be expressed in terms of \( T \) so that (10) is completely a differential equation w.r.t. \( T \).

2.3. Analytical resolution. In this section we show how to compute the expression of \( H_\ell, \ell \geq 0 \), in terms of the EGF \( T \) of rooted hypertrees. We note that the equation (10) for \( \ell \geq 0 \) allows us to compute recursively the expression of \( H_j \) for successive values of \( j \). Thus, for each step, we have to solve a differential equation of order one in the variable \( T \) to get the expression of \( H_j \) which verifies the condition that \( H_j|_{T=0} = 0 \).

Lemma 2.6. Let us define \( \theta \) as
\[ \theta = 1 - \frac{T^{b-1}}{(b-2)!} \]
For all \( j \geq -1 \) and for all \( k \geq 0 \), there is a function \( f_{j,k} \) such that
\[ \frac{d^k}{dz^k} H_j(z) = \frac{f_{j,k}(\theta)}{z^k T^j} \]
Denoting \( f_j \equiv f_{j,0} \), in particular we have
\[ H_j = \frac{f_j(\theta)}{T^j} \]

Proof. From (11) and by the chain rule for differential we deduce (13):
\[ f_{j,k+1}(\theta) = -(b-1) \frac{f_{j,k}(\theta)}{\theta} + (b-1)f_{j,k}(\theta) - j \frac{f_{j,k}(\theta)}{\theta} - kf_{j,k}(\theta) \]
The change of variable given by (12) allow us to deduce that \( f_{\ell}(\theta) \) satisfies:
\[ \frac{df_{\ell}(\theta)}{-(b-2)!} = \left( \sum_{(m_{jk}) \in S_{\ell,b}} \prod_{j,k} \frac{1}{m_{jk}!} \left( \frac{f_{j,k}(\theta)}{k!} \right)^{m_{jk}} - \frac{1}{b-1} ((\ell - b + 1)f_{\ell-b+1,0}(\theta) + f_{\ell-b+1,1}(\theta)) \right) d\theta \]

3. On the form of the EGFs \( H_\ell \)

In order to establish the forms of the EGFs \( H_\ell \), we introduce some definitions.

Definitions. The degree of a vertex \( v \) is the number of the hyperedges that contain \( v \). A special hyperedge is one that contains 3 or more vertices of degree at least 2. A special vertex is either a vertex that belongs to a special hyperedge or a vertex of degree \( \geq 3 \). A pendant hyperedge is one where there are \((b-1)\) vertices of degree 1 of degree 1. In the following, we call path a sequence of hyperedges. A path is also characterized by a starting vertex that belongs to the first hyperedge and by an ending vertex that belongs to the last hyperedge, and the sequence of hyperedges defining a path is such that each hyperedge contains exactly \((b-2)\) vertices of degree 1 in the hypergraph and where any pair of successive hyperedges share exactly one vertex that is not the starting nor the ending vertex of the path. We distinguish four kind of paths:
- \( \alpha \)-path: a path that starts from and ends to the same special vertex, there are at least 2 hyperedges in an \( \alpha \)-path and if there are exactly 2 hyperedges in an \( \alpha \)-path then it is said to be minimal.
- \( \beta \)-path: a path that connects 2 special vertices such that if any hyperedge in the path is broken, these 2 special vertices become disconnected, there is at least 1 hyperedge in such a path; a single
hyperedge $\beta$-path is said to be minimal.

- $\gamma$-path: a path that joins 2 special vertices such that these vertices remain connected even if this path is broken, there is at least 1 hyperedge in such a path; a single hyperedge $\gamma$-path is said to be minimal.

A basic hypergraph is an unlabeled hypergraph that can be obtained from a labeled hypergraph by the following procedure:
- Discard the labels.
- Remove recursively pendant hyperedges.
- Shrink paths to a minimal special path of the same kind.

Thus, basic hypergraph has the same excess as any hypergraph from which it may be obtained: in the procedure we have described each time a hyperedge is removed (this happens when shrinking a path), it is just as if we have removed $(b-1)$ vertices. Furthermore, basic hypergraph has, for a given kind of paths, as much number of this kind of paths as in any (original) labeled hypergraph from which it may be obtained.

Let us enumerate the number of hypergraphs from which a fixed basic hypergraph with excess $\ell$ can be obtained. Let $J$ be the EGF of such hypergraphs. Let $m$ be the number of vertices in the basic hypergraph and respectively $c_\alpha$, $c_\beta$ and $c_\gamma$ be the number of $\alpha$-, $\beta$- and $\gamma$-paths, then

$$J = \frac{1}{g} \frac{T^m}{\theta^p}$$

where $g$ is the number of automorphisms (e.g. [14]) of the basic hypergraph and $p = c_\alpha + c_\beta + c_\gamma$ is the number of $\alpha$-, $\beta$- or $\gamma$-paths. The proof of this relation is immediate since “original” hypergraphs are obtained by rooting $m$ rooted hypertrees in the basic hypergraph and by re-inserting $p$ “chains” of eventually zero length in $\alpha$-, $\beta$- and $\gamma$-paths of the basic hypergraph. Thus, each hypergraph is obtained $g$ times because of the number of choices where the $m$ rooted hypertrees can be fixed. Furthermore, with $s$ denoting the number of hyperedges in the considered basic hypergraph, there is a positive rational $\lambda$ such that:

$$J = \frac{1}{g} \frac{T^{s(b-1) - \ell}}{\theta^p} = \frac{(1 - \theta)^s}{T^{(b-1)\ell} \theta^p},$$

and necessary $s \geq \left\lceil \frac{\ell + 1}{b-1} \right\rceil$.

**Lemma 3.1.** For any basic hypergraph with excess $\ell$, the total number of $\alpha$-, $\beta$- and $\gamma$-paths verifies

$$c_\alpha + c_\beta + c_\gamma \leq 3\ell$$

**Proof.** Let $B_0$ be the hypergraph induced by the special vertices in the basic hypergraph, let $m_0$ be the number of special vertices and $r_0$ be the number of special hyperedges then

$$m_0 + c_\alpha + 2(b-2)c_\alpha + (b-2)c_\beta + (b-2)c_\gamma + \ell = (b-1) \left( r_0 + 2c_\alpha + c_\beta + c_\gamma \right).$$

Thus, $m_0 - r_0(b-1) + \ell = c_\alpha + c_\beta + c_\gamma$ and

$$-\text{excess}(B_0) + \ell = p.$$
where only the vertices of degree at least 2 are represented. The basic hypergraph that can be obtained from the hypergraph in the figure above is only with special vertices of degree 3 and the hypergraph induced by these vertices consists of exactly $2 \times \ell$ isolated vertices (because the excess of the hypergraph is \(\ell\)). Thus, $p \leq 3\ell$.

\begin{lemma}
\label{lem3.2}
\[H_\ell = \frac{f_\ell(\theta)}{T^\ell} \] with $f_\ell$ a polynomial of maximum degree \(\left\lfloor \frac{\ell + 1}{b - 1} \right\rfloor + 1\).
\end{lemma}

\begin{proof}
A matrix in $S^*_\ell$ corresponds to constructions as the one we have described in lemma \ref{lem2.3}. After having assigned an arbitrary order to the marked hypergraphs used in such a construction, let:

- the $i$-th hypergraph be of excess $\ell_i$ such that $\ell_i + 1 = q_i(b - 1) + r_i$ with $q_i, r_i \geq 0$ and $r_i < b - 1$.

As in the proof of lemma \ref{lem2.3}, we get here $\ell + 1 = q(b - 1) + r = \sum_{i=1}^N (q_i(b - 1) + r_i - 1 + k_i)$ since $\sum_{i=1}^N (r_i - 1 + k_i) \geq 0$, we deduce that $q \geq \sum_{i=1}^N q_i$. So,

\[q + 1 \geq \sum_{i=1}^N q_i + 1.\]

The lemma follows, since the summation in the right-hand side maximizes the degree of $\theta$ in $f_\ell(\theta)$.
\end{proof}

Using lemmas \ref{lem3.1} and \ref{lem3.2} with combinatorial identities, we obtain the following theorem and its corollary about the forms of the EGFs $H_\ell$.

\begin{theorem}
The EGF of connected $b$-uniform hypergraphs with excess $\ell$ can be put into the form

\[H_\ell = \frac{(1 - \theta)^{\left\lfloor \frac{\ell + 1}{b - 1} \right\rfloor}}{T^\ell} \sum_{p=0}^{3\ell} \frac{A_\ell p}{\theta^p} \left(1 - \frac{\theta}{\theta}\right)^p\]

with the coefficients $A_\ell p$ being rational.
\end{theorem}

\begin{corollary}
The EGF of connected $b$-uniform hypergraphs with excess $\ell \geq 1$ can be rewritten as

\[H_\ell = \frac{1}{T^\ell} \sum_{j=-3\ell}^{3\ell} c_j(\ell, b) \theta^j,\]

where $c_j(\ell, b) \in \mathbb{Q}$.
\end{corollary}

The proofs of theorem \ref{thm3.3} and corollary \ref{cor3.4} are omitted in this extended abstract.

\section{Asymptotic results}

4.1. **Wright-like inequalities for hypergraphs.** In order to compute the asymptotic number of connected $\ell$-excess hypergraphs of a given size, we need the following result which gives the first two terms of $H_\ell$. Let us recall that $\theta = 1 - T^{b-1}/(b - 2)!$. 

\[\]
Lemma 4.1. Developing the two first coefficients of the partial fraction form of $H_\ell$, we get for $\ell \geq 1$

\[(21) \quad T(z)^\ell H_\ell(z) = \frac{\lambda_\ell (b-1)^{2\ell}}{3 \ell T(z)^{2\ell}} - \frac{(\kappa_\ell - \nu_\ell (b-2))((b-1)^{2\ell-1})}{(3 \ell - 1) T(z)^{2\ell-1}} + \sum_{j=-3\ell+2}^{b\ell+1} c_j(\ell, b) \theta(z)^j.\]

In (22), $(\lambda_\ell)_{\ell \in \mathbb{N}}$ is defined recursively by $\lambda_0 = \frac{1}{2}$ and

\[(22) \quad \lambda_\ell = \frac{1}{2} \lambda_{\ell-1}(3\ell - 1) + \frac{1}{2} \ell \lambda_{\ell-1-t}, \quad (\ell \geq 1).\]

Similarly, define $(\nu_\ell)_{\ell \geq 1}$, $(\mu_\ell)_{\ell \geq 0}$ and $(\kappa_\ell)_{\ell \geq 1}$ as follows: $\nu_1 = \frac{5}{12}$ and

\[(23) \quad \nu_\ell = \frac{1}{2} \lambda_{\ell-1} + \frac{1}{6}(3\ell - 4)(3\ell - 2)\lambda_{\ell-2} + \frac{1}{2} \sum_{t=0}^{\ell-2} (3\ell + 2)\lambda_t \lambda_{\ell-2-t} + \frac{1}{6} \sum_{s=0}^{\ell-2} \sum_{t=0}^{s} \lambda_s \lambda_t \lambda_{\ell-2-s-t} \quad (\ell \geq 2).\]

\[(24) \quad \kappa_\ell = \frac{1}{2} ((3\ell - 2)\mu_{\ell-1} + (3b\ell - b - 2\ell)\lambda_{\ell-1}) + \sum_{t=0}^{\ell-1} \mu_t \lambda_{\ell-1-t}.\]

$\mu_0 = b - 1$ and for $\ell \geq 1$, $\mu_\ell$ is defined by

\[(25) \quad \mu_\ell = \kappa_\ell - \nu_\ell (b-2) + \lambda_\ell (b - \frac{2}{3}) \quad (\ell \geq 1).\]

Sketch of proof. Use the differential equation (10) given in corollary 2.5 with corollary 3.4, mainly focusing on the ‘first two terms’ of $H_\ell$ after a bit of standard algebra we get (21).

We are now ready to state similar inequalities such as those obtained by Wright in [29]. If $A$ and $B$ are two formal power series such that for all $n \geq 0$ we have $[z^n] A(z) \leq [z^n] B(z)$ then we denote this relation $A \preceq B$ (or $A(z) \preceq B(z)$).

Lemma 4.2. For any $\ell \geq 1$, $H_\ell$ satisfies

\[(26) \quad \frac{\lambda_\ell (b-1)^{2\ell}}{3 \ell T(z)^{2\ell} \theta(z)^{2\ell}} - \frac{(\kappa_\ell - \nu_\ell (b-2))((b-1)^{2\ell-1})}{(3 \ell - 1) T(z)^{2\ell-1} \theta(z)^{2\ell-1}} \leq H_\ell(z) \leq \frac{\lambda_\ell (b-1)^{2\ell}}{3 \ell T(z)^{2\ell} \theta(z)^{2\ell}},\]

where $(\lambda_\ell)_{\ell \in \mathbb{N}}$, $(\kappa_\ell)_{\ell \in \mathbb{N}}$, and $(\nu_\ell)_{\ell \in \mathbb{N}}$, are defined as in lemma 4.1.

The proof of this lemma will be provided in the full paper.

The following lemma gives the order of magnitude of the two first coefficients of the partial fraction form of $H_\ell$.

Lemma 4.3. We have

\[(27) \quad \lambda_\ell = 3 \left(\frac{3}{2}\right) \frac{\ell!}{2\pi} \left(1 + O\left(\frac{1}{\ell}\right)\right),\]

\[(28) \quad \left|\kappa_\ell - \nu_\ell (b-2)\right| = O(\ell \lambda_\ell).\]

Proof. To prove (27), it suffices to remark that $\lambda_\ell = 3 \ell b \ell$ where the sequence $(b_k)$ corresponds to the Wright’s coefficients defined in [27, eq. (3.2)]. Therefore, by the proof of Lambert Meertens reported in [23] (see also Vobly [25], [27]) holds. The remaining proof of (28) is technical and is omitted in this extended abstract. □
4.2. A lemma from contour integration. In order to get rid of the asymptotic behavior of the coefficients of $H_t(z)$, we need a last intermediate step. Define $h_n(a,\beta)$ as follows

$$h_n(a,\beta) = \frac{1}{T(z)^a} \left(1 - T(z)^{b-1} \right)^{3a+\beta} \sum_{n \geq 0} h_n(a,\beta) \frac{z^n}{n!} \left(1 - (b-1)u_0 \right)^{(1-\beta)}$$

The following lemma is an application of the saddle point method which is well suited to cope with our analysis:

**Lemma 4.4.** Let $a \equiv a(n)$ be such that $a(b-1) \to 0$ but $\frac{a(b-1)n}{\ln n^2} \to \infty$ and let $\beta$ be a fixed number. Then $h_n(a,\beta)$ defined in (29) satisfies

$$h_n(a,\beta) = \frac{n!}{2\pi i} T(z)^a \left(1 - T(z)^{b-1} \right)^{3a+\beta} \int_{c-i\infty}^{c+i\infty} \frac{dz}{(b-1)^{n+1}} \exp(n\Phi(u_0)) \left(1 + O\left(\sqrt{\frac{a}{a(b-1)u_0}}\right) + O\left(\frac{1}{\sqrt{a(b-1)u_0}}\right)\right),$$

where

$$\Phi(u) = -\left(\frac{a+1}{b-1}\right) \ln u - 3a \ln (1 - (b-1)u)$$

and

$$u_0 = \frac{3/2 ab - a + 1 - 1/2 \sqrt{\Delta}}{b-1}$$

with $\Delta = 9a^2b^2 - 12a^2b + 12ab + 4a^2 - 12a$.

**Proof.** Cauchy’s integral formula gives

$$h_n(a,\beta) = \frac{n!}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{dz}{(b-1)^{n+1}} \exp(n\Phi(u_0)) \left(1 + O\left(\frac{1}{\sqrt{a(b-1)u_0}}\right)\right),$$

Note that the radius of convergence of the series $T(z)$ is given by $\exp(-1/(b-1))$. We make the substitution $u = T(z)^{b-1} / (b-1)!$ and get successively

$$T(z) = \left(\frac{b-1}{b-1} \right)^n (1 - (b-1)u), \quad z = \left(\frac{b-1}{b-1} \right)u e^{-u}, \quad \text{and}$$

$$dz = \left(\frac{b-1}{b-1} \right)^n (b-1)! e^{-u} du.$$

From (32), we then obtain

$$\Phi(u) = -\left(\frac{a+1}{b-1}\right) \ln u - 3a \ln (1 - (b-1)u).$$

The big power in the integrand, viz. $\exp(n\Phi(u_0))$, suggests us to use the saddle point method. Investigating the roots of $\Phi'(u)$, we find two saddle points, $u_0 = 3/2ab - a + 1 - 1/2 \sqrt{\Delta}$ and $u_1 = 3/2ab - a + 1 + 1/2 \sqrt{\Delta}$ with $\Delta = 9a^2b^2 - 12a^2b + 12ab + 4a^2 - 12a$.

Moreover, we have $\Phi''(u) = \frac{a^2+1}{(b-1)^2} + 3 \frac{a(-b+1)^2}{(b-1)^2}$, so that for $u \notin \{0,1/(b-1)\}$, $\Phi''(u) > 0$. The main point of the application of the saddle point method here is that $\Phi'(u_0) = 0$ and $\Phi''(u_0) > 0$, hence $n\Phi(u_0 \exp(i\pi))$ is well approximated by $n\Phi(u_0) - n u_0^2 \Phi''(u_0) \frac{z^2}{z}$ in the vicinity of $\tau = 0$. If we integrate (32) around a circle passing vertically through $u = u_0$ in the $z$-plane, we obtain

$$h_n(a,\beta) = \frac{n!}{2\pi i} \int_{-\pi}^{\pi} (1 - (b-1)u_0 e^{i\tau})^{1-\beta} (b-1) \exp(n\Phi(u_0 e^{i\tau})) \, d\tau$$
where
\begin{equation}
\Phi(u_0e^{i\tau}) = u_0 \cos \tau + iu_0 \sin \tau - \frac{a + 1}{b - 1} \ln u_0 - i\frac{a + 1}{b - 1} \tau - 3a \ln(1 - (b - 1)u_0e^{i\tau}).
\end{equation}

Denoting by $\Re(z)$ the real part of $z$, if $f(\tau) = \Re(\Phi(u_0e^{i\tau}))$ we have
\begin{equation}
f(\tau) = u_0 \cos \tau - \frac{a + 1}{b - 1} \ln u_0 - 3a \ln u_0 - 3a \ln(b - 1) = \frac{3a}{2} \ln \left(1 + \frac{1}{(b - 1)^2u_0^2} - \frac{2 \cos \tau}{(b - 1)u_0}\right).
\end{equation}

It comes
\begin{equation}
f'(\tau) = \frac{d}{d\tau} \Re(h(u_0e^{i\tau})) = -u_0 \sin \tau - \frac{3a \sin \tau}{u_0(b - 1)} - \frac{2 \cos \tau}{(b - 1)u_0}.
\end{equation}

Therefore, if $\tau = 0 f'(\tau) = 0$. Also, $f(\tau)$ is a symmetric function of $\tau$ and in $[-\pi, -\tau_0] \cup [\tau_0, \pi]$, for any given $\tau_0 \in (0, \pi)$, and $f(\tau)$ takes its maximum value for $\tau = \tau_0$. Since $|\exp(\Phi(u))| = \exp(\Re(\Phi(u)))$, when splitting the integral in (35) into three parts, viz. $\int_{-\tau_0}^{\tau_0} + \int_{\tau_0}^{\pi} + \int_{\pi}^{-\tau_0}$, we know that it suffices to integrate from $-\tau_0$ to $\tau_0$, for a convenient value of $\tau_0$, because the others can be bounded by the magnitude of the integrand at $\tau_0$. In fact, we have
\begin{equation}
\Phi(u_0e^{i\theta}) = \Phi(u_0) + \sum_{p \geq 2} \phi_p(e^{i\theta} - 1)^p
\end{equation}
where $\phi_p = \frac{u_0}{p!} \Phi^{(p)}(u_0)$. We easily compute $\Phi^{(p)}(u_0) = (-1)^p(p - 1)! \left(\frac{a + 1}{(b - 1)u_0^p} + \frac{3a(1-b)^p}{(1-(b-1)u_0)^p}\right)$, for $p \geq 2$. Whenever $ab \to 0$, we have
\begin{equation}
(b - 1)u_0 = 1 - \sqrt{3(b - 1)} a + (3/2 b - 1) a + O \left(b^{3/2}a^{3/2}\right).
\end{equation}

Therefore, we obtain after a bit of algebra
\begin{equation}
|\phi_p| \leq O \left(\frac{2^p}{a^{p-1}(b - 1)^{p}}\right), \quad \text{as } a(b-1) \to 0.
\end{equation}

On the other hand,
\begin{equation}
|e^{i\tau} - 1| = \sqrt{2(1 - \cos \tau)} < \tau, \quad \tau > 0.
\end{equation}

Thus, the summation in (35) can be bounded for values of $\tau$ and $a$ such that $\tau \to 0$, $ab \to 0$ ($a \to 0$) but $\frac{1}{\sqrt{a}} \to 0$ and we have
\begin{equation}
\left|\sum_{p \geq 4} \phi_p(e^{i\tau} - 1)^p\right| \leq \sum_{p \geq 4} |\phi_p|^p \leq \sum_{p \geq 4} \frac{2^p \tau^p}{a^{p-1}(b - 1)^p} = O \left(\frac{\tau^4}{a(b-1)}\right).
\end{equation}

It follows that for $\tau \to 0$, $a(b-1) \to 0$ and $\frac{\tau}{\sqrt{a(b-1)}} \to 0$, $\Phi(u_0e^{i\tau})$ can be rewritten as
\begin{equation}
\Phi(u_0e^{i\tau}) = \Phi(u_0) - \frac{1}{(b - 1)} \left(1 - \frac{\sqrt{a}}{\sqrt{3(b - 1)}} \frac{3b - 4}{2} + \frac{9b^2 - 12b + 4}{12(b - 1)} a\right) \tau^2
\end{equation}
\begin{equation}
- \frac{i}{(b - 1)} \left(1 - \frac{(3b - 4)\sqrt{a}}{2\sqrt{3(b - 1)}} + \frac{9b^2 - 12b + 4}{12(b - 1)} a\right) \tau^3 + O \left(\frac{\tau^4}{a(b-1)}\right).
\end{equation}

Therefore, if $a(b-1) \to 0$ but $\frac{(b-1)n}{\ln u_0} \to \infty$, if we let $\tau_0 = \frac{\ln n}{\sqrt{n}u_0^2\Phi''(u_0)}$ (with $u_0^2\Phi''(u_0) = (\frac{2}{a-1})^2 + O(\sqrt{a(b-1)})$) we can remark (as already said) that it suffices to integrate (35) from $-\tau_0$ to $\tau_0$, using the magnitude of the integrand at $\tau_0$ to bound the resulting error. In fact,
\begin{equation}
\left|\left((1 - (b - 1)u_0e^{i\tau_0})^{1-\beta}\exp\left(n\Phi(u_0e^{i\tau_0}) - nu_0 + \frac{n(a + 1)}{(b - 1)} \ln u_0 + 3an \ln(1 - (b - 1)u_0)\right)\right| =
\end{equation}
\begin{equation}
\left((1 - (b - 1)u_0e^{i\tau_0})^{1-\beta}\exp\left(-\frac{n}{2}a^2\Phi''(u_0)\tau_0^2 + O\left(n\frac{\tau_0^4}{a(b-1)}\right)\right)\right) = O\left(e^{\frac{(n\ln)^2}{2\tau_0^2}}\right).
\end{equation}
The rest of the proof is now standard application of the saddle point method (see for instance De Bruijn [8, Chapters 5 & 6]) and is omitted in this extended abstract. After a bit of algebra, one gets the formula (44).

4.3. Asymptotic number of connected hypergraphs. We are now ready to state the main result of this section.

Theorem 4.5. For \( \ell \equiv \ell(n) \) such that \( \ell = o\left(\sqrt[n]{n}\right) \) as \( n \to \infty \), the number of connected \( b \)-uniform hypergraphs built with \( n \) vertices and having excess \( \ell \) satisfies

\[
(46) \quad \sqrt{\frac{3}{2\pi}} \frac{(b - 1)^{\frac{1}{b}}}{12^\frac{1}{2\pi} \ell^{\frac{1}{2}} \left((b - 2)!\right)^{\frac{1}{2\pi}}} \exp\left(\frac{n}{b - 1} - n\right) \left(1 + O\left(\frac{1}{\sqrt{\ell}}\right) + O\left(\frac{b}{\sqrt{n}}\right)\right).
\]

We urge the reader to compare the methods and results obtained by Karoński and Łuczak in [20] with ours. In particular, the authors of [20] obtained results concerning various kinds of hypergraphs (smooth hypergraphs, clean hypergraphs, etc.). Unlike their results, where the excesses are of order \( \log n / \log \log n \), the theorem above states that the three variables \( n, \ell \) and \( b \) can tend together to infinity but (46) remains valid whenever \( \ell = o\left(\sqrt{n}\right) \). Note also that by setting \( b = 2 \) in (46), we retrieve Wright’s formula for graphs [2]. We remark also that the powerful methods developed in [3] and in [24] can be used to extend the validity of our asymptotic result.

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