Coloring Meyniel graphs in linear time

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Abstract

A Meyniel graph is a graph in which every odd cycle of length at least five has two chords. We present a linear-time algorithm that colors optimally the vertices of a Meyniel graph and finds a clique of maximum size.

Keywords: Perfect graphs, Meyniel graphs, coloring, algorithm

1 Introduction

This paper deals with the graphs in which every odd cycle of length at least five has at least two chords. Meyniel [13, 14] proved that such graphs are perfect [17], and it has become customary to call them *Meyniel graphs*. These graphs were also studied by Markosyan and Karapetyan [12] who also proved their perfectness. Chordal graphs (graphs in which every cycle of length at least four has a chord), bipartite graphs, and more generally i-triangulated graphs and parity graphs (see [4]) are examples of Meyniel graphs. Burlet and Fonlupt [4] gave a polynomial-time recognition algorithm for Meyniel graphs, and later Roussel and Rusu [21] gave another, faster, such algorithm, whose complexity is $\mathcal{O}(m(m+n))$ for a graph with n vertices and m edges.

A *coloring* of the vertices of a graph is a mapping that assigns one color to each vertex in such a way that any two adjacent vertices receive distinct colors. A coloring is *optimal* if it uses as few colors as possible. The *chromatic number* $\chi(G)$ of a graph G is the number of colors used by an optimal coloring. We are interested in polynomial-time algorithms that color the vertices of a Meyniel graphs optimally. Hoàng [11] gave such an algorithm, with complexity $\mathcal{O}(n^8)$,

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which uses the so-called amalgam decomposition from [4]. Hertz [9] devised a coloring algorithm that works in time $\mathcal{O}(nm)$ and produces an optimal coloring for Meyniel graphs. The optimality is based on the concept of even pair [7] (which we recall formally below) : at each step Hertz's algorithm finds an even pair of vertices in the graph and contracts them. Later, Roussel and Rusu [22] defined a coloring algorithm called LexColor (for Lexicographic Color) wich works in time $\mathcal{O}(n^2)$. In the case of a Meyniel graph, this algorithm colors the vertices optimally, without contracting even pairs. It "simulates" such a contraction and its optimality follows from that of Hertz's algorithm. LexColor is based on LexBFS (for Lexicographic Breadth First Search), which was originally invented by Rose, Tarjan and Lueker [19] to find a simplicial elimination ordering in a chordal graph.

We propose here an algorithm that we call MCColor (for Maximum Constraint Color), which will color any graph in time $\mathcal{O}(n+m)$. This algorithm is a simplification of LexColor. Just like LexColor, in the case of a Meyniel graph MCColor produces an optimal coloring by "simulating" the contraction of even pairs. MCColor is based on the algorithm MCS (for Maximum Cardinality Search) due to Tarjan and Yannakakis [25], which is a simplification of LexBFS and can also be used to find a simplicial elimination ordering in a chordal graph. MCColor can also be viewed as a simplified version of the famous coloring heuristic DSatur (for Degree Saturation) due to Brélaz [3].

2 Algorithm MCColor

Our algorithm is a rather simple version of the greedy coloring algorithm. Colors are viewed as integers $1, 2, \ldots$ At each step, the algorithm considers, for every uncolored vertex x, the number of colors that appear in the neighbourhood of x, selects an uncolored vertex for which this number is maximum (this vertex is the most "constrained"), assigns to this vertex the smallest color not present in its neighbourhood, and iterates this procedure until every vertex is colored. More formally:

Algorithm MCColor

Input: A graph G with n vertices.

Output: A coloring of the vertices of G.

Initialization: For every vertex x of G do $label(x) := \emptyset$;

General step: For $i = 1, \ldots, n$ do:
1. Choose an uncolored vertex x that maximizes $|label(x)|$;
2. Color x with the smallest color in $\{1, 2, \ldots, n\} \setminus label(x)$;
3. For every uncolored neighbour y of x, add $color(x)$ to $label(y)$.

For the sake of clarity let us compare our algorithm with LexColor and DSatur. Algorithm LexColor [22, 23] assigns to every vertex v a vector whose i-th component is the rank in the i-th color of the first neighbour of v colored i. At each step, these vectors must be updated and reordered lexicographically, and they are used to select the next vertex to be colored. This leads to the $\mathcal{O}(n^2)$ complexity. In Roussel's PhD thesis [23], the question is posed of finding a simplification of LEXCOLOR that would insure linear time complexity. Our main contribution is to answer this question positively. Our simplification consists in assigning a simple label which must be ordered according only to its size, thus doing away with the lexicographic aspects. Brélaz's algorithm DSATUR [3] selects a "most saturated" vertex, where the saturation is the number of colors in the neighbourhood, and ties are broken by choosing a most saturated vertex of maximum degree. Our algorithm selects vertices according to the same criterion of saturation, but ties are broken arbitrarily.

3 Even pairs and Meyniel graphs

Before proving the main result, we need to recall some notation and definitions. An *even pair* in a graph G is a pair of non-adjacent vertices such that every chordless path between them has even length (number of edges). A survey on even pairs is given in [7]. In a graph G, the neighbourhood of a vertex v is denoted by $N(v)$. Given two vertices x, y in G, the operation of *contracting* them means removing x and y and adding one vertex with an edge to each vertex of $N(x) \cup N(y)$. The next lemma states an essential result about even pairs.

Lemma 1 ([8, 15]) *The graph G' obtained from a graph G by contracting an even pair of G satisfies $\omega(G') = \omega(G)$ and $\chi(G') = \chi(G)$.*

A graph is *even contractile* [2] if there is a sequence of graphs $G_0, \ldots, G_k$ $(k \geq 0)$ such that $G_0 = G$, G_k is a clique, and if $k > 0$ then for $i = 1, \ldots, k$ the graph G_i is obtained from G_{i-1} by contracting an even pair of G_{i-1}. A graph is *perfectly contractile* [2] if every induced subgraph of G is even contractile. Note that if a graph G is even contractile, with the above notation, then its chromatic number $\chi(G)$ can be computed using the sequence of graphs $G_0, \ldots, G_k$ and Lemma 1. However, finding such a sequence may be a difficult task from the algorithmic point of view. See [7] for more insight on this question.

In the case of Meyniel graphs, Meyniel himself [15] showed that every such graph either is a clique or contains an even pair. However, there are Meyniel graphs that are not cliques and do not contain any even pair whose contraction yields a Meyniel graph (see [9]). This could have made it difficult to prove that Meyniel graphs are even contractile. Hertz [9] side-stepped this problem by defining the larger class of quasi-Meyniel graphs. A graph G is a *quasi-Meyniel graph* [9] if G contains no odd chordless cycle on at least five vertices and G has a vertex, called a *pivot*, that is an endvertex of every edge that is the unique chord of

an odd cycle of G. Hertz [9] proved that every quasi-Meyniel graph that is not a clique has an even pair whose contraction yields a quasi-Meyniel graph, and thus that every quasi-Meyniel graph is perfectly contractile; indeed his proof is the polynomial-time algorithm mentioned in the introduction, which colors every Meyniel graph optimally through a sequence of even-pair contractions. De Figueiredo and Vušković [6] found a polynomial-time algorithm to decide if a graph G is quasi-Meyniel and, if it is, to give a pivot of G. One can observe that:
(a) Every Meyniel graph is a quasi-Meyniel graph.
(b) In a Meyniel graph every vertex is a pivot.
(c) If G is a quasi-Meyniel graph and z is a pivot, then $G \setminus z$ is a Meyniel graph.

We can run algorithm MCColor on any quasi-Meyniel graph with a given pivot, only imposing that the first vertex to be colored is the given pivot. By observations (a) and (b), any application of the algorithm on a Meyniel graph is simply an application on the graph viewed as a quasi-Meyniel and with an arbitrary vertex as pivot.

4 Optimality

In this section we prove that Algorithm MCColor can color every quasi-Meyniel graph G with $\omega(G)$ colors, where $\omega(G)$ is the maximum size of a clique in G. This will prove the optimality of the algorithm (and also the perfectness of G). Our proof follows the same steps as Roussel and Rusu's proof [22] that their algorithm LexColor is optimal on quasi-Meyniel graphs. Just like in [22], the optimality of our algorithm will follow from the fact that each color class produced by the algorithm corresponds to the contraction of some even pairs.

Theorem 1 *When Algorithm* MCColor *is applied on a quasi-Meyniel graph G with a given pivot, coloring the pivot first, it uses exactly $\omega(G)$ colors.*

As in [22], say that a path $P = v_0\text{-}v_1\text{-}v_2\text{-}\cdots\text{-}v_p$ is *quasi-chordless* if it has at most one chord and, if it has one, then this chord is $v_{j-1}v_{j+1}$ with $1 < j < p-1$ (so the endvertices of P are not incident to the chord). We will use the following lemma which is essentially from [22] (the first item of the lemma was also proved for Meyniel graphs in [14, 18]).

Lemma 2 ([22]) *Let G be a quasi-Meyniel graph. Let $P = v_0\text{-}v_1\text{-}\cdots\text{-}v_{2k}\text{-}v_{2k+1}$ (with $k \geq 0$) be a quasi-chordless odd path in G. Suppose that v_0 is a pivot of G, and that there is a vertex w adjacent to both v_0, v_{2k+1}. Then:*

- *If P is chordless, then w is adjacent to every vertex of P.*

- *If P has a chord $v_{j-1}v_{j+1}$ (with $1 < j < 2k$), then w is adjacent to every vertex of $P \setminus v_j$.* □

Proof of Theorem 1. Let G be a quasi-Meyniel graph on which Algorithm MCColor is applied, so that a given pivot is the first vertex to be colored. Let l be the total number of colors used by the algorithm. For each color $c \in \{1, \ldots, l\}$ let k_c be the number of vertices colored c. Therefore every vertex of G can be renamed x_c^i, where $c \in \{1, \ldots, l\}$ is the color assigned to the vertex by the algorithm and $i \in \{1, \ldots, k_c\}$ is the integer such that x_c^i is the i-th vertex colored c. Thus $V(G) = \{x_1^1, x_1^2, \ldots, x_1^{k_1}, x_2^1, \ldots, x_2^{k_2}, \ldots, x_l^1, \ldots, x_l^{k_l}\}$.

Define a sequence of graphs and vertices as follows. Put $G_1^1 = G$ and $w_1^1 = x_1^1$ (which is a pivot of G). For $i = 2, \ldots, k_1$, call G_1^i the graph obtained from G_1^{i-1} by contracting w_1^{i-1} and x_1^i into a new vertex w_1^i colored with the color 1. In the graph $G_1^{k_1}$, we remark that $w_1^{k_1}$ is adjacent to all other vertices of $G_1^{k_1}$; for otherwise, there is a vertex y that is not adjacent to $w_1^{k_1}$, which means that y has no neighbour of color 1, so y should have received color 1, a contradiction. Let us call simply w_1 the vertex $w_1^{k_1}$.

The sequence continues as follows. For each $c \in \{2, \ldots, l\}$, put $G_c^1 = G_{c-1}^{k_{c-1}}$ and $w_c^1 = x_c^1$. For $i = 2, \ldots, k_c$, call G_c^i the graph obtained from G_c^{i-1} by contracting vertices w_c^{i-1} and x_c^i into a new vertex w_c^i colored with the color c. In $G_c^{k_c}$, we can again remark that $w_c^{k_c}$ is adjacent to all other vertices of $G_c^{k_c}$, for the same reason as above, and we call simply w_c the vertex $w_c^{k_c}$. So the last graph in the sequence, $G_l^{k_l}$, is a clique of size l with vertices $w_1, \ldots, w_l$, where each w_c is obtained by the contraction of the vertices of color c.

Lemma 3 *For every $c \in \{1, \ldots, l\}$ and $i \in \{1, \ldots, k_c - 1\}$, if G_c^i is a quasi-Meyniel graph and w_c^i is a pivot of G_c^i, then there is no quasi-chordless odd path from w_c^i to x_c^{i+1} in G_c^i.*

Proof. Suppose on the contrary that there exists a quasi-chordless odd path $P = v_0\text{-}v_1\text{-}\cdots\text{-}v_{2k}\text{-}v_{2k+1}$ from $v_0 = w_c^i$ to $v_{2k+1} = x_c^{i+1}$ in G_c^i. We have $k > 0$ since w_c^i, x_c^{i+1} are not adjacent. Note that every vertex of P has a non-neighbour in G_c^i. Put $W_1 = \emptyset$ and $W_c = \{w_1, \ldots, w_{c-1}\}$ if $c \geq 2$, and recall that any $w \in W_c$ is a vertex of G_c^i that is adjacent to all vertices of $G_c^i \setminus w$. So P contains no vertex of W_c. We know that every vertex of $G_c^i \setminus W_c$ will have a color from $\{c, c+1, \ldots, l\}$ when the algorithm terminates. So, if $c \geq 2$, every vertex v of $G_c^i \setminus W_c$ (in particular every vertex of P) satisfies $label(v) \supseteq \{1, 2, \ldots, c-1\}$.

Let us consider the situation when the algorithm selects x_c^{i+1}. Let U be the set of vertices that are already colored at that moment. For any $X \subseteq V$, let $color(X)$ be the set of colors of the vertices of $X \cap U$. So for every vertex $v \in V \setminus U$ we have $label(v) = color(N(v))$. Put $T = \{v \in N(x_c^{i+1}) \cap U,\ color(v) \geq c+1\}$. We have $|label(x_c^{i+1})| = (c-1) + |color(T)|$. Every vertex of T is adjacent to at least one vertex colored c in G and thus is adjacent to w_c^i in G_c^i. Specify one vertex v_r of P as follows: put $r = 3$ if v_1v_3 is a chord of P, else put $r = 2$. Note that v_r is not adjacent to v_0 and $v_r \neq x_c^{i+1}$ since P is quasi-chordless. Since every vertex of T is adjacent to both v_0, v_{2k+1}, by Lemma 2 every vertex of T is adjacent to v_1 and v_r.

Suppose v_1 is not colored yet. Since $label(v_1) \supseteq \{1, 2, \ldots, c-1\}$ if $c \geq 2$, and $N(v_1) \supseteq T \cup \{v_0\}$ and v_0 has color c, we have $|label(v_1)| = |color(N(v_1))| \geq c + |color(T)| > |label(x_c^{i+1})|$, which contradicts the fact that the algorithm is about to color x_c^{i+1}. So v_1 is already colored; moreover $color(v_1) \notin \{1, \ldots, c\} \cup color(T)$.

Suppose v_r is not colored yet. Since $label(v_r) \supseteq \{1, 2, \ldots, c-1\}$ if $c \geq 2$ and v_r is adjacent to all of $T \cup \{v_1\}$, we have $|label(v_r)| = |color(N(v_r))| \geq (c-1) + |color(T \cup \{v_1\})| = c + |color(T)| > |label(x_c^{i+1})|$, again a contradiction. So v_r is already colored. However, v_r is not adjacent to w_c^i, so c is the smallest color available for v_r; but this contradicts the definition of w_c^i and x_c^{i+1}. This completes the proof of Lemma 3. □

Lemma 4 *For every color $c \in \{1, \ldots, l\}$ and integer $i \in \{0, 1, \ldots, k_c - 1\}$, the following two properties hold:*

(A_i) If $i \geq 1$, then w_c^i and x_c^{i+1} form an even pair of G_c^i.

(B_i) G_c^{i+1} is a quasi-Meyniel graph and w_c^{i+1} is a pivot of G_c^{i+1}.

Proof. Let $c \in \{1, \ldots, l\}$. We show by induction on i that (A_i) and (B_i) hold.

Property (A_0) holds by vacuity. Property (B_0) is just the general assumption when $c = 1$, so suppose $c \geq 2$. In the graph G_c^1, every vertex w_h with $h \in \{1, \ldots, c-1\}$ is adjacent to all other vertices of the graph; moreover $G_c^1 \setminus \{w_1, \ldots, w_{c-1}\}$ is a Meyniel graph by observation (c) above, since it is a subgraph of $G \setminus x_1^1$ and x_1^1 is a pivot of G. It follows that G_c^1 is actually a Meyniel graph and so w_c^1 is a pivot of this graph.

Now suppose that $i \geq 1$ and that (A_{i-1}) and (B_{i-1}) hold. Lemma 3 implies immediately that (A_i) holds. To prove (B_i), suppose on the contrary that G_c^{i+1} is not a quasi-Meyniel graph with pivot w_c^{i+1}. This means that G_c^{i+1} contains an odd cycle C of length at least 5, with vertices $v_1, \ldots, v_{2k+1}$ ($k \geq 2$) and edges v_iv_{i+1} modulo $2k+1$, such that either C is chordless or C has exactly one chord and w_c^{i+1} is not an endvertex of that chord. By taking such a C as short as possible, we may assume that if it has a chord then this chord is $v_{j-1}v_{j+1}$ for some $j \in \{1, \ldots, 2k+1\}$. Then w_c^{i+1} must be in C, for otherwise C would be a cycle in $G_c^i \setminus w_c^i$, contradicting (B_{i-1}). So we may assume $w_c^{i+1} = v_1$. A vertex $x \in \{w_c^i, x_c^{i+1}\}$ cannot be adjacent to both v_2, v_{2k+1} in G_c^i, for otherwise replacing w_c^{i+1} by x in C gives an odd cycle in G_c^i that contradicts (B_{i-1}). So we may assume that w_c^i is adjacent to v_2 and not to v_{2k+1} and that x_c^{i+1} is adjacent to v_{2k+1} and not to v_2. If C has a chord v_2v_{2k+1}, then w_c^i-v_2-v_{2k+1}-x_c^{i+1} is a chordless odd path from w_c^i to x_c^{i+1} in G_c^i, which contradicts Lemma 3.

So either C is chordless or its unique chord is $v_{j-1}v_{j+1}$ with $j \in \{3, \ldots, 2k\}$. But then w_c^i-v_2-$\cdots$-v_{2k+1}-x_c^{i+1} is a quasi-chordless odd path from w_c^i to x_c^{i+1} in G_c^i, which contradicts Lemma 3. So (B_i) holds. This completes the proof of Lemma 4. $\square$

Lemma 4 implies that in the sequence $G = G_1^1, \ldots, G_l^{k_l}$, each graph other than the first one is obtained from its predecessor by contracting an even pair of the predecessor. Then Lemma 1 applied successively along the sequence implies that $\omega(G) = \omega(G_l^{k_l})$ and $\chi(G) = \chi(G_l^{k_l})$; but $\chi(G_l^{k_l}) = \omega(G_l^{k_l}) = l$ since $G_l^{k_l}$ is a clique of size l; so the algorithm does color the input graph optimally with $\omega(G)$ colors. $\square$

5 Complexity

We now analyze the complexity of Algorithm MCColor. Let the input be a graph $G = (V, E)$ with $n = |V|$ and $m = |E|$. We assume that, for every vertex x, we are given the set $N(x)$ as a list, of size $degree(x)$. So the total size of the input is $n + m$.

We will not compute explicitly the set $label(x)$ for $x \in V$. Instead, we consider the counter $|label(x)|$ for every $x \in V$ and maintain an $n \times n$ array A such that, for every vertex $x \in V$ and every color i, the entry $A(x, i)$ is set to 1 if x has a neighbour of color i, else it is set to 0.

Ordering the vertices according to the value of $|label(x)|$ can be done with the usual techniques, such as bucket sort [1]: For each $j = 0, 1, \ldots, n-1$, we maintain the set L_j of the uncolored vertices x such that $|label(x)| = j$. This set is implemented as a doubly linked list, where each element also points to the head of the list, which is the integer j. The heads of the non-empty L_j's are themselves put in decreasing order into a doubly linked list M.

During the initialization step we must initialize each L_j, M and the array A. All vertices are put into L_0, and L_0 is the only element of M. Thus the initialization of the L_j's and of M takes time $\mathcal{O}(n)$. Initializing every entry of A would take time $\mathcal{O}(n^2)$; but we can skip this by using an argument from [1, ex. 2.12] (see also [24, p. 9]), which ensures that only those entries that are actually accessed during the algorithm are initialized, on the first occasion they are accessed. It will be obvious below that for every vertex x at most $degree(x) + 1$ entries $A(x, .)$ are accessed. So the total time for initializing A will be $\mathcal{O}(m + n)$.

Consider line 1 of the general step. Using the data structure we just described, this line can be done in constant time for each vertex: we get the largest integer j in M and the first vertex x in L_j, remove x from L_j, and, if L_j becomes empty, remove j from M. So the total time over all vertices is $\mathcal{O}(n)$.

Now consider line 2. Given x, we scan the entries $A(x, 1)$, $A(x, 2)$, ..., until we find the first entry $A(x, c)$ that is not equal to 1, and we assign color c to

x. Since each entry equal to 1 corresponds to a color assigned to a different neighbour of x, we will scan at most $degree(x) + 1$ entries for each x. So the total time over all vertices is $\mathcal{O}(m + n)$.

Finally consider line 3. For every neighbour y of x we check whether $A(y, c)$ is equal to 1 or not. If it is 1 we do nothing, else we update vertex y: we set $A(y, c) := 1$, move y from the list L_j that contains it to the list L_{j+1}, and update M accordingly (i.e., if $j + 1$ was not in M we insert it between j and the predecessor of j, and if L_j becomes empty we remove j from M). Using the data structure this takes constant time for each y. So line 3 takes time $\mathcal{O}(degree(x))$. Note that it happens only once for each x. So the total complexity over all vertices is $\mathcal{O}(m)$.

In conclusion, the total running time of the algorithm is $\mathcal{O}(n + m)$.

MCColor may fail to color a quasi-Meyniel graph G optimally if the first vertex is not a pivot. (We can still get an optimal coloring of G by applying the algorithm on each vertex as the first vertex, but this multiplies the complexity by n.) However, in a Meyniel graph every vertex is a pivot so MCColor need only be applied once to obtain an optimal coloring in linear time.

6 Finding a maximum clique

We can extend our algorithm by another greedy algorithm, which, in the case of a quasi-Meyniel graph, will produce in linear time a clique of maximum size. This idea is implicit in [22] but the algorithmic aspects were not worked out there. Let G be any graph given with a coloring of its vertices using l colors. Then we can apply the following algorithm to build a set Q:

> Algorithm Q
>
> *Input:* A graph G and a coloring of its vertices using l colors.
>
> *Output:* A set Q that consists of l vertices of G.
>
> *Initialization:* Set $Q := \emptyset$, $c := l$, and for every vertex x set $q(x) := 0$;
>
> *General step:* While $c \neq 0$ do:
> Pick a vertex x of color c that maximizes $q(x)$, do $Q := Q \cup \{x\}$, for every neighbour y of x do $q(y) := q(y) + 1$, and do $c := c - 1$.

Algorithm Q can be implemented in time $\mathcal{O}(m + n)$. To do this, at Step c we need only scan the vertices of color c, keep one vertex of color c that maximizes the counter q, and update the counter of the neighbours of that vertex.

We claim that when the input consists of a quasi-Meyniel graph G with the coloring produced by MCColor, then the output Q is a clique of size l. Actually this will be true in a more general framework.

Lemma 5 *Let G be a graph given with a coloring of its vertices using l colors. Call its vertices $x_1^1, x_1^2, \ldots, x_1^{k_1}, x_2^1, \ldots, x_2^{k_2}, \ldots, x_l^1, \ldots, x_l^{k_l}$, so that vertices of subscript c have color c. Define the corresponding sequence of graphs G_c^i and vertices w_c^i ($1 \leq c \leq l$, $1 \leq i \leq k_c$) obtained by successive contractions as in the preceding section. Suppose that for each color $c = 1, \ldots, l-1$, we have:*

(i) Every vertex of color strictly greater than c has a neighbour of color c,

(ii) For each $i = 1, \ldots, k_c - 1$, the graph G_c^i contains no chordless path on four vertices whose endvertices are w_c^i and x_c^{i+1}.

Let Q be a clique whose vertices have colors strictly greater than c for some $c \in \{1, \ldots, l-1\}$. Then there is a vertex of color c that is adjacent to all of Q.

Proof. For $i = 1, \ldots, k_c$, consider the following Property P_i: "In the graph G_c^i, vertex w_c^i is adjacent to all of Q." Note that Property P_{k_c} holds by (i) and by the definition of $w_c^{k_c}$. We may assume that Property P_1 does not hold, for otherwise the lemma holds with vertex $x_c^1 = w_c^1$. So there is an integer $i \in \{2, \ldots, k_c\}$ such that P_i holds and P_{i-1} does not. Then, in the graph G_c^{i-1}, vertex x_c^i must be adjacent to all of Q, for otherwise Q contains vertices a, b such that a is adjacent to w_c^{i-1} and not to x_c^i and b is adjacent to x_c^i and not to w_c^{i-1}, and then the path w_c^{i-1}-a-b-x_c^i contradicts (ii). So the lemma holds with vertex x_c^i. □

Lemma 6 *Let G be a quasi-Meyniel graph, and $x_1^1, x_1^2, \ldots, x_1^{k_1}, x_2^1, \ldots, x_2^{k_2}, \ldots, x_l^1, \ldots, x_l^{k_l}$ be a coloring produced by Algorithm MCColor, where the first vertex x_1^1 is a pivot of G. Then, when Algorithm Q is run on this input it produces a clique of size $\omega(G)$.*

Proof. Consider the set Q maintained during Algorithm Q. We claim that at the end of step $c = l, l-1, \ldots, 1$, the set Q is a clique of size $l - c + 1$ that contains one vertex of each color $c, \ldots, l$. This is clear when $c = l$. At the general step, Lemma 5 ensures that there exists a vertex of color $c - 1$ that is adjacent to all of Q. So Algorithm Q will select such a vertex, add it to Q, and so the claim remains true at the end of that step. Thus the algorithm ends with a clique Q of size l. Since G admits a coloring of size l, we have $l = \chi(G) = \omega(G)$. □

We can observe that the hypothesis of Lemma 6 actually yields some slightly stronger properties:
(a) For any color c, every vertex of color c lies in a clique of size c; and more generally, every clique whose smallest color is c is included in a clique that contains a vertex of each color $1, \ldots, c$. This is a consequence of Lemma 5 which can be derived just like Lemma 6. A coloring that has this property is called *strongly canonical* in [10].

(b) The set of vertices of color 1 is a stable set that intersects all maximal cliques of G. This too can be derived easily from Lemma 5. Such a set is called a *good stable set* in [11]. Moreover, since any vertex can be used as the first vertex in MCColor, we obtain that every vertex of a Meyniel graph lies in a good stable set; this is one of the main theoretical results in [11], where an $\mathcal{O}(n^7)$ algorithm was given to find such a set, while our algorithm produces a good stable set in time $\mathcal{O}(m+n)$.

7 Comments

Algorithm MCColor can be applied on any graph, thus producing a coloring that may be non-optimal. An example is the graph $\overline{P}_6$ whose vertices are u, v, w, x, y, z and whose non-edges are uv, vw, wx, xy, yz. A possible application of the algorithm produces the ordering v, y, x, z, u, w with the corresponding colors $1, 2, 2, 3, 1, 4$ although the graph has chromatic number 3. Since $\overline{P}_6$ is in many families of perfect graphs (such as brittle graphs, weakly chordal graphs, perfectly orderable graphs, etc; see [17] for the definitions), our algorithm will not perform optimally on these classes. We remark that $\overline{P}_6$ is also a graph which the algorithms LexColor [22] and LexBFS + Color (which consists simply of the greedy algorithm applied on an ordering produced by LexBFS) [20] may fail to color optimally.

The ordering produced by MCColor is not necessarily a *perfect ordering* in the sense of Chvátal [5], since there are Meyniel graphs that do not admit any perfect ordering (see e.g. Graph 'E' in Fig. 7.1 of [17]).

Our algorithm is "robust" [16] in the sense that if the input graph is not Meyniel and the output coloring is not optimal, it can detect this fault. To do this we apply MCColor followed by Algorithm Q and we check (in linear time) whether Q is a clique. If Q is a clique, the coloring is optimal since it uses l colors and Q has size l. If Q is not a clique, the input graph is not Meyniel since our algorithm is optimal on Meyniel graphs.

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