An output feedback for a chain of integrator suitable for the use of dynamic scaling

Vincent Andrieu, Laurent Praly, Alessandro Astolfi

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V. Andrieu,* L. Praly†and A. Astolfi‡

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Abstract

An output feedback for a chain of integrator is given in this paper. This one is such that the closed loop system obtained is homogeneous in the bi-limit. Furthermore the associated Lyapunov function satisfies a specific property which is useful in the context of dynamic scaling as used in (Andrieu at al., 2008, Automatica).

1 Introduction

We consider a system, with state $\mathfrak{X} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ described by :

$$\dot{\mathfrak{X}} = \mathcal{S}\,\mathfrak{X} + \mathcal{B}u \quad , \qquad y = x_1 \ , \tag{1}$$

where y is the output, $\mathcal{S}$ in $\mathbb{R}^{n\times n}$ is the left shift matrix defined as :

$$\mathcal{S}\,\mathfrak{X} = (x_2, \ldots, x_n, 0)^T \ ,$$

and $\mathcal{B}$ in $\mathbb{R}^n$ is the vector defined by

$$\mathcal{B} = (0, \ldots, 1)^T$$

In the whole paper we refer to tools developed within the framework of homogeneity in the bi-limit, notion introduced and studied in (Andrieu et al., 2008, SICON). We give in Section 2 a brief summary and definition.

Selecting $\mathfrak{d}_0 = 0$ and arbitrary degrees $0 \leq \mathfrak{d}_\infty < \frac{1}{n-1}$, homogeneity in the bi-limit (see Definition 3 below) with degrees $\mathfrak{d}_0$ and $\mathfrak{d}_\infty$ is obtained for system (1) provided the weights

*V. Andrieu is LAAS-CNRS, University of Toulouse, 31077 Toulouse, France (vincent.andrieu@gmail.com)

†L. Praly is with CAS – Ecole des Mines de Paris, 35, rue St-Honore, 77305 Fontainebleau CEDEX, France, (laurent.praly@ensmp.fr)

‡A. Astolfi is with the Electrical and Electronic Engineering Dept, Imperial College, London, SW7 2AZ, UK (a.astolfi@ic.ac.uk) and with Dipartimento di Informatica Sistemi e Produzione, University of Rome Tor Vergata, Via del Politecnico 1, 00133 Roma, Italy

$r_0 = (r_{0,1}, \dots, r_{0,n})$ and $r_\infty = (r_{\infty,1}, \dots, r_{\infty,n})$ are :

$$r_{0,i} = 1 \ , \quad r_{\infty,i} = 1 \ - \ \mathfrak{d}_\infty \, (n-i) \ . \tag{2}$$

In (Andrieu et al., 2008, SICON), we have proposed an output feedback for system (1) given by :

$$u = \phi(\hat{\mathfrak{X}}) \ , \quad \dot{\hat{\mathfrak{X}}} = \mathcal{S}\,\hat{\mathfrak{X}} \ + \ \mathcal{B}\,\phi(\hat{\mathfrak{X}}) \ + \ K(\hat{x}_1 - x_1) \ , \tag{3}$$

where $\hat{\mathfrak{X}} = (\hat{x}_1, \dots, \hat{x}_n)$ in $\mathbb{R}^n$, and K and ϕ are respectively a homogeneous in the bi-limit vector field and function with weights r_0 and r_∞, and degrees $\mathfrak{d}_0$ and $\mathfrak{d}_\infty$ (respectively $1+\mathfrak{d}_0$ and $1+\mathfrak{d}_\infty$). Setting :

$$E \ = \ (e_1 \dots, e_n)^T \ = \ \hat{\mathfrak{X}} - \mathfrak{X}$$

the chain of integrator (1) with the controller (3) can be described by :

$$\left\{ \begin{array}{lcl} \dot{\hat{\mathfrak{X}}} & = & \mathcal{S}\,\hat{\mathfrak{X}} \ + \ \mathcal{B}\,\phi(\hat{\mathfrak{X}}) \ + \ K(e_1) \\ \dot{E} & = & \mathcal{S}\,E \ + \ K(e_1) \ . \end{array} \right. \tag{4}$$

In (Andrieu et al., 2008, SICON), the design of K and ϕ is done recursively, following an observer / controller approach in such a way that there exists a homogeneous in the bi-limit Lyapunov function U of degree $\mathfrak{d}_U$ which time derivative along the trajectories of system (4) satisfies for some positive real number c_1 :

$$\dot{\overbrace{U(\hat{\mathfrak{X}}, E)}} \ \leq \ -c_1 \left(U(\hat{\mathfrak{X}}, E) + U(\hat{\mathfrak{X}}, E)^{\frac{\mathfrak{d}_U + \mathfrak{d}_\infty}{\mathfrak{d}_U}} \right) \ . \tag{5}$$

In (Andrieu et al., 2008, IEEE-Tac), to synthesize a control law in the context of trajectory tracking, we combine this tool with dynamic scaling. To do so, we need a specific property on the Lyapunov function U. This property is a homogeneous in the bi-limit version of the one given in the seminal paper on high-gain dynamic scaling (Praly, 2003, equation (16)) or in (Krishnamurthy et al., 2003, Lemma A1) and also used in the context of observer design in (Andrieu et al., 2008, Automatica). Namely, given a diagonal matrix

$$D = \mathrm{diag}(d_1, \dots, d_n) \ ,$$

with $d_j > 0$, the function ϕ and the vector field K have to be selected such that the associated Lyapunov function U satisfies (5), and also :

$$\frac{\partial U}{\partial E}(\hat{\mathfrak{X}}, E)\, D\, E \ + \ \frac{\partial U}{\partial \hat{\mathfrak{X}}}(\hat{\mathfrak{X}}, E)\, D\, \hat{\mathfrak{X}} \ \geq \ c_2\, U(\hat{\mathfrak{X}}, E) \ , \tag{6}$$

for some positive real number c_2. This is summarized in the following Theorem which proof will constitute the main (only) contribution of this paper.

Theorem 1 *Let $\mathfrak{d}_U$ be a positive real number satisfying $\mathfrak{d}_U \geq 2 + \mathfrak{d}_\infty$. There exists a homogeneous in the bi-limit function $\phi : \mathbb{R}^n \to \mathbb{R}$ with associated triples $(r_0, \mathfrak{d}_0, \phi_0)$ and $(r_\infty, \mathfrak{d}_\infty, \phi_\infty)$ and a homogeneous in the bi-limit vector field $K : \mathbb{R} \to \mathbb{R}^n$, with associated triples $(r_0, \mathfrak{d}_0, K_0)$ and $(r_\infty, \mathfrak{d}_\infty, K_\infty)$ and a positive definite, proper and C^1 function $U : \mathbb{R}^{2n} \to \mathbb{R}_+$, homogeneous in the bi-limit with associated triples $(r_0, \mathfrak{d}_U, U_0)$ and $(r_\infty, \mathfrak{d}_U, U_\infty)$, such that :*

1. *The functions U_0 and U_∞ are positive definite and proper and the functions $\frac{\partial U}{\partial e_j}$ and $\frac{\partial U}{\partial \hat{x}_j}$*

are homogeneous in the bi-limit with approximating functions respectively $\frac{\partial U_0}{\partial e_j}$, $\frac{\partial U_\infty}{\partial e_j}$, $\frac{\partial U_0}{\partial \hat{x}_j}$, *and* $\frac{\partial U_\infty}{\partial \hat{x}_j}$.

2. *There exist two positive real numbers* c_1 *and* c_2 *such that (5) and (6) are satisfied.*

The proof of this Theorem was omitted in (Andrieu et al., 2008, IEEE-Tac) due to space limitation and is given in Section 3. Section 2 gives some prerequisite needed to address this proof.

2 Some prerequisite

The proof of this Theorem needs some prerequisite. Indeed, we recall the definition of homogeneity in the bi-limit, introduced in (Andrieu et al., 2008, SICON), and give some related properties.

Given a vector $r = (r_1, \ldots, r_n)$ in $(\mathbb{R}_+/\{0\})^n$, we define the dilation of a vector x in $\mathbb{R}^n$ as

$$\lambda^r \diamond x \;=\; (\lambda^{r_1} x_1, \, \ldots, \, \lambda^{r_n} x_n)^T \quad .$$

Definition 1 (Homogeneity in the 0-limit)

- *A continuous function* $\phi \;: \mathbb{R}^n \to \mathbb{R}$ *is said homogeneous in the* 0*-limit with associated triple* $(r_0, \mathfrak{d}_0, \phi_0)$, *where* r_0 *in* $(\mathbb{R}_+/\{0\})^n$ *is the weight,* $\mathfrak{d}_0$ *in* $\mathbb{R}_+$ *the degree and* $\phi_0 \;: \mathbb{R}^n \to \mathbb{R}$ *the approximating function, respectively, if* ϕ_0 *is continuous and not identically zero and, for each compact set* C *in* $\mathbb{R}^n$ *and each* $\varepsilon > 0$, *there exists* λ^* *such that we have :*

$$\max_{x \in C} \left| \frac{\phi(\lambda^{r_0} \diamond x)}{\lambda^{\mathfrak{d}_0}} - \phi_0(x) \right| \;\leq\; \varepsilon \quad \forall \lambda \in (0, \lambda^*].$$

- *A vector field* $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ *is said homogeneous in the* 0*-limit with associated triple* $(r_0, \mathfrak{d}_0, f_0)$, *where* $f_0 = \sum_{i=1}^n f_{0,i} \frac{\partial}{\partial x_i}$, *if, for each* i *in* $\{1, \ldots, n\}$, *the function* f_i *is homogeneous in the* 0*-limit with associated triple* $(r_0, \mathfrak{d}_0 + r_{0,i}, f_{0,i})$[1].

Definition 2 (Homogeneity in the ∞-limit)

- *A continuous function* $\phi \;: \mathbb{R}^n \to \mathbb{R}$ *is said homogeneous in the* ∞*-limit with associated triple* $(r_\infty, \mathfrak{d}_\infty, \phi_\infty)$ *where* r_∞ *in* $(\mathbb{R}_+/\{0\})^n$ *is the weight,* $\mathfrak{d}_\infty$ *in* $\mathbb{R}_+$ *the degree and* $\phi_\infty \;: \mathbb{R}^n \to \mathbb{R}$ *the approximating function, respectively, if* ϕ_∞ *is continuous and not identically zero and, for each compact set* C *in* $\mathbb{R}^n$ *and each* $\varepsilon > 0$, *there exists* λ^* *such that we have :*

$$\max_{x \in C} \left| \frac{\phi(\lambda^{r_\infty} \diamond x)}{\lambda^{\mathfrak{d}_\infty}} - \phi_\infty(x) \right| \;\leq\; \varepsilon \quad \forall \lambda \in [\lambda^*, +\infty) \; .$$

[1] In the case of a vector field the degree $\mathfrak{d}_0$ can be negative as long as $\mathfrak{d}_0 + r_{0,i} \geq 0$, for all $1 \leq i \leq n$.

- *A vector field $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ is said homogeneous in the ∞-limit with associated triple $(r_\infty, \mathfrak{d}_\infty, f_\infty)$, with $f_\infty = \sum_{i=1}^n f_{\infty,i} \frac{\partial}{\partial x_i}$, if, for each i in $\{1, \dots, n\}$, the function f_i is homogeneous in the ∞-limit with associated triple $(r_\infty, \mathfrak{d}_\infty + r_{\infty,i}, f_{\infty,i})$.*

Definition 3 (Homogeneity in the bi-limit)
A continuous function $\phi : \mathbb{R}^n \to \mathbb{R}$ (or a vector field f) is said homogeneous in the bi-limit if it is homogeneous in the 0-limit and homogeneous in the ∞-limit.

The following propositions are proved, or are direct consequences of results, in (Andrieu et al., 2008, SICON).

Proposition 1 *Let η and μ be two continuous homogeneous in the bi-limit functions with weights r_0 and r_∞, degrees $d_{\eta,0}$, $d_{\eta,\infty}$ and $d_{\mu,0}$, $d_{\mu,\infty}$, and continuous approximating functions η_0, η_∞, μ_0, μ_∞.*

1. *The function $x \mapsto \eta(x)\mu(x)$ is homogeneous in the bi-limit with associated triples $(r_0, d_{\eta,0}+ d_{\mu,0}, \eta_0\, \mu_0)$ and $(r_\infty, d_{\eta,\infty} + d_{\mu,\infty}, \eta_\infty\, \mu_\infty)$.*

2. *If the degrees satisfy $d_{\eta,0} \geq d_{\mu,0}$ and $d_{\eta,\infty} \leq d_{\mu,\infty}$ and the functions μ, μ_0 and μ_∞ are positive definite then there exists a positive real number c satisfying :*
$$\eta(x) \ \leq \ c\, \mu(x) \quad , \forall\, x \ \in \ \mathbb{R}^n \ .$$

Proposition 2 *If $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\zeta : \mathbb{R} \to \mathbb{R}$ are homogeneous in the 0-limit functions, with weights $r_{\phi,0}$ and $r_{\zeta,0}$, degrees $d_\phi = r_{\zeta,0}$ and d_ζ in $\mathbb{R}_+$, and approximating functions ϕ_0 and ζ_0, then $\zeta \circ \phi$ is homogeneous in the 0-limit with weight $r_{\phi,0}$, degree d_ζ, and approximating function $\zeta_0 \circ \phi_0$. The same result holds for the cases of homogeneity in the ∞-limit and in the bi-limit.*

Proposition 3 *Suppose η and μ are two functions homogeneous in the bi-limit, with weights r_0 and r_∞, degrees $\mathfrak{d}_0$ and $\mathfrak{d}_\infty$, and such that the approximating functions, denoted η_0 and η_∞, and, μ_0 and μ_∞ are continuous. If $\mu(x) \geq 0$ and*
$$\begin{array}{rcl} \{\ x \in \mathbb{R}^n \setminus \{0\}\ , \quad \mu(x) = 0\ \} & \Rightarrow & \eta(x) > 0\ , \\ \{\ x \in \mathbb{R}^n \setminus \{0\}\ , \quad \mu_0(x) = 0\ \} & \Rightarrow & \eta_0(x) > 0\ , \\ \{\ x \in \mathbb{R}^n \setminus \{0\}\ , \quad \mu_\infty(x) = 0\ \} & \Rightarrow & \eta_\infty(x) > 0\ , \end{array}$$
then there exists a strictly positive real number k^* such that, for all $k \geq k^*$, the functions $\eta(x) + k\,\mu(x)$, $\eta_0(x) + k\,\mu_0(x)$ and $\eta_\infty(x) + k\,\mu_\infty(x)$ are positive definite.

3 Proof of Theorem 1

3.1 Main proof

The proof of this theorem is based on two other results. The first one is associated to the observer part of the output feedback controller and has already been stated in (Andrieu at al., 2008, Hal) in a more general version. The other one is associated to the state-feedback part and constitutes one of the contribution of this paper.

Theorem 2 ((Andrieu et al., 2008, Hal)) *Given $\mathfrak{d}_\infty$ in $[0, \frac{1}{n-1})$, let $\mathfrak{d}_W$ be a positive real number satisfying $\mathfrak{d}_W \geq 2 + \mathfrak{d}_\infty$. There exist a vector field $K \ : \mathbb{R} \to \mathbb{R}^n$ which is homogeneous in the bi-limit with associated weights r_0 and r_∞, and a positive definite, proper and C^1 function $W \ : \mathbb{R}^n \to \mathbb{R}_+$, homogeneous in the bi-limit with associated triples $(r_0, \mathfrak{d}_W, W_0)$ and $(r_\infty, \mathfrak{d}_W, W_\infty)$, such that the following holds.*

1. *The functions W_0 and W_∞ are positive definite and proper and, for each j in $\{1, \ldots, n\}$, the function $\frac{\partial W}{\partial e_j}$ is homogeneous in the bi-limit with approximating functions $\frac{\partial W_0}{\partial e_j}$ and $\frac{\partial W_\infty}{\partial e_j}$.*
2. *There exist two positive real numbers c_1 and c_2 such that we have, for all E in $\mathbb{R}^n$,*

$$\frac{\partial W}{\partial E}(E) \ (\mathcal{S}\, E + K(e_1)) \leq -c_3 \left(W(E) + W(E)^{\frac{d_W + \mathfrak{d}_\infty}{d_W}} \right) \ , \tag{7}$$

$$\frac{\partial W}{\partial E}(E)\, D\, E \ \geq \ c_4\, W(E) \ , \tag{8}$$

The proof of this Theorem has been given in (Andrieu et al., 2008, Hal).

The other Theorem needed to prove the main result can be stated as follows :

Theorem 3 *Given $\mathfrak{d}_\infty$ in $[0, \frac{1}{n-1})$, let $\mathfrak{d}_V$ be a positive real number satisfying $\mathfrak{d}_V \geq 2 + \mathfrak{d}_\infty$. There exist a function $\phi \ : \mathbb{R}^n \to \mathbb{R}$ which is homogeneous in the bi-limit with associated weights r_0 and r_∞ and degrees 1 and $1 + \mathfrak{d}_\infty$, and a positive definite, proper and C^1 function $V \ : \mathbb{R}^n \to \mathbb{R}_+$, homogeneous in the bi-limit with associated triples $(r_0, \mathfrak{d}_V, V_0)$ and $(r_\infty, \mathfrak{d}_V, V_\infty)$, such that the following holds.*

1. *The functions V_0 and V_∞ are positive definite and proper and, for each j in $\{1, \ldots, n\}$, the function $\frac{\partial V}{\partial x_j}$ is homogeneous in the bi-limit with approximating functions $\frac{\partial V_0}{\partial x_j}$ and $\frac{\partial V_\infty}{\partial x_j}$.*
2. *There exist two positive real numbers c_1 and c_2 such that we have, for all $\mathfrak{X}$ in $\mathbb{R}^n$,*

$$\frac{\partial V}{\partial \mathfrak{X}}(\mathfrak{X}) \ (\mathcal{S}\, \mathfrak{X} + \mathcal{B}\, \phi(\mathfrak{X})) \leq -c_5 \left(V(\mathfrak{X}) + V(\mathfrak{X})^{\frac{\mathfrak{d}_V + \mathfrak{d}_\infty}{\mathfrak{d}_V}} \right) \ , \tag{9}$$

$$\frac{\partial V}{\partial \mathfrak{X}}(\mathfrak{X})\, D\, \mathfrak{X} \ \geq \ c_6\, V(\mathfrak{X}) \ , \tag{10}$$

The proof of this Theorem is given in Section 3.2.

With these two theorems in hand the proof follows the line of the one given in (Andrieu et al., 2008, SICON) and is a direct application of Proposition 3. Indeed, consider a positive real number $\mathfrak{d}_\infty$ in $[0, \frac{1}{n-1})$, a diagonal matrix D and a sufficiently big positive real number d_U. Employing Theorem 2 and 3 we get the existence of a homogeneous in the bi-limit vector field

K, three homogeneous in the bi-limit functions ϕ, V and W and four positive real numbers c_3, c_4, c_5 and c_6 such that (7), (8), (9) and (10) are satisfied.

Let $U\ : \mathbb{R}^{2n} \to \mathbb{R}_+$ be the candidate Lyapunov function for the closed loop system (4) defined as :

$$U(\hat{\mathfrak{X}}, E) \ = \ V(\hat{\mathfrak{X}}) \ + \ \mathfrak{c}\, W(E) \ , \tag{11}$$

where $\mathfrak{c}$ is a positive real number to tuned.

Note that this function is homogeneous in the bi-limit with appropriate weights and degrees and with approximating functions

$$\begin{aligned} U_0(\hat{\mathfrak{X}}, E) &= V_0(\hat{\mathfrak{X}}) \ + \ \mathfrak{c}\, W_0(E) \ , \\ U_\infty(\hat{\mathfrak{X}}, E) &= V_\infty(\hat{\mathfrak{X}}) \ + \ \mathfrak{c}\, W_\infty(E) \ . \end{aligned}$$

Moreover, due to the property of V and W exhibited in point 1) of Theorems 2 and 3, for all j in $\{1, \ldots, n\}$, we get that the homogeneous approximating functions of $\frac{\partial U}{\partial \hat{x}_j}$ are $\frac{\partial U_0}{\partial \hat{x}_j}$ and $\frac{\partial U_\infty}{\partial \hat{x}_j}$, while the homogeneous approximating functions of $\frac{\partial U}{\partial e_j}$ are $\frac{\partial U_0}{\partial e_j}$ and $\frac{\partial U_\infty}{\partial e_j}$. Hence point 1) of Theorem 1 is satisfied.

Moreover, along the trajectory of the system (4), the function U satisfies :

$$\dot{\overbrace{U(\hat{\mathfrak{X}}, E)}} \ = \ -\eta(\hat{\mathfrak{X}}, E) \ - \ \mathfrak{c}\, \mu(E) \tag{12}$$

where μ and η are two homogeneous in the bi-limit functions with weights (r_0, r_∞) and degrees $\mathfrak{d}_U$ and $\mathfrak{d}_U + \mathfrak{d}_\infty$ defined as,

$$\begin{aligned} \mu(E) \ &= \ \frac{\partial W}{\partial E}(E)\, (\mathcal{S}\, E \ + \ K(e_1)) \ , \\ \eta(\hat{\mathfrak{X}}, E) \ &= \ \frac{\partial V}{\partial \hat{\mathfrak{X}}}(\hat{\mathfrak{X}}) \left(\mathcal{S}\, \hat{\mathfrak{X}} \ + \ \mathcal{B}\, \phi(\hat{\mathfrak{X}}) \ + \ K(e_1) \right) \ . \end{aligned}$$

With (7), the function μ and its homogeneous approximations μ_0 and μ_∞ are negative definite. Furthermore, with (9), we have for all $\hat{\mathfrak{X}} \neq 0$:

$$\eta(\hat{\mathfrak{X}}, 0) \ < \ 0 \ , \quad \eta_0(\hat{\mathfrak{X}}, 0) \ < \ 0 \ , \quad \eta_\infty(\hat{\mathfrak{X}}, 0) \ < \ 0 \ .$$

Consequently, Proposition 3 yields the existence of a positive real number $\mathfrak{c}^*$ such that for all $\mathfrak{c} \geq \mathfrak{c}^*$ the function $\dot{\overbrace{U(\hat{\mathfrak{X}}, E)}}$ and its homogeneous approximation is negative definite. With Proposition 1, point 2), we conclude that there exists c_1 such that the inequality (5) is satisfied.

Finally, with (8) and (10) we get that the homogeneous in the bi-limit function

$$(\hat{\mathfrak{X}}, E) \mapsto \frac{\partial U}{\partial E}(\hat{\mathfrak{X}}, E)\, D\, E \ + \ \frac{\partial U}{\partial \hat{\mathfrak{X}}}(\hat{\mathfrak{X}}, E)\, D\, \hat{\mathfrak{X}}$$

and its homogeneous approximations are positive definite. Hence, employing Proposition (1) we get a positive real number c_2 such that (6) is satisfied.

3.2 Proof of Theorem 3

The proof we propose here is an adaptation of the one given in (Andrieu et al., 2008, SICON). It is done by induction. To do so we use notations with an index showing the value from which we stop counting. For instance $\mathfrak{X}_i = (x_1, \ldots, x_i)^T$ denotes a state vector in $\mathbb{R}^i$ and $\mathcal{B}_i$ is a vector in $\mathbb{R}^i$ defined as

$$\mathcal{B}_i = (0, \ldots, 0, 1)^T .$$

Also, $\mathcal{S}_i$ is the left shift matrix of dimension i, i.e.

$$\mathcal{S}_i \mathfrak{X}_i = (x_2, \ldots, x_i, 0)^T .$$

Proposition 4 *Let $\mathfrak{d}_V$ be a positive real number satisfying $\mathfrak{d}_V \geq 2 + \mathfrak{d}_\infty$ and $d_1, \ldots, d_i$ be i positive real numbers. Suppose there exist a C^1 homogeneous in the bi-limit function $\phi_i : \mathbb{R}^i \to \mathbb{R}$ of degrees 1 and $r_{\infty,i} + \mathfrak{d}_\infty$ with approximating functions $\phi_{i,0}$ and $\phi_{i,\infty}$, and a positive definite, proper and C^1 function homogeneous in the bi-limit $V_i : \mathbb{R}^i \to \mathbb{R}_+$, with associated triples $(r_0, \mathfrak{d}_V, V_{i,0})$ and $(r_\infty, \mathfrak{d}_V, V_{i,\infty})$ such that the following holds :*

1. *for all j in $\{1, \ldots, i\}$, the function $\frac{\partial \phi_i}{x_j}$ is homogeneous in the bi-limit with approximating functions $\frac{\partial \phi_{i,0}}{x_j}$ and $\frac{\partial \phi_{i,\infty}}{x_j}$*

2. *the function $V_{i,0}$ and $V_{i,\infty}$ are positive definite and proper and for all j in $\{1, \ldots, i\}$, the functions $\frac{\partial V_i}{\partial x_j}$ are homogeneous in the bi-limit with approximating functions $\frac{\partial V_{i,0}}{\partial x_j}$ and $\frac{\partial V_{i,\infty}}{\partial x_j}$.*

3. *There exist positive real numbers $\underline{c}_5$ and $\underline{c}_6$, such that for all $\mathfrak{X}_i$ in $\mathbb{R}^i$:*

$$\sum_{j=1}^{1} d_j \frac{\partial V_i}{\partial x_j}(\mathfrak{X}_i)\, x_j \;\geq\; \underline{c}_5\, V_i(\mathfrak{X}_i) \;, \tag{13}$$

$$\frac{\partial V_i}{\partial \mathfrak{X}_i}(\mathfrak{X}_i)\,(\mathcal{S}_i \mathfrak{X}_i + \mathcal{B}_i\, \phi_i(\mathfrak{X}_i)) \leq -\underline{c}_6 \left(V_i(\mathfrak{X}_i) + V_i(\mathfrak{X}_i)^{\frac{\mathfrak{d}_V + \mathfrak{d}_\infty}{\mathfrak{d}_V}} \right) \tag{14}$$

Then, for any positive real number d_{i+1}, there exist a C^1 homogeneous in the bi-limit function $\phi_{i+1} : \mathbb{R}^{i+1} \to \mathbb{R}$, with degree 1 and $r_{\infty,i+1} + \mathfrak{d}_\infty$ and a positive definite, proper and C^1 function $V_{i+1} : \mathbb{R}^{i+1} \to \mathbb{R}_+$ homogeneous in the bi-limit with associated triples $(r_0, \mathfrak{d}_V, V_{i+1,0})$ and $(r_\infty, \mathfrak{d}_V, V_{i+1,\infty})$ such that the following holds :

1. *for all j in $\{1, \ldots, i+1\}$, the function $\frac{\partial \phi_{i+1}}{x_j}$ is homogeneous in the bi-limit with approximating functions $\frac{\partial \phi_{i+1,0}}{x_j}$ and $\frac{\partial \phi_{i+1,\infty}}{x_j}$.*

2. *The functions $V_{i+1,0}$ and $V_{i+1,\infty}$ are positive definite and proper and for all j in $\{1, \ldots, i+1\}$, the functions $\frac{\partial V_{i+1}}{\partial x_j}$ are homogeneous in the bi-limit with approximating functions $\frac{\partial V_{i+1,0}}{\partial x_j}$ and $\frac{\partial V_{i+1,\infty}}{\partial x_j}$.*

3. There exists a positive real number $\bar{c}_5$ and $\bar{c}_6$ such that for all $\mathfrak{X}_{i+1}$ in $\mathbb{R}^{i+1}$,

$$\sum_{j=1}^{i+1} \mathfrak{b}_j \, \frac{\partial V_{i+1}}{\partial x_j}(\mathfrak{X}_{i+1}) \, x_j \;\geq\; \bar{c}_5 \, V_{i+1}(\mathfrak{X}_{i+1}) \; . \tag{15}$$

$$\frac{\partial V_{i+1}}{\partial \mathfrak{X}_{i+1}}(\mathfrak{X}_{i+1}) \; (\mathcal{S}_{i+1}\mathfrak{X}_{i+1} + \mathcal{B}_{i+1}\phi_{i+1}(\mathfrak{X}_{i+1})) \leq -\bar{c}_6 \left(V_{i+1}(\mathfrak{X}_{i+1}) + V_{i+1}(\mathfrak{X}_{i+1})^{\frac{\mathfrak{d}_V + \mathfrak{d}_\infty}{\mathfrak{d}_V}} \right) \tag{16}$$

Proof : The proof is divided in three steps.

1. Construction of the Lyapunov function. Consider the candidate Lyapunov function $V_{i+1} \; : \mathbb{R}^{i+1} \to \mathbb{R}_+$ defined by :

$$V_{i+1}(\mathfrak{X}_{i+1}) \;=\; V_i(\mathfrak{X}_i) \;+\; \sigma \, L(\mathfrak{X}_{i+1}) \; ,$$

where σ is a positive real number which will be specified later and L is the function defined as :

$$L(\mathfrak{X}_{i+1}) \;=\; |\phi_i(\mathfrak{X}_i) - x_{i+1}|^{\mathfrak{d}_V} \;+\; |\phi_i(\mathfrak{X}_i) - x_{i+1}|^{\frac{\mathfrak{d}_V}{r_{\infty,i+1}}} \; .$$

The function V_{i+1} is positive definite and proper. Moreover, the function $\phi_i(\mathfrak{X}_i)$ being C^1 and since $\mathfrak{d}_V \geq 2$, it yields that the function V_{i+1} is C^1 . Finally, for each j in $\{1, \ldots, i+1\}$, the function $\frac{\partial V_{i+1}}{\partial x_j}$ is homogeneous in the bi-limit with associated triples $\left(r_0, \mathfrak{d}_V - 1, \frac{\partial V_{i+1,0}}{\partial x_j}\right)$, $\left(r_\infty, d_V - r_{\infty,j}, \frac{\partial V_{i+1,\infty}}{\partial x_j}\right)$. Consequently Point 2) of Proposition 4 is satisfied.

2. Properties of the Lyapunov function. Let d_{i+1} be any positive real number. Let $J \; : \mathbb{R}^{i+1} \to \mathbb{R}$ be the function defined as :

$$J(\mathfrak{X}_i, x_{i+1}) \;=\; \sum_{j=1}^{i+1} d_j \, \frac{\partial L}{\partial x_j}(\mathfrak{X}_{i+1}) \, x_j \; .$$

The functions V_i and J are homogeneous in the bi-limit with associated weights 1 and r_∞, and unique degree $\mathfrak{d}_V$. By assumption V_i is positive definite and the same holds for its homogeneous approximations in the 0-limit and in the ∞-limit and we have :

$$\begin{aligned} J(0, x_{i+1}) \;&=\; d_{i+1} \left[\mathfrak{d}_V |x_{i+1}|^{\mathfrak{d}_V} + \tfrac{\mathfrak{d}_V}{r_{\infty,i}} |x_{i+1}|^{\frac{\mathfrak{d}_V}{r_{\infty,i}}} \right] \; , \\ &>\; 0 \quad \forall x_{i+1} \neq 0 \; . \end{aligned}$$

It follows that the assumptions of Proposition 3 are satisfied with $\mu = V_i$ and $\eta = J$. Hence, with $\underline{c}_5$ given in (13), there exists a positive real number σ such that the functions $\underline{c}_5 V_{i,0} + \sigma J_0$, $\underline{c}_5 V_{i,\infty} + \sigma J_\infty$ and $\underline{c}_5 V_i + \sigma J$ are continuous and positive definite in $\mathfrak{X}_{i+1}$. But then, from Proposition 1.2, there exists a positive real number $\bar{c}_5$ satisfying :

$$\tfrac{1}{\bar{c}_5} \left[\underline{c}_5 V_i + \sigma J \right] \;\geq\; V_{i+1} \; .$$

Since assumption (13) gives readily, for all $\mathfrak{X}_{i+1}$ in $\mathbb{R}^{i+1}$,

$$\sum_{j=1}^{i+1} d_j \, \frac{\partial V_i}{\partial x_j}(\mathfrak{X}_{i+1}) \, x_j \geq \; \underline{c}_5 \, V_i(\mathfrak{X}_i) \;+\; \sigma \, J(\mathfrak{X}_i, x_{i+1}) \; ,$$

we have established inequality (15) of Proposition 4.

3. Construction of the function ϕ_{i+1}. Given a real number σ, we define the homogeneous in the bi-limit function ϕ_{i+1} as :

$$\phi_{i+1}(\mathfrak{X}_i) \;=\; -k\left[(x_{i+1}-\phi_i(\mathfrak{X}_i)) \;+\; (x_{i+1}-\phi_i(\mathfrak{X}_i))^{\frac{r_{\infty,i}+\mathfrak{d}_\infty}{r_{\infty,i}}}\right] \;,$$

With Propositions 1 and 2 it is a homogeneous in the bi-limit function with degrees 1 and $r_{\infty,i}+\mathfrak{d}_\infty$.

Moreover, with Proposition 2 and the properties of the function ϕ_i, for all j in $\{1,\ldots,i+1\}$, the function $\frac{\partial\phi_{i+1}}{x_j}$ is homogeneous in the bi-limit with approximating functions $\frac{\partial\phi_{i+1,0}}{x_j}$ and $\frac{\partial\phi_{i+1,\infty}}{x_j}$. Hence point 1) of the Theorem is satisfied.

We show now that by selecting k large enough we can satisfy (16). Note that :

$$\frac{\partial V_{i+1}}{\partial \mathfrak{X}_{i+1}}(\mathfrak{X}_{i+1})\;[\mathcal{S}_{i+1}\,\mathfrak{X}_{i+1}+\mathcal{B}_{i+1}\,\phi_{i+1}(\mathfrak{X}_{i+1})] \;=\; T_1(\mathfrak{X}_{i+1})-k\,T_2(\mathfrak{X}_{i+1})\;,$$

with the functions T_1 and T_2 defined as :

$$T_1(\mathfrak{X}_{i+1}) \;=\; \frac{\partial V_{i+1}}{\partial \mathfrak{X}_i}(\mathfrak{X}_{i+1})\;[\mathcal{S}_i\,\mathfrak{X}_i + B_i x_{i+1})]$$

$$T_2(\mathfrak{X}_{i+1}) \;=\; \sigma\frac{\partial L}{\partial x_{i+1}}(\mathfrak{X}_{i+1})\phi_{i+1}(\mathfrak{X}_{i+1})\;,$$

where,

$$\frac{\partial L}{\partial x_{i+1}}(\mathfrak{X}_{i+1}) \;=\; \left(\mathfrak{d}_V\,(x_{i+1}\;-\;\phi_i(\mathfrak{X}_i))^{\mathfrak{d}_V-1}\;+\;\frac{\mathfrak{d}_V}{r_{\infty,i+1}}\,(x_{i+1}\;-\;\phi_i(\mathfrak{X}_i))^{\frac{\mathfrak{d}_V-r_{\infty,i+1}}{r_{\infty,i+1}}}\right)\phi_{i+1}(\mathfrak{X}_{i+1})\;.$$

By definition of homogeneity in the bi-limit and Proposition 2, T_1 and T_2 are homogeneous in the bi-limit with weights r_0 and r_∞, and degrees $\mathfrak{d}_V$. Moreover, since $\phi_{i+1}(\mathfrak{X}_{i+1})$ has the same sign as $x_{i+1}-\phi_i(\mathfrak{X}_i)$, $T_2(\mathfrak{X}_{i+1})$ is non-negative for all $\mathfrak{X}_{i+1}$ in $\mathbb{R}^{i+1}$ and as $\phi_{i+1}(\mathfrak{X}_{i+1})=0$ only if $x_{i+1}-\phi_i(\mathfrak{X}_i)=0$ we get :

$$T_2(\mathfrak{X}_{i+1})=0 \qquad\Longrightarrow\qquad x_{i+1}\;=\;\phi_i(\mathfrak{X}_i)\;.$$

and,

$$x_{i+1}\;=\;\phi_i(\mathfrak{X}_i) \quad\Longrightarrow\quad T_1(\mathfrak{X}_{i+1})\;=\;\frac{\partial V_i}{\partial \mathfrak{X}_i}(\mathfrak{X}_i)\,[\mathcal{S}_i\,\mathfrak{X}_i+B_i\phi_i(\mathfrak{X}_i)]\;.$$

Consequently, equations (14) yield :

$$\left\{\mathfrak{X}_{i+1}\in\mathbb{R}^{i+1}\setminus\{0\}\;:\;T_2(\mathfrak{X}_{i+1})=0\right\} \quad\subseteq\quad \left\{\mathfrak{X}_{i+1}\in\mathbb{R}^{i+1}\;:\;T_1(\mathfrak{X}_{i+1})<0\right\}\;.$$

The same implication holds for the homogeneous approximations of the two functions at infinity and around the origin. Hence, by Proposition 3, there exists $k^*>0$ such that, for all $k\geq k^*$, we have for all $\mathfrak{X}_{i+1}\neq 0$:

$$\begin{aligned}
\frac{\partial V_{i+1}}{\partial \mathfrak{X}_{i+1}}(\mathfrak{X}_{i+1})\;[\mathcal{S}_{i+1}\,\mathfrak{X}_{i+1}+B_{i+1}\phi_{i+1}(\mathfrak{X}_{i+1})] &\;<\; 0\;,\\
\frac{\partial V_{i+1,0}}{\partial \mathfrak{X}_{i+1}}(\mathfrak{X}_{i+1})\;[\mathcal{S}_{i+1}\,\mathfrak{X}_{i+1}+B_{i+1}\phi_{i+1,0}(\mathfrak{X}_{i+1})] &\;<\; 0\;,\\
\frac{\partial V_{i+1,\infty}}{\partial \mathfrak{X}_{i+1}}(\mathfrak{X}_{i+1})\,[\mathcal{S}_{i+1}\,\mathfrak{X}_{i+1}+B_{i+1}\phi_{i+1,\infty}(\mathfrak{X}_{i+1})] &\;<\; 0\;.
\end{aligned}$$

Employing Proposition 1, this implies that there exists $\bar{c}_6$ such that (16) is satisfied. Hence Proposition 4 is proved. □

To construct the Lyapunov function V and the vector field ϕ, which prove Theorem 3, it is sufficient to iterate the construction proposed in Proposition 4 starting from

$$\phi_1(x_1) = -x_1 - -x_1^{\frac{r_{\infty,2}}{r_{\infty,1}}} \ , \quad V_1(x_1) = |x_1|^{\mathfrak{d}_V} \ ,$$

Note that with these datas, Assumptions of Proposition 4 are satisfied.

We apply this Proposition recursively for i ranging from 1 to $n-1$.

4 Conclusion

In this short paper we have given a proof to a result stated in (Andrieu et al., 2008, IEEE-Tac).

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