Unsolvable one-dimensional lifting problems for congruence lattices of lattices

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UNIVERSAL LIFTING PROBLEMS FOR
CONGRUENCE LATTICES OF LATTICES

JIŘÍ TŮMA AND FRIEDRICH WEHRUNG

Abstract. Let \( S \) be a distributive \( \{ \lor, 0 \} \)-semilattice. In a previous paper, the second author proved the following result:

\[
\text{Suppose that } S \text{ is a lattice. Let } K \text{ be a lattice, let } \varphi : \text{Con}_c K \to S \text{ be a } \{ \lor, 0 \} \text{-homomorphism. Then } \varphi \text{ is, up to isomorphism, of the form } \text{Con}_c f, \text{ for a lattice } L \text{ and a lattice homomorphism } f : K \to L.
\]

In the statement above, \( \text{Con}_c K \) denotes as usual the \( \{ \lor, 0 \} \)-semilattice of all finitely generated congruences of \( K \).

We prove here that this statement characterizes \( S \) being a lattice.

Introduction

The Congruence Lattice Problem (CLP in short) asks whether for any distributive \( \{ \lor, 0 \} \)-semilattice \( S \), there exists a lattice \( L \) such that \( \text{Con}_c L \cong S \). While this problem is still unsolved, many related problems have been solved. Among these, we mention the following, due to G. Grätzer and E.T. Schmidt, see [4, 5], and also [6] for a survey about this and related problems.

Theorem 1. Let \( S \) be a finite distributive \( \{ \lor, 0 \} \)-semilattice, let \( K \) be a finite lattice, let \( \varphi : \text{Con}_c K \to S \) be a \( \{ \lor, 0 \} \)-homomorphism. Then there are a finite lattice \( L \), a lattice homomorphism \( f : K \to L \), and an isomorphism \( \alpha : \text{Con}_c L \to S \) such that \( \alpha \circ \text{Con}_c f = \varphi \).

In the statement of Theorem 1, \( \text{Con}_c f \) denotes the map from \( \text{Con}_c K \) to \( \text{Con}_c L \) that with any congruence \( \alpha \) of \( K \) associates the congruence of \( L \) generated by all the pairs \( \langle f(x), f(y) \rangle \) where \( \langle x, y \rangle \in \alpha \).

In [10], the second author proves that provided that \( S \) is a lattice, all finiteness assumptions in Theorem 1 can be dropped, that is:

Theorem 2. Let \( S \) be a distributive lattice with zero, let \( K \) be a lattice, let \( \varphi : \text{Con}_c K \to S \) be a \( \{ \lor, 0 \} \)-homomorphism. Then \( \varphi \) can be “lifted”, that is, there are a lattice \( L \), a lattice homomorphism \( f : K \to L \), and an isomorphism \( \alpha : \text{Con}_c L \to S \) such that \( \alpha \circ \text{Con}_c f = \varphi \).

In the result of Theorem 2, instead of lifting a distributive \( \{ \lor, 0 \} \)-semilattice \( S \) (with respect to the \( \text{Con}_c \) functor), we lift a \( \{ \lor, 0 \} \)-homomorphism \( \varphi : \text{Con}_c K \to S \).
S. For this reason, we shall call such a statement “one-dimensional Congruence Lattice Problem”, in short 1-CLP. With this terminology, the usual CLP would have to be called 0-CLP. By replacing $K$ by a truncated $n$-dimensional cube (diagram) of lattices, we can define the $n$-CLP, for any positive integer $n$. It turns out that this problem is interesting only for $n \in \{0, 1, 2\}$. Indeed, it follows from [8] that the 3-CLP holds only for trivial $S$—but much more is proved in [8], while the result about 3-CLP follows from a trivial (and unpublished) example of the second author. The 2-CLP is another matter (far less trivial than 3-CLP but still far easier than 1-CLP), which will be considered elsewhere.

Our main result (see Theorem A) states that for a given distributive $\{\lor, 0\}$-semilattice $S$, Theorem 2 characterizes $S$ being a lattice. This solves also a problem formulated by H. Dobbertin in the (yet unpublished) monograph [2], see Corollary 1.4. In fact, our approach is inspired by Dobbertin’s solution for the particular case of his own problem where $S$ is primely generated, see Theorem 15 in [1]. It gives, for a distributive $\{\lor, 0\}$-semilattice $S$ that is not a lattice, the construction of a Boolean algebra $B$ of size at most $2^{2^{|S|}}$ and a $\{\lor, 0\}$-homomorphism $\varphi$: $\text{Con}_\omega B \to S$ that cannot be “lifted” as in Theorem 2.

Even in the particular case where $S = D$, the simplest distributive $\{\lor, 0\}$-semilattice that is not a lattice, see Section 2, it has been an open problem, stated at the end of Section 1 in [1], whether the size of $B$ can be reduced from $2^{61}$ to $8_1$ (without the Continuum Hypothesis). We solve this affirmatively in Theorem B. This also gives us that there are a Boolean algebra $B$ of size $8_1$ and a $\{\lor, 0\}$-homomorphism $\varphi$: $\text{Con}_\omega B \to D$ that cannot be lifted, see Corollary 2.4.

We use standard notation and terminology. For a partially ordered set $(P, \leq)$ and for $a \in P$, we put

$$(a) = \{x \in P \mid x \leq a\}.$$  

We denote by $\omega$ the set of all natural numbers, and by $\omega_1$ the first uncountable ordinal.

1. Characterization of distributive $\{\lor, 0\}$-semilattices with 1-CLP

The main lemma of this section is the following.

**Lemma 1.1.** Let $S$ be a distributive $\{\lor, 0\}$-semilattice, let $a_0, a_1 \in S$ be such that the set $Q = \{a_0\} \cup \{a_1\}$ has no largest element.

There are a Boolean algebra $B$ and a $\{\lor, 0\}$-homomorphism $\mu: B \to S$ such that the following holds:

(a) $\mu(1) = a_0 \lor a_1$;

(b) there are no maps $\mu_0, \mu_1: B \to S$ that satisfy the following properties:

(i) $\mu(x) = \mu_0(x) \lor \mu_1(x)$, for all $x \in B$,

(ii) $\mu_0$ and $\mu_1$ are order-preserving,

(iii) $\mu_\ell(1) \leq a_\ell$, for all $\ell < 2$.

**Proof.** Let $\kappa$ be the minimum size of a cofinal subset of $Q$, and pick a cofinal subset $\{x_\xi \mid \xi < \kappa\}$ of $Q$. So $\kappa$ is an infinite cardinal. We define recursively a map $f: \kappa \to \kappa$ by the rule

$$f(\alpha) = \min\{\xi < \kappa \mid x_\xi \notin \text{Id}\{x_{f(\beta)} \mid \beta < \alpha\}\}$$  \hspace{1cm} (1.1)

for all $\alpha < \kappa$, where $\text{Id} X$ denotes the ideal of $S$ generated by a subset $X$ of $S$. Let $\beta < \alpha$. Then, by (1.1), $x_{f(\alpha)} \notin \text{Id}\{x_{f(\gamma)} \mid \gamma < \alpha\}$, so $f(\alpha) \neq f(\beta)$. Moreover,
\(x_{f(\alpha)} \notin \text{Id}\{ x_{f(\gamma)} \mid \gamma < \beta \}\), so \(f(\beta) \leq f(\alpha)\), whence \(f(\beta) < f(\alpha)\). So \(f\) is strictly increasing.

For \(\alpha < \kappa\), we put \(q_\alpha = x_{f(\alpha)}\) and \(Q_\alpha = \text{Id}\{ q_\beta \mid \beta < \alpha \}\). By (1.1), \(q_\alpha \notin Q_\alpha\) for all \(\alpha < \kappa\). Furthermore, all the sets \(Q_\alpha\) are ideals of \(Q\) and \(Q_\alpha \subset Q_\beta\) whenever \(\alpha < \beta\). Finally, for \(\alpha < \beta\), the relation \(q_\alpha \in Q_\beta\) holds. (Otherwise \(x_{f(\alpha)} \notin Q_\beta = \text{Id}\{ x_{f(\gamma)} \mid \gamma < \beta \}\), thus, by (1.1), \(f(\beta) \leq f(\alpha)\), a contradiction since \(f\) is strictly increasing.) Hence \(\bigcup_{\alpha < \kappa} Q_\alpha = Q\).

For \(x \in Q\), we denote by \(\|x\|\) the least \(\alpha < \kappa\) such that \(x \in Q_\alpha\). Observe that the following obvious properties hold:

\[
\|q_\alpha\| = \alpha + 1, \quad \text{for all } \alpha < \kappa, \quad (1.2)
\]

\[
\|x \vee y\| = \|x\| \vee \|y\|, \quad \text{for all } x, y \in Q. \quad (1.3)
\]

Now pick a partition \(\kappa = \bigcup_{\alpha < \kappa} Z_\alpha\) of \(\kappa\) into sets \(Z_\alpha\) such that \(|Z_\alpha| = \kappa\) for all \(\alpha < \kappa\). Define ideals \(I, I_0, I_1\) of the Boolean algebra \(B = P(\kappa)\) as follows:

\[
I = \{ X \subseteq \kappa \mid X \text{ finite} \},
\]

\[
I_0 = \text{ideal of } B \text{ generated by } \{ Z_\alpha \mid \alpha < \kappa \},
\]

\[
I_1 = \{ X \subseteq \kappa \mid X \cap Z_\alpha \text{ is finite for every } \alpha < \kappa \}.
\]

It is obvious that \(I = I_0 \cap I_1\), and that \(\kappa \notin I_0 \cup I_1\). We define a map \(\mu: B \to S\) by the following rule:

\[
\mu(X) = \begin{cases} 
\bigvee_{\alpha < X} q_\alpha, & \text{if } X \text{ is finite,} \\
\alpha_\ell, & \text{if } X \in I_\ell \setminus I, \text{ for } \ell < 2, \\
\alpha_0 \vee \alpha_1, & \text{if } X \notin I_0 \cup I_1.
\end{cases}
\]

So \(\mu\) is a \(\{ \vee, 0 \}\)-homomorphism from \(B\) to \(S\) with \(\mu(1) = \alpha_0 \vee \alpha_1\).

Now suppose that \(\mu_0, \mu_1: B \to S\) satisfy (i)–(iii) above. For \(\alpha < \kappa\), \(\mu_1(Z_\alpha) \leq \mu(Z_\alpha) = \alpha_0\) (because \(Z_\alpha \in I_0 \setminus I\)), and \(\mu_1(Z_\alpha) \leq \mu_1(\kappa) \leq \alpha_1\) (by the assumption (iii)), hence \(\mu_1(Z_\alpha) \in Q\). Hence, since \(Z_\alpha\) is a cofinal subset of \(\kappa\), there exists \(\xi_\alpha \in Z_\alpha\) such that \(\alpha \vee \|\mu_1(Z_\alpha)\| \leq \xi_\alpha\). We put \(Z = \{ \xi_\alpha \mid \alpha < \kappa \}\). Observe that \(Z \in I_1 \setminus I\), hence \(\mu(Z) = \alpha_1\). So \(\mu_0(Z) \leq \mu(Z) = \alpha_1\) on the one hand, and \(\mu_0(Z) \leq \mu_0(\kappa) = \alpha_0\) on the other hand, thus \(\mu_0(Z) \in Q\). Put \(\beta = \|\mu_0(Z)\|\). Then

\[
\xi_\beta + 1 = \|q_{\xi_\beta}\| \quad \text{(by (1.2))}
\]

\[
= \|\mu(\{ \xi_\beta \})\| \quad \text{(by the definition of } \mu \text{)}
\]

\[
= \|\mu_0(\{ \xi_\beta \})\| \vee \|\mu_1(\{ \xi_\beta \})\| \quad \text{(by (i) and (1.3))}
\]

\[
\leq \|\mu_0(Z)\| \vee \|\mu_1(Z_\beta)\| \quad \text{(by (iii))}
\]

\[
= \beta \vee \|\mu_1(Z_\beta)\|
\]

\[
\leq \xi_\beta,
\]

a contradiction. \(\square\)

In order to formulate Corollary 1.3, we recall the following definition, used in particular in [9]. It generalizes the classical definition of a weakly distributive homomorphism presented in [7].

**Definition 1.2.** Let \(S\) and \(T\) be join-semilattices, let \(a \in S\). A join-homomorphism \(\mu: S \to T\) is **weakly distributive** at \(a\), if for all \(b_0, b_1 \in T\) such that \(\mu(a) = b_0 \lor b_1\), there are \(a_0, a_1 \in S\) such that \(a = a_0 \lor a_1\) and \(\mu(a_\ell) \leq b_\ell\) for all \(\ell < 2\).
Corollary 1.3. Let $S$ be a $\{\lor, 0\}$-semilattice that is not a lattice. There exist a Boolean algebra $B$ and a $\{\lor, 0\}$-homomorphism $\varphi : \text{Con}_c B \to S$ such that there are no lattice $L$, no lattice homomorphism $f : B \to L$ and no $\{\lor, 0\}$-homomorphism $\alpha : \text{Con}_c L \to S$ that satisfy the following properties:

(i) $\alpha$ is weakly distributive at $\Theta_L(f(0_B), f(1_B))$.

(ii) $\varphi = \alpha \circ \text{Con}_c f$.

Proof. By assumption, there exist $a_0, a_1 \in S$ such that $Q = (a_0] \cap (a_1]$ has no largest element. We consider $B, \mu$ as in Lemma 1.1. Since the lattice $B$ is Boolean, the rule $x \mapsto \Theta_B(0_B, x)$ defines an isomorphism $\pi : B \to \text{Con}_c B$. We put $\varphi = \mu \circ \pi^{-1}$.

So suppose that $L, f$, and $\alpha$ are as above. Observe that $\alpha \Theta_L(f(0_B), f(1_B)) = \alpha \circ (\text{Con}_c f)(\Theta_B(0_B, 1_B)) = \varphi \Theta_B(0_B, 1_B) = \mu(1_B) = a_0 \lor a_1$, thus, since $\alpha$ is weakly distributive at $\Theta_L(f(0_B), f(1_B))$, there are $\Psi_0, \Psi_1 \in \text{Con}_c L$ such that $\Psi_0 \lor \Psi_1 = \Theta_L(f(0_B), f(1_B))$ and $\alpha(\Psi_\ell) \leq a_\ell$, for all $\ell < 2$. Thus there are positive integers $n$ and a decomposition

$$f(0_B) = t_0 \leq t_1 \leq \cdots \leq t_{2n} = f(1_B) \quad (1.4)$$

in $L$ such that the relations

$$t_{2i} \equiv t_{2i+1} \pmod{\Psi_0},$$

$$t_{2i+1} \equiv t_{2i+2} \pmod{\Psi_1}$$

hold for all $i < n$. For $x \in B$, we put

$$\mu_0(x) = \bigvee_{i < n} \alpha \Theta_L(t_{2i} \land f(x), t_{2i+1} \land f(x)),$$

$$\mu_1(x) = \bigvee_{i < n} \alpha \Theta_L(t_{2i+1} \land f(x), t_{2i+2} \land f(x)).$$

We verify that conditions (i)–(iii) of Lemma 1.1 are satisfied, thus causing a contradiction.

Condition (i). For $x \in B$, we get

$$\mu_0(x) \lor \mu_1(x) = \bigvee_{i < 2n} \alpha \Theta_L(t_i \land f(x), t_{i+1} \land f(x))$$

$$= \alpha \Theta_L(f(0_B) \land f(x), f(1_B) \land f(x)) \quad \text{(by (1.4))}$$

$$= \alpha \Theta_L(f(0_B), f(x))$$

$$= \varphi(\Theta_B(0, x))$$

$$= \mu(x).$$

Condition (ii). For $x \leq y$ and $i < n$, the relation

$$\Theta_L(t_{2i} \land f(x), t_{2i+1} \land f(x)) \leq \Theta_L(t_{2i} \land f(y), t_{2i+1} \land f(y))$$

holds (because $f(x) \leq f(y)$), thus $\mu_0(x) \leq \mu_0(y)$. So $\mu_0$ is order-preserving. The proof that $\mu_1$ is order-preserving is similar.

Condition (iii). For $i < n$, $\Theta_L(t_{2i}, t_{2i+1}) \subseteq \Psi_0$, thus $\alpha \Theta_L(t_{2i}, t_{2i+1}) \leq \alpha(\Psi_0) \leq a_0$, whence $\mu_0(1) = \bigvee_{i < n} \alpha \Theta_L(t_{2i}, t_{2i+1}) \leq a_0$. Similarly, $\mu_1(1) \leq a_1$.

This contradicts, by Lemma 1.1, the existence of $L, f$, and $\alpha$. \qed
Theorem A. Let $S$ be a distributive $\{\lor, 0\}$-semilattice. Then the following are equivalent:

(i) For any lattice $K$ and any $\{\lor, 0\}$-homomorphism $\varphi: \text{Con}_c K \to S$, there are a lattice $L$, a lattice homomorphism $f: K \to L$, and an isomorphism $\alpha: \text{Con}_c L \to S$ such that $\varphi = \alpha \circ \text{Con}_c f$.

(ii) $S$ is a lattice.

Proof. (ii)$\Rightarrow$(i) follows from Theorem C in [10].

(i)$\Rightarrow$(ii) is a particular case of Corollary 1.3. □

With the terminology mentioned in the Introduction, this proves that 1-CLP holds at $S$ iff $S$ is a lattice, for any distributive $\{\lor, 0\}$-semilattice.

We also mention the following immediate consequence of Corollary 1.3, that solves (positively) the problem, stated by Dobbertin in [2], whether “strongly measurable semilattices are lattices”:

Corollary 1.4. Let $S$ be a distributive $\{\lor, 0\}$-semilattice. Then the following are equivalent:

(i) For any Boolean algebra $B$, any $\{\lor, 0\}$-homomorphism $\mu: B \to S$, and any $a_0, a_1 \in S$ such that $\mu(1_B) = a_0 \lor a_1$, there are $\{\lor, 0\}$-homomorphisms $\mu_0, \mu_1: B \to S$ such that $\mu = \mu_0 \lor \mu_1$ and $\mu_\ell(1_B) = a_\ell$, for all $\ell < 2$.

(ii) $S$ is a lattice.

Proof. (ii)$\Rightarrow$(i) is proved in Corollary 10 of [1], see also [2].

(i)$\Rightarrow$(ii) follows immediately from Corollary 1.3. □

2. A counterexample of size $\aleph_1$

Throughout this section, we shall denote by $D$ the $\{\lor, 0\}$-semilattice defined as $D = \omega \cup \{a_0, a_1, \infty\}$, with $\omega$ a $\{\lor, 0\}$-subsemilattice of $D$, $\omega < a_\ell < \infty$ for all $\ell < 2$, and $\infty = a_0 \lor a_1$, see Figure 1.

![Figure 1. The semilattice D](image)

Now we shall construct a Boolean algebra $B$. By Cantor’s Theorem, $\aleph_1 \leq 2^{\aleph_0}$, thus there exists a one-to-one map $f: \omega_1 \to \mathcal{P}(\omega)$ (where $\mathcal{P}(\omega)$ denotes the powerset of $\omega$). We define a map $g: \omega_1 \times \omega_1 \to \omega$ by the rule

$$g(\xi, \eta) = \begin{cases} \text{least } n < \omega \text{ such that } f(\xi) \cap (n + 1) \neq f(\eta) \cap (n + 1), & \text{if } \xi \neq \eta, \\ 0, & \text{if } \xi = \eta. \end{cases}$$
Lemma 2.1. Let $n < \omega$, let $X$ be a subset of $\omega_1$. If $g(\xi, \eta) < n$ for all $\xi, \eta \in X$, then $|X| \leq 2^n$.

Proof. Let $p$ be the map from $X$ to $\mathcal{P}(n)$ defined by the rule

$$p(\xi) = f(\xi) \cap n, \quad \text{for all } \xi \in X.$$ 

(We identify $n$ with $\{0, 1, \ldots, n - 1\}$.) If $|X| > 2^n$, then there are $\xi, \eta \in X$ such that $\xi \neq \eta$ and $p(\xi) = p(\eta)$. Hence $g(\xi, \eta) \geq n$, by the definition of $g$, a contradiction.

Definition 2.2. We denote by $B$ the Boolean algebra defined by generators $u_{0,\xi}$ and $u_{1,\xi}$, for $\xi < \omega_1$, and $v_n$, for $n < \omega$, and the following relations:

$$u_{0,\xi} \land u_{1,\xi} \leq v_{g(\xi, \eta)}, \quad \text{for all } \xi, \eta < \omega_1.$$ 

(2.1)

Furthermore, we put $w_n = \bigvee_{k \leq n} v_k$, for all $n < \omega$.

Lemma 2.3. $u_{0,\xi} \land u_{1,\eta} \leq w_n$ iff $g(\xi, \eta) \leq n$, for all $\xi, \eta < \omega_1$ and all $n < \omega$.

Proof. If $g(\xi, \eta) \leq n$, then $u_{0,\xi} \land u_{1,\eta} \leq w_n$ by (2.1).

Conversely, suppose that $u_{0,\xi} \land u_{1,\eta} \leq w_n$. We define elements $u_{0,\xi}', u_{1,\eta}',$ and $v_k$ of the two-element Boolean algebra $2$, for $\xi', \eta' < \omega_1$ and $k < \omega$, as follows:

$$u_{0,\xi}' = u_{1,\eta}' = 1; \quad (2.2)$$

$$u_{0,\xi}' = 0, \quad \text{for all } \xi' < \omega_1 \text{ such that } \xi' \neq \xi; \quad (2.3)$$

$$u_{1,\eta}' = 0, \quad \text{for all } \eta' < \omega_1 \text{ such that } \eta' \neq \eta; \quad (2.4)$$

$$v_{g(\xi, \eta)} = 1; \quad (2.5)$$

$$v_k = 0, \quad \text{for all } k < \omega \text{ such that } k \neq g(\xi, \eta). \quad (2.6)$$

Let $\xi', \eta' < \omega_1$. If $\xi' = \xi$ and $\eta' = \eta$, then $u_{0,\xi}' \land u_{1,\eta}' = 1 - v_{g(\xi, \eta)}$. Otherwise, $u_{0,\xi}' \land u_{1,\eta}' = 0 \leq v_{g(\xi', \eta')}$. So the elements $u_{0,\xi}', u_{1,\eta}',$ and $v_k$, for $\xi', \eta' < \omega_1$ and $k < \omega$, verify the inequalities (2.1). Therefore, there exists a homomorphism of Boolean algebras $\varphi: B \to 2$ such that

$$\varphi(u_{\ell,\xi'}) = u_{0,\xi}', \quad \text{for all } \xi' < \omega_1 \text{ and } \ell < 2,$$

$$\varphi(v_k) = v_k, \quad \text{for all } k < \omega.$$ 

In particular, by assumption, $u_{0,\xi}' \land u_{1,\eta}' \leq \bigvee_{k \leq n} v_k$, that is, $\bigvee_{k \leq n} v_k = 1$. Therefore, by (2.6), $g(\xi, \eta) \leq n$.

Theorem B. There exist a Boolean algebra $B$ of size $\mathfrak{c}$ and a $\{\lor, 0\}$-homomorphism $\mu: B \to D$ such that the following holds:

(a) $\mu(1_B) = \infty$;

(b) there are no maps $\mu_0, \mu_1: B \to D$ that satisfy the following properties:

(i) $\mu(x) = \mu_0(x) \lor \mu_1(x)$, for all $x \in B$,

(ii) $\mu_0$ and $\mu_1$ are order-preserving,

(iii) $\mu_\ell(1) \leq a_\ell$, for all $\ell < 2$.

Proof. Let $B$ be the Boolean algebra constructed in Definition 2.2. It is clear that $|B| = \mathfrak{c}$. We define ideals $I_0$, $I_1$, and $I$ of $B$, as follows:

$I = \text{ideal of } B \text{ generated by } \{u_{\ell,\xi} \mid \xi < \omega_1\} \cup \{v_k \mid k < \omega\}$, for all $\ell < 2$,

$I = \text{ideal of } B \text{ generated by } \{v_k \mid k < \omega\}$. 

Problem 1. Let one-to-one every lattice $K \bigcup$ properties:

\[
\{ \bigvee \mu \}
\]

Claim 1.

Proof. As in the proof of Corollary 1.3.

(i) is trivial.

Proof of Claim. (i) is trivial.

(ii) Let $\xi < \omega$. Then

\[
\mu_1(u_{0,\xi}) \leq \mu(u_{0,\xi}) \leq a_0
\]

(by assumption (i))

while also $\mu_1(u_{0,\xi}) \leq a_1$ by assumptions (ii) and (iii). Therefore, $\mu_1(u_{0,\xi}) \leq n$ for some $n < \omega$. This proves that $\omega_1 = \bigcup_{n < \omega} X_n$. The proof that $\omega_1 = \bigcup_{n < \omega} Y_n$ is similar. □ Claim 1.

Now we put $Z_n = X_n \cap Y_n$, for all $n < \omega$. It follows from Claim 1 that $\omega_1 = \bigcup_{n < \omega} Z_n$. In particular, one of the $Z_n$ should be infinite (and even uncountable). We fix such an $n$. For all $\xi, \eta \in Z_n$, $\mu_1(u_{0,\xi}) \leq n$ and $\mu_0(u_{1,\eta}) \leq n$, thus, by assumptions (i) and (ii), $\mu(u_{0,\xi} \land u_{1,\eta}) \leq n$, that is, $u_{0,\xi} \land u_{1,\eta} \leq w_n$. Thus, by Lemma 2.3, $g(\xi, \eta) \leq n$. Hence, by Lemma 2.1, $Z_n$ is finite, a contradiction. □

Corollary 2.4. There exist a Boolean algebra $B$ of size $\aleph_1$ and a $\{ \lor, 0 \}$-homomorphism $\varphi: \text{Con}_c B \to D$ such that there are no lattice $L$, no lattice homomorphism $f: B \to L$ and no $\{ \lor, 0 \}$-homomorphism $\alpha: \text{Con}_c L \to D$ that satisfy the following properties:

(i) $\alpha$ is weakly distributive at $\Theta_L(f(0_B), f(1_B))$.

(ii) $\varphi = \alpha \circ \text{Con}_c f$.

Proof. As in the proof of Corollary 1.3. □

3. Open problems

The main result of Theorem A states that the possibility, for a given distributive $\{ \lor, 0 \}$-semilattice $S$, to lift every $\{ \lor, 0 \}$-homomorphism $\text{Con}_c K \to S$ for any lattice $K$ is equivalent to $S$ being a lattice. The maps considered in the proof of this result are not one-to-one. This leaves open the following question:

Problem 1. Let $S$ be a distributive $\{ \lor, 0 \}$-semilattice. When is it possible to lift every one-to-one $\{ \lor, 0 \}$-homomorphism $\varphi: \text{Con}_c K \to S$, for any lattice $K$?
By Theorem C of [10], the condition that $S$ be a lattice is sufficient. Is this condition also necessary?

**Problem 2.** Let $K$ be a lattice, let $S$ be a distributive $\{\lor, 0\}$-semilattice, let $\varphi : \operatorname{Con}c K \to S$ be a distributive $\{\lor, 0\}$-homomorphism. Can $\varphi$ be lifted?

Recall (see [7]) that for $\{\lor, 0\}$-semilattices $S$ and $T$, a homomorphism $\varphi : S \to T$ is distributive, if $\varphi$ is surjective and $\ker \varphi$ is a directed union of the form $\bigcup_{i \in I} \ker s_i$, where $s_i$ is a closure operator on $S$ for all $i$. The result of Corollary 1.3 is of no help for solving Problem 2, because the contradiction follows there from the failure of $\alpha$ to be (weakly) distributive.

**Problem 3.** Let $K$ be a countable lattice, let $S$ be a countable distributive $\{\lor, 0\}$-semilattice. Can every $\{\lor, 0\}$-homomorphism from $\operatorname{Con}c K$ to $S$ be lifted?

For countable $S$, not every $\{\lor, 0\}$-homomorphism from $\operatorname{Con}c K$ to $S$ can be lifted as a rule, even for $K$ of size $\aleph_1$ (this follows from Corollary 2.4). However, the problem is still open for countable $K$.

Our last problem is more oriented to axiomatic set theory. It originates in the observation that the construction of the Boolean algebra of the proof of Theorem A does not rely on the Axiom of Choice (but it has size the continuum), while the construction of the Boolean algebra of the proof of Theorem B does not rely on the Continuum Hypothesis (but it relies on the Axiom of Choice, in the form of the existence of a one-to-one map from $\omega_1$ into $\mathcal{P}(\omega)$).

**Problem 4.** Can one prove Theorem B by using neither the Axiom of Choice nor the Continuum Hypothesis?

References

[10] , Forcing extensions of partial lattices, manuscript.