Rational and algebraic series in combinatorial enumeration

Mireille Bousquet-Mélou

Abstract. Let $\mathcal{A}$ be a class of objects, equipped with an integer size such that for all n the number a_n of objects of size n is finite. We are interested in the case where the generating function $\sum_n a_n t^n$ is rational, or more generally algebraic. This property has a practical interest, since one can usually say a lot on the numbers a_n, but also a combinatorial one: the rational or algebraic nature of the generating function suggests that the objects have a (possibly hidden) structure, similar to the *linear structure* of words in the rational case, and to the *branching structure* of trees in the algebraic case. We describe and illustrate this combinatorial intuition, and discuss its validity. While it seems to be satisfactory in the rational case, it is probably incomplete in the algebraic one. We conclude with open questions.

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1. Introduction

The general topic of this paper is the enumeration of discrete objects (words, trees, graphs...) and more specifically the *rational* or *algebraic* nature of the associated generating functions. Let $\mathcal{A}$ be a class of discrete objects equipped with a size:

$$\begin{array}{rcl} size: \mathcal{A} & \rightarrow & \mathbb{N} \\ A & \mapsto & |A|. \end{array}$$

Assume that for all n, the number a_n of objects of size n is finite. The *generating function of the objects of* $\mathcal{A}$, *counted by their size*, is the following formal power series in the indeterminate t:

$$A(t) := \sum_{n\geq 0} a_n t^n = \sum_{A\in\mathcal{A}} t^{|A|}. \quad (1)$$

To take a very simple example, if $\mathcal{A}$ is the set of words on the alphabet $\{a, b\}$ and the size of a word is its number of letters, then the generating function is $\sum_{n\geq 0} 2^n t^n = 1/(1-2t)$.

Generating functions provide both a tool for solving counting problems, and a concise way to encode their solution. Ideally, one would probably dream of finding a closed formula for the numbers a_n. But the world of mathematical objects would be extremely poor if this was always possible. In practise, one is usually happy with an expression of the generating function $A(t)$, or even with a recurrence relation defining the sequence a_n, or a functional equation defining $A(t)$.

Enumerative problems arise spontaneously in various fields of mathematics, computer science, and physics. Among the most generous suppliers of such problems, let us cite discrete probability theory, the analysis of the complexity of algorithms [56, 44], and the discrete models of statistical physics, like the famous Ising model [5]. More generally, counting the objects that occur in one's work seems to answer a natural curiosity. It helps to understand the objects, for instance to appreciate how restrictive are the conditions that define them. It also forces us to get some understanding of the *structure* of the objects: an enumerative result never comes for free, but only after one has elucidated, at least partly, what the objects really are.

We focus in this survey on objects having a rational, or, more generally, algebraic generating function. Rational and algebraic formal power series are well-behaved objects with many interesting properties. This is one of the reasons why several classical textbooks on enumeration devote one or several chapters to these series [43, 74, 75]. These chapters give typical examples of objects with a rational [resp. algebraic] generating function (GF). After a while, the collection of these examples builds up a general picture: one starts thinking that yes, all these objects have something in common in their structure. At the same time arises the following question: do all objects with a rational [algebraic] GF look like that? In other words, what does it mean, what does it suggest about the objects when they are counted by a rational [algebraic] GF ?

This question is at the heart of this survey. For each of the two classes of series under consideration, we first present a general family of enumerative problems whose solution falls invariably in this class. These problems are simple to describe: the first one deals with walks in a directed graph, the other with plane trees. Interestingly, these families of objects admit alternative descriptions in language theoretic terms: they correspond to *regular languages*, and to *unambiguous context-free languages*, respectively. The words of these languages have a clear recursive structure, which explains directly the rationality [algebraicity] of their GF.

The series counting words of a regular [unambiguous context-free] language are called $\mathbb{N}$-*rational* [$\mathbb{N}$-*algebraic*]. It is worth noting that a rational [algebraic] series with non-negative coefficients is not necessarily $\mathbb{N}$-rational [$\mathbb{N}$-algebraic]. Since we want to appreciate whether our two generic classes of objects are good representatives of objects with a rational [algebraic] GF, the first question to address is the following: do we always fall in the class of $\mathbb{N}$-rational [$\mathbb{N}$-algebraic] series when we count objects with a rational [algebraic] GF? More informally, do these objects exhibit a structure similar to the structure of regular [context-free] languages? Is such a structure usually clearly visible? That is to say, is it easy to feel, to predict rationality [algebraicity]?

We shall see that the answer to all these questions tends to be *yes* in the rational case (with a few warnings...) but is probably *no* in the algebraic case. In particular, the rich world of *planar maps* (planar graphs embedded in the sphere) abounds in candidates for non-$\mathbb{N}$-algebraicity. The algebraicity of the associated GFs has been known for more than 40 years (at least for some families of maps), but it is only in the past 10 years that a general combinatorial explanation of this algebraicity has emerged. Moreover, the underlying constructions are more general that those allowed in context-free descriptions, as they involve taking *complements*.

Each of the main two sections ends with a list of questions. In particular, we present at the end of Section 3 several counting problems that are simple to state and have an algebraic GF, but for reasons that remain mysterious.

The paper is sometimes written in an informal style. We hope that this will not stop the reader. We have tried to give precise references where he/she will find more details and more material on the topics we discuss. In particular, this survey borrows a lot to two books that we warmly recommend: Stanley's *Enumerative Combinatorics* [74, 75], and Flajolet & Sedgewick's *Analytic Combinatorics* [43].

Notation and definitions. Given a (commutative) ring R, we denote by $R[t]$ the ring of polynomials in t having coefficients in R. A *Laurent series* in t is a series of the form $A(t) = \sum_{n \geq n_0} a_n t^n$, with $n_0 \in \mathbb{Z}$ and $a_n \in R$ for all n. If $n_0 \geq 0$, we say that $A(t)$ is a *formal power series*. The coefficient of t^n is denoted $a_n := [t^n]A(t)$. The set of Laurent series forms a ring, and even a field if R is a field. The *quasi-inverse* of $A(t)$ is the series $A^*(t) := 1/(1 - A(t))$. If $A(t)$ is a formal power series with constant term 0, then $A^*(t)$ is a formal power series too.

In most occasions, the series we consider are GFs of the form (1) and thus have rational coefficients. However, we sometimes consider refined enumeration problems, in which every object A is *weighted*, usually by a monomial $w(A)$ in some additional indeterminates $x_1, \ldots, x_m$. The weighted GF is then $\sum_{A \in \mathcal{A}} w(A) t^{|A|}$, so that the coefficient ring is $\mathbb{Q}[x_1, \ldots, x_m]$ rather than $\mathbb{Q}$.

We denote $[\![k]\!] = \{1, 2, \ldots, k\}$. We use the standard notation $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{P} := \{1, 2, 3, \ldots\}$.

2. Rational generating functions

2.1. Definitions and properties. The Laurent series $A(t)$ with coefficients in the field R is said to be *rational* if it can be written in the form

$$A(t) = \frac{P(t)}{Q(t)}$$

where $P(t)$ and $Q(t)$ belong to $R[t]$.

There is probably no need to spend a lot of time explaining why such series are simple and well-behaved. We refer to [74, Ch. 4] and [43, Ch. IV] for a survey of their properties. Let us review briefly some of them, in the case where $R = \mathbb{Q}$. The set of (Laurent) rational series is closed under sum, product, derivation, reciprocals – but *not under integration* as shown by $A(t) = 1/(1 - t)$. The coefficients a_n of a rational series $A(t)$ satisfy a linear recurrence relation with constant coefficients:

for n large enough,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

The partial fraction expansion of $A(t)$ provides a closed form expression of these coefficients of the form:

$$a_n = \sum_{i=0}^{k} P_i(n)\mu_i^n \tag{2}$$

where the μ_i are the reciprocals of the roots of the denominator $Q(t)$, and the P_i are polynomials. In particular, if $A(t)$ has non-negative integer coefficients, its radius of convergence ρ is one its the poles (Pringsheim) and the "typical" asymptotic behaviour of a_n is

$$a_n \sim \kappa \rho^{-n} n^d \tag{3}$$

where $d \in \mathbb{N}$ and κ is an algebraic number. The above statement has to be taken with a grain of salt: *all* poles of minimal modulus may actually contribute to the dominant term in the asymptotic expansion of a_n, as indicated by (2).

Let us add that Padé approximants allow us to *guess* whether a generating function whose first coefficients are known is likely to be rational. For instance, given the 10 first coefficients of the series

$$A(t) = t+2\,t^2+6\,t^3+19\,t^4+61\,t^5+196\,t^6+629\,t^7+2017\,t^8+6466\,t^9+20727\,t^{10}+O\left(t^{11}\right),$$

it is easy to conjecture that actually

$$A(t) = \frac{t\,(1-t)^3}{1-5\,t+7\,t^2-4\,t^3}.$$

Padé approximants are implemented in most computer algebra packages. For instance, the relevant MAPLE command is `convert/ratpoly`.

2.2. Walks on a digraph. We now introduce our typical "rational" objects. Let $G = (V, E)$ be a directed graph with (finite) vertex set $V = [\![p]\!]$ and (directed) edge set $E \subset V \times V$. A *walk of length* n on G is a sequence of vertices $w = (v_0, v_1, \ldots, v_n)$ such that for all i, the pair (v_i, v_{i+1}) is an edge. Such a walk *goes from* v_0 *to* v_n. We denote $|w| = n$. Now assign to each directed edge e a weight (an indeterminate) x_e. Define the weight x_w of the walk w as the product of the weights of the edges it visits: more precisely,

$$x_w = \prod_{i=0}^{n-1} x_{(v_i, v_{i+1})}.$$

See Fig. 1(a) for an example. Let X denote the (weighted) adjacency matrix of G: for i and j in $[\![p]\!]$, the entry $X_{i,j}$ is x_e if $(i, j) = e$ is an edge of G and 0 otherwise. Let $W_{i,j}(t)$ be the weighted generating function of walks going from i to j:

$$W_{i,j}(t) = \sum_{w: i \rightsquigarrow j} x_w t^{|w|}.$$

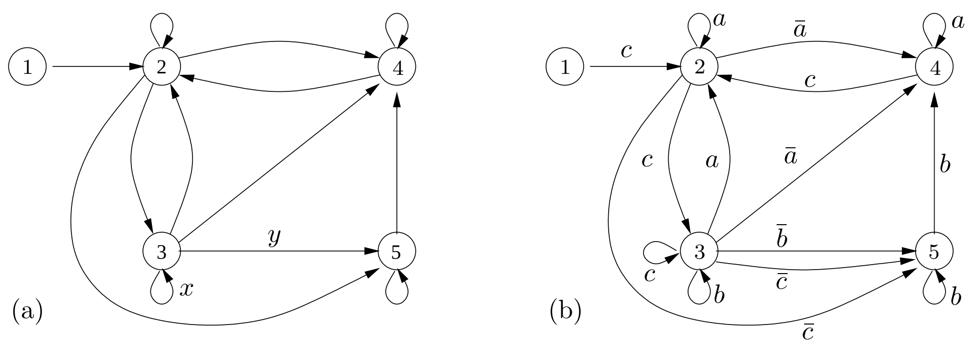


Figure 1. (a) A weighted digraph. The default value of the weight is 1. (b) A deterministic automaton on the alphabet $\{a, b, c, \bar{a}, \bar{b}, \bar{c}\}$. The initial state is 1 and and the final states are 2 and 3.

It is well-known, and easy to prove, that $W_{i,j}$ is a rational function in t with coefficicients in $\mathbb{Q}[x_e, e \in E]$ (see [74, Thm. 4.7.1]).

Theorem 2.1. *The series $W_{i,j}(t)$ is the (i,j)-entry in the matrix $(1 - tX)^{-1}$.*

This theorem reduces the enumeration of walks on a digraph to the calculation of the inverse of a matrix with polynomial coefficients. It seems to be little known in the combinatorics community that this inverse matrix can be computed by studying the elementary cycles of the digraph G. This practical tool relies on Viennot's theory of *heaps of pieces* [81]. Since it is little known, and often convenient, let us advertise it here. It will be illustrated further down.

An *elementary cycle* of G is a closed walk $w = (v_0, v_1, \ldots, v_{n-1}, v_0)$ such that $v_0, \ldots, v_{n-1}$ are distinct. It is defined up to a cyclic permutation of the v_i. That is, $(v_1, v_2, \ldots, v_{n-1}, v_0, v_1)$ is the same cycle as w. A collection $\gamma = \{\gamma_1, \ldots, \gamma_r\}$ of (elementary) cycles is *non-intersecting* if the γ_i are pairwise disjoint. The weight x_γ of γ is the product of the weights of the γ_i. We denote $|\gamma| = \sum |\gamma_i|$.

Proposition 2.2 ([81]). *The generating function of walks going from i to j reads*

$$W_{i,j}(t) = \frac{N_{i,j}}{D},$$

where

$$D = \sum_{\gamma=\{\gamma_1,\ldots,\gamma_r\}} (-1)^r x_\gamma t^{|\gamma|} \quad \text{and} \quad N_{i,j} = \sum_{w;\gamma=\{\gamma_1,\ldots,\gamma_r\}} (-1)^r x_w x_\gamma t^{|w|+|\gamma|}.$$

The polynomial D is the alternating generating function of non-intersecting collections of cycles. In the expression of N, γ a non-intersecting collection of cycles and w a self-avoiding walk going from i to j, disjoint from the cycles of γ.

To illustrate this result, let us determine the generating function of walks going from 1 to 2 and from 1 to 3 on the digraph of Fig. 1(a). This graph contains 4 cycles of length 1, 2 cycles of length 2, 2 cycles of length 3 and 1 cycle of length 4. By forming all non-intersecting collections of cycles, one finds:

$$D(t) = 1-(3+x)t+(3+3x-2)t^2+(-1-3x+3+x-2)t^3+(x-1-x+1+x-y)t^4$$
$$= 1-(3+x)t+(1+3x)t^2-2xt^3+(x-y)t^4.$$

There is only one self-avoiding walk (SAW) going from 1 to 2, and one SAW going from 1 to 3 (via the vertex 2). The collections of cycles that do not intersect these walks are formed of loops, which gives

$$N_{1,2} = t(1-t)^2(1-xt) \quad \text{and} \quad N_{1,3} = t^2(1-t)^2.$$

Hence the generating function of walks that start from 1 and end at 2 or 3 is:

$$W_{1,2} + W_{1,3} = \frac{N_{1,2}+N_{1,3}}{D} = \frac{t(1-t)^2(1+t-xt)}{1-(3+x)t+(1+3x)t^2-2xt^3+(x-y)t^4}. \quad (4)$$

2.3. Regular languages and automata. There is a very close connection between the collection of walks on a digraph and the words of *regular languages*. Let $\mathcal{A}$ be an *alphabet*, that is, a finite set of symbols (called *letters).* A *word* on $\mathcal{A}$ is a sequence $u = u_1u_2\cdots u_n$ of letters. The number of occurrences of the letter a in the word u is denoted $|u|_a$. The *product* of two words $u_1u_2\cdots u_n$ and $v_1v_2\cdots v_m$ is the concatenation $u_1u_2\cdots u_nv_1v_2\cdots v_m$. The empty word is denoted ϵ. A *language* on $\mathcal{A}$ is a set of words. We define two operations on languages:

– the product $\mathcal{L}\mathcal{K}$ of two languages $\mathcal{L}$ and $\mathcal{K}$ is the set of words uv, with $u \in \mathcal{L}$ and $v \in \mathcal{K}$; this product is easily seen to be associative,

– the star $\mathcal{L}^*$ of the language $\mathcal{L}$ is the union of all languages $\mathcal{L}^k$, for $k \geq 0$. By convention, $\mathcal{L}^0$ is reduced to the empty word ϵ.

A *finite state automaton* on $\mathcal{A}$ is a digraph (V,E) *with possibly multiple edges*, together with:
– a labelling of the edges by letters of $\mathcal{A}$, that is to say, a function $L : E \rightarrow \mathcal{A}$,
– an initial vertex i,
– a set $V_f \subset V$ of final vertices.
The vertices are usually called the *states* of the automaton. The automaton is *deterministic* if for every state v and every letter a, there is at most one edge labelled a starting from v.

To every walk on the underlying multigraph, one associates a word on the alphabet $\mathcal{A}$ by reading the letters met along the walk. The language $\mathcal{L}$ *recognized* by the automaton is the set of words associated with walks going from the initial state i to one of the states of V_f. For $j \in V$, let $\mathcal{L}_j$ denote the set of words associated with walks going from i to j. These sets admit a recursive description. For the automaton of Fig. 1(b), one has $\mathcal{L} = \mathcal{L}_2 \cup \mathcal{L}_3$ with

$$\begin{array}{lll} \mathcal{L}_1 &=& \{\epsilon\}, \\ \mathcal{L}_2 &=& \mathcal{L}_1c \cup \mathcal{L}_2a \cup \mathcal{L}_3a \cup \mathcal{L}_4c, \\ \mathcal{L}_3 &=& \mathcal{L}_2c \cup \mathcal{L}_3b \cup \mathcal{L}_3c, \end{array} \qquad \begin{array}{lll} \mathcal{L}_4 &=& \mathcal{L}_2\bar{a} \cup \mathcal{L}_3\bar{a} \cup \mathcal{L}_4a \cup \mathcal{L}_5b, \\ \mathcal{L}_5 &=& \mathcal{L}_2\bar{c} \cup \mathcal{L}_3\bar{b} \cup \mathcal{L}_3\bar{c} \cup \mathcal{L}_5b. \end{array}$$

Remarkably, there also exists a non-recursive combinatorial description of the languages that are recognized by an automaton [52, Thms. 3.3 and 3.10].

Theorem 2.3. *Let $\mathcal{L}$ be a language on the alphabet $\mathcal{A}$. There exists a finite state automaton that recognizes $\mathcal{L}$ if and only if $\mathcal{L}$ can be expressed in terms of finite languages on $\mathcal{A}$, using a finite number of unions, products and stars of languages.*

If these conditions hold, $\mathcal{L}$ is said to be regular*. Moreover, there exists a* deterministic *automaton that recognizes $\mathcal{L}$.*

Regular languages and walks on digraphs. Take a deterministic automaton, and associate with it a weighted digraph as follows: the vertices are those of the automaton, and for all vertices j and k, if m edges go from j to k in the automaton, they are replaced by a *single* edge labelled m in the digraph. For instance, the automaton of Fig. 1(b) gives the digraph to its left, with $x = y = 2$. Clearly, the length GF of words of $\mathcal{L}$ is the GF of (weighted) walks of this digraph going from the initial vertex i to one of the final vertices of V_f. For instance, according to (4), the length GF of the language recognized by the automaton of Fig. 1(b) is

$$A(t) = \frac{t\,(1-t)^3}{1-5\,t+7\,t^2-4\,t^3}. \tag{5}$$

Take a regular language $\mathcal{L}$ recognized by a deterministic automaton $\mathcal{A}$. There exists another deterministic automaton that recognizes $\mathcal{L}$ and does not contain multiple edges. The key is to create a state (j, a) for every edge labelled a ending at j in the automaton $\mathcal{A}$. The digraph associated with this new automaton has all its edges labelled 1, so that there exists a length preserving bijection between the words of $\mathcal{L}$ and the walks on the digraph going from a specified initial vertex v_0 to one of the vertices of a given subset V_f of vertices.

Conversely, starting from a digraph with all edges labelled 1, together with a specified vertex v_0 and a set V_f of final vertices, it is easy to construct a regular language that is in bijection with the walks of the graph going from v_0 to V_f (consider the automaton obtained by labelling all edges with distinct letters). This shows that *counting words of regular languages is completely equivalent to counting walks in digraphs.* In particular, the set of rational series obtained in both types of problems coincide, and have even been given a name:

Definition 2.4. A series $A(t) = \sum_{n\geq 0} a_n t^n$ with coefficients in $\mathbb{N}$ is said to be $\mathbb{N}$-*rational* if there exists a regular language having generating function $A(t) - a_0$.

The description of regular languages given by Theorem 2.3 implies that the set of $\mathbb{N}$-rational series contains the smallest set of series containing $\mathbb{N}[t]$ and closed under sum, product and quasi-inverse. The converse is true [71, Thm. II.5.1]. There exists a simple way to decide whether a given rational series with coefficients in $\mathbb{N}$ is $\mathbb{N}$-rational [71, Thms. II.10.2 and II.10.5].

Theorem 2.5. *A series $A(t) = \sum_{n\geq 0} a_n t^n$ with coefficients in $\mathbb{N}$ is $\mathbb{N}$-rational if and only if there exists a positive integer p such that for all $r \in \{0, \ldots, p\}$, the series*

$$A_{r,p}(t) := \sum_{n\geq 0} a_{np+r} t^n$$

has a unique singularity of minimal modulus (called dominant*).*

There exist rational series with non-negative integer coefficients that are *not* $\mathbb{N}$-rational. For instance, let α be such that $\cos\alpha = 3/5$ and $\sin\alpha = 4/5$, and define $a_n = 25^n \cos(n\alpha)^2$. It is not hard to see that a_n is a non-negative integer. The associated series $A(t)$ reads

$$A(t) = \frac{1 - 2t + 225t^2}{(1 - 25t)(625t^2 + 14t + 1)}.$$

It has 3 distinct dominant poles. As α is not a rational multiple of π, the same holds for all series $A_{0,p}(t)$, for all values of p. Thus $A(t)$ is not $\mathbb{N}$-rational.

2.4. The combinatorial intuition of rational generating functions.

We have described two families of combinatorial objects that naturally yield rational generating functions: walks in a digraph and words of regular languages. We have, moreover, shown that the enumeration of these objects are equivalent problems. It seems that these families convey the "right" intuition about objects with a rational GF. By this, we mean informally that:

(*i*) "every" family of objects with a rational GF has actually an $\mathbb{N}$-rational GF,

(*ii*) for almost all families of combinatorial objects with a rational GF, it is easy to foresee that there will be a bijection between these objects and words of a regular language.

Point (*ii*) means that most of these families $\mathcal{F}$ have a clear *automatic structure*, similar to the automatic structure of regular languages: roughly speaking, the objects of $\mathcal{F}$ can be constructed recursively using unions of sets and concatenation of *cells* (replacing letters). A more formal definition would simply paraphrase the definition of automata.

Point (*i*) means simply that I have never met a counting problem that would yield a rational, but not $\mathbb{N}$-rational GF. This includes problems coming from algebra, like growth functions of groups. On the contrary, Point (*ii*) only concerns purely combinatorial problems (but I do not want to be asked about the border between combinatorics and algebra). It admits very few counter-examples. Some will be discussed in Section 2.5. For the moment, let us illustrate the two above statements by describing the automatic structure of certain classes of objects (some being rather general), borrowed from [74, Ch. 4].

2.4.1. Column-convex polyominoes.

A *polyomino* is a finite union of cells of the square lattice, whose interior is connected. Polyominoes are considered up to a translation. A polyomino is *column-convex* (cc) if its intersection with every vertical line is connected. Let a_n be the number of cc-polyominoes having n cells, and let $A(t)$ be the associated generating function. We claim that these polyominoes have an automatic structure.

Consider a cc-polyomino P having n cells. Let us number these cells from 1 to n as illustrated in Fig. 2. The columns are visited from left to right. In the first

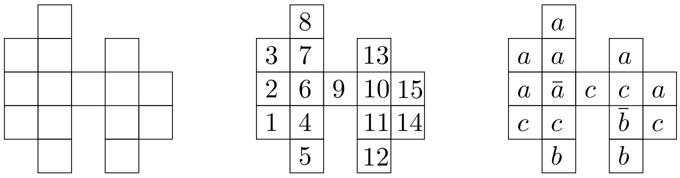


Figure 2. A column-convex polyomino, with the numbering and encoding of the cells.

column, cells are numbered from bottom to top. In each of the other columns, the lowest cell that has a left neighbour gets the smallest number; then the cells lying below it are numbered from top to bottom, and finally the cells lying above it are numbered from bottom to top. Note that for all i, the cells labelled $1, 2, \dots, i$ form a cc-polyomino. This will be essential in our description of the automatic structure of these objects. Associate with P the word $u = u_1 \cdots u_n$ on the alphabet $\{a, b, c\}$ defined by

– $u_i = c$ (like Column) if the ith cell is the first to be visited in its column,
– $u_i = b$ (like Below) if the ith cell lies below the first visited cell of its column,
– $u_i = a$ (like Above) if the ith cell lies above the first visited cell of its column.

Then, add a bar on the letter u_i if the ith cell of P has a South neighbour, an East neighbour, but no South-East neighbour. (In other words, the barred letters indicate where to start a new column, when the bottommost cell of this new column lies above the bottommost cell of the previous column.) This gives a word v on the alphabet $\{a, b, c, \bar{a}, \bar{b}, \bar{c}\}$. It is not hard to see that the map that sends P on the word v is a size-preserving bijection between cc-polyominoes and words recognized by the automaton of Fig. 1(b). Hence by (5), the generating function of column-convex polyominoes is [76]:

$$A(t) = \frac{t\,(1-t)^3}{1 - 5\,t + 7\,t^2 - 4\,t^3}.$$

2.4.2. P-partitions. A *partition* of the integer n into at most k parts is a non-decreasing k-tuple $\lambda = (\lambda_1, \dots, \lambda_k)$ of nonnegative integers that sum to n. This classical number-theoretic notion is generalized by the notion of *P-partitions.* Let P be a *natural* partial order on $[\![k]\!]$ (by *natural* we mean that if $i < j$ in P, then $i < j$ in $\mathbb{N}$). A *P-partition* of n is a k-tuple $\lambda = (\lambda_1, \dots, \lambda_k)$ of nonnegative integers that sum to n and satisfy $\lambda_i \le \lambda_j$ if $i \le j$ in P. Thus when P is the natural total order on $[\![k]\!]$, a P-partition is simply a partition[1].

We are interested in the following series:

$$F_P(t) = \sum_{\lambda} t^{|\lambda|},$$

where the sum runs over all P-partitions and $|\lambda| = \lambda_1 + \cdots + \lambda_k$ is the *weight* of λ.

[1] A P-partition is usually defined as an *order-reversing* map from $[\![k]\!]$ to $\mathbb{N}$ [74, Section 4.5]. Both notions are of course completely equivalent.

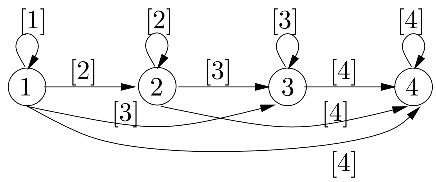
The case of ordinary partitions is easy to analyze: every partition can be written in a unique way as a linear combination

$$c_1\lambda^{(1)} + \cdots + c_k\lambda^{(k)} \tag{6}$$

where $\lambda^{(i)} = (0, 0, \ldots, 0, 1, 1, \ldots, 1)$ has exactly i parts equal to 1 and $c_i \in \mathbb{N}$. The weight of $\lambda^{(i)}$ is i, and one obtains:

$$F_P(t) = \frac{1}{(1-t)(1-t^2)\cdots(1-t^k)}. \tag{7}$$

The automatic structure of (ordinary) partitions is transparent: since they are constructed by adding a number of copies of $\lambda^{(1)}$, then a number of copies of $\lambda^{(2)}$, and so on, there is a size preserving bijection between these partitions and walks starting from 1 and ending anywhere in the following digraph:



Note that this graph corresponds to $k = 4$, and that an edge labelled $[\ell]$ must be understood as a *sequence* of ℓ edges. These labels do *not* correspond to multiplicities. Observe that the only cycles in this digraph are loops. This, combined with Proposition 2.2, explains the factored form of the denominator of (7).

Consider now the partial order on $[\![4]\!]$ defined by $1 < 3, 2 < 3$ and $2 < 4$. The partitions of weight at most 2 are

$$(0,0,0,0), (0,0,1,0), (0,0,0,1), (1,0,1,0), (0,0,1,1), (0,0,2,0), (0,0,0,2),$$

so that $F_P(t) = 1 + 2t + 4t^2 + O(t^3)$. If one is brave enough to list P-partitions of weight at most 20, the Padé approximant of the truncated series thus obtained is remarkably simple:

$$F_P(t) = \frac{1 + t + t^2 + t^3 + t^4}{(1-t)(1-t^2)(1-t^3)(1-t^4)} + O(t^{21}),$$

and allows one to make a (correct) conjecture.

It turns out that the generating function of P-partitions is always a rational series of denominator $(1-t)(1-t^2)\ldots(1-t^k)$. Moreover, P-partitions obey our general intuition about objects with a rational GF. The following proposition, illustrated below by an example, describes their automatic structure: the set of P-partitions can be partitioned into a finite number of subsets; in each of these subsets, partitions have a structure similar to (6). Recall that a *linear extension* of P is a bijection σ on $[\![k]\!]$ such that $\sigma(i) < \sigma(j)$ if $i < j$ in P.

Proposition 2.6 ([74], Section 4.5). *Let P be a natural order on $[\![k]\!]$.*

For every P-partition λ, there exists a unique linear extension σ of P such that for all i, $\lambda_{\sigma(i)} \leq \lambda_{\sigma(i+1)}$, the inequality being strict if $\sigma(i) > \sigma(i+1)$. We say that λ is compatible *with σ.*

Given a linear extension σ, the P-partitions that are compatible with σ can be written in a unique way as a linear combination with coefficients in $\mathbb{N}$:

$$\lambda^{(\sigma,0)} + c_1\lambda^{(\sigma,1)} + \cdots + c_k\lambda^{(\sigma,k)} \tag{8}$$

where $\lambda^{(\sigma,0)}$ is the smallest P-partition compatible with σ:

$$\lambda^{(\sigma,0)}_{\sigma(j)} = |\{i < j : \sigma(i) > \sigma(i+1)\}| \qquad \text{for } 1 \leq j \leq k,$$

and for $1 \leq i \leq k$,

$$(\lambda^{(\sigma,i)}_{\sigma(1)}, \ldots, \lambda^{(\sigma,i)}_{\sigma(k)}) = (0, 0, \ldots, 0, 1, 1, \ldots, 1)$$

has exactly i parts equal to 1. Thus the GF of these P-partitions is

$$F_{P,\sigma}(t) = \frac{t^{e(\sigma)}}{(1-t)(1-t^2)\ldots(1-t^k)}$$

where $e(\sigma)$ is a variant of the Major index *of σ:*

$$e(\sigma) = \sum_{i:\sigma(i)>\sigma(i+1)} (k-i).$$

Example. Let us return to the order $1 < 3$, $2 < 3$ and $2 < 4$. The 5 linear extensions are $1234, 2134, 1243, 2143$ and 2413. Take $\sigma = 2143$. The P-partitions λ that are compatible with σ are those that satisfy $\lambda_2 < \lambda_1 \leq \lambda_4 < \lambda_3$. The smallest of them is thus $\lambda^{(\sigma,0)} = (1, 0, 2, 1)$. Then $\lambda^{(\sigma,1)} = (0, 0, 1, 0)$, $\lambda^{(\sigma,2)} = (0, 0, 1, 1)$, $\lambda^{(\sigma,3)} = (1, 0, 1, 1)$ and $\lambda^{(\sigma,4)} = (1, 1, 1, 1)$.

2.4.3. Integer points in a convex polyhedral cone ([74], Sec. 4.6). Let $\mathcal{H}$ be a finite collection of linear half-spaces of $\mathbb{R}^m$ of the form $c_1\alpha_1 + \cdots + c_m\alpha_m \geq 0$, with $c_i \in \mathbb{Z}$. We are interested in the set $\mathcal{E}$ of *non-negative integer points* $\alpha = (\alpha_1, \ldots, \alpha_m)$ lying in the intersection of those half-spaces. For instance, we could have the following set $\mathcal{E}$, illustrated in Fig. 3(a):

$$\mathcal{E} = \{(\alpha_1, \alpha_2) \in \mathbb{N}^2 : 2\alpha_1 \geq \alpha_2 \text{ and } 2\alpha_2 \geq \alpha_1\}. \tag{9}$$

Numerous enumerative problems (including P-partitions) can be formulated in terms of linear inequalities as above. The generating function of $\mathcal{E}$ is

$$E(t) = \sum_{\alpha \in \mathcal{E}} t^{|\alpha|},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_m$. In the above example, $E(t) = 1 + t^2 + 2t^3 + t^4 + 2t^5 + 3t^6 + 2t^7 + O(t^8)$.

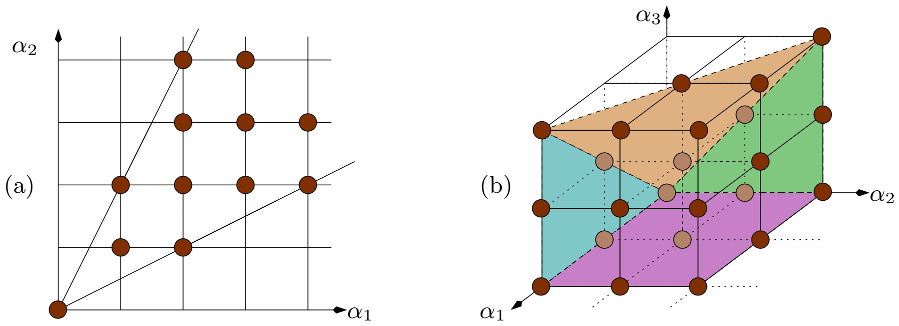


Figure 3. Integer points in a polyhedral cone.

The set $\mathcal{E}$ is a monoid (it is closed under summation). In general, it is not a *free* monoid. Geometrically, the set $\mathcal{C}$ of non-negative *real* points in the intersection of the half-spaces of $\mathcal{H}$ forms a *pointed convex polyhedral cone* (the term *pointed* means that it does not contain a line), and $\mathcal{E}$ is the set of integer points in $\mathcal{C}$.

The simplicial case. In the simplest case, the cone $\mathcal{C}$ is simplicial. This implies that the *monoid* $\mathcal{E}$ is *simplicial*, meaning that there exists linearly independent vectors $\alpha^{(1)}, \ldots, \alpha^{(k)}$ such that

$$\mathcal{E} = \left\{ \alpha \in \mathbb{N}^m : \alpha = q_1 \alpha^{(1)} + \cdots + q_k \alpha^{(k)} \text{ with } q_i \in \mathbb{Q},\ q_i \geq 0 \right\}.$$

This is the case in Example (9), with $\alpha^{(1)} = (1,2)$ and $\alpha^{(2)} = (2,1)$. The *interior* of $\mathcal{E}$ (the set of points of $\mathcal{E}$ that are not on the boundary of $\mathcal{C}$) is then

$$\overline{\mathcal{E}} = \left\{ \alpha \in \mathbb{N}^m : \alpha = q_1 \alpha^{(1)} + \cdots + q_k \alpha^{(k)} \text{ with } q_i \in \mathbb{Q}, q_i > 0 \right\}. \tag{10}$$

Then there exists a finite subset $\mathcal{D}$ of $\mathcal{E}$ [resp. $\overline{\mathcal{D}}$ of $\overline{\mathcal{E}}$] such that every element of $\mathcal{E}$ [resp. $\overline{\mathcal{E}}$] can be written uniquely in the form

$$\alpha = \beta + c_1 \alpha^{(1)} + \cdots + c_k \alpha^{(k)}, \tag{11}$$

with $\beta \in \mathcal{D}$ [resp. $\beta \in \overline{\mathcal{D}}$] and $c_i \in \mathbb{N}$ [74, Lemma 4.6.7]. In our running example (9), taken with $\alpha^{(1)} = (1,2)$ and $\alpha^{(2)} = (2,1)$, one has $\mathcal{D} = \{(0,0),(1,1),(2,2)\}$ while $\overline{\mathcal{D}} = \{(1,1),(2,2),(3,3)\}$. Compare (11) with the structure found for P-partitions (8). Thus $\mathcal{E}$ and $\overline{\mathcal{E}}$ have an automatic structure and their GFs read

$$E(t) = \frac{\sum_{\beta \in \mathcal{D}} t^{|\beta|}}{\prod_{i=1}^k (1 - t^{|\alpha^{(i)}|})} \qquad \left[\text{resp.} \quad \bar{E}(t) = \frac{\sum_{\beta \in \overline{\mathcal{D}}} t^{|\beta|}}{\prod_{i=1}^k (1 - t^{|\alpha^{(i)}|})} \right].$$

In Example (9), one thus obtains

$$E(t) = \frac{1 + t^2 + t^4}{(1-t^3)^2} = \frac{1 - t + t^2}{(1-t)(1-t^3)} \quad \text{and} \quad \bar{E}(t) = t^2 E(t).$$

The general case. The set $\mathcal{E}$ can always be partitioned into a finite number of sets $\overline{\mathcal{F}}$ of the form (10), where $\mathcal{F}$ is a simplicial monoid [74, Ch. 4, Eq. (24)]. Thus $\mathcal{E}$, as a finite union of sets with an automatic structure, has an automatic structure as well. The associated generating function $E(t)$ is $\mathbb{N}$-rational, with a denominator which is a product of cyclotomic polynomials.

Consider, for example, the set

$$\mathcal{E} = \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 : \alpha_3 \leq \alpha_1 + \alpha_2\}.$$

The cone $\mathcal{C}$ of non-negative *real* points α satisfying $\alpha_3 \leq \alpha_1 + \alpha_2$ is *not* simplicial, as it has 4 faces of dimension 2, lying respectively in the hyperplanes $\alpha_i = 0$ for $i = 1, 2, 3$ and $\alpha_3 = \alpha_1 + \alpha_2$ (Fig. 3(b)). But it is the union of two simplicial cones $\mathcal{C}_1$ and $\mathcal{C}_2$, obtained by intersecting $\mathcal{C}$ with the half-spaces $\alpha_1 \geq \alpha_3$ and $\alpha_1 \leq \alpha_3$, respectively. Let $\mathcal{E}_1$ [resp. $\mathcal{E}_2$] denote the set of integer points of $\mathcal{C}_1$ [resp. $\mathcal{C}_2$].

The fastest way to obtain the generating function $E(t)$ is to write

$$E(t) = E_1(t) + E_2(t) - E_{12}(t) \tag{12}$$

where $E_{12}(t)$ counts integer points in the intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$ (that is, in the plane $\alpha_1 = \alpha_3$). Since $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_1 \cap \mathcal{E}_2$ are simplicial cones (of dimension 3, 3 and 2 respectively), the method presented above for simplicial cones applies. Indeed, $\mathcal{E}_1$ [resp. $\mathcal{E}_2$; $\mathcal{E}_{12}$] is the set of linear combinations (with coefficients in $\mathbb{N}$) of $(1, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 0)$ [resp. $(1, 0, 1)$, $(0, 1, 0)$ and $(0, 1, 1)$; $(1, 0, 1)$ and $(0, 1, 0)$]. This implies:

$$E(t) = \frac{1}{(1-t)^2(1-t^2)} + \frac{1}{(1-t)(1-t^2)^2} - \frac{1}{(1-t)(1-t^2)} = \frac{1+t+t^2}{(1-t)(1-t^2)^2}.$$

However, the "minus" sign in (12) prevents us from seeing directly the automatic nature of $\mathcal{E}$ (the difference of $\mathbb{N}$-rational series is not always $\mathbb{N}$-rational). This structure only becomes clear when we write $\mathcal{E}$ as the disjoint union of the interiors of all simplicial monoids induced by the triangulation of $\mathcal{C}$ into $\mathcal{C}_1$ and $\mathcal{C}_2$. These monoids are the integer points of the faces (of all possible dimensions) of $\mathcal{C}_1$ and $\mathcal{C}_2$. As there are 12 such faces (more precisely, 1 [resp. 4, 5, 2] faces of dimension 0 [resp. 1, 2, 3]), this gives $\mathcal{E}$ as the disjoint union of 12 sets having an automatic structure of the form (10).

2.5. Rational generating functions: more difficult questions.

2.5.1. Predicting rationality. We wrote in Section 2.4 that it is usually easy to foresee, to predict when a class of combinatorial objects has a rational GF. There are a few exceptions. Here is one of the most remarkable ones.

Example 2.7 (Directed animals). A *directed animal with a compact source of size* k is a finite set of points A on the square lattice $\mathbb{Z}^2$ such that:

– the points $(-i, i)$ for $0 \leq i < k$ belong to A; they are called the *source points*,

– all the other points in A can be reached from one of the source points by a path made of North and East steps, having all its vertices in A.

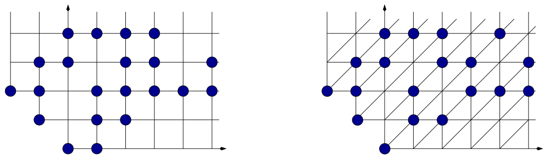


Figure 4. Compact-source directed animals on the square and triangular lattices.

See Fig. 4 for an illustration. A similar notion exists for the triangular lattice. It turns out that these animals have extremely simple generating functions [50, 10].

Theorem 2.8. *The number of compact-source directed animals of cardinality* n *is* 3^{n-1} *on the square lattice, and* 4^{n-1} *on the triangular lattice.*

The corresponding GFs are respectively $t/(1-3t)$ and $t/(1-4t)$, and are as rational as a series can be. There is at the moment no simple combinatorial intuition as to why these animals have rational GFs. A bijection between square lattice animals and words on a 3-letter alphabet was described in [50], but it does not shed a clear light on the structure of these objects. Still, there is now a convincing explanation of the *algebraicity* of these series (see Section 3.4.2).

Example 2.9 (The area under Dyck paths). Another family of (slightly less natural) examples is provided by the enumeration of points lying below certain lattice paths. For instance, let us call *Dyck path of length* $2n$ any path P on $\mathbb{Z}^2$ formed of steps $(1,1)$ and $(1,-1)$, that starts from $(0,0)$ and ends at $(2n,0)$ without ever hitting a point with a negative ordinate. The *area* below P is the number of non-negative integer points (i,j), with $i \leq 2n$, lying weakly below P (Fig. 5). It turns out that the sum of the areas of Dyck paths of length $2n$ is simply

$$\sum_{P:|P|=2n} a(P) = 4^n.$$

Again, the rationality of the associated generating function does not seem easy to predict, but there are good combinatorial reasons explaining why it is *algebraic*. See [33, 65] for a direct explanation of this result, references, and a few variations on this phenomenon, first spotted by Kreweras [58].

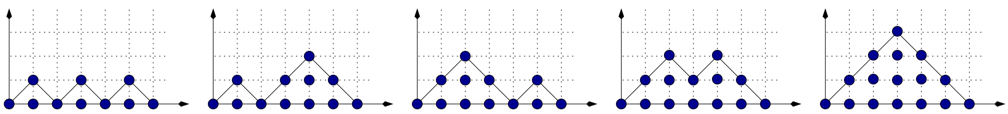


Figure 5. The 5 Dyck paths of length 6 and the $4^3 = 64$ points lying below.

Finally, let us mention that our optimistic statement about how easy it is to predict the rationality of a generating function becomes less and less true as we move from purely combinatorial problems to more algebraic ones. For instance, it is not especially easy to foresee that a group has an automatic structure [39]. Let us give also an example coming from number theory. Let $P(x) \equiv P(x_1, \ldots, x_r)$ be a polynomial with integer coefficients, and take p a prime. For $n \geq 0$, let a_n be the number of $x \in (\mathbb{Z}/p^n\mathbb{Z})^r$ such that $P(x) = 0 \mod p^n$. Then the generating function $\sum_n a_n t^n$ is rational. A related result holds with p-adic solutions [37, 53].

2.5.2. Computing a rational generating function. Let us start with an elementary, but important observation. Many enumerative problems, including some very hard, can be *approximated* by problems having a rational GF. To take one example, consider the notoriously difficult problem of counting *self-avoiding polygons* (elementary cycles) on the square lattice. It is easy to convince oneself that the generating function of SAP lying in a horizontal strip of height k is rational for all k. This does not mean that it will be easy (or even possible, in the current state of affairs) to compute the corresponding generating function when $k = 100$. Needless to say, there is at the moment no hope to express this GF for a *generic* value of k. The generating function of SAP having $2k$ horizontal steps can also be seen to be rational. Moreover, these SAP can be described in terms of linear inequalities (as in Section 2.4.3), which implies that the denominator of the corresponding series G_k is a product of cyclotomic polynomials. But again, no one knows what this series is for a generic value of k, or even for $k = 100$. Still, some progress have been made recently, since it has been proved that the series G_k have more and more poles as k increases, which means that their denominators involve infinitely many cyclotomic polynomials [68]. This may be considered as a *proof of the difficulty* of this enumerative problem [51].

In general, computing the (rational) generating function of a family of objects depending on a parameter k may be non-obvious, if not difficult, even if the objects are clearly regular, and even if the final result turns out to be nice. A classical example is provided by the growth functions of Coxeter groups [61]. Here is a more combinatorial example. A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is said to be a *k-Lecture Hall partition* (k-LHP) if

$$0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \cdots \leq \frac{\lambda_k}{k}.$$

Since these partitions are defined by linear inequalities, it follows from Section 2.4.3 that their weight generating function is rational, with a denominator formed of cyclotomic polynomials. Still, there is no clear reason to expect that [15]:

$$\sum_{\lambda\ k-\mathrm{LHP}} t^{|\lambda|} = \frac{1}{(1-t)(1-t^3)\cdots(1-t^{2k-1})}.$$

Several proofs have been given for this result and variations on it. See for instance [16, 35] and references in the latter paper. Some of these proofs are based on a bijection between lecture hall partitions and partitions into parts taken in $\{1, 3, \ldots, 2k-1\}$, but these bijections are never really simple [82, 40].

2.5.3. $\mathbb{N}$-rationality. As we wrote in Section 2.4, we do not know of a counting problem that would yield a rational, but not $\mathbb{N}$-rational series. It would certainly be interesting to find one (even if it ruins some parts of this paper).

Let us return to Soittola's criterion for $\mathbb{N}$-rationality (Theorem 2.5). It is not always easy to prove that a rational series has non-negative coefficients. For instance, it was conjectured in [46] that for any odd k, the number of partitions of n into parts taken in $\{k, k+1, \ldots, 2k-1\}$ is a non-decreasing function of n, for $n \geq 1$. In terms of generating functions, this means that the series

$$q + \frac{1-q}{(1-q^k)(1-q^{k+1})\cdots(1-q^{2k-1})}$$

has non-negative coefficients. This was only proved recently [67]. When k is even, a similar result holds for the series

$$q + \frac{1-q}{(1-q^k)(1-q^{k+1})\cdots(1-q^{2k})(1-q^{2k+1})}.$$

Once the non-negativity of the coefficients has been established, it is not hard to prove that these series are $\mathbb{N}$-rational. This raises the question of finding a family of combinatorial objects that they count.

3. Algebraic generating functions

3.1. Definitions and properties. The Laurent series $A(t)$ with coefficients in the field R is said to be *algebraic* (over $R(t)$) if it satisfies a non-trivial *algebraic* equation:

$$P(t, A(t)) = 0$$

where P is a bivariate polynomial with coefficients in R.

We assume below that $R = \mathbb{Q}$. Again, the set of algebraic Laurent series possesses numerous interesting properties [75, Ch. 6], [43, Ch. VII]. It is closed under sum, product, derivation, reciprocals, but not under integration. These closure properties become effective using either the theory of elimination or Gröbner bases, which are implemented in most computer algebra packages. The coefficients a_n of an algebraic series $A(t)$ satisfy a linear recurrence relation with polynomial coefficients: for n large enough,

$$p_0(n)a_n + p_1(n)a_{n-1} + p_2(n)a_{n-2} + \cdots + p_k(n)a_{n-k} = 0.$$

Thus the first n coefficients can be computed using a linear number of operations.

There is no systematic way to express the coefficients of an algebraic series in closed form. Still, one can sometimes apply the *Lagrange inversion formula*:

Proposition 3.1. *Let Φ and Ψ be two formal power series and let $U \equiv U(t)$ be the unique formal power series with no constant term satisfying*

$$U = t\Phi(U).$$

Then for $n > 0$, the coefficient of t^n in $\Psi(U)$ is:

$$[t^n]\Psi(U) = \frac{1}{n}[t^{n-1}]\left(\Psi'(t)\Phi(t)^n\right).$$

Given an algebraic equation $P(t, A(t)) = 0$, one can decide whether there exists a series $U(t)$ and two *rational* series Φ and Ψ satisfying

$$U = t\Phi(U) \quad \text{and} \quad A = \Psi(U). \tag{13}$$

Indeed, such series exist if and only if the *genus* of the curve $P(t, a)$ is zero [1, Ch. 15]. Moreover, both the genus and a parametrization of the curve in the form (13) can be determined algorithmically.

Example 3.2 (Finding a rational parametrization). The following algebraic equation was recently obtained [22], after a highly non-combinatorial derivation, for the GF of certain planar graphs carrying a *hard-particle configuration*:

$$\begin{aligned} 0 = 23328\, t^6 A^4 + 27\, t^4\, (91 - 2088\, t)\, A^3 + t^2\, \left(86 - 3951\, t + 46710\, t^2 + 3456\, t^3\right) A^2 \\ + \left(1 - 69\, t + 1598\, t^2 - 11743\, t^3 - 14544\, t^4\right) A - 1 + 66\, t - 1495\, t^2 + 11485\, t^3 + 128\, t^4. \end{aligned} \tag{14}$$

The package `algcurves` of Maple, and more precisely the commands `genus` and `parametrization`, reveal that a rational parametrization is obtained by setting

$$t = -3\, \frac{(3\,U + 7)\left(9\,U^2 + 33\,U + 37\right)}{(3\,U + 1)^4}.$$

Of course, this is just the net result of Maple, which is not necessarily very meaningful for combinatorics. Still, starting from this parametrization, one obtains after a few attempts an alternative parametrizing series V with positive coefficients:

$$V = \frac{t}{(1 - 2V)(1 - 3V + 3V^2)}. \tag{15}$$

The main interest of such a parametrization for this problem does *not* lie in the possibility of applying the Lagrange inversion formula. Rather, it suggests that a more combinatorial approach exists, based on the enumeration of certain *trees*, in the vein of [19, 27]. It also gives a hint of what these trees may look like.

Another convenient tool borrowed from the theory of algebraic curves is the possibility to explore all branches of the curve $P(t, A(t)) = 0$ in the neighbourhood of a given point t_0. This is based on Newton's polygon method. All branches have a *Puiseux expansion*, that is, an expansion of the form:

$$A(t) = \sum_{n \ge n_0} a_n (t - t_0)^{n/d}$$

with $n_0 \in \mathbb{Z}$, $d \in \mathbb{P}$. The coefficients a_n belong to $\mathbb{C}$ (in general, to an algebraic closure of the ground field). These expansions can be computed automatically using standard software. For instance, the Maple command `puiseux` of the `algcurves` package tells us that (14) has a unique solution that is a formal power series, the other three solutions starting with a term t^{-2}.

Such Puiseux expansions are crucial for studying the *asymptotic behaviour* of the coefficients of an algebraic series $A(t)$. As in the rational case, one has first to locate the singularities of $A(t)$, considered as a function of a complex variable t. These singularities are found among the roots of the *discriminant* and of the leading coefficient of $P(t, a)$ (seen as a polynomial in a). The singular expansion of $A(t)$ near its singularities of smallest modulus can then be converted, using certain *transfer theorems*, into an asymptotic expansion of the coefficients [42, 43, VII.4].

Example 3.3 (Asymptotics of the coefficients of an algebraic series). Consider the series $V(t)$ defined by (15). Its singularities lie among the roots of the discriminant

$$\Delta(t) = -3 + 114t - 4635t^2 + 55296t^3.$$

Only one root is real. Denote it $t_0 \sim 0.065$. The modulus of the other two roots is smaller than t_0, so they could, in theory, be candidates for singularities. However, $V(t)$ has non-negative coefficients, and this implies, by Pringsheim's theorem, that one of the roots of minimal modulus is real and positive. Hence $V(t)$ has a unique singularity, lying at t_0. A Puiseux expansion at this point gives

$$V(t) = c_0 - c_1\sqrt{1 - t/t_0} + O(t - t_0),$$

for some explicit (positive) algebraic numbers c_0 and c_1, which translates into

$$[t^n]V(t) = \frac{c_1}{2\sqrt{\pi}} t_0^{-n} n^{-3/2} \left(1 + o(1)\right).$$

The determination of asymptotic expansions for the coefficients of algebraic series is probably not far from being completely automated, at least in the case of series with non-negative coefficients [31, 43]. The "typical" behaviour is

$$a_n \sim \frac{\kappa}{\Gamma(d+1)} \rho^{-n} n^d, \tag{16}$$

where κ is an algebraic number and $d \in \mathbb{Q} \setminus \{-1, -2, -3, \ldots\}$. Compare with the result (3) obtained for rational series. Again, the above statement is not exact, as the contribution of *all* dominant singularities must be taken into account. See [43, Thm. VII.6] for a complete statement.

Let us add that, again, one can guess if a series $A(t)$ given by its first coefficients satisfies an algebraic equation $P(t, A(t)) = 0$ of a given bi-degree (d, e). The guessing procedure requires to know at least $(d+1)(e+1)$ coefficients, and amounts to solving a system of linear equations. It is implemented in the package `Gfun` of Maple [72]. For instance, given the 10 first coefficients of the series $V(t)$ satisfying $V(0) = 0$ and (15), one automatically conjectures (15).

3.2. Plane trees. Our typical "algebraic" objects will be (plane) trees. Let us begin with their usual intuitive recursive definition. A *tree* is a graph formed of a distinguished vertex (called the *root*) to which are attached a certain number (possibly zero) of trees, ordered from left to right. The number of these trees is

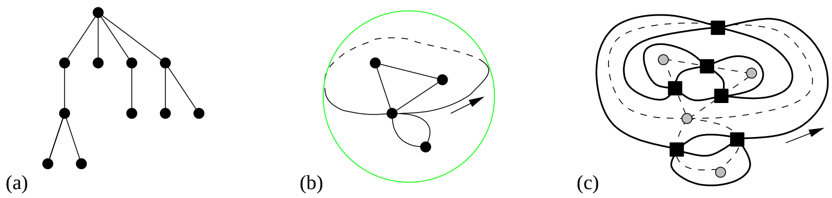


Figure 6. (a) A plane tree. (b) A rooted planar map. (c) The corresponding 4-valent map (thick lines).

the *degree* of the root. The roots of these trees are the *children* of the root. A more rigorous definition describes a tree as a finite set of words on the alphabet $\mathbb{P}$ satisfying certain conditions [63]. We hope that our less formal definition and Fig. 6(a) suffice to understand what we mean. The vertices of a tree are often called *nodes*. Nodes of degree 0 are called *leaves*, the others are called *inner* nodes.

The enumeration of classes of trees yields very often algebraic equations. Let us consider for instance the *complete binary trees*, that is, the trees in which all vertices have degree 0 or 2 (Fig. 12). Let a_n be the number of such trees having n leaves. Then, by looking at the two (sub)trees of the root, one gets, for $n > 1$:

$$a_n = \sum_{k=1}^{n-1} a_k a_{n-k}.$$

The initial condition is $a_1 = 1$. In terms of GFs, this gives $A(t) = t + A(t)^2$, which is easily solved:

$$A(t) = \frac{1-\sqrt{1-4t}}{2} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} t^{n+1}. \tag{17}$$

More generally, many algebraic series obtained in enumeration are given as the first component of the solution of a system of the form

$$A_i = P_i(t, A_1, \dots, A_k), \tag{18}$$

for some polynomials $P_i(t, x_1, \dots, x_k)$ having coefficients in $\mathbb{Z}$. This system is said to be *proper* if P_i has no constant term ($P_i(0, \dots, 0) = 0$) and does not contain any linear term x_i. It is *positive* if the coefficients of the P_i are non-negative. For instance,

$$A_1 = t^2 + A_1 A_2 \quad \text{and} \quad A_2 = 2tA_1^3$$

is a proper positive system. The system is *quadratic* if every $P_i(t, x_1, \dots, x_k)$ is a linear combination of the monomials t and $x_\ell x_m$, for $1 \leq \ell \leq m \leq k$.

Theorem 3.4 ([75], Thm. 6.6.10 and [71], Thm. IV.2.2). *A proper algebraic system has a unique solution $(A_1, \dots, A_k)$ in the set of formal power series in t with no constant term. This solution is called the* canonical *solution of the system. The series A_1 is also the first component of the solution of*

– *a proper quadratic system,*

– *a proper system of the form* $B_i = tQ_i(t, B_1, \ldots, B_\ell)$*, for* $1 \le i \le \ell$*.*

These two systems can be chosen to be positive if the original system is positive.

Proof. Let us prove the last property, which we have not found in the above references. Assume A_1 satisfies (18) and that this system is quadratic. The ith equation reads $A_i = m_i t + n_i A_{\sigma(i)} A_{\tau(i)}$. Rewrite each monomial $A_i A_j$ as tU_{ij} and add the equations $U_{ij} = t\left(m_i m_j + m_i n_j U_{\sigma(j)\tau(j)} + m_j n_i U_{\sigma(i)\tau(i)} + n_i n_j U_{\sigma(i)\tau(i)} U_{\sigma(j)\tau(j)}\right)$. The new system has the required properties. □

Definition 3.5. A series $A(t)$ is $\mathbb{N}$-*algebraic* if it has coefficients in $\mathbb{N}$ and if $A(t) - A(0)$ is the first component of the solution of a proper positive system.

Proper positive systems like (18) can always be given a combinatorial interpretation in terms of trees. Every vertex of these trees carries a label (i, c) where $i \in [\![k]\!]$ and $c \in \mathbb{P}$. We say that i is the *type* of the vertex and that c is its *colour*. The *type* of a tree is the type of its root. Write $A_0 = t$, so that $A_i = P_i(A_0, A_1, \ldots, A_k)$. Let $\mathcal{A}_0$ be the set reduced to the tree with one node, labelled $(0, 1)$. For $i \in [\![k]\!]$, let $\mathcal{A}_i$ be the set of trees such that

– the root has type i,

– the types of the subtrees of the root, visited from left to right, are $0, \ldots, 0, 1, \ldots, 1, \ldots, k, \ldots, k$, in this order,

– if exactly e_j children of the root have type j, the colour of the root is any integer in the interval $[1, m]$, where m is the coefficient of $x_0^{e_0} \cdots x_k^{e_k}$ in $P_i(x_0, \ldots, x_k)$.

Then it is not hard to see that $A_i(t)$ is the generating function of trees of type i, counted by the number of leaves. This explains why trees will be, in the rest of this paper, our typical "algebraic" objects.

3.3. Context-free languages.

As in the case of rational (and, more precisely, $\mathbb{N}$-rational) series, there exists a family of languages that is closely related to algebraic series. A *context-free grammar* G consists of

– a set $\mathcal{S} = \{S_1, \ldots, S_k\}$ of *symbols*, with one distinguished symbol, say, S_1,

– a finite alphabet $\mathcal{A}$ of *letters*, disjoint from $\mathcal{S}$,

– a set of *rewriting rules* of the form $S_i \to w$ where w is a non-empty word on the alphabet $\mathcal{S} \cup \mathcal{A}$.

The grammar is *proper* if there is no rule $S_i \to S_j$. The language $\mathcal{L}(G)$ generated by G is the set of words *on the alphabet* $\mathcal{A}$ that can be obtained from S_1 by applying iteratively the rewriting rules. A language is *context-free* is there exists a context-free grammar that generates it. In this case there exists also a proper context-free grammar that generates it.

Example 3.6 (Dyck words). Consider the grammar G having only one symbol, S, alphabet $\{a, b\}$, and rules $S \to ab + abS + aSb + aSbS$ (which is short for $S \to ab$, $S \to abS$, $S \to aSb$, $S \to aSbS$). It is easy to see that $\mathcal{L}(G)$ is the set of non-empty words u on $\{a, b\}$ such that $|u|_a = |u|_b$ and for every prefix v of u, $|v|_a \ge |v|_b$. These words, called *Dyck words*, provide a simple encoding of the Dyck paths met in Example 2.9.

A *derivation tree* associated with G is a plane tree in which all inner nodes are labelled by symbols, and all leaves by letters, in such a way that if a node is labelled S_i and its children $w_1, \ldots, w_k$ (from left to right), then the rewriting rule $S_i \to w_1 \cdots w_k$ is in the grammar. If the root is labelled S_1, then the word obtained by reading the labels of the leaves in prefix order (*i.e.*, from left to right) belongs to the language generated by G. Conversely, for every word w in $\mathcal{L}(G)$, there exists at least one derivation tree with root labelled S_1 that gives w. The grammar is said to be *unambiguous* if every word of $\mathcal{L}(G)$ admits a unique derivation tree.

Assume G is proper. For $1 \le i \le k$, let $A_i(t)$ be the generating function of derivation trees rooted at S_i, counted by the number of leaves. With each rule r, associate the monomial $M(r) = x_0^{e_0} \cdots x_k^{e_k}$ where e_0 [resp. e_i, with $i > 0$] is the number of letters of $\mathcal{A}$ [resp. occurrences of S_i] in the right-hand side of r. Then the series $A_1, \ldots, A_k$ form the canonical solution of the proper positive system (18), with

$$P_i(x_0, x_1, \ldots, x_k) = \sum_r M(r),$$

where the sum runs over all rules r with left-hand side S_i.

Conversely, starting from a positive system $B_i = tQ_i(t, B_1, \ldots, B_k)$ and its canonical solution, it is always possible to construct an unambiguous grammar with symbols $S_1, \ldots, S_k$ such that B_i is the generating function of derivation trees rooted at S_i (the idea is to introduce a new letter a_i for each occurrence of t). In view of Theorem 3.4 and Definition 3.5, this gives the following alternative characterization of $\mathbb{N}$-algebraic series:

Proposition 3.7. *A series $A(t)$ is $\mathbb{N}$-algebraic if and only if only $A(0) \in \mathbb{N}$ and there exists an unambiguous context-free language having generating function $A(t) - A(0)$.*

3.4. The combinatorial intuition of algebraic generating functions.

We have described two families of combinatorial objects that naturally yield algebraic GFs: plane trees and words of unambiguous context-free languages. We have, moreover, shown a close relationship between these two types of objects. These two families convey the standard intuition of what a family with an algebraic generating function looks like: the algebraicity suggests that it *may* (or should...) be possible to give a recursive description of the objects based on disjoint union of sets and concatenation of objects. Underlying such a description is a context-free grammar. This intuition is the basis of the so-called Schützenberger methodology, according to which the “right” combinatorial way of proving algebraicity is to describe a bijection between the objects one counts and the words of an unambiguous context-free language. This approach has led in the 80’s and 90’s to numerous satisfactory explanations of the algebraicity of certain series, and we describe some of them in this subsection. Let us, however, warn the reader that the similarities with the rational case will stop here. Indeed, it seems that the “context-free” intuition is far from explaining all algebraicity phenomena in enumerative combinatorics. In particular,

(i) it is very likely that many families of objects have an algebraic, but not $\mathbb{N}$-algebraic generating function,

(ii) there are many families of combinatorial objects with an algebraic GF that do not exhibit a clear "context-free" structure, based on union and concatenation. For several of these families, there is just no explanation of this type, be it clear or not.

This will be discussed in the next subsections. For the moment, let us illustrate the "context-free" intuition.

3.4.1. Walks on a line. Let $\mathcal{S}$ be a finite subset of $\mathbb{Z}$. Let $\mathcal{W}$ be the set of walks on the line $\mathbb{Z}$ that start from 0 and take their steps in $\mathcal{S}$. The *length* of a walk is its number of steps. Let $\mathcal{W}_k$ be the set of walks ending at position k. For $k \geq 0$, let $\mathcal{M}_k$ be the subset of $\mathcal{W}_k$ consisting of walks that never visit a negative position, and let $\mathcal{M}$ be the union of the sets $\mathcal{M}_k$. In probabilistic terms, the walks in $\mathcal{M}$ would be called *meanders* and the walks of $\mathcal{M}_0$ *excursions.* Of course, a walk is simply a sequence of steps, hence a word on the alphabet $\mathcal{S}$. Thus the sets of walks we have defined can be considered as languages on this alphabet.

Theorem 3.8. *The language $\mathcal{W}$ is simply $\mathcal{S}^*$ and is thus regular. The languages $\mathcal{M}$, $\mathcal{W}_k$ and $\mathcal{M}_k$ are unambiguous context-free for all k.*

Proof. We only describe the (very simple) case $\mathcal{S} = \{+1, -1\}$, to illustrate the ideas that are involved in the construction of the grammar. We encode the steps $+1$ by the letter a, the steps -1 by b, and introduce some auxiliary languages:

- $\mathcal{M}_0^-$, the subset of $\mathcal{W}_0$ formed of walks that never visit a positive position,
- $\mathcal{W}_0^+$ [resp. $\mathcal{W}_0^-$], the subset of $\mathcal{W}_0$ formed of walks that start with a [resp. b].

The language $\mathcal{M}_0$ will be generated from the symbol M_0, and similarly for the other languages. By looking at the first time a walk of $\mathcal{M}_0$ [resp. $\mathcal{M}_0^-$] reaches position 0 after its first step, one obtains

$$M_0 \rightarrow a(1+M_0)b(1+M_0) \quad \text{and} \quad M_0^- \rightarrow b(1+M_0^-)a(1+M_0^-).$$

By considering the last visit to 0 of a walk of $\mathcal{M}_k$, one obtains, for $k > 0$:

$$M_k \rightarrow (1+M_0)a\,(1_{k=1} + M_{k-1})\,.$$

This is easily adapted to general meanders:

$$M \rightarrow M_0 + (1+M_0)a\,(1+M)\,.$$

Considering the first step of a walk of $\mathcal{W}_0$ gives

$$W_0 \rightarrow W_0^+ + W_0^- \quad \text{with} \quad W_0^+ \rightarrow M_0(1+W_0^-) \quad \text{and} \quad W_0^- \rightarrow M_0^-(1+W_0^+).$$

Finally, for $k > 0$, looking at the first visit at 1 [resp. -1] of a walk of $\mathcal{W}_k$ [resp. $\mathcal{W}_{-k}$] yields

$$W_k \rightarrow (1+M_0^-)a\,(1_{k=1} + W_{k-1}) \quad \big[\text{resp.} \quad W_{-k} \rightarrow (1+M_0)b\,\big(1_{k=1} + W_{-(k-1)}\big)\big]\,.$$

For a general set of steps $\mathcal{S}$, various grammars have been described for the languages $\mathcal{M}_k$ of meanders [38, 60, 59]. For $\mathcal{W}_k$, we refer to [59, Section 4] where the (representative) case $\mathcal{S} = \{-2, -1, 0, 1, 2\}$ is treated. $\square$

Theorem 3.8 is often described in terms of walks in $\mathbb{Z}^2$ starting from $(0, 0)$ and taking their steps in $\{(1, j), j \in \mathcal{S}\}$. The conditions on the positions of the walks that lead to the definition of $\mathcal{M}_k$ and $\mathcal{W}_k$ are restated in terms of conditions on the *ordinates* of the vertices visited by the walk. A harmless generalization is obtained by taking steps in a finite subset $\mathcal{S}$ of $\mathbb{P} \times \mathbb{Z}$. A walk is still encoded by a word on the alphabet $\mathcal{S}$. The languages $\mathcal{W}_k$ remain unambiguous context-free. If each step (i, j) is, moreover, weighted by a rational number $w_{i,j}$, then the generating function of walks of $\mathcal{W}$, counted by the coordinates of their endpoint, is

$$W(t, s) = \frac{1}{1 - \sum_{(i,j)\in\mathcal{S}} w_{i,j} t^i s^j}.$$

The generating function $W_k(t)$ that counts (weighted) walks ending at ordinate k is the coefficient of s^k in $W(t, s)$. Since $\mathcal{W}_k$ is unambiguous context-free, the series $W_k(t)$ is algebraic. This gives a combinatorial explanation of the following result [75, Thm. 6.3.3].

Theorem 3.9 (Diagonals of rational series). *Let $A(x, y) = \sum_{m,n\geq 0} a_{m,n} x^m y^n$ be a series in two variables x and y, with coefficients in $\mathbb{Q}$, that is rational. Then the diagonal of A, that is, the series $\Delta A(t) = \sum_{n\geq 0} a_{n,n} t^n$, is algebraic.*

Proof. By linearity, it suffices to consider the case

$$A(x, y) = \frac{x^a y^b}{1 - \sum_{0\leq m,n\leq d} c_{m,n} x^m y^n},$$

with $c_{0,0} = 0$. Set $x = ts$ and $y = t/s$. The diagonal of A satisfies

$$\Delta A(t^2) = [s^0]A(ts, t/s) = t^{a+b}[s^{b-a}]\frac{1}{1 - \sum_{0\leq m,n\leq d} c_{m,n} t^{m+n} s^{m-n}},$$

which is algebraic as it counts weighted paths in $\mathcal{W}_{b-a}$, for a certain set of steps. Hence $\Delta A(t)$ is algebraic too. $\square$

The converse of Theorem 3.9 holds: every series $B(t)$ that is algebraic over $\mathbb{Q}(t)$ is the diagonal of a bivariate rational series $A(x, t)$ [70].

Note. If one is simply interested in obtaining a set of algebraic equations defining the GFs of the sets $\mathcal{M}_k$ and $\mathcal{W}_k$, a more straightforward approach is to use a partial fraction decomposition (for $\mathcal{W}_k$) and the kernel method (for $\mathcal{M}_k$). See [75, 6.3], and [17, Example 3].

3.4.2. Directed animals. Let us move to an example where a neat context-free exists, but is uneasy to discover. We return to the *directed animals* defined in Section 2.5.1. As discussed there, there is no simple explanation as to why the number of compact-source animals is so simple (Theorem 2.8). Still, there is a convincing explanation for the *algebraicity* of the corresponding series: directed animals have, indeed, a context-free structure. This structure was discovered a few years after the proof of Theorem 2.8, with the development by Viennot of the theory of *heaps* [81], a geometric version of partially commutative monoids [30]. Intuitively, a heap is obtained by dropping vertically some solid pieces, the one after the other. Thus, a piece lies either on the "floor" (then it is said to be *minimal*), or covers, at least partially, another piece.

Directed animals *are*, in essence, heaps. To see this, replace every point of the animal by a *dimer* (Fig. 7). Note that if the animal has a unique source, the associated heap has a unique minimal piece. Such heaps are named *pyramids*.

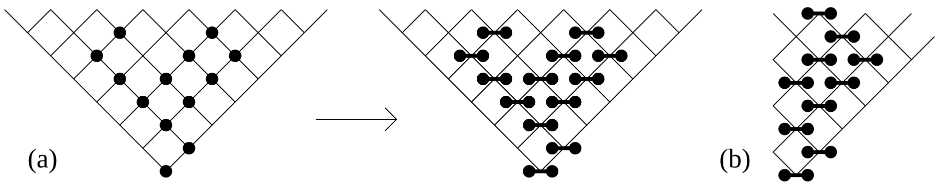


Figure 7. (a) A directed animal and the associated pyramid. (b) A half-pyramid.

What makes heaps interesting here is that there exists a *monoid structure* on the set of heaps: The product of two heaps is obtained by putting one heap above the other and dropping its pieces. This product is the key in our context-free description of directed animals.

Let us begin with the description of pyramids (one-source animals). A pyramid is either a *half-pyramid* (Fig. 7(b)), or the product of a half-pyramid and a pyramid (Fig. 8, top). Let $P(t)$ denote the GF of pyramids counted by the number of dimers, and $H(t)$ denote the GF of half-pyramids. Then $P(t) = H(t)(1 + P(t))$. Now, a half-pyramid may be reduced to a single dimer. If it has several dimers, it is the product of a single dimer and of one or two half-pyramids (Fig. 8, bottom), which implies $H(t) = t + tH(t) + tH^2(t)$.
A trivial computation finally provides the GF of directed (single-source) animals:

$$P(t) = \frac{1}{2}\left(\sqrt{\frac{1+t}{1-3t}} - 1\right) \qquad \left(\text{while } H(t) = \frac{1-t-\sqrt{(1+t)(1-3t)}}{2t}\right).$$

The enumeration of compact-source directed animals is equivalent to the enumeration of heaps having a compact *basis* (the minimal dimers are adjacent). The generating function of heaps having a compact basis formed with k dimers is $P(t)H(t)^{k-1}$ (Fig. 9), which implies that the generating function of compact-source animals is

$$\frac{P(t)}{1-H(t)} = \frac{t}{1-3t}.$$

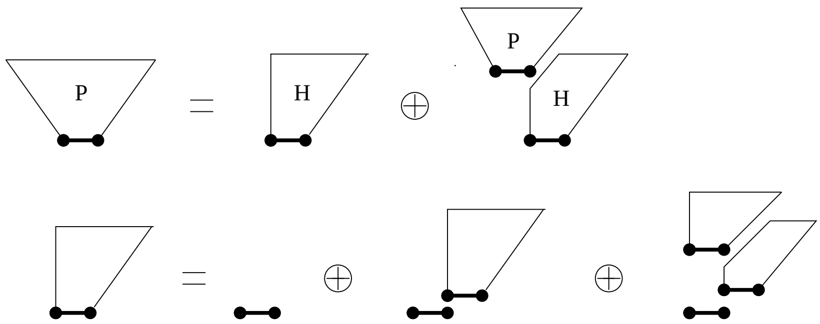


Figure 8. Decomposition of pyramids (top) and half-pyramids (bottom).

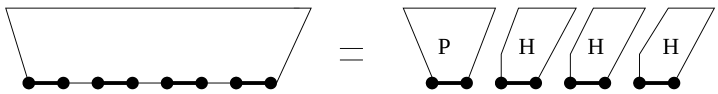


Figure 9. Decomposition of heaps having a compact basis.

3.5. The world of planar maps. We have seen in Section 3.2 that plane trees are the paradigm for objects with an algebraic generating function. A more general family of plane objects seems to be just as deeply associated with algebraic series, but for reasons that are far more mysterious: *planar maps*.

A (planar) map is a proper embedding of a planar graph in the sphere (Fig. 6(b)). In order to avoid symmetries, all the maps we consider are *rooted*: this means that one edge is distinguished and oriented. Maps are only considered up to a continuous deformation of the sphere. A map induces a 2-cell decomposition of the sphere: the cells of dimension 0 [resp. 1, 2] are called *vertices* [resp. *edges*, *faces*]. Hence plane trees are maps with a single face.

The interest for the enumeration of planar maps dates back to the early 60's, in connection with the 4-colour theorem. The first results are due to Tutte [77, 78, 79]. Ten to fifteen years later, maps started to be investigated independently in theoretical physics, as a model for 2-dimensional *quantum gravity* [28, 9]. However, neither the recursive approach used by Tutte and his disciples, nor the physics approach based on matrix integrals were able to explain in a combinatorially satisfactory way the following observations:

– the generating functions of many classes of planar maps are algebraic,

– the associated numbers are often irritatingly simple.

Let us illustrate this with three examples:

1. General maps. The number of planar maps having n edges is [80]:

$$g_n = \frac{2.3^n}{(n+1)(n+2)}\binom{2n}{n}. \tag{19}$$

The associated generating function $G \equiv G(t) = \sum_{n\geq 0} g_n t^n$ satisfies:

$$-1 + 16t + (1 - 18t)G + 27t^2 G^2 = 0. \tag{20}$$

2. Loopless triangulations. The number of loopless *triangulations* (maps in which all faces have degree 3) having $2n + 2$ faces is [62]:

$$t_n = \frac{2^n}{(n+1)(2n+1)} \binom{3n}{n}.$$

The associated generating function $T \equiv T(t) = \sum_n t_n t^n$ satisfies

$$1 - 27t + (-1 + 36t)T - 8tT^2 - 16t^2 T^3 = 0.$$

3. Three-connected triangulations. The number of 3-connected triangulations having $2n + 2$ faces is [77]:

$$m_n = \frac{2}{(n+1)(3n+2)} \binom{4n+1}{n}.$$

The associated generating function $M \equiv M(t) = \sum_n t_n t^n$ satisfies

$$-1 + 16t + (1 - 20t)M + (3t + 8t^2)M^2 + 3t^2 M^3 + t^3 M^4 = 0.$$

These maps are in bijection with rooted *maximal* planar simple graphs (graphs with no loop nor multiple edge that lose planarity as soon as one adds an edge).

At last, in the past ten years, a general combinatorial picture has emerged, suggesting that maps are, in essence, unrooted plane trees. In what follows, we illustrate on the example of general maps the main three approaches that now exist, and give references for further developments of these methods.

3.5.1. The recursive approach. We leave to the reader to experience personally that maps do not have an obvious context-free structure. Still, maps *do* have a simple recursive structure, based on the deletion of the root-edge. However, in order to exploit this structure, one is forced to keep track of the degree of the *root-face* (the face lying to the right of the root edge). The decomposition illustrated in Fig. 10 leads in a few lines to the following equation:

$$G(u, t) = 1 + tu^2 G(u, t)^2 + tu\, \frac{uG(u, t) - G(1, t)}{u - 1}, \tag{21}$$

where $G(u, t)$ counts planar maps by the number of edges (t) and the degree of the root-face (u).

It can be checked that the above equation defines $G(u, t)$ uniquely as a formal power series in t (with polynomial coefficients in u). However, it is not clear on the equation why $G(1, t)$ (and hence $G(u, t)$) are algebraic. In his original paper, Tutte first guessed the value of $G_1(t) := G(1, t)$, and then proved the existence of a series $G(u, t)$ that fits with $G_1(t)$ when $u = 1$, and satisfies the above equation. Still,

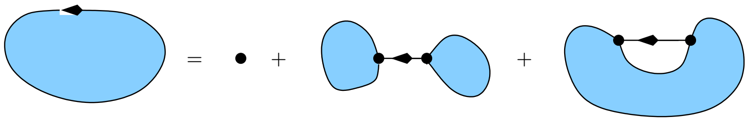


Figure 10. Tutte's decomposition of rooted planar maps.

a bit later, Brown came with a *method* for solving (21): the so-called *quadratic method* [29, 49, Sec. 2.9]. Write (21) in the form $(2aG(u,t)+b)^2=\delta$, where a, b and δ are polynomials in t, u and $G_1(t)$. That is,

$$\left(2tu^2(u-1)G(u,t)+tu^2-u+1\right)^2 = 4t^2u^3(u-1)G_1+(1-u)^2-4tu^4+6tu^3+u^4t^2-2tu^2.$$

It is not hard to see, even without knowing the value of $G(u,t)$, that there exists a (unique) formal power series in t, say $U \equiv U(t)$, that cancels the left-hand side of this equation. That is,

$$U = 1 + tU^2 + 2tU^2(U-1)G(U,t).$$

This implies that the series U is a *double root* of the polynomial δ that lies on the right-hand side. The discriminant of this polynomial (in u) thus vanishes: this gives the algebraic equation (20) satisfied by $G(1,t)$.

The enumeration of many other families of planar maps can also be attacked by a recursive description based on the deletion of an edge (or vertex, or face...). See for instance [62] for 2-connected triangulations, or [6] for maps with prescribed face degrees. (For maps with high connectivity, like 3-connected triangulations, an additional *composition formula* is often required [77, 3].) The resulting equations are usually of the form

$$P(F(u), F_1, \ldots, F_k, t, u) = 0, \tag{22}$$

where $F(u) \equiv F(t,u)$, the main generating function, is a series in t with polynomial coefficients in u, and $F_1, \ldots, F_k$ are series in t only, independent of u. Brown's *quadratic method* applies as long as the degree in $F(u)$ is 2 (for the linear case, see the *kernel method* in [17, 2]). Recently, it was understood how these equations could be solved in full generality [22]. Moreover, the solution of any (well-founded) equation of the above type was shown to be algebraic. This provides two types of enumerative results:

– the proof that many map generating functions are algebraic: it now suffices to exhibit an equation of the form (22), or to explain why such an equation *exists*,

– the solution of previously unsolved map problems (like the enumeration of hard-particle configurations on maps, which led to (14), or that of triangulations with high vertex degrees [8]).

3.5.2. Matrix integrals. In the late 70's, it was understood by a group of physicists that certain matrix integral techniques coming from quantum field theory could be used to attack enumerative problems on maps [28, 9]. This approach proved to be extremely efficient (even if it is usually not fully rigorous). The first step is fairly automatized, and consists in converting the description of maps into a certain integral. For instance, the relevant integral for the enumeration of 4-valent maps (maps in which all vertices have degree 4) is

$$Z(t,N) = \frac{2^{N(N-1)/2}}{(2\pi)^{N^2/2}} \int dH e^{\mathrm{tr}\left(-H^2/2 + tH^4/N\right)},$$

where the integration space is that of hermitian matrices H of size N, equipped with the Lebesgue measure $dH = \prod dx_{kk} \prod_{k<\ell} dx_{k\ell} dy_{k\ell}$ with $h_{k\ell} = x_{k\ell} + iy_{k\ell}$. As there is a classical bijection between 4-valent maps with n vertices and planar maps with n edges (Fig. 6(c)), we are still dealing with our reference problem: the enumeration of general planar maps. The connection between the above integral and maps is

$$G(t) = tE'(t) \quad \text{with} \quad E(t) = \lim_{N\to\infty} \frac{1}{N^2} \log Z(t,N).$$

Other map problems lead to integrals involving several hermitian matrices [55]. We refer to [83] for a neat explanation of the encoding of map problems by integrals, and to [45, 41] (and references therein) for the evaluation of integrals.

3.5.3. Planar maps and trees. We finally come to a combinatorial explanation of the formula/equation for g_n and $G(t)$. Take a plane binary tree with n (inner) nodes, planted at a leaf, and add to every inner node a new distinguished child, called a *bud*. At each node, we have three choices for the position of the bud (Fig. 11(a)). The new tree, called *budding tree*, has now n buds and $n+2$ leaves. Now start from the root and walk around the tree in counterclockwise order, paying attention to the sequence of buds and leaves you meet. Each time a bud is immediately followed by a leaf in this sequence, match them by forming a new edge (Fig. 11(b)) and then go on walking around the plane figure thus obtained. At the end, exactly two leaves remain unmatched. Match them together and orient this final edge in one of the two possible ways. Also, mark the face to the left of the matching edge that ends at the root-leaf.

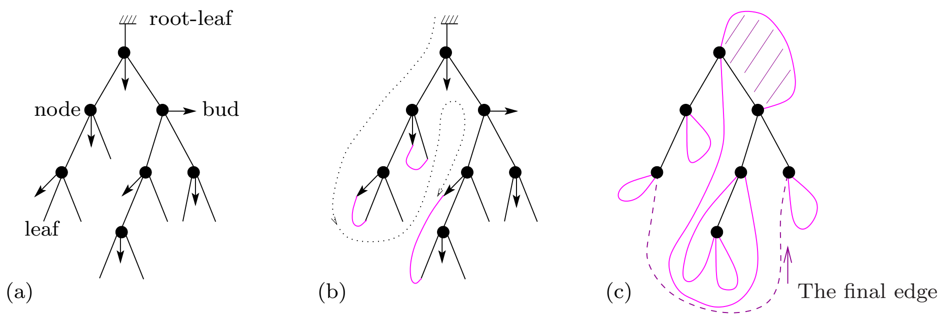


Figure 11. (a). A budding tree. (b) An intermediate step in the matching procedure. (c) The resulting 4-valent map, with its marked face.

Theorem 3.10 ([73]). *The above correspondence is a bijection between pairs* (T, ϵ) *where* T *is a budding tree having* n *inner nodes and* $\epsilon \in \{0, 1\}$*, and* 4*-valent maps with* n *vertices and a marked face.*

The value of ϵ tells how to orient the final matching edge. Schaeffer first used this bijection to explain combinatorially the formula (19). Indeed, the number of budding trees with n inner nodes is clearly $3^n\binom{2n}{n}/(n+1)$ (see (17)), while the number of 4-valent maps with n vertices and a marked face is $(n+2)g_n$. Eq. (19) follows.

Later, it was realized that this construction could also be used to explain the algebraicity of the series $G(t)$ [23]. Say that a budding tree is *balanced* if the root-leaf is not matched by a bud. Take such a tree, match all buds, and orient the final edge from the root-leaf to the other unmatched leaf. This gives a bijection between balanced budding trees and 4-valent maps. We thus have to count balanced trees, or, equivalently, the unbalanced ones. By re-rooting them at the bud that matches the root-leaf, one sees that they are in bijection with a node attached to three budding trees. This gives

$$G(t) = B(t) - tB(t)^3, \qquad \text{where} \qquad B(t) = 3t(1 + B(t))^2$$

counts budding trees by (inner) nodes. The above construction involves taking a *difference* of $\mathbb{N}$-algebraic series, which needs not be $\mathbb{N}$-algebraic. We actually conjecture that the series $G(t)$ is not $\mathbb{N}$-algebraic (see Section 3.6.4).

There is little doubt that the above construction (once described in greater detail...) explains in a very satisfactory way both the simplicity of the formula giving g_n and the algebraicity of $G(t)$. Moreover, this is not an *ad hoc*, isolated magic trick: over the past ten years, it was realized that this construction is one in a family of constructions of the same type, which apply to numerous families of maps (Eulerian maps [73], maps with prescribed vertex degrees [23], constellations [18], bipartite maps with prescribed degrees [19], maps with higher connectivity [66, 47]). Definitely, these constructions reveal a lot about the combinatorial nature of planar maps.

To conclude this section, let us mention that a different combinatorial construction for general planar maps, discovered in the early 80's [34], has recently been simplified [32] and adapted to other families of maps [36, 54, 25, 26]. It is a bit less easy to handle than the one based on trees with buds, but it allows one to keep track of the distances between some vertices of the map. This has led to remarkable connections with a random probability distribution called the *Integrated SuperBrownian Excursion* [32]. A third type of construction has emerged even more recently [7] for 2-connected triangulations, but no ones knows at the moment whether it will remain isolated or is just the tip of another iceberg.

3.6. Algebraic series: some questions. We begin with three simple classes of objects that have an algebraic GF, but for reasons that remain mysterious. We then discuss a possible criterion (or necessary condition) for $\mathbb{N}$-algebraicity, and finally the algebraicity of certain hypergeometric series.

3.6.1. Kreweras' words and walks on the quarter plane. Let $\mathcal{L}$ be the set of words u on the alphabet $\{a, b, c\}$ such that for every prefix v of u, $|v|_a \geq |v|_b$ and $|v|_a \geq |v|_c$. These words encode certain walks on the plane: these walks start at $(0, 0)$, are made of three types of steps, $a = (1, 1)$, $b = (-1, 0)$ and $c = (0, -1)$, and never leave the first quadrant of the plane, defined by $x, y \geq 0$. The *pumping lemma* [52, Thm. 4.7], applied to the word $a^n b^n c^n$, shows that the language $\mathcal{L}$ is not context-free. However, its generating function *is* algebraic. More precisely, let us denote by $\ell_{i,j}(n)$ the number of words u of $\mathcal{L}$ of length n such that $|u|_a - |u|_b = i$ and $|u|_a - |u|_c = j$. They correspond to walks of length n ending at position (i, j). Then the associated three-variable generating function is

$$L(u, v; t) = \sum_{i,j,n} \ell_{i,j}(n) u^i v^j t^n = \frac{(1/W - \bar{u})\sqrt{1 - uW^2} + (1/W - \bar{v})\sqrt{1 - vW^2}}{uv - t(u + v + u^2v^2)} - \frac{1}{uvt}$$

where $\bar{u} = 1/u$, $\bar{v} = 1/v$ and $W \equiv W(t)$ is the unique power series in t satisfying $W = t(2 + W^3)$. Moreover, the number of walks ending at $(i, 0)$ is remarkably simple:

$$\ell_{i,0}(3n + 2i) = \frac{4^n(2i + 1)}{(n + i + 1)(2n + 2i + 1)} \binom{2i}{i} \binom{3n + 2i}{n}.$$

The latter formula was proved in 1965 by Kreweras, in a fairly complicated way [57]. This rather mysterious result has attracted the attention of several combinatorialists since its publication [14, 48, 64]. The first combinatorial explanation of the above formula (in the case $i = 0$) has just been found by Bernardi [7].

Walks in the quarter plane do not always have an algebraic GF: for instance, the number of *square lattice walks* (with North, South, East and West steps) of size $2n$ that start and end at $(0, 0)$ and remain in the quarter plane is

$$\frac{1}{(2n + 1)(2n + 4)} \binom{2n + 2}{n + 1}^2 \sim \frac{4^{2n+1}}{\pi n^3},$$

and this asymptotic behaviour prevents the corresponding generating function from being algebraic (see (16)). The above formula is easily proved by looking at the projections of the walk onto the horizontal and vertical axes.

3.6.2. Walks on the slit plane. Take now *any* finite set of steps $\mathcal{S} \subset \mathbb{Z} \times \{-1, 0, 1\}$ (we say that these steps have *small height variations*). Let $s_{i,j}(n)$ be the number of walks of length n that start from the origin, consist of steps of $\mathcal{S}$, never return to the non-positive horizontal axis $\{(-k, 0), k \geq 0\}$, and end at (i, j). Let $S(u, v; t)$ be the associated generating function:

$$S(u, v; t) = \sum_{i,j \in \mathbb{Z}, n \geq 0} s_{i,j}(n) u^i v^j t^n.$$

Then this series is *always* algebraic, as well as the series $S_{i,j}(t) := \sum_n s_{i,j}(n)t^n$ that counts walks ending at (i, j) [13, 20]. For instance, when $\mathcal{S}$ is formed of the usual square lattice steps (North, South, West and East), then

$$S(u,v;t)=\frac{\left(1-2t(1+\bar u)+\sqrt{1-4t}\right)^{1/2}\left(1+2t(1-\bar u)+\sqrt{1+4t}\right)^{1/2}}{1-t(u+\bar u+v+\bar v)}$$

with $\bar u = 1/u$ and $\bar v = 1/v$. Moreover, the number of walks ending at certain specific points is remarkably simple. For instance:

$$s_{1,0}(2n+1)=C_{2n+1},\quad s_{0,1}(2n+1)=4^nC_n,\quad s_{-1,1}(2n)=C_{2n},$$

where $C_n=\binom{2n}{n}/(n+1)$ is the nth Catalan number, which counts binary trees (17), Dyck words, and numerous other combinatorial objects [75, Ch. 6]. The first of these three identities has been proved combinatorially [4]. The others still defeat our understanding.

3.6.3. Embedded binary trees. We consider again the complete binary trees met at the beginning of Section 3.2. Let us associate with each (inner) node of such a tree a label, equal to the difference between the number of right steps and the number of left steps one takes when going from the root to the node. In other words, the label of the node is its abscissa in the natural integer embedding of the tree (Fig. 12).

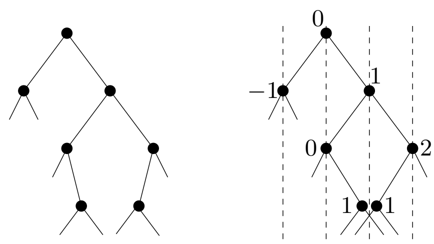


Figure 12. The integer embedding of a binary tree.

Let $S_j \equiv S_j(t,u)$ be the generating function of binary trees counted by the number of nodes (variable t) and the number of nodes at abscissa j (variable u). Then for all $j\in\mathbb{Z}$, this series is algebraic of degree (at most) 8 (while $S_j(t,1)$ is quadratic) [12]. Moreover, for $j\ge 0$,

$$S_j=T\,\frac{(1+\mu Z^j)(1+\mu Z^{j+5})}{(1+\mu Z^{j+2})(1+\mu Z^{j+3})},$$

where

$$T=1+tT^2,\qquad Z=t\,\frac{\left(1+Z^2\right)^2}{(1-Z+Z^2)},$$

and $\mu\equiv\mu(t,u)$ is the unique formal power series in t satisfying

$$\mu=(u-1)\frac{Z(1+\mu Z)^2(1+\mu Z^2)(1+\mu Z^6)}{(1+Z)^2(1+Z+Z^2)(1-Z)^3(1-\mu^2Z^5)}.$$

Why is that so? This algebraicity property holds as well for other families of labelled trees [12, 24]. From these series, one can derive certain limit results on the distribution of the number of nodes at abscissa $\lfloor \lambda n^{1/4} \rfloor$ in a random tree with n nodes [12]. These results provide some information about the law of the *integrated super-Brownian excursion* [12, 21].

3.6.4. $\mathbb{N}$-algebraicity. $\mathbb{N}$-algebraic series have been defined in Section 3.2 in terms of positive proper algebraic systems. The author has been unable to find in the literature a criterion, or even a necessary condition for an algebraic series with coefficients in $\mathbb{N}$ to be $\mathbb{N}$-algebraic. Nor even an algebraic series with coefficients in $\mathbb{N}$ that would *not* be $\mathbb{N}$-algebraic (together with a proof of this statement...).

A partial answer could be provided by the study of the possible asymptotic behaviour of coefficients of $\mathbb{N}$-algebraic series. It is very likely that not all behaviours of the form (16) are possible. An important result in this direction states that, if a proper positive system (18) is *strongly connected*, the nth coefficient of, say, A_1 follows the general pattern (16), but with $d = -3/2$ [43, Thm. VII.7]. The system is *strongly connected* if, roughly speaking, the expression of every series A_i involves (possibly after a few iterations of the system) every other series A_j. For instance, the system defining the walks ending at 0 in Section 3.4.1 reads

$$M_0 = t^2(1 + M_0)^2 \quad \text{and} \quad W_0 = M_0(2 + W_0).$$

This system is not strongly connected, as M_0 does not involve W_0. Accordingly, the number of $2n$-step walks returning to 0 is $\binom{2n}{n} \sim \kappa 4^n n^{-1/2}$.

If one can rule out the possibility that $d = -5/2$ for $\mathbb{N}$-algebraic series, then this will prove that most map generating functions are not $\mathbb{N}$-algebraic (see the examples in Section 3.5).

3.6.5. Some algebraic hypergeometric series. Consider the following series:

$$F(t) = \sum_{n \geq 0} f_n t^n = \sum_{n \geq 0} \frac{\prod_{i=1}^{d} (a_i n)!}{\prod_{j=1}^{e} (b_j n)!} t^n,$$

where $a_1, \ldots, a_d, b_1, \ldots, b_e$ are positive integers. This series is algebraic for some values of the a_i's and b_j's, as shown by the case

$$\sum_{n \geq 0} \frac{(2n)!}{n!^2} t^n = \frac{1}{\sqrt{1-4t}}.$$

Can we describe all algebraic cases? Well, one can easily obtain some necessary conditions on the sequences a and b by looking at the asymptotics of f_n. First, an algebraic power series has a finite, positive radius of convergence (unless it is a polynomial). This, combined with Stirling's formula, gives at once

$$a_1 + \cdots + a_d = b_1 + \cdots + b_e. \tag{23}$$

Moreover, by looking at the dominant term in the asymptotic behaviour of f_n, and comparing with (16), one obtains that either $e = d$, or $e = d + 1$. The case $d = e$

only gives the trivial solution $F(t) = 1/(1-t)$, and the complete answer to this problem is as follows [11, 69]:

Theorem 3.11. *Assume* (23) *holds, and* $F(t) \neq 1/(1-t)$. *The series* $F(t)$ *is algebraic if and only if* $f_n \in \mathbb{N}$ *for all* n *and* $e = d + 1$.

Here are some algebraic instances:

$$f_n = \frac{(6n)!(n)!}{(3n)!(2n)!^2}, \quad f_n = \frac{(10n)!(n)!}{(5n)!(4n)!(2n)!}, \quad f_n = \frac{(20n)!(n)!}{(10n)!(7n)!(4n)!}.$$

The degree of these series is rather big: 12 [resp. 30] for the first [second] series above. This theorem provides a collection of algebraic series with nice integer coefficients: are these series $\mathbb{N}$-algebraic? Do they count some interesting objects?

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CNRS, LaBRI, Université Bordeaux 1
351 cours de la Libération, 33405 Talence Cedex, France
E-mail: bousquet@labri.fr