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# Evaluation for moments of a ratio with application to regression estimation

**Paul Doukhan**

*LS-CREST, Timbre J340  
3, avenue Pierre Larousse,  
92240 Malakoff*

*and*

*SAMOS-MATISSE-CES  
Université Paris 1-Panthéon-Sorbonne.  
90 Rue de Tolbiac,  
75634 Paris Cedex 13,  
France.*

*e-mail: paul.doukhan@free.fr*

**and**

**Gabriel Lang**

*UMR MIA 518  
AgroParisTech  
19 avenue du Maine  
75732 PARIS CEDEX 15,  
France.*

*e-mail: gabriel.lang@engref.agroparistech.fr*

*This work is dedicated to the memory of Gérard Collomb.*

**Abstract:** Many problems involve ratios in probability or in statistical applications. We aim at approximating the moments of such ratios under specific assumptions. Using ideas from Collomb (1977) [7], we propose sharper bounds for the moments of randomly weighted sums which also may appear as a ratio of two random variables. Suitable applications are given in more detail here in the fields of functional estimation, in finance and for censored data analysis. Several weak dependence dependence situations are considered.

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## 1. Introduction

We consider statistics with the form of a ratio. This situation arises naturally in the following simple and generic example. Let  $(V_i, W_i)_{i \geq 0}$  be a stationary sequence with values in a finite space  $\mathcal{V} \times \mathcal{W}$ , with  $\mathcal{V} \subset \mathbb{R}$ . A conditional expectation writes

$$\mathbb{E}(V_0 | W_0 = w) = \frac{\sum_{v \in \mathcal{V}} v \mathbb{P}(\{v, w\})}{\sum_{v \in \mathcal{V}} \mathbb{P}(\{v, w\})}.$$

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In order to recover several examples,

- we first go on with discrete random variables; the previous expression is empirically estimated from a sample  $(V_i, W_i)_{1 \leq i \leq n}$  by the random quantity

$$\widehat{\mathbb{E}}(V_0|W_0 = w) = \frac{\frac{1}{n} \sum_{i=1}^n V_i \cdot \mathbb{I}\{W_i = w\}}{\frac{1}{n} \sum_{i=1}^n \mathbb{I}\{W_i = w\}},$$

- the case of real valued data is more involved, here  $\mathbb{P}(\{v, w\})$  has no rigorous meaning but standard smoothing techniques authorize to consider extensions; we replace  $\mathbb{I}\{W_i = w\}$  by an approximate Dirac measure  $\delta_n(w, W_i)$ , then the previous expression takes the form of the Nadaraya-Watson kernel estimator considered below.

Many statistics take the form of a ratio  $\widehat{R}_n = \widehat{N}_n / \widehat{D}_n$ , where  $\widehat{N}_n$  and  $\widehat{D}_n$  are empirical quantities with known distributions and moments, we set:

$$\widehat{R}_n = \frac{\widehat{N}_n}{\widehat{D}_n}, \quad \widehat{D}_n = \frac{1}{n} \sum_{i=1}^n U_{i,n}, \quad \widehat{N}_n = \frac{1}{n} \sum_{i=1}^n U_{i,n} V_{i,n}. \quad (1)$$

Examples of this situation are:

- Functional estimation of a conditional expectation:  $U_{i,n} = K((X_i - x)/h_n)/h_n^d$  and  $V_{i,n} = Y_i$  for a stationary process  $(X_i, Y_i) \in \mathbb{R}^d \times \mathbb{R}$ ; here  $\widehat{R}_n = \widehat{r}(x)$  is an estimator for  $r(x) = \mathbb{E}(Y_i|X_i = x)$  and usual assumptions are  $h_n \rightarrow 0$  and  $nh_n^d \rightarrow \infty$  as  $n \rightarrow \infty$ . See Tsybakov (2004) [32] for a general setting and Ango Nze and Doukhan (2004) [3] for dependent data cases.
- Computation of empirical means for censored data: let the censoring  $U_i = C_i \in \{0, 1\}$  be independent of a process  $(V_i)$ , then  $V_i$  is observed if  $C_i = 1$  and not observed else.

$$\widehat{R}_n = \frac{1}{\#\{i \in \{1, n\} | C_i = 1\}} \sum_{1 \leq i \leq n, C_i = 1} V_i,$$

A special case for this situation is the estimation of covariances under censoring where  $V_i = X_i X_{i+\ell}$ . Under stationarity, covariance functions write  $\gamma_X(\ell) = \gamma_Y(\ell)/(\gamma_C(\ell) + \mathbb{E}C_0^2)$  where  $Y_i = C_i X_i$  is observed (Bahamonde et al. (2008) [5]). Furthermore, moments of the empirical covariances are used to reconstruct the periodogram from the censored data.

- General weighted sums

$$\widehat{R}_n = \left( \sum_{1 \leq i \leq n} U_i V_i \right) / \left( \sum_{1 \leq i \leq n} U_i \right),$$

may be used to model various quantities like prices with prices per unit  $V_i$  and volumes  $U_i$ , as in Jaber and Salameh (1995) [24].

Various alternative questions also involve a division:

- Functional estimation of point processes. A compound Poisson processes (cpp) writes  $\xi = \sum_{j=1}^N \alpha_j \delta_{X_j}$  for some random triple  $(N, (\alpha_j)_{j \geq 1}, (X_j)_{j \geq 1})$  with  $N \in \{1, 2, \dots\}$ ,  $X_j \in \mathbb{R}^d$ . For a sequences of mixing couples of cpp  $(\xi_i, \eta_i)_{i \geq 1}$  with  $\mu = \mathbb{E}\eta_1 \ll \nu = \mathbb{E}\xi_1$ , Bensaïd and Fabre (2007) [6] estimate the Radon Nykodym density  $\varphi = d\mu/d\nu$  with kernel estimates  $\varphi_n = g_n/f_n$  with  $f_n(x) = \sum_{i=1}^n \eta_i \star K_n(x)$ ,  $g_n(x) = \sum_{i=1}^n \xi_i \star K_n(x)$ , where e.g.  $\xi \star K_n(x) = \sum_{j=1}^N \alpha_j K_n(X_j - x)$  with  $K_n(x) = K(x/h)/h^d$ , the procedure is thus analogue to Nadaraya Watson estimator. In order to bound quadratic errors of this ratio it is assumed in [6] that  $|\varphi_n(x)| \leq C$  which is a heavy assumption easily relaxed by using our result.
- Self-normalized sums see e.g. de la Peña et al. (2007) [13],
- Simulation of Markov chains, and Monte-Carlo MC technique widely developed in Robert et al.'s monograph (1999) [27]; a more precise reference is Li et al (2006) [25], see the relations (45), (46) and (47) which explicitly involve ratios for reducing the dimensionality in a nonparametric problem,
- Particle filtering is considered from the theoretical viewpoint in Del Moral and Miclo (2000) [14] and for applications to change-point problems in Fearnhead and Liu (2007) [22].

Deducing the convergence in probability of the ratio from the convergence of the denominator and numerator is straightforward, however in some statistical problem,  $\mathbb{L}^p$ -convergence needs to be checked. But evaluating the moments of  $\widehat{R}_n$  is a much more difficult problem, even if one knows sharp bounds of moments for both the numerator and the denominator. Curiously, we were not able to find much references for this subject; a first method is to compute the exact distribution of the ratio; a nice example of this situation is obtained by Spiegelmann and Sachs (1980) [29], who compute the moments of a Nadaraya-Watson regression estimator with  $\{0, 1\}$ -valued kernels; in this case, independence allows to use binomial based distributions: computations are thus exact. But this computation is usually difficult to handle. An alternative is the expansion in Collomb (1977) [7]. We adressed this problem for dependent data frame, in the paper Collomb and Doukhan (1983) [8], published after Gérard Collomb's death. In [7] and [8], Collomb assumed that convergence rates in  $\mathbb{L}^q$  for  $q > 2p$  are known for the denominator.

This limitation is avoided here by using an interpolation technique, and we shall only assume such rates for some  $q > p$ . With notation (1) we set

$$N_n = \mathbb{E}\widehat{N}_n, \quad D_n = \mathbb{E}\widehat{D}_n, \quad \text{and} \quad R_n = \frac{N_n}{D_n}. \quad (2)$$

We aim at providing  $\mathbb{L}^p$ -rates of convergence to 0 of the expression

$$\Delta_n = \widehat{R}_n - R_n. \quad (3)$$

In some cases, the expectations  $N_n$  and  $D_n$  are constant. In the other cases that we present, we only have convergence to some constants,  $N_n \rightarrow N$  and  $D_n \rightarrow D$

as  $n \rightarrow \infty$ , thus moments of ratio may be proved to converge with the bound

$$\left\| \widehat{R}_n - \frac{N}{D} \right\|_p \leq \|\Delta_n\|_p + \left| \frac{N}{D} - R_n \right|.$$

Convergence in probability or *a.s.* is immediate but to obtain moment bounds, one needs to divide at least by a nonzero expression; for simplicity, we definitely assume it to be nonnegative and fix  $U_{i,n} \geq 0$ . In this case, an alternative viewpoint is to represent the previous expression as a weighted sum

$$\widehat{R}_n = \sum_{i=1}^n w_{i,n} V_{i,N}, \quad w_{i,n} = \frac{U_{i,n}}{\sum_{j=1}^n U_{j,n}} \geq 0, \quad \sum_{i=1}^n w_{i,n} = 1,$$

it is thus clear that the previous representation of  $\widehat{R}_n$  belongs to the convex hull of  $(U_{i,n})_{1 \leq i \leq n}$ .

The paper is organized as follows. Section 2 is devoted to the main lemma and comments. The two following sections are dedicated to applications of it. Section 3 is concerned with simple weighted sums while Section 4 considers the behavior of the Nadaraya-Watson kernel regression estimation  $\widehat{r}_n(x)$  of a regression function  $r(x) = \mathbb{E}(Y|X = x)$ ; it is divided into two subsections. A first subsection directly applies the lemma to provide the minimax bound  $\|\widehat{r}_n(x) - r(x)\|_p = \mathcal{O}(n^{-\frac{\rho}{2\rho+d}})$  in an estimation problem with dimension  $d$  and regularity  $\rho$  less than 2). A second subsection makes use of the same ideas with slight modification to derive the (also minimax) bound  $\|\sup_{x \in B} |\widehat{r}_n(x) - r(x)|\|_p = \mathcal{O}((n/\log n)^{-\frac{\rho}{2\rho+d}})$  over a suitable compact subset  $B \subset \mathbb{R}^d$ , see Stone (1980) [30] and (1982) [31]. We derive our result under various dependence assumptions namely we consider independent, strongly mixing, absolutely regular or weakly dependent (either causal or noncausal) sequences in all those questions. The last section includes the proofs; it is organized in four subsections where we develop the proofs for the main lemma, for weighted sums, for moments of Nadaraya-Watson estimation and for sup bounds of this estimator, respectively.

## 2. Main Lemma

We are now in position to formulate our main result.

**Lemma 1** *Assume that  $\|\widehat{N}_n - N_n\|_p \leq v_n$  and  $\|\widehat{D}_n - D_n\|_q \leq v_n$  for some  $q > p$ . If moreover  $\|U_{i,n} V_{i,n}\|_r \leq C_n$  and  $\|V_{i,n}\|_s \leq c_n$  where  $q/p - q/r \geq 1$ ,  $1/p > 1/q + 1/s$ , then*

$$D_n \|\Delta_n\|_p \leq \left( 1 + \frac{|N_n|}{D_n} + \frac{|N_n|^\beta v_n^{1-\beta}}{D_n} + \frac{C_n^\beta v_n^{1-\beta}}{D_n} + \frac{v_n^\alpha c_n n^{\frac{1}{s}}}{D_n^\alpha} \right) v_n.$$

Here  $\alpha, \beta$  are chosen from the data  $p, q, r, s$  by setting

$$\alpha = q \left( \frac{1}{p} - \frac{1}{s} - \frac{1}{q} \right) \leq 1 \leq \beta^{-1} = q \left( \frac{1}{p} - \frac{1}{r} \right)$$

The message of this lemma is the following. We essentially need to know the rate for the moments with order slightly greater than  $p$  for the denominator, and for  $p$ -th order moments of the numerator to derive a bound of the rate for ratio moments.

**Remarks.** In all our examples,  $c_n \equiv c$  will be a constant.

- If  $C_n \equiv C$  is also a constant, then we assume that  $r = \frac{pq}{q-p}$ , so that  $\beta = 1$ . In this case, large values of  $q$  give  $r$  close (and larger) than  $p$ ; if now  $q > p$  is very close to  $p$  then  $r$  need to be very large and  $s$  is even larger.  
*This is the situation for weighted sums or censored data questions.*  
 Here  $V_{i,n} = V_i$  and  $0 \leq U_{i,n} = U_i$ , and the sequence  $(U_i, V_i)$  is stationary. Moreover  $v_n = c/\sqrt{n}$  and thus  $\Delta_n = \mathcal{O}(n^{-1/2})$  if  $\alpha = 2/s$ . This condition writes  $s = p(q+2)/(q-p)$  as proved in the forthcoming Theorem 1.
- If now the sequence  $C_n$  is not bounded, in order to control the corresponding term, we shall need  $\beta < 1$  or  $1/p > 1/q + 1/r$ . The order  $q$  of the moment of the denominator should be larger as well as the order of the moments of the variables  $V_{i,n}$ , e.g. in the case of functional estimation. Here  $v_n = c/\sqrt{nh_n^d} \gg 1/\sqrt{n}$  as  $h_n \rightarrow_{n \rightarrow \infty} 0$  and  $nh_n^d \rightarrow_{n \rightarrow \infty} \infty$ .
- *Orlicz spaces.* Instead of  $\mathbb{L}^q$  norms we may consider Orlicz norms and ask only for  $x^p \log^q x$ -order moments of the denominator. Exponential moments of the variables  $V_{i,n}$  will thus be needed because of the relations (4), (15) and of the Pisier technique inequalities (16).
- *Suprema.* The same equations (4), (15) and (16) will be also use to derive bounds of suprema for moments for expressions involving an additional parameter; an emblematic example of this situation is regression estimation given in details below.

We consider two distinct classes of applications in Sections 3 and 4 devoted respectively to *weighted sums* and *nonparametric regression*. The following inequality is essential to bound the uniform rates of convergence of a Nadaraya-Watson regression estimator and it is thus set as a specific Lemma:

**Lemma 2** Let  $0 < \alpha < 1$ ,

$$D_n |\Delta_n| \leq |\hat{N}_n - N_n| + \frac{|\hat{N}_n|}{D_n} |\hat{D}_n - D_n| + \max_{1 \leq i \leq n} |V_{i,n}| \frac{|\hat{D}_n - D_n|^{1+\alpha}}{|D_n|^\alpha} \quad (4)$$

Inequality (4) also implies tails bounds for  $\Delta_n$ 's distribution.

### 3. Weighted sums

We consider here the simplest situation for which Lemma 1 applies. Let  $(U_i, V_i)_{i \in \mathbb{Z}}$  be a stationary sequence and set  $\hat{D}_n = 1/n \sum_{i=1}^n U_i$ ,  $\hat{N}_n = 1/n \sum_{i=1}^n U_i V_i$  then  $N_n = N = \mathbb{E}U_1 V_1$ ,  $D_n = D = \mathbb{E}U_1$  and  $\hat{R}_n = \hat{N}_n/\hat{D}_n$ ,  $R_n = R = N/D$ .

**Theorem 1** Let  $(U_i, V_i)_{i \in \mathbb{Z}}$  be a stationary sequence with  $U_i \geq 0$  (a.s.). Let  $0 < p < q$  and assume that for  $r = \frac{pq}{q-p}$  and  $s = \frac{p(q+2)}{q-p}$ :

$$\|U_i V_i\|_r \leq c, \quad \|V_i\|_s \leq c.$$

Assume that the dependence structure of the sequence  $(U_i, V_i)_{i \in \mathbb{Z}}$  is such that

$$\|\widehat{D}_n - D\|_q \leq \frac{C}{\sqrt{n}}, \quad \|\widehat{N}_n - N\|_p \leq \frac{C}{\sqrt{n}} \quad (5)$$

then  $\|\widehat{R}_n - R\|_p = \mathcal{O}(1/\sqrt{n})$ .

In the following cases, we assume that  $\|V_i\|_s \leq c$  and  $\|U_i V_i\|_r \leq c$  and prove that (5) holds. Denote  $Z_i = U_i V_i - \mathbb{E}U_i V_i$ . For simplicity we will often assume  $\|U_i\|_\infty < \infty$ ,  $\|V_i\|_r < \infty$ .

### 3.1. Independent case

Assume that  $(U_i, V_i)$  is i.i.d. Assume that  $\|U_0\|_q \leq c$  and  $\|U_0 V_0\|_p \leq c$ . From the Marcinkiewitz-Zygmund inequality for independent variables, for  $2 \leq q \leq r$ , we get

$$\begin{aligned} \mathbb{E} \left| \widehat{D}_n - D \right|^q &\leq C_q \mathbb{E} |U_1|^q n^{-\frac{q}{2}} \leq C n^{-\frac{q}{2}}, \\ \mathbb{E} \left| \widehat{N}_n - N \right|^p &\leq C_p \mathbb{E} |U_1 V_1|^p n^{-\frac{p}{2}} \leq C n^{-\frac{p}{2}}, \end{aligned}$$

and (5) holds. Now Hölder inequality implies those relations if  $\|U_0\|_q$ , and  $\|V_0\|_{\frac{qp}{q-p}} < \infty$ .

### 3.2. Strong mixing case

Denote  $(\alpha_i)_{i \in \mathbb{N}}$  the strong mixing coefficient sequence of the stationary sequence  $(U_i, V_i)_{i \in \mathbb{N}}$ .

**Proposition 1** Assume that for  $r' > q$ ,  $\|U_0\|_{r'} \leq c$ . Relation (5) holds if  $\alpha_i = \mathcal{O}(i^{-\alpha})$  with

$$\alpha > \left( \frac{p}{2} \cdot \frac{r}{r-p} \right) \vee \left( \frac{q}{2} \cdot \frac{r'}{r'-p} \right),$$

### 3.3. Causal weak dependence

Define the  $\gamma$  coefficient of dependence of a centered sequence  $(W_i)_{i \in \mathbb{N}}$  with values in  $\mathbb{R}^d$  by

$$\gamma_i = \sup_{k \geq 0} \|\mathbb{E}(W_{i+k} | \mathcal{M}_k)\|_1$$

**Proposition 2** Assume that  $\|U_0\|_\infty \leq c$ . Relation (5) holds if the sequence of coefficients  $\gamma$  associated to the stationary sequence  $(W_i)_{i \in \mathbb{N}} = (U_i, V_i)_{i \in \mathbb{N}}$  is such that  $\gamma_i = \mathcal{O}(i^{-\gamma})$  with

$$\gamma > \left( \frac{p}{2} \cdot \frac{r-1}{r-p} \right) \vee \frac{q}{2}.$$

### 3.4. Non causal weak dependence

Here we consider non causal weakly dependent stationary sequences of bounded variables and assume that  $q$  and  $p$  are integers. A sequence  $(W_i)_{i \in \mathbb{N}}$  is said to be  $\lambda$ -weakly dependent if there exists a sequence  $(\lambda(i))_{i \in \mathbb{N}}$  decreasing to zero at infinity such that:

$$\begin{aligned} & \left| \text{Cov} \left( g_1(W_{i_1}, \dots, W_{i_u}), g_2(W_{j_1}, \dots, W_{j_v}) \right) \right| \\ & \leq (u \text{Lip } g_1 + v \text{Lip } g_2 + uv \text{Lip } g_1 \text{Lip } g_2) \lambda(k), \end{aligned}$$

for any  $u$ -tuple  $(i_1, \dots, i_u)$  and any  $v$ -tuple  $(j_1, \dots, j_v)$  with  $i_1 \leq \dots \leq i_u < i_u + k \leq j_1 \leq \dots \leq j_v$  where  $g_1, g_2$  are two real functions of  $\Lambda^{(1)} = \{g_1 \in \Lambda \mid \|g_1\|_\infty \leq 1\}$  respectively defined on  $\mathbb{R}^{Du}$  and  $\mathbb{R}^{Dv}$  ( $u, v \in \mathbb{N}^*$ ). Recall here that  $\Lambda$  is the set of functions with  $\text{Lip } g_1 < \infty$  for some  $u \geq 1$ , with

$$\text{Lip } g_1 = \sup_{(x_1, \dots, x_u) \neq (y_1, \dots, y_u)} \frac{|g_1(y_1, \dots, y_u) - g_1(x_1, \dots, x_u)|}{|y_1 - x_1| + \dots + |y_u - x_u|}.$$

The monograph Dedecker *et al.* (2007) [11] details weak dependence concepts, as well as extensive models and results.

**Proposition 3** Assume that  $p$  and  $q \geq 2$  are even integers. Assume that the stationary sequence  $(U_i, V_i)_{i \in \mathbb{N}}$  is  $\lambda$ -weakly dependent. Assume that  $Z_0 = U_0 V_0 - \mathbb{E}U_0 V_0$  is bounded by  $M$ . Relation (5) holds if  $\lambda(i) = \mathcal{O}(i^{-\lambda})$  with  $\lambda > \frac{q}{2}$ .

#### Remarks

- Unbounded random variables may also be considered under an additional concentration inequality ( $\mathbb{P}(Z_i \in (x, x+y)) \leq C y^a$  for some  $a > 0$ ) and Theorem 3 and Lemma 1 from Doukhan and Louhichi(1999) [17], imply that the same relation holds if  $\mathbb{E}|Z_i|^{q+\delta} < \infty$ ,  $\sup_{x,i} \mathbb{P}(Z_i \in (x, x+y)) \leq$

$$C y^a, (\forall y > 0), \text{ and } \sum_{n=1}^{\infty} n^{q \frac{q+\delta}{2\delta} - 1} \lambda^{\frac{a}{2+a}}(n) < \infty.$$

- Non-integer moments  $q \in (2, 3)$  are considered in Doukhan and Wintenberger (2007) (see [21], Lemma 4), and the same inequality holds if  $\mathbb{E}|Z_i|^{q'} < \infty$  with  $q' = q + \delta$ , and  $\lambda(i) = \mathcal{O}(i^{-\lambda})$  with  $\lambda > 4 + 2/q'$  for  $q$  small enough:

$$q \leq 2 + \frac{1}{2} \left( \sqrt{(q' + 4 - 2\lambda)^2 + 4(\lambda - 4)(q' - 2) - 2} + q' + 4 - 2\lambda \right) \quad (\leq q').$$



#### 4. Regression estimation

We now use a measurable bounded function  $K : \mathbb{R}^d \rightarrow \mathbb{R}$ . Consider a stationary process  $(X_i, Y_i) \in \mathbb{R}^d \times \mathbb{R}$ . We now set for some  $h = h_n$ :

$$U_{i,n} = U_{i,n}(x) = \frac{1}{h^d} K \left( \frac{X_i - x}{h} \right), \quad \text{and} \quad V_{i,n} = Y_i,$$

where  $h$  tends to zero as  $n$  tends to infinity. Then  $\widehat{R}_n = \widehat{r}(x)$  is the Nadaraya-Watson estimator of  $r(x) = \mathbb{E}(Y_i | X_i = x)$ . Independently from the dependence structure of the process  $(X_i, Y_i)$ , we first introduce regularity conditions:

- (A1) For the point  $x$  of interest: the functions  $f, g$  are  $k$ -times continuously differentiable around  $x$  and  $(\rho - k)$ -Hölderian, where  $k < \rho$  is the largest possible integer.
- (A2) The function  $K$  is Lipschitz, admits a compact support, and satisfies  $K(u) \geq 0$  ( $\forall u \in \mathbb{R}^d$ ), and

$$\begin{aligned} \int_{\mathbb{R}^d} K(u) du &= 1, \\ \int_{\mathbb{R}^d} u_1^{\ell_1} \cdots u_d^{\ell_d} K(u) du &= 0, \text{ if } 0 < \ell_1 + \cdots + \ell_d < k. \end{aligned}$$

Moment and conditional moment conditions are also needed:

- (A3) For the point  $x$  of interest, there exist  $r$  and  $s$ , with  $r \leq s$  such that:
  1.  $\|Y_0\|_s = c < \infty$ ,
  2.  $\bar{g}_1(x) = \int y f(x, y) dy$  is a function bounded around the point  $x$ ,
  3.  $g_r(x) = \int |y|^r f(x, y) dy$  is a function bounded around the point  $x$ .
  4.  $G(x', x'')$  is bounded around the point  $(x, x)$ , where  $G(x', x'') = \sup_i f_i(x', x'')$  and  $f_i(x', x'')$  denotes the joint density of  $(X_0, X_i)$ .

#### Remarks

- First notice that the last condition holds immediately for independent sequences  $(X_i)$  with a locally bounded marginal density.
- An alternative condition involving local uniform boundedness of

$$(x', x'') \mapsto H(x', x'') = \sup_i \int |yy'| f_i(x', y'; x'', y'') dy' dy''$$

where now  $f_i(x', y'; x'', y'')$  denotes the joint density of  $(X_0, Y_0; X_i, Y_i)$ , yields sharper results but such conditions are much more complicated to check so that we avoided this assumption. This condition holds for independent sequences if  $g_1$  is locally bounded.

We quote here that the assumption  $K \geq 0$  implies that we are not able to control biases if  $\rho > 2$ . In order to make (A2) possible we thus need  $\rho \leq 2$  hence  $k = 1$  or  $2$ , and the previous vanishing moment condition holds with  $k = 2$  in case

of a symmetric kernel. Without this nonnegativity condition the moments of numerator and denominator are still controlled but we definitely cannot handle the moments of our ratio.

#### 4.1. Moment estimation

We now consider the quantity

$$\delta_n(x) = \|\widehat{r}(x) - r(x)\|_p, \quad \forall x \in \mathbb{R}^d.$$

We recall below the usual assumptions for  $\mathbb{L}^p$ -consistency of such estimators are  $h = h_n \rightarrow 0$  and  $nh_n^d \rightarrow \infty$  as  $n \rightarrow \infty$  and the rate is  $v_n = c/\sqrt{nh_n^d}$ .

**(A4)** For the point  $x$  of interest and for some  $q > p$ , there exists a constant  $c > 0$  with:

1.  $\|\widehat{f}(x) - \mathbb{E}\widehat{f}(x)\|_q \leq c/\sqrt{nh_n^d}$ ,
2.  $\|\widehat{g}(x) - \mathbb{E}\widehat{g}(x)\|_p \leq c/\sqrt{nh_n^d}$ .

which implies that the usual rate  $\|\widehat{r}(x) - r(x)\|_p = \mathcal{O}(n^{-\frac{p}{2\rho+d}})$  is obtained if  $h_n$  is well chosen.

**Proposition 4 (Bias)** *Assume that  $f$  is bounded below around the point  $x$  and **(A1)** and **(A2)**, then*

$$\left| r(x) - \frac{\mathbb{E}\widehat{g}(x)}{\mathbb{E}\widehat{f}(x)} \right| = \mathcal{O}(h_n^p), \quad \forall x \in \mathbb{R}^d.$$

**Proposition 5** *Assume that  $f$  is bounded below around the point  $x$  and that assumptions **(A2)**, **(A3)** and **(A4)** hold, then*

$$\left\| \widehat{r}(x) - \frac{\mathbb{E}\widehat{g}(x)}{\mathbb{E}\widehat{f}(x)} \right\|_p \leq C \left( 1 + h^{d\beta(\frac{1}{r}-1)} (nh^d)^{\frac{\beta-1}{2}} + (nh^d)^{-\frac{\alpha}{2}} n^{\frac{1}{s}} \right) v_n.$$

with  $\alpha = q\left(\frac{1}{p} - \frac{1}{s} - \frac{1}{q}\right)$  and  $\beta = \frac{pr}{q(r-p)}$ .

Those two propositions easily imply the following result. The optimal window width  $h \sim n^{-\frac{1}{2\rho+d}}$  equilibrates both expressions to get the minimax rate  $v_n \sim n^{-\frac{p}{2\rho+d}}$ .

**Theorem 2** *Choose the window width  $h_n = Cn^{-\frac{1}{d+2\rho}}$  for a constant  $C > 0$ . Assume that  $f$  is bounded below around the point  $x$  and that assumptions **(A1)**, **(A3)** and **(A4)** hold for*

$$\frac{pd(r-1)}{qr-pq-pr} \vee \frac{pd}{qs-pq-ps-2p} \leq \rho. \quad (6)$$

*Then there exists a constant  $C > 0$  with:  $\|\widehat{r}(x) - r(x)\|_p \leq Cv_n$ .*

Now we consider specific dependence structures to get moments inequality for  $\widehat{f}$  and  $\widehat{g}$ , in order to check **(A4)**. From now on, fix  $x$  and write:

$$\widehat{g}(x) - \mathbb{E}\widehat{g}(x) = \frac{1}{nh^d} \sum_i Z_i, \quad Z_i = K_i Y_i - \mathbb{E}K_i Y_i, \quad K_i = K\left(\frac{X_i - x}{h}\right).$$

#### 4.1.1. Independence

**Proposition 6** *Assume that  $(X_i, Y_i)$  is i.i.d. and **(A2)**, **(A3)** with  $r = q$  hold then **(A4)** moment inequalities hold.*

#### 4.1.2. Noncausal weak dependence

We now work as in Doukhan and Louhichi (1999) [18], excepted for the necessary truncation used in Ango-Nze and Doukhan (1998) [2].

**Proposition 7** *Assume that  $(X_i, Y_i)$  is  $\lambda$ -weakly dependent and **(A2)**, **(A3)** hold. Assume that  $\|Y_0\|_s < \infty$  for some  $s > 2p$ . If  $\lambda(i) = \mathcal{O}(i^{-\lambda})$ , with  $\lambda > \frac{r(2r(s-p)+2p-s)}{(r-p)(s-2p)(r-1)}(p-1) \vee \frac{2(d-1)}{d}(q-1)$  then **(A4)** moment inequalities hold.*

#### 4.1.3. Strong mixing

**Proposition 8** *Assume that **(A2)**, **(A3)** with  $r > q$  hold and  $(X_i, Y_i)$  is  $\alpha$ -mixing with  $\alpha_i = \mathcal{O}(i^{-\alpha})$ , we also suppose, either*

- $\alpha > \left((q-1)\frac{r}{r-q}\right) \vee \frac{4sr-2s-4r}{(r-2)(s-4)}$  and  $h \sim n^{-a}$  with  $ad \leq \frac{1-2/p}{3-2/r}$ , or
- $p, q$  are even integers and  $\alpha > \frac{r}{2} \frac{s-2p}{s-p} \left(1 - \frac{1}{p}\right)$ ,

*then **(A4)** moment inequalities hold.*

#### Remarks

- In the first item, the previous limitation on  $h$  writes  $p \geq d/\rho + 2$  if one makes use of the window width  $h \sim n^{-\frac{1}{2\rho+d}}$ , optimal with respect to power loss functions; this loss does not appear for integral order moments of the second item.
- Note that for the case of absolute regularity a nice argument in Viennet (1997) [33] provides sharp bounds for specific integrals of the second order moments of such expressions. We do not derive them here even if integrated square errors have specific interpretations: indeed higher order moments are also needed in our case.

#### 4.2. Uniform mean estimates

We now investigate uniform bounds:

$$\delta_n(K) = \left\| \sup_{x \in B} |\widehat{r}(x) - r(x)| \right\|_p \quad (7)$$

In this setting, Ango Nze and Doukhan (1996) [1] prove the needed results under mixing assumptions, Ango Nze *et al.* (2002) [4] and Ango Nze and Doukhan (2004) [3] both propose the bounds to conclude under weak dependence. For this the lemma needs to be rephrased by replacing  $\widehat{N}_n - N_n$  and  $\widehat{D}_n - D_n$  by suprema of those expressions over  $x \in B$  for some compact subset  $B \subset \mathbb{R}^d$ . We now introduce the necessary conditions for this.

**(A5)** The condition **(A1)** holds for each  $x \in B$ .

**(A6)** The condition **(A3)** holds for each  $x \in B$ .

**(A7)** For some  $q > p$  and  $w_n = \frac{\sqrt{\log n}}{\sqrt{nh^d}} = v_n \sqrt{\log n}$ , there exists a constant  $c > 0$  with:

1.  $\left\| \sup_{x \in B} |\widehat{f}(x) - \mathbb{E}\widehat{f}(x)| \right\|_q \leq cw_n$ ,
2.  $\left\| \sup_{x \in B} |\widehat{g}(x) - \mathbb{E}\widehat{g}(x)| \right\|_p \leq cw_n$ .

Here again we begin with two preliminary propositions before stating our main result.

**Proposition 9 (Uniform bias)** *Assume that  $f$  is bounded below over an open neighborhood of  $B$  and **(A2)** and **(A5)**, then*

$$\sup_{x \in B} \left| r(x) - \frac{\mathbb{E}\widehat{g}(x)}{\mathbb{E}\widehat{f}(x)} \right| = \mathcal{O}(h_n^\rho).$$

**Proposition 10** *Assume that  $f$  is bounded below over an open neighborhood of  $B$  and that assumptions **(A2)**, **(A6)** and **(A7)** hold, then*

$$\left\| \sup_{x \in B} \left| \widehat{r}(x) - \frac{\mathbb{E}\widehat{g}(x)}{\mathbb{E}\widehat{f}(x)} \right| \right\|_p \leq C \left( 1 + h^{d\beta(\frac{1}{r}-1)} (nh^d)^{\frac{\beta-1}{2}} + (nh^d)^{-\frac{\alpha}{2}} n^{\frac{1}{s}} \right) w_n.$$

with  $\alpha = q\left(\frac{1}{p} - \frac{1}{s} - \frac{1}{q}\right)$  and  $\beta = \frac{pr}{q(r-p)}$ .

The following theorem results from the two previous propositions.

**Theorem 3** *Let  $(X_t, Y_t)$  be a stationary sequence. Assume that conditions **(A2)**, **(A5)**, **(A6)** and **(A7)** hold for some  $s > 2p$ . Then,*

$$\left\| \sup_{x \in B} |\widehat{r}(x) - r(x)| \right\|_p \leq C \left( \frac{\log n}{n} \right)^{\frac{p}{2p+d}}. \quad (8)$$

The end of the section is devoted to check sufficient conditions for **(A7)** to hold. The proofs do not use the positivity of  $K$  so that arbitrary values for  $\rho$  and  $k$  are possible. We thus derive conditions for the following relations:

$$\left\| \sup_{x \in B} |\widehat{g}(x) - \mathbb{E}\widehat{g}(x)| \right\|_p \leq C \frac{\sqrt{\log n}}{\sqrt{nh^d}}, \quad (9)$$

$$\left\| \sup_{x \in B} |\mathbb{E}\widehat{g}(x) - g(x)| \right\|_p \leq Ch^\rho. \quad (10)$$

If those conditions hold then the following relation equilibrates the bias and the fluctuations of the estimators

$$h \sim \left( \frac{\log n}{n} \right)^{\frac{1}{2\rho+d}}. \quad (11)$$

#### 4.2.1. Independence

First, we evaluate the uniform bound for the moments under independence.

**Proposition 11** *Let  $(X_t, Y_t)_{t \in \mathbb{N}}$  be an i.i.d. sequence. Assume that conditions **(A2)**, **(A5)**, **(A6)** and **(A7)** hold for some  $s > 2p$ ,  $\rho > dp/(s-2p)$  then (9-10) hold hence (11) yields the bounds*

$$\left\| \sup_{x \in B} |\widehat{f}(x) - f(x)| \right\|_p \leq C \left( \frac{\log n}{n} \right)^{\frac{\rho}{2\rho+d}}, \quad (12)$$

$$\left\| \sup_{x \in B} |\widehat{g}(x) - g(x)| \right\|_p \leq C \left( \frac{\log n}{n} \right)^{\frac{\rho}{2\rho+d}}. \quad (13)$$

#### 4.2.2. Absolute regularity

**Proposition 12** *Let  $(X_i, Y_i)_{i \in \mathbb{N}}$  be an absolute regular (also called  $\beta$ -mixing) sequence. Assume that conditions **(A2)**, **(A5)**, **(A6)** and **(A7)** hold for some  $s > 2p$ ,  $\rho > dp/(s-2p)$ . Assume that the mixing coefficients satisfy  $\beta_i = \mathcal{O}(i^{-\beta})$  with  $\beta > \frac{sp + (2s-p)d}{\rho(s-2p) - pd} \vee \left(1 + \frac{2d}{\rho}\right)$ , then relations (9-10) both hold, hence a choice (11) yields the bounds (12-13).*

#### 4.2.3. Strong mixing

Using Fuk-Nagaev inequality in Rio (2000) [28] also yields an analogous result.

**Proposition 13** *Assume that the process  $(X_i, Y_i)_{i \in \mathbb{N}}$  is stationary and strongly mixing with  $\alpha_i = \mathcal{O}(i^{-\alpha})$  for  $\alpha > \frac{3\rho s + 2ds + d\rho s - 4pp - 3dp - d\rho p}{dp - \rho(s-2p)} \vee 2 \frac{s-1}{s-2}$ . Assume that conditions **(A2)**, **(A5)**, **(A6)** and **(A7)** hold for some  $s > 2p$ ,  $\rho > dp/(s-2p)$  then (9-10) hold hence (11) yields the bounds (12-13).*

#### 4.2.4. Non causal weak dependence

**Proposition 14** Assume that the process  $(X_i, Y_i)_{i \in \mathbb{Z}}$  is stationary and  $\lambda$ -weakly dependent with  $\lambda(i) = \mathcal{O}(e^{-\lambda i^b})$  with  $b > 0$ . Assume that conditions **(A2)**, **(A5)**, **(A6)** and **(A7)** hold for some  $s > 2p$ ,  $\rho > dp/(s - 2p)$  then (9-10) hold hence (11) yields the bounds (12-13).

**Remark** Other dependence settings may also be addressed, for example the  $\phi$ -mixing case considered in Dedecker (2001) [9] and the use of coupling in weakly dependent sequences by Dedecker and Prieur (2005) [12] both yield suitable exponential inequalities to complete analogous results.

## 5. Proofs

In the proofs,  $C > 0$  will denote a constant which may change from one line to another.

### 5.1. Proof of the main Lemma 1

Setting  $z = (D_n - \widehat{D}_n)/D_n$ , we rewrite:

$$\Delta_n = \frac{\widehat{N}_n}{D_n} \cdot \frac{1}{1-z} - \frac{N_n}{D_n} = \frac{\widehat{N}_n}{D_n} \left( \frac{1}{1-z} - 1 \right) + \frac{\widehat{N}_n - N_n}{D_n} \quad (14)$$

then

$$\frac{1}{1-z} - 1 = \frac{z}{1-z} = z + \frac{z^2}{1-z}$$

hence for each  $\alpha \in [0, 1]$ ,

$$\begin{aligned} \left| \frac{1}{1-z} - 1 \right| &\leq |z| \left( \frac{1}{|1-z|} \wedge \left( 1 + \frac{|z|}{|1-z|} \right) \right) \\ &\leq |z| + \frac{|z|}{|1-z|} (1 \wedge |z|) \\ &\leq |z| + \frac{|z|^{1+\alpha}}{|1-z|} \end{aligned}$$

which implies

$$\begin{aligned} D_n |\Delta_n| &\leq |\widehat{N}_n - N_n| + |\widehat{N}_n| |z| + |\widehat{N}_n| \frac{|z|^{1+\alpha}}{|1-z|} \\ &\leq |\widehat{N}_n - N_n| + \frac{|\widehat{N}_n|}{D_n} \cdot |\widehat{D}_n - D_n| + \left| \frac{\widehat{N}_n}{D_n} \frac{1}{1-z} \right| \cdot \frac{|\widehat{D}_n - D_n|^{1+\alpha}}{|D_n|^\alpha} \\ &\leq |\widehat{N}_n - N_n| + \frac{|\widehat{N}_n|}{D_n} \cdot |\widehat{D}_n - D_n| + \left| \frac{\widehat{N}_n}{\widehat{D}_n} \right| \cdot \frac{|\widehat{D}_n - D_n|^{1+\alpha}}{|D_n|^\alpha} \\ &\leq |\widehat{N}_n - N_n| + \frac{|\widehat{N}_n|}{D_n} \cdot |\widehat{D}_n - D_n| + \max_{1 \leq i \leq n} |V_{i,n}| \cdot \frac{|\widehat{D}_n - D_n|^{1+\alpha}}{|D_n|^\alpha} \end{aligned}$$

*Proof of Lemma 2* Together with Hölder inequality with exponents  $1/a+1/b = 1$  we derive,

$$\begin{aligned}
D_n \|\Delta_n\|_p &\leq \|\widehat{N}_n - N_n\|_p \\
&\quad + \left\| \frac{|\widehat{N}_n|}{D_n} \cdot |\widehat{D}_n - D_n| \right\|_p + \left\| \max_{1 \leq i \leq n} |V_{i,n}| \cdot \frac{|\widehat{D}_n - D_n|^{1+\alpha}}{|D_n|^\alpha} \right\|_p \\
&\leq v_n + \frac{1}{D_n} \|\widehat{N}_n\|_{pa} \cdot \|\widehat{D}_n - D_n\|_{pb} \\
&\quad + \left\| \max_{1 \leq i \leq n} |V_{i,n}| \cdot \frac{|\widehat{D}_n - D_n|^{1+\alpha}}{|D_n|^\alpha} \right\|_p \tag{15}
\end{aligned}$$

Now, the assumption  $\|U_{i,n} V_{i,n}\|_r \leq C_n$  implies  $\|\widehat{N}_n\|_r \leq C_n$ . The second term in the RHS of inequality (15) is bounded using the property  $\|\widehat{N}_n\|_{pa} \leq |N_n| + \|\widehat{N}_n - N_n\|_{pa}$ . Consider now some  $\beta \in [0, 1]$  and  $u, v \geq 0$  such that  $1/u + 1/v = 1$  to be determined later then  $|\widehat{N}_n - N_n| = |\widehat{N}_n - N_n|^\beta |\widehat{N}_n - N_n|^{1-\beta}$ ; Hölder inequality implies if we choose  $upa(1-\beta) = p$  and  $vpa\beta = r$

$$\begin{aligned}
\|\widehat{N}_n - N_n\|_{pa} &\leq \|\widehat{N}_n - N_n\|_{upa(1-\beta)}^{1-\beta} \|\widehat{N}_n - N_n\|_{vpa\beta}^\beta \\
&\leq \|\widehat{N}_n - N_n\|_p^{1-\beta} \|\widehat{N}_n - N_n\|_r^\beta \\
&\leq \|\widehat{N}_n - N_n\|_p^{1-\beta} \|\widehat{N}_n - N_n\|_r^\beta \\
&\leq v_n^{1-\beta} (|N_n|^\beta + C_n^\beta),
\end{aligned}$$

thus

$$\|\widehat{N}_n\|_{pa} \leq |N_n| + v_n^{1-\beta} (|N_n|^\beta + C_n^\beta),$$

by setting  $b = q/p$  we derive  $a = (q-p)/q$  hence  $\frac{upq}{q-p}(1-\beta) = p$ , and  $\frac{vpq}{q-p}\beta = r$ , which may be rewritten as  $\frac{1}{u} = \frac{q}{q-p}(1-\beta)$ , and  $\frac{1}{v} = \frac{pq}{r(q-p)}\beta$ . With the relation  $1/u + 1/v = 1$  we find  $\beta = pr/q(r-p)$ . The second term in the RHS of inequation (15) is thus bounded as follows,

$$\frac{1}{D_n} \|\widehat{N}_n\|_{pa} \|\widehat{D}_n - D_n\|_{pb} \leq \frac{1}{D_n} (|N_n| + v_n^{1-\beta} (|N_n|^\beta + C_n^\beta)) v_n.$$

The last term in relation (15) is more difficult to handle; it may be bounded using Hölder inequality with exponents  $1/a + 1/b = 1$  and

$$\begin{aligned}
\left\| \max_{1 \leq i \leq n} |V_{i,n}| \cdot \frac{|\widehat{D}_n - D_n|^{1+\alpha}}{|D_n|^\alpha} \right\|_p &\leq \frac{1}{|D_n|^\alpha} \left\| \max_{1 \leq i \leq n} |V_{i,n}| \right\|_{pa} \|\widehat{D}_n - D_n\|_{pb}^{1+\alpha} \\
&\leq \frac{1}{|D_n|^\alpha} \left( \mathbb{E} \max_{1 \leq i \leq n} |V_{i,n}|^{pa} \right)^{\frac{1}{pa}} v_n^{1+\alpha}
\end{aligned}$$

if  $q \geq pb(1+\alpha)$  or equivalently if  $q - p(1+\alpha) \geq q/a$ . We need an argument of Pisier (1978) [26] written as follows: assume that  $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is convex and

non decreasing then

$$\begin{aligned} \varphi\left(\mathbb{E}\max_i |V_{i,n}|^{pa}\right) &\leq \mathbb{E}\varphi\left(\max_i |V_{i,n}|^{pa}\right) \\ &\leq \mathbb{E}\sum_i \varphi(|V_{i,n}|^{pa}) \\ &\leq \sum_i \mathbb{E}\varphi(|V_{i,n}|^{pa}) \end{aligned} \quad (16)$$

Hence  $\mathbb{E}\max_i |V_{i,n}|^{pa} \leq (nc)^{pa/s}$  with  $\varphi(x) = x^{s/pa}$ . Now the bound in the RHS of (15) can be precised as  $v_n^{1+\alpha}(nc)^{\frac{1}{s}}/|D_n|^\alpha$  if  $s \geq pa$ ; we use for this  $1 - \frac{p}{q}(1+\alpha) \geq \frac{1}{a} \geq \frac{p}{s}$  if  $\alpha > 0$  is small enough with  $\frac{1}{p} \geq \frac{1+\alpha}{q} + \frac{1}{s}$ .  $\square$

### 5.1.1. Proofs for the subsection 3

*Proof of Theorem 1.* Set  $\Delta_n = \widehat{R}_n - R$ . Refer to Lemma 1: here  $r$  and  $s$  are such that that  $\beta = 1$ , and  $\alpha = 2/s$ . Because  $v_n = cn^{-1/2}$ , we get

$$D\|\Delta_n\|_p \leq \left(1 + 2\frac{N}{D} + \frac{2c}{D} + \frac{cn^{-\frac{\alpha}{2} + \frac{1}{s}}}{D^\alpha}\right)v_n,$$

where the last term in the parenthesis is bounded with respect to  $n$  implying

$$\|\widehat{R}_n - R\|_p = \mathcal{O}(1/\sqrt{n}).$$

*Proof of Proposition 1.* Denote  $\alpha^{-1}(u) = \sum_{i \geq 0} \mathbb{I}\{u < \alpha_i\}$ , and  $Q_Z$  the generalized inverse of the tail function  $x \mapsto \mathbb{P}(|Z_0| > x)$ .

First recall that the mixing coefficient of  $(Z_i)_{i \in \mathbb{N}}$  are bounded by the sequence  $\alpha_i$ . Theorem 2.5 in Rio (2000) [28] shows that

$$\left\|\sum_{i=1}^n Z_i\right\|_p \leq \sqrt{2pn} \left(\int_0^1 (\alpha^{-1}(u) \wedge n)^{\frac{p}{2}} Q_Z^p(u) du\right)^{\frac{1}{p}}.$$

But as  $\|Z_0\|_r \leq c$ ,  $\mathbb{E}|Z_0|^r = \int_0^\infty Q_Z^r(u) du \leq c^r$ . Define  $a = r/p$  and  $b = r/(r-p)$ . Then by Hölder inequality

$$\left\|\sum_{i=1}^n Z_i\right\|_p \leq \sqrt{2pn} \left(\int_0^\infty (\alpha^{-1}(u) \wedge n)^{\frac{pb}{2}} du\right)^{\frac{1}{bp}} \left(\int_0^\infty Q_Z^r(u) du\right)^{\frac{1}{ap}}.$$

From  $\alpha > pb/2$  the first integral is finite and,  $\|\widehat{N}_n - N\|_p \leq \frac{C}{\sqrt{n}}$ . For  $Y_n = U_n - \mathbb{E}U_n$  and  $b' = r'/(r' - q)$ , we get similarly that  $\|\widehat{D}_n - D\|_q \leq \frac{C}{\sqrt{n}}$ .  $\square$



*Proof of Proposition 2* If  $\|U_i\|_\infty \leq c$  then to the 1-dimensional sequences  $(Y_i) = (U_i - \mathbb{E}U_i)$  and  $(Z_i) = (U_i V_i - \mathbb{E}U_i V_i)$  are  $\gamma$ -weakly dependent with coefficients having the same decay rate (up to the constant  $c$ ). Denote  $G_Z$  the inverse of  $x \mapsto \int_0^x Q_Z(u) du$ . Corollary 5.3, page 124 in Dedecker *et al.* (2007) [11] states that

$$\left\| \sum_{i=1}^n Z_i \right\|_p \leq \sqrt{2pn} \left( \int_0^{\|Z\|_1} (\gamma^{-1}(u) \wedge n)^{\frac{p}{2}} Q_Z^{p-1} \circ G_Z(u) du \right)^{\frac{1}{p}}.$$

Quote that  $\mathbb{E}|Z|^r = \int_0^\infty Q_Z^{r-1} \circ G_Z(u) du \leq c^r$ . Define  $a = (r-1)/(p-1)$  and  $b = (r-1)/(r-p)$ . Then by Hölder inequality

$$\left\| \sum_{i=1}^n Z_i \right\|_p \leq \sqrt{2pn} \left( \int_0^\infty (\gamma^{-1}(u) \wedge n)^{\frac{pb}{2}} du \right)^{\frac{1}{bp}} \left( \int_0^\infty Q_Z^{r-1} \circ G_Z(u) du \right)^{\frac{1}{ap}}.$$

Because  $\gamma > \frac{pb}{2}$ , the first integral is finite and  $\|\widehat{N}_n - N\|_p \leq \frac{C}{\sqrt{n}}$ . Similarly

$$\begin{aligned} \left\| \sum_{i=1}^n Y_i \right\|_q &\leq \sqrt{2qn} \left( \int_0^{\|Y\|_1} (\gamma^{-1}(u) \wedge n)^{\frac{q}{2}} Q_Y^{q-1} \circ G_Y(u) du \right)^{\frac{1}{q}} \\ &\leq \sqrt{2qn} (2c)^{q-1} \left( \int_0^{2c} (\gamma^{-1}(u) \wedge n)^{\frac{q}{2}} du \right)^{\frac{1}{q}}. \end{aligned}$$

Because  $\gamma > \frac{q}{2}$ , the first integral is finite and  $\|\widehat{D}_n - D\|_q \leq \frac{C}{\sqrt{n}}$ .

*Proof of Proposition 3.* Recall that if  $(U_i, V_i)$  is  $\lambda$ -weakly dependent with  $\lambda(k) \leq Ck^{-\lambda}$ , and uniformly bounded then  $(Z_i)$  is also  $\lambda$ -weakly dependent with  $\lambda_Z(k) \leq Ck^{-\lambda}$ . Doukhan and Louhichi [17] show that there exists a constant  $C > 0$  such that if the random variables  $Z_i$  are bounded by some  $M \geq 1$ :

$$\left\| \sum_{i=1}^n Z_i \right\|_q^q \leq \frac{C(2q-2)!}{(q-1)!} \left\{ \left( n \sum_{k=0}^{n-1} \lambda_Z(k) \right)^{\frac{q}{2}} \vee \left( M^{q-2} n \sum_{k=0}^{n-1} (k+1)^{q-2} \lambda_Z(k) \right) \right\}.$$

Because  $\lambda_Z(k) \leq Ck^{-\lambda}$  with  $\lambda > \frac{q}{2}$ , then the second term is negligible as  $n$  tends to infinity and the first sum over  $k$  is bounded so that  $\left\| \sum_{i=1}^n Z_i \right\|_q \leq Cn^{\frac{1}{2}}$ ,

and thus  $\|\widehat{N}_n - N\|_q \leq \frac{C}{\sqrt{n}}$ .  $\square$

### 5.1.2. Proofs for the subsection 4.1

*Proof of Lemma 4.* The previous convergences  $\mathbb{E}\widehat{f}(x) \rightarrow f(x)$  and  $\mathbb{E}\widehat{g}(x) \rightarrow r(x)f(x)$  are controlled by  $\mathcal{O}(h_n^\rho)$  under  $\rho$ -regularity conditions **(A1)** (see

Ango-Nze and Doukhan (2004) [3]). Write

$$\left| r(x) - \frac{\mathbb{E}\widehat{g}(x)}{\mathbb{E}\widehat{f}(x)} \right| \leq \frac{|g(x) - \mathbb{E}\widehat{g}(x)|}{f(x)} + |\mathbb{E}\widehat{g}(x)| \frac{|\mathbb{E}\widehat{f}(x) - f(x)|}{f(x)\mathbb{E}\widehat{f}(x)}.$$

Since  $f$  is bounded below by 0 around  $x$ ,  $\mathbb{E}\widehat{f}(x)$  is also bounded below by 0, and we get the result.  $\square$

*Proof of Proposition 5.* The condition ((A3)-1) gives  $\|V_{i,n}\|_s < \infty$  and  $\|U_{i,n}V_{i,n}\|_r = \mathcal{O}(h^{d(\frac{1}{r}-1)})$  for  $r < s$  whenever ((A3)-3) holds. Indeed  $\|V_{i,n}\|_s = \|Y_0\|_s = c < \infty$ , we get for  $r \leq s$

$$\begin{aligned} \mathbb{E}|U_{i,n}V_{i,n}|^r &\leq \int \int h^{-rd} K^r((X_i - x)/h) |y|^r f(x, y) dx dy \\ &\leq \int h^{-rd} K^r((X_i - x)/h) g_r(x) dx \\ &\leq \|g_r\|_\infty \int h^{-rd} K^r((X_i - x)/h) dx \\ &\leq Ch^{(1-r)d}. \end{aligned}$$

Set  $K_{h_n}(\cdot) = h_n^{-d}K(\cdot/h_n)$  and denote by  $\star$  the convolution. The expressions write with  $D_n = \mathbb{E}\widehat{f}(x) = \mathbb{E}U_{i,n} = f \star K_{h_n}(x) \rightarrow f(x)$ , the marginal density of  $X_0$ , and  $N_n = \mathbb{E}\widehat{g}(x) = \mathbb{E}U_{i,n}V_{i,n} = (rf) \star K_{h_n}(t) \rightarrow r(x)f(x)$ . From Lemma 1, we get

$$D_n \left\| \widehat{r}(x) - \frac{\mathbb{E}\widehat{g}(x)}{\mathbb{E}\widehat{f}(x)} \right\|_p \leq \left( 1 + \frac{|N_n|}{D_n} + \frac{|N_n|^\beta v_n^{1-\beta}}{D_n} + \frac{C_n^\beta v_n^{1-\beta}}{D_n} + \frac{v_n^\alpha c_n n^{\frac{1}{s}}}{D_n^\alpha} \right) v_n,$$

where  $D_n$ ,  $N_n$  and  $c_n$  are equivalent to constants,  $C_n \equiv Ch^{d(1/r-1)}$  and  $v_n \equiv C(nh^d)^{-1/2}$ . Substituting the orders in the different terms, we get the result.  $\square$

*Proof of Theorem 2.* From the preceding propositions, we get

$$\|\widehat{r}(x) - r(x)\|_p \leq Ch_n^\rho + C \left( 1 + h^{d\beta(\frac{1}{r}-1)} (nh^d)^{\frac{\beta-1}{2}} + (nh^d)^{-\frac{\alpha}{2}} n^{\frac{1}{s}} \right) v_n. \quad (17)$$

Note that  $h_n^\rho = Cv_n$ . The expression in parenthesis is bounded if

$$0 \leq \frac{\beta d(1-r)}{r} + (1-\beta)\rho, \quad 0 \leq \frac{\alpha\rho}{d+2\rho} - \frac{1}{s}.$$

These conditions correspond to (6).  $\square$

*A bound of interest which does not need dependence.* The proofs of the propositions under different kinds of dependence make use of a common bound that

holds in all cases. For a positive integer  $k$ , we define the coefficient of weak dependence as non-decreasing sequences  $(C_{k,q})_{q \geq 2}$  such that

$$C_{k,q} = \sup |\text{Cov}(Z_{i_1} \cdots Z_{i_m}, Z_{i_{m+1}} \cdots Z_{i_q})|, \quad (18)$$

where the supremum is taken over all  $\{i_1, \dots, i_q\}$  such that  $1 \leq i_1 \leq \dots \leq i_q$  and  $m, k$  satisfy  $i_{m+1} - i_m = k$ . Independently of the dependence structure, we get a bound for  $C_{k,q}$ .

**Lemma 3** *Assume (A3), (A4), then  $C_{k,p} \leq Ch^{\frac{2d(s-p)}{s}}$ .*

*Proof.* Define  $\{i_1, \dots, i_p\}$  as a sequence that attains the sup defining  $C_{k,p}$ .

$$\begin{aligned} C_{k,p} &= |\mathbb{E}Z_{i_1} \cdots Z_{i_p}| \\ &\leq 2^p \mathbb{E}|Y_{i_1} K_{i_1} \cdots Y_{i_p} K_{i_p}| \\ &\leq 2^p \left( \mathbb{E}|Y_{i_1} \cdots Y_{i_p}|^{\frac{s}{p}} \right)^{\frac{p}{s}} \left( \max_{i_1, i_2} \mathbb{E}(K_{i_1} K_{i_2})^{\frac{s}{s-p}} \right)^{1-\frac{p}{s}} \\ &\leq 2^p \|Y_0\|_s^p \left( \int \int K^{\frac{s}{s-p}} \left( \frac{x-u}{h} \right) K^{\frac{s}{s-p}} \left( \frac{x-t}{h} \right) G(u, t) du dt \right)^{1-\frac{p}{s}} \\ &\leq 2^p \|Y_0\|_s^p h^{2d(1-\frac{p}{s})} \end{aligned}$$

The claim in the remark following (A3) is based on the point there is no need to use Hölder inequality if  $H$  is bounded around  $(x, x)$  and thus  $C_{k,p} \leq Ch^{2d}$ . Now  $C_{k,q} \leq Ch^{2d}$  also hold for the denominator (we also may set  $Y_i \equiv 1$ ).  $\square$

*Proof of Proposition 6.* From the Rosenthal inequality for independent variables there exist constants  $C_q \geq 0$  only depending on  $q$  (see e.g Figiel *et al.* (1997) [23] for more details concerning the constants),

$$\mathbb{E} \left| \sum_{i=1}^n Z_i \right|^q \leq C_q \left( \left| \sum_{i=1}^n \mathbb{E}Z_i^2 \right|^{\frac{q}{2}} + \sum_{i=1}^n \mathbb{E}|Z_i|^q \right).$$

Here  $\mathbb{E}Z_i^q \leq 2^q \mathbb{E}|K_i Y_i|^q$ . In the beginning of the proof of Proposition 5, we get  $\mathbb{E}|K_i Y_i|^q \leq \|g_q\|_\infty \|K\|_q^q h^d$ , and we deduce  $\mathbb{E} \left| \sum Z_i \right|^q \leq C \left( (nh^d)^{\frac{q}{2}} + nh^d \right)$ , and:

$$\|\hat{g}(x) - \mathbb{E}\hat{g}(x)\|_q \leq \frac{C}{\sqrt{nh^d}}.$$

The case of the denominator is obtained by setting  $Y_i \equiv 1$ .

*Proof of Proposition 7* We first establish a Rosenthal inequality for weakly dependent variables. For any integer  $p \geq 2$ ,  $\mathbb{E} \left( \sum_{i=0}^n Z_i \right)^p \leq p! A_p$  where

$$A_p = \sum_{1 \leq i_1 \leq \dots \leq i_p \leq n} |\mathbb{E}(Z_{i_1} \cdots Z_{i_p})|. \quad (19)$$

Then

$$A_p \leq n \sum_{k=0}^n (k+1)^{p-2} C_{k,p} + \sum_{l=2}^{p-2} A_l A_{p-l}. \quad (20)$$

To bound the sum with coefficients  $C_{k,p}$ , we shall use the following Lemma.

**Lemma 4 (Doukhan & Neumann [19], Lemma 10-(11))** *Assume that the stationary sequence  $(Z_n)_{n \in \mathbb{Z}}$  is  $\lambda$ -weakly dependent and satisfies  $\mu = \|Z_0\|_r \leq 1$  for some  $r > p$  then*

$$C_{k,p} \leq 2^{p+3} p^4 \mu^{\frac{r(p-1)}{r-1}} \lambda(k)^{\frac{r-p}{r-2}}.$$

Recall that if  $(X_i, Y_i)$  is  $\lambda$ -weakly dependent with  $\lambda(k) \leq Ck^{-\lambda}$ , then  $Z_i$  is  $\lambda$ -weakly dependent with  $\lambda(k) \leq Ck^{-\frac{\lambda(r-2)}{r}}$  (see Proposition 2.1 p. 12 in our (2007)'s monograph [11]). From Lemma 4 we get  $C_{k,p} \leq Ch^d \frac{p-1}{r-1} k^{-\lambda(1-\frac{p}{r})}$ .

We define conditions on the dependence coefficients that ensure that the sum to control is of the same order of magnitude than its first term, that is  $\mathcal{O}(h^d)$ . From Lemma 3, for any  $0 < b < 1$ :

$$\begin{aligned} C_{k,p} &\leq Ch^d \left( h^{-d \frac{r-p}{r-1}} k^{-\frac{\lambda(r-p)}{r}} \wedge h^{d(1-\frac{2p}{s})} \right) \\ &\leq Ch^d \times k^{-\frac{\lambda b(r-p)}{r}} h^{d(-b \frac{r-p}{r-1} + (1-b)(1-\frac{2p}{s}))} \end{aligned}$$

Choosing  $b < \frac{(s-2p)(r-1)}{2r(s-p)+2p-s}$ , the exponent of  $h$  in the parenthesis is positive. Because  $\lambda > \frac{1}{b} \cdot \frac{r(p-1)}{r-p}$ , the sum over  $k$  (in inequality (20)) converges and is less

than a constant, say  $a_p$ . We get  $A_p \leq a_p n h^d + \sum_{l=2}^{p-2} A_l A_{p-l}$ , and from  $A_2 \leq a_2 n h^d$ ,

we deduce by induction on  $p$  that  $A_p \leq c_p (n h^d)^{p/2}$  for a sequence  $c_p$  which may also depend on  $x$ , hence:  $\|\hat{g}(x) - \mathbb{E}\hat{g}(x)\|_p \leq c_p(x) (n h^d)^{-\frac{p}{2}}$ .

The case of the denominator is obtained by setting  $Y_i \equiv 1$ . In this case we also quote that the bound in Lemma 3 writes  $C_{k,q} \leq Ch^{2d}$  since no Hölder inequality is needed anymore.

$$\begin{aligned} C_{k,q} &\leq Ch^d (h^{-2+d} k^{-\lambda} \wedge h^d) \\ &\leq Ch^d \times k^{-\lambda b} h^{(-b(d-2)+(1-b)d)} \end{aligned}$$

Choosing  $b < d/(2d-2)$ , the exponent of  $h$  in the parenthesis is positive. Because  $\lambda > q-1/b$ , the sum over  $k$  (in inequality (20)) converges.  $\square$

*Proof of Proposition 8.* • Under the first set of conditions, Rio [28], Theorem 6.3 states the following Rosenthal inequality:

$$\mathbb{E} \left| \sum Z_i \right|^p \leq a_p \left( \sum_i \sum_j |\text{Cov}(Z_i, Z_j)| \right)^{\frac{p}{2}} + n b_p \int_0^1 (\alpha^{-1}(u) \wedge n)^{p-1} Q_Z^p(u) du$$

We use Lemma 3 to prove that the first term is  $\mathcal{O}((nh^d)^{\frac{p}{2}})$ . From Davydov inequality we get a second bound for the covariance

$$\begin{aligned} |\text{Cov}(Z_0, Z_i)| &\leq 6\alpha_i^{\frac{r-2}{r}} \|Y_0 K_0\|_r^2 \leq C\alpha_i^{\frac{r-2}{r}} h^{2d(\frac{1}{r}-1)} \\ \text{Var}(Z_0) &\sim h^d g_2(x) \int K^2(u) du \end{aligned}$$

hence

$$\left| \sum_i \sum_j \text{Cov}(Z_i, Z_j) - \sum_i \text{Var}(Z_i) \right| \leq nh^d \sum_i \alpha_i^{\frac{r-2}{r}} h^{d(\frac{2}{r}-3)} \wedge h^{d(1-\frac{4}{s})}$$

Thus considering some  $0 < b < 1$ ,

$$\sum_i \alpha_i^{1-\frac{2}{r}} h^{d(\frac{2}{r}-3)} \wedge h^{d(1-\frac{4}{s})} \leq h^{d(2b(\frac{1}{r}+\frac{2}{s}-2)+1-\frac{4}{s})} \sum_i \alpha_i^{b(1-\frac{2}{r})}.$$

This last term tends to 0 if  $b > \frac{sr-4r}{4sr-2s-4r}$  and  $\alpha > \frac{r}{b(r-2)}$ . This is possible if  $\alpha > \frac{r(4sr-2s-4r)}{(r-2)(sr-4r)}$ .

Consider the second term, apply Hölder inequality with exponents  $r/(r-p)$  and  $r/p$ :  $n \int_0^1 (\alpha^{-1}(u) \wedge n)^{p-1} Q_Z^p(u) du \leq n \left( \int_0^1 (\alpha^{-1}(u))^{\frac{r(p-1)}{r-p}} du \right)^{\frac{r-p}{r}} \|Z\|_r^p$ .

The first integral is convergent as soon as  $\alpha > \frac{r(p-1)}{r-p}$ , and from assumption **(A3)**-3,  $\|Z\|_r^r \leq Ch^{d(1-r)}$  so that this second term is negligible if  $nh^{dp(1-r)/r} \leq C(nh^d)^{p/2}$  hence if the sequence  $n^{r(2-p)} h^{dp(2-3r)}$  is bounded we obtain the desired bound.

Consider now the denominator.

$$\left| \sum_i \sum_j \text{Cov}(K_i, K_j) - \sum_i \text{Var}(K_i) \right| \leq nh^d \sum_i \alpha_i h^{-d} \wedge h^d$$

Thus  $\sum_i \alpha_i h^{-d} \wedge h^d \leq h^{d(1-2b)} \sum_i \alpha_i^b$  for  $0 < b < 1$ . This last term tends to 0 if  $b > 1/2$ , which implies  $\alpha > 2$ .

Consider the second term,

$$n \int_0^1 (\alpha^{-1}(u) \wedge n)^{q-1} Q_{K_0}^q(u) du \leq n \left( \int_0^1 (\alpha^{-1}(u))^{\frac{r(q-1)}{r-q}} du \right)^{\frac{r-q}{r}} \|K_0\|_r^q$$

The first integral is finite if  $\alpha > \frac{r(q-1)}{r-q}$ . Analogously the second term is negligible as soon as  $n^{r(2-q)} h^{dq(2-r)}$  is a bounded sequence.

Hence if  $h \sim n^{-a}$  a monotonicity argument shows that the previous bounds need  $ad \leq \frac{1-2/p}{3-2/r}$ .

• Under the second set of conditions, we use the idea in the proof of Proposition 7 (this idea was initiated in Doukhan & Portal, 1983, [20]). Here we use again

relations (20) and now expression (19) is again bounded by using the alternative bound of  $C_{k,p}$  which writes  $|\mathbb{E}Z_{i_1} \cdots Z_{i_p}|$  for a suitable sequence  $i_1 \leq \cdots \leq i_p$  with  $i_{u+1} - i_u = k$ . Then Davydov inequality (see Theorem 3(i) in Doukhan, 1994, [15]) and Hölder inequality together entail

$$C_{k,p} \leq 6\alpha_k^{\frac{p}{r}} \|Z_{i_1} \cdots Z_{i_u}\|_{\frac{r}{u}} \|Z_{i_{u+1}} \cdots Z_{i_p}\|_{\frac{r}{p-u}} \leq 6\alpha_k^{\frac{p}{r}} \|Z_0\|_r^p$$

and  $C_{k,p} \leq Ch^d (\alpha_k^{\frac{p}{r}} h^{-d}) \wedge h^{d(1-\frac{2p}{s})} \leq Ch^d \alpha_k^{\frac{p}{r}} h^{d(1-\frac{2p}{s}-2b(1-\frac{p}{s}))}$  from Lemma 3 if  $0 \leq b \leq 1$ . Then setting  $b = (s-2p)/(2(s-p))$ ,

$$n \sum_{k=0}^n (k+1)^{p-2} C_{k,p} = \mathcal{O}(nh^d), \quad \text{if } \alpha > \frac{r}{2} \frac{s-2p}{s-p} \left(1 - \frac{1}{p}\right).$$

The case of the denominator is exactly analogous and we replace here  $p$  by  $q$ ,  $s$  by  $\infty$  and in order to let the previous condition unchanged we replace  $r$  by  $r'$  with  $\frac{r'}{2} \left(1 - \frac{1}{q}\right) = \frac{r}{2} \frac{s-2p}{s-p} \left(1 - \frac{1}{p}\right)$ .  $\square$

### 5.1.3. Proofs for the subsection 4.2

*Proof of Proposition 9.* The previous convergences  $\mathbb{E}\hat{f}(x) \rightarrow f(x)$  and  $\mathbb{E}\hat{g}(x) \rightarrow r(x)f(x)$  are uniformly controlled by  $\mathcal{O}(h_n^\rho)$  under  $\rho$ -regularity conditions **(A5)** from the continuity of derivatives over the considered sets and from a standard compactness argument, see Ango-Nze and Doukhan (2004) [3]. Note, indeed, that if  $V \ni x$  denotes an open set over which the previous assumptions **(A1)** hold and such that  $\inf_B f > 0$ , then for each open set  $W$  with  $\overline{W} \subset V$  the previous relation holds uniformly over  $W$ . Hence under **(A5)**, if  $V$  denotes an open set with  $B \subset V$  such that the assumptions **(A1)** still hold the the bounds for biases hold uniformly over  $B$ . We thus proceed as in Proposition 4 to conclude.  $\square$

*Proof of Proposition 10.* From Lemma 2, we get

$$\begin{aligned} \inf_B D_n \cdot \sup_{x \in B} \left| \hat{r}(x) - \frac{\mathbb{E}\hat{g}(x)}{\mathbb{E}\hat{f}(x)} \right| &\leq \sup_B |\hat{N}_n - N_n| + \sup_B \frac{|\hat{N}_n|}{D_n} \sup_B |\hat{D}_n - D_n| \\ &\quad + \max_{1 \leq i \leq n} |Y_i| \sup_B \frac{|\hat{D}_n - D_n|^{1+\alpha}}{|D_n|^\alpha}. \end{aligned}$$

Along the same line of the proof of Lemma 1, substituting the supremum to the variables we get

$$\begin{aligned} \inf_{x \in B} D_n(x) \cdot \left\| \sup_{x \in B} \left| \hat{r}(x) - \frac{\mathbb{E}\hat{g}(x)}{\mathbb{E}\hat{f}(x)} \right| \right\|_p &\leq \\ C \left( 1 + \frac{\sup_B |N_n|}{\inf_B D_n} + \frac{\sup_B |N_n|^\beta w_n^{1-\beta}}{\inf_B D_n} + \frac{C_n^\beta w_n^{1-\beta}}{\inf_B D_n} + \frac{w_n^\alpha c_n n^{\frac{1}{s}}}{\inf_B D_n^\alpha} \right) w_n & \end{aligned}$$

where  $D_n$ ,  $N_n$  and  $c_n$  are equivalent to constants, and  $C_n \equiv Ch^{d(1/r-1)}$ . Substituting the orders to the expressions, we get the result.  $\square$

*Proof of the inequalities following Theorem 3* For (9), let  $M > 0$  and consider the truncated modification of  $Y_i$ :  $\tilde{Y}_i = Y_i \mathbb{I}\{|Y_i| \leq M\} - M \mathbb{I}\{Y_i < -M\} + M \mathbb{I}\{Y_i > M\}$ . Define  $\tilde{g}(x) = \frac{1}{nh^d} \sum_{i=1}^n \tilde{Y}_i K_i$  then, from the Markov inequality,

$$\left\| \sup_{x \in \mathbb{R}^d} |\hat{g}(x) - \tilde{g}(x)| \right\|_p^p \leq \frac{1}{h^{dp}} \mathbb{E}|Y|^p \mathbb{I}\{|Y| > M\} \leq \frac{M^{p-s}}{h^{dp}} \mathbb{E}|Y|^s.$$

With the choice (11), in order to bound conveniently this term we assume

$$M \geq Ch^{-\frac{p}{s-p}(\rho+d)}. \quad (21)$$

Denote  $Z(x) = \sqrt{nh^d} |\tilde{g}(x) - \mathbb{E}\tilde{g}(x)|$ . Below we shall need (uniform) bounds of

$$\begin{aligned} \text{Var } Z(x) &= \frac{1}{nh^d} \sum_{|i| < n} (n - |i|) \text{Cov}(\tilde{Y}_0 K_0, \tilde{Y}_i K_i) \\ &\leq \frac{2}{h^d} \sum_{i=0}^{n-1} |\text{Cov}(\tilde{Y}_0 K_0, \tilde{Y}_i K_i)| \\ &\leq \frac{2}{h^d} \sum_{i=0}^{n-1} \Gamma_i(x) \end{aligned}$$

A first bound of the covariance  $\Gamma_i(x)$  for  $i \neq 0$  has nothing to do with weak dependence conditions and from Lemma 3: there exists a constant  $C$  which may change from line to line such that

$$\begin{aligned} \Gamma_i(x) &\leq Ch^{2d(1-\frac{2}{s})} \\ \Gamma_0(x) &\sim h^d g_2(x) \int K^2(u) du, \quad \text{around the point } x. \end{aligned} \quad (22)$$

For independent random sequences,  $\Gamma_0(x)$  is the only nonzero term. The second bound of  $\Gamma_i(x)$  for  $i > 0$  is based on dependence conditions.

*Proof of Proposition 11.* Proposition 9 proves (10). In order to prove (9), from the Bernstein inequality for independent bounded variables, we get

$$\mathbb{P}(Z(x) > u) \leq 2 \exp \left( - \frac{u^2}{2 \left( \text{Var}(Z(x)) + \frac{2Mu \|K\|_\infty}{3\sqrt{nh^d}} \right)} \right). \quad (23)$$

From (22),  $\text{Var}(Z(x))$  is bounded by a constant Because  $K$  is Lipschitz kernel,  $x \mapsto Z(x)$  is a Lipschitz function and:

$$|Z(x) - Z(y)| \leq 2\sqrt{nh^d} M h^{-(d+1)} \text{Lip } K \|x - y\|_1$$

Define a regular partition  $(B_i)_{i=1, \dots, \nu}$  of diameter  $\delta$  over  $B$ , let  $x_i$  be the center of  $B_i$ .

$$\sup_{x \in B_j} |K_i(x) - K_i(x_j)| \leq \frac{c\delta}{h} \mathbb{I}\{|x_j - X_i| \leq 2hR\}$$

if  $\delta < M_K h$ , where  $M_K$  is the size of the support of  $K$ . Define

$$\bar{g}(x) = \frac{1}{nh^d} \sum |\tilde{Y}_i| \mathbb{I}\{|x - X_i| \leq 2hR\}$$

and

$$\bar{Z}(x) = \sqrt{nh^d} |\bar{g}(x) - \mathbb{E}\bar{g}(x)|.$$

Write  $\tilde{g}(x) - \mathbb{E}\tilde{g}(x) = \tilde{g}(x) - \tilde{g}(x_j) + \mathbb{E}(\tilde{g}(x_j) - \tilde{g}(x)) + \tilde{g}(x_j) - \mathbb{E}\tilde{g}(x_j)$ .

For  $x \in B_j$ ,  $|\tilde{g}(x) - \tilde{g}(x_j)| \leq \frac{c\delta}{h} \bar{g}(x_j)$  so that

$$|\tilde{g}(x) - \mathbb{E}\tilde{g}(x)| \leq \frac{c\delta}{h} (\bar{g}(x_j) + \mathbb{E}\bar{g}(x_j)) + |\tilde{g}(x_j) - \mathbb{E}\tilde{g}(x_j)|.$$

Denote  $Z_j = \sup_{x \in B_j} Z(x)$ :

$$|Z_j| \leq \frac{c\delta}{h} |\bar{Z}(x_j)| + \frac{2c\delta\sqrt{nh^d}}{h} \mathbb{E}\bar{g}(x_j) + |Z(x_j)|.$$

Note that since  $\mathbb{E}\bar{g}(x)$  tends to  $\bar{g}_1(x)$  as  $h$  tends to 0, so that  $2c\delta\mathbb{E}\bar{g}(x)/h \leq t/(3\sqrt{nh^d})$  holds if  $\delta \leq c'th/\sqrt{nh^d}$  for a suitable constant  $c' > 0$ ; this condition holds for the following choice (considered only for large values of  $t > 0$ ),

$$\delta = \frac{Ch}{\sqrt{nh^d}}, \quad \text{and} \quad t \geq t_0 > 0. \quad (24)$$

Denote  $Z = \sup_{x \in B} Z(x)$ . For  $t > t_0$ :

$$\begin{aligned} \mathbb{P}(Z > t) &= \mathbb{P}(\max_{1 \leq j \leq \nu} Z_j > t) \\ &\leq \nu \max_{1 \leq j \leq \nu} \mathbb{P}(Z_j > t) \\ &\leq \nu \max_{1 \leq j \leq \nu} \left\{ \mathbb{P}\left(|\tilde{Z}(x_j)| > \frac{t}{3}\right) + \mathbb{P}\left(|\bar{Z}(x_j)| > \frac{th}{3c\delta}\right) \right\} \\ &\leq \nu \max_{1 \leq j \leq \nu} \left\{ \mathbb{P}\left(|\tilde{Z}(x_j)| > \frac{t}{3}\right) + \mathbb{P}\left(|\bar{Z}(x_j)| > \frac{t}{3}\right) \right\} \end{aligned} \quad (25)$$

$$\leq 4\nu \exp\left(-\frac{at^2}{1 + tM(nh^d)^{-1/2}}\right), \quad (26)$$

from the relation  $h/\delta \rightarrow \infty$ , we indeed assume that  $h/\delta \geq 1$  in order to derive relation (25), for some  $a > 0$

Now

$$\mathbb{E}Z^p \leq T^p + p \int_T^\infty \mathbb{P}(Z > t) t^{p-1} dt$$

Choose  $T = \sqrt{A \log n}$ , quote that the function  $u \mapsto \exp\left(-\frac{at^2}{1+tu}\right)$  is nonincreasing, then assuming that

$$\frac{M}{\sqrt{nh^d}} \leq \frac{1}{T} = \frac{1}{\sqrt{A \log n}} \quad (27)$$



That is, referring to the choice for  $h$ :

$$M \leq \frac{h^{-\rho}}{\sqrt{A}} \tag{28}$$

we derive

$$\mathbb{E}Z^p \leq T^p + 4p\nu \int_T^\infty t^{p-1} \exp\left(-\frac{at^2}{1+t/T}\right) dt \tag{29}$$

$$\leq T^p + 4p\nu \int_T^\infty t^{p-1} \exp\left(-\frac{atT}{2}\right) dt, \tag{30}$$

and using the incomplete gamma function expansion for  $x > 2p$ :

$$\int_x^\infty u^{p-1} e^{-u} du \leq 2x^{p-1} e^{-x}$$

then setting  $u = atT/2$  in the previous inequality we obtain

$$\begin{aligned} \mathbb{E}Z^p &\leq T^p + 4p\nu \left(\frac{2}{aT}\right)^p \int_{aT^2/2}^\infty u^{p-1} e^{-u} du \\ &\leq T^p + \frac{8p}{a} \nu T^{p-1} n^{-aA/2} \end{aligned}$$

With  $\nu \sim \delta^{-d}$  and relation (24) we see that the second term is negligible with respect to the first one if  $A$  is chosen large enough. Then  $\mathbb{E}Z^p = \mathcal{O}((\log n)^{p/2})$ , and  $\|Z\|_p = \sqrt{nh^d} \|\sup_{x \in B} |\tilde{g}(x) - \mathbb{E}\tilde{g}(x)|\|_p = \mathcal{O}((\log n)^{1/2})$ . Now we check that the conditions (21) and (28) on  $M$  are compatible. This possibly holds if  $\rho > dp/(s - 2p)$ .  $\square$

*Proof of Proposition 12.* First we define a truncation level  $M$  satisfying (21) and consider the truncated variable  $\tilde{g}(x)$ . A strong coupling argument by Berbee (1979) yields a Bernstein type inequality. With Theorem 4 of Doukhan (1994) [15], we recall analogously to (23) that there exists some  $\theta, \lambda, \mu > 0$  and there exists an event  $A_n$  (which does not depend neither on  $x$  nor on  $u$ ) with:

$$\mathbb{P}((Z(x) > u) \cap A_n) \leq 4 \exp\left(-\frac{\lambda u^2}{2\left(\text{Var}(Z(x)) + \frac{2Mqu\|K\|_\infty}{3\sqrt{nh^d}}\right)}\right), \tag{31}$$

with  $\mathbb{P}(A_n^c) \leq \mu\beta_{q\theta}$ . We first check that  $\text{Var}(Z(x))$  is bounded by a constant independent of  $x$ .

$$|\text{Var}(Z(x)) - h^{-d}\Gamma_0(x)| \leq \frac{2}{h^d} \sum_{i=1}^{n-1} \Gamma_i(x) \tag{32}$$

In fact using  $\beta$ -mixing does do improve on the results so this variance will be bounded under strong mixing and the relation  $\alpha_i \leq \beta_i$  will allow to conclude

here and in the forthcoming section dedicated to strong mixing.

Recall, with  $\alpha_0 = 1$ , that  $\Gamma_i(x) \leq \int_0^{\alpha_i} Q^2(x)dx$ . Now inequality (31) is exactly (23) substituting  $qM$  to  $M$ . Following the line of the proof in the independent case we get  $\mathbb{P}((Z > t) \cap A_n) \leq 4\nu \exp\left(-\frac{at^2}{1 + tqM(nh^d)^{-1/2}}\right)$ .

The condition (21) on  $M$  now writes

$$qM \leq \frac{h^{-\rho}}{\sqrt{A}} \quad (33)$$

and the end of the proof remains unchanged. We get

$$\sqrt{nh^d} \left\| \sup_{x \in B} |\hat{g}(x) - \mathbb{E}\hat{g}(x)| \right\|_p \leq C \sqrt{\frac{\log n}{nh^d}} + \mathbb{E}Z^p \mathbb{1}_{A_n^c}.$$

Then, using the trivial bound  $\mathbb{E}Z^p \mathbb{1}_{A_n^c} \leq \|Z\|_\infty^p \mathbb{P}(A_n^c)$ ,

$$\sqrt{nh^d} \left\| \sup_{x \in B} |\hat{g}(x) - \mathbb{E}\hat{g}(x)| \right\|_p \leq C \sqrt{\frac{\log n}{nh^d}} + 2M \|K\|_\infty \frac{\sqrt{n}}{h^{d/2}} \mu \beta_{q\theta}. \quad (34)$$

Define  $q = n^\gamma$ , with  $0 < \gamma < 1$  the compatibility of the inequalities concerning  $M$  demands  $n^{-\gamma} h^{-\rho} > h^{-\frac{p(\rho+d)}{s-p}}$ . This holds if  $\gamma = \frac{\rho(s-2p) - pd}{(2\rho+d)(s-p)}$ . Choosing  $M = Ch^{-\frac{p}{s-p}(\rho+d)}$ , the second term of (34) writes

$$\frac{Mn}{\sqrt{nh^d}} \beta_{q\theta} = C \frac{\log(n)^{\frac{p}{s-p}(\rho+d)}}{\sqrt{nh^d}} n^{1-\beta\gamma + \frac{p}{s-p}(\rho+d)}.$$

It is negligible with respect to the second term as soon as  $\beta > \frac{s\rho + (2s-p)d}{\rho(s-2p) - pd}$ .  $\square$

*Proof of Proposition 13* As in the independent case, we chose a truncation level  $M = \left(\frac{n}{\log n}\right)^{\frac{(\rho+d)p}{(2\rho+d)(s-p)}}$  satisfying relation (21) and define  $Z(x)$  with respect to the truncated process  $\tilde{g}$ . Fuk-Nagaev inequality leads to:

$$\mathbb{P}(Z(x) > u) \leq 4 \left(1 + \frac{u^2}{16r \text{Var}(Z(x))}\right)^{-r/2} + \frac{16nM \|K\|_\infty}{u\sqrt{nh^d}} \alpha \left(\frac{u\sqrt{nh^d}}{4M \|K\|_\infty r}\right).$$

Adapting the proof of Proposition 8, we check that  $\text{Var}(Z(x))$  is bounded by a constant independent of  $x$ . Following the proof in the independent case, we get:

$$\mathbb{P}(Z > t) \leq c_0 \nu \left(1 + \frac{t^2}{c_1 r}\right)^{-\frac{r}{2}} + c_2 \nu n \left(\frac{M}{t\sqrt{nh^d}}\right)^{1+\alpha} r^\alpha,$$

for suitable constants  $c_0, c_1$  and  $c_2$ . We choose  $T = \sqrt{A \log n}$  and  $r = bT^2$ . Then  $\sqrt{\frac{\log n}{nh^d}} = \left(\frac{n}{\log n}\right)^{-\frac{\rho}{2\rho+d}}$ . With this choice:  $\frac{MT}{\sqrt{nh^d}} = \sqrt{A} \left(\frac{n}{\log n}\right)^{\frac{dp-\rho(s-2p)}{(2\rho+d)(s-p)}}$ .

$$\begin{aligned} \mathbb{E}Z^p &\leq T^p + c_0 \nu \int_T^\infty t^{p-1} \left(1 + \frac{t^2}{c_1 b T^2}\right)^{-bT^2/2} dt \\ &\quad + c_2 b^\alpha \nu n \left(\frac{MT}{\sqrt{nh^d}}\right)^{1+\alpha} T^{\alpha-1} \int_T^\infty t^{p-2-\alpha} dt \\ &\leq T^p + c_0 \nu T^p \int_1^\infty u^{p-1} \left(1 + \frac{u^2}{c_1 b}\right)^{-bT^2/2} du \\ &\quad + \frac{c_2 A^{\frac{1+\alpha}{2}} b^\alpha}{\alpha - p + 1} \nu n T^{p-2} \left(\frac{n}{\log n}\right)^{\frac{(1+\alpha)(dp-\rho(s-2p))}{(2\rho+d)(s-p)}} \\ &\leq T^p + c_0 \nu (c_1 b)^{bT^2/2} T^p \int_1^\infty u^{p-1-bT^2} du \\ &\quad + \frac{c_2 A^{\frac{1+\alpha}{2}} b^\alpha}{\alpha - p + 1} \nu n T^{p-2} \left(\frac{n}{\log n}\right)^{\frac{(1+\alpha)(dp-\rho(s-2p))}{(2\rho+d)(s-p)}} \\ &\leq T^p + c_0 \frac{\nu T^p (c_1 b)^{bT^2/2}}{bT^2 - p + 1} + \frac{c_2 A^{\frac{1+\alpha}{2}} b^\alpha}{\alpha - p + 1} \nu n T^{p-2} \left(\frac{n}{\log n}\right)^{\frac{(1+\alpha)(dp-\rho(s-2p))}{(2\rho+d)(s-p)}} \end{aligned}$$

Setting  $b = 1/(ec_1)$ ,  $(c_1 b)^{bT^2/2} \leq n^{-bA/2}$ , the second term is negligible for a suitable choice of  $A$ . Now

$$\nu n = n \left(\frac{n}{\log n}\right)^{\frac{d(\rho+1)}{2\rho+d}} (\log n)^{\frac{d}{2}}.$$

Thus, if  $dp > \rho(s-2p)$  and  $1+\alpha > \frac{(2\rho+2d+dp)(s-p)}{dp-\rho(s-2p)}$ , the third term is negligible. The end of the proof follows the lines of the independent case.  $\square$

*Proof of Proposition 14.* Choose

$$M = \left(\frac{n}{\log n}\right)^{\frac{(\rho+d)p}{(2\rho+d)(s-p)}}$$

and define  $Z(x)$  as in the preceding sections. Proposition 8 and Theorem 1 in Doukhan and Neumann (2007) [19] prove that:

$$\mathbb{P}(Z(x) > u) \leq c_0 \exp\left(-\frac{c_1 u^2}{\text{Var}(Z(x)) + \left(\frac{Mu\|K\|_\infty}{\sqrt{nh^d}}\right)^{\frac{b}{b+2}}}\right),$$

for suitable constants  $c_0$  and  $c_1$ . We first check that  $\text{Var}(Z(x))$  is bounded by a constant independent of  $x$ . The generic term in (32) is bounded by using weak

dependence and the fact that the function  $u \mapsto K((x-u)/h)$  is  $C/h$ -Lipschitz:

$$\Gamma_i(x) \leq C\left(\frac{M}{h} + \frac{1}{h^2}\right)\lambda(i) \leq C\left(h^{-1-\frac{(\rho+d)p}{s-p}} + h^{-2}\right)\lambda(i),$$

Thus considering some  $0 < \alpha < 1$ , up to a constant, the RHS of (32) is bounded above by:

$$\sum_i h^d \wedge (h^{-(d+b)}\lambda(i)) \leq h^{d(1-2\alpha)-b\alpha} \sum_i \lambda^\alpha(i),$$

in the previous relation  $b = 2$  or  $b = 1 + (\rho + d)p/(s - p)$  respectively if  $s \geq (\rho + d + 1)p$  or  $s < (\rho + d + 1)p$ . Taking  $\alpha < d/(2d + b)$  and noting that  $\sum_i \lambda^\alpha(i) < \infty$  the corresponding sum is negligible; thus  $\text{Var } Z(x) \sim g_2(x) \int K^2(u)du$ .

Then, following the same lines as in the independent case, we get:

$$\mathbb{P}(Z > t) \leq 2c_0\nu \exp\left(-\frac{at^2}{1 + (tM(nh^d)^{-1/2})^{\frac{b}{b+2}}}\right),$$

for some  $a > 0$ . Choosing  $T = \sqrt{A \log n}$ ,  $MT/\sqrt{nh^d} < 1$ .

$$\begin{aligned} \mathbb{E}Z^p &\leq T^p + 4p\nu \int_T^\infty t^{p-1} \exp\left(-\frac{at^2}{1 + (t/T)^{\frac{b}{b+2}}}\right) dt \\ &\leq T^p + 4p\nu \int_T^\infty t^{p-1} \exp\left(-\frac{aT^{\frac{b}{b+2}}}{2} t^{\frac{b+4}{b+2}}\right) dt \end{aligned} \quad (35)$$

and then setting  $u = \frac{aT^{\frac{b}{b+2}}}{2} t^{\frac{b+4}{b+2}}$  in the previous inequality (35) we obtain

$$\begin{aligned} \mathbb{E}Z^p &\leq T^p + 4p\nu \left(\frac{2}{aT^{\frac{b}{b+2}}}\right)^p \int_{\frac{a}{2}T^2}^\infty u^{\frac{(b+2)p}{b+4}-1} e^{-u} du \\ &\leq T^p + 4p \left(\frac{2}{a}\right)^{\frac{2p}{b+4}-1} \nu T^{\frac{(b^2+4b+8)p}{(b+4)(b+2)}-1} n^{-aA/2} \end{aligned}$$

With  $\nu \sim \delta^{-d}$  and relation (24) we see that the second term is negligible with respect to the first one if  $A$  is chosen large enough. The end of the proof is exactly the same as above.  $\square$

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