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To cite this version:
Lazhar Dhaouadi. Prolate Spheroidal Wave Functions In q-Fourier Analysis. 2008. <hal-00163624v2>

HAL Id: hal-00163624
https://hal.archives-ouvertes.fr/hal-00163624v2
Submitted on 9 Apr 2008

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Prolate Spheroidal Wave Functions In
\(q\)-Fourier Analysis

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Abstract

In this paper we introduce a new version of the Prolate spheroidal wave function using standard methods of \(q\)-calculus and we formulate some of its properties. As application we give a \(q\)-sampling theorem which extrapolates functions defined on \(q^n\) and \(0 < q < 1\).

Keywords : \(q\)-Prolate spheroidal wave function, \(q\)-sampling,

2000 AMS Mathematics Subject Classification—Primary 33D15,47A05.

1 Introduction

The prolate spheroidal wave functions, which are a special case of the spheroidal wave functions, possess a very surprising and unique property [7]. They are an orthogonal basis of both \(L^2(-1,1)\) and the Paley-Wiener space of bandlimited functions. They also satisfy a discrete orthogonality relation. No other system of classical orthogonal functions is known to possess this strange property. We prove that there are new systems possessing this property in \(q\)-Fourier analysis. In the following we discuss some properties of the \(q\)-Prolate spheroidal wave function using new developments and technics in \(q\)-Fourier analysis. In particular we prove that these functions forms an orthogonal basis of the \(q\)-Paley-Wiener space \(PW_{q,a}^v\). Finally and as application we give a constructive \(q\)-sampling formula having as sampling points \(q^n\) where \(n \in \mathbb{Z}\). In the end, we cit the reference [4], where the reproducing kernel for the \(q\)-Paley-Wiener space was already discussed, and the explicit formula for the kernel was given, similar to the formula in Remark [3]. However, the paper [4] proceeds with a \(q\)-sampling theorem which
extrapolates functions defined on the zeros of the $q$-Bessel function. These zeros are given in the following form

$$\{q^{-n+\epsilon_n}\}_{n\in\mathbb{N}},$$

where $0 < \epsilon_n < 1$, but it is not explicitly evaluated.

## 2 Preliminary

Throughout this paper we consider $0 < q < 1$ and we adopt the standard conventional notations of [3]. We put

$$\mathbb{R}_q = \{\pm q^n, \; n \in \mathbb{Z}\}, \quad \mathbb{R}_q^+ = \{q^n, \; n \in \mathbb{Z}\},$$

and if $a = q^n$, $n \in \mathbb{Z}$ put

$$[0, a]_q = \{q^s, \; s \in \mathbb{Z}, \; s \geq n\}.$$

For complex $z$, let

$$ (z; q)_0 = 1, \quad (z; q)_n = \prod_{i=0}^{n-1} (1 - \frac{z}{q^i}), \quad n = 1\ldots\infty. $$

Jackson’s $q$-integral in the interval $[0, a]$ and in the interval $[0, \infty]$ are defined, respectively, by (see [3])

$$\int_0^a f(x) d_q x = (1 - q)a \sum_{n=0}^{\infty} q^n f(aq^n),$$

$$\int_0^\infty f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} q^n f(q^n).$$

For $v > -1$, let $L_{q,p,v}$ be the space of even functions $f$ defined on $\mathbb{R}_q$ such that

$$\|f\|_{q,p,v} = \left[ \int_0^\infty |f(x)|^p x^{2v+1} d_q x \right]^{1/p} < \infty.$$

The set $L_{q,2,v}$ is an Hilbert space with the inner product

$$\langle f, g \rangle = \int_0^\infty f(t) g(t) t^{2v+1} d_q t.$$
We consider $L_{q,v,a}$ the space of function defined on $[0,a]_q$ which satisfies
\[
\int_0^a |f(x)|^2 x^{2v+1} d_q x < \infty,
\]
and $L_{v,a}^{q,v}$ the subspace of $L_{q,2,v}$ given by the natural embedding of $L_{q,v,a}$ in $L_{q,2,v}$.

The normalized Hahn-Exton $q$-Bessel function of order $v > -1$ (see [6]) is defined by
\[
j_v(z,q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q,q)_n(q^{v+1},q)_n} z^{2n}.
\]
It is an entire analytic function in $z$.

**Proposition 1** For $\Re(v) > -1$, $a > 0$ and $y,z \in \mathbb{C}\backslash \{0\}$ we have
\[
\int_0^a j_v(yt, q^2) j_v(zt, q^2) t^{2v+1} d_q t
= \frac{1 - q}{1 - q^{2v+2}} a^{2v+2} y^2 j_{v+1}(ay, q^2) j_v(az^{-1} q^{-1} y, q^2) - z^2 j_{v+1}(aq^{-1} y, q^2) j_v(az^{-1} q^{-1} y, q^2).
\]

**Proof.** See [5] (Proposition 1.3) ■

The following results in this section were proved in [2].

**Proposition 2**
\[
|j_v(q^n, q^2)| \leq \left( -q^2:q^2 \right)_\infty \left( -q^{2v+2}:q^2 \right)_\infty \left( q^{n(2v+1)} \right) \begin{cases} 1 & \text{if } n \geq 0 \\ q^{n^2+(2v+1)n} & \text{if } n < 0 \end{cases}.
\]

The $q$-Bessel Fourier transform $F_{q,v}$ introduced in [3,4] as follow
\[
F_{q,v} f(x) = c_{q,v} \int_0^\infty f(t) j_v(zt, q^2) t^{2v+1} d_q t,
\]
where
\[
c_{q,v} = \frac{1}{1 - q} \left( q^{2v+2}, q^2 \right)_\infty
\]
The $q$–Bessel translation operator is defined as follows:
\[
T_{q,v}^y f(y) = c_{q,v} \int_0^\infty F_{q,v}(f)(t) j_v(xt, q^2) j_v(yt, q^2) t^{2v+1} d_q t, \quad \forall x, y \in \mathbb{R}_q, \forall f \in L_{q,1,v},
\]
Recall that $T_{q,x}^v$ is said positive if $T_{q,x}^v f \geq 0$ for $f \geq 0$. In the following we tack $q \in Q_v$ where

$$Q_v = \{ q \in [0,1], \quad T_{q,x}^v \text{ is positive for all } x \in \mathbb{R} \}.$$ 

The $q$–convolution product of both functions $f, g \in L_{q,1,v}$ is defined by

$$f \ast_q g(x) = c_{q,v} \int_{0}^{\infty} T_{q,x}^v f(y) g(y) y^{2v+1} d_q y.$$ 

**Theorem 1** The operator $F_{q,v}$ satisfying

1. For all functions $f \in L_{q,2,v}$, $F_{q,v}^2 f(x) = f(x)$, $\forall x \in \mathbb{R}_q$.
2. For all functions $f, g \in L_{q,2,v}$, $\langle F_{q,v} f, g \rangle = \langle f, F_{q,v} g \rangle$.
3. For all functions $f \in L_{q,2,v}$, $\| F_{q,v} f \|_{q,v,2} = \| f \|_{q,v,2}$.
4. For all functions $f, g \in L_{q,1,v}$,

$$F_{q,v} (f \ast_q g)(x) = F_{q,v} f(x) \times F_{q,v} g(x), \quad \forall x \in \mathbb{R}_q.$$ 

In the end we consider $PW_{q,a}^v$ the $q$-Paley Wiener space

$$PW_{q,a}^v = \left\{ f(x) = \int_{a}^{0} u(t) j_v(xt, q^2) t^{2v+1} d_q t, \quad u \in L_{q,a}^v \right\},$$

the set of $q$-bandlimited signal.

### 3 Main Results

We introduce the $q$-analogue of the Prolate Spheroidal Wave Functions $\psi_i$ as the eigenfunction of the integral operator $T_{a}^v$ acting on the Hilbert space $L_{q,v,a}$ as follows

$$T_{a}^v u(x) = c_{q,v} \int_{0}^{a} u(t) j_v(xt, q^2) t^{2v+1} d_q t,$$

then we have

$$T_{a}^v \psi_i = \lambda_i \psi_i.$$ 

It’s easy to see that the operator $T_{a}^v$ is symmetric and compact

$$\int_{0}^{a} T_{a}^v u(t) w(t) t^{2v+1} d_q t = \int_{0}^{a} u(t) T_{a}^v w(t) t^{2v+1} d_q t,$$

then the sequence $\{ \psi_i \}_{i \in \mathbb{N}}$ form an orthogonal basis of the Hilbert space $L_{q,v,a}$ and any eigenvalue $\lambda_i$ is real.
Proposition 3 The sequence of eigenvalue \( \{ \lambda_i \}_{i \in \mathbb{N}} \) satisfying
\[
\lambda_0^2 \geq \lambda_1^2 \geq \ldots > 0.
\]

**Proof.** The operator \( T_a^v \) is compact, then the spectrum is a countably infinite subset of \( \mathbb{R} \) (\( T_a^v \) is symmetric) which has 0 as its only limit point. If we denote by \( \Lambda = \{ \lambda_0, \lambda_1, \ldots \} \),
the spectrum of \( T_a^v \) then we can write
\[
|\lambda_0| \geq |\lambda_1| \geq \ldots \geq 0.
\]
To finish the proof, if suffice to prove that 0 \( \notin \) \( \Lambda \). In fact if \( T_a^v \psi = 0 \) then \( \mathcal{F}_{q,v} \psi \) is an entire function which vanishes on \([0, a]\). By the identity theorem for analytic functions, \( \mathcal{F}_{q,v} \psi = 0 \) everywhere and thus \( \psi = 0 \). ■

**Remark 1** Consider the operator
\[
k_a^v = T_a^v \circ T_a^v,
\]
then \( K_a^v \) is an integral operator acting on the Hilbert space \( L_{q,v,a} \) as follows
\[
k_a^v u(x) = \int_0^a u(y)k(x, y)y^{2v+1} dq_y,
\]
where
\[
k(x, y) = c_{q,v}^2 \int_0^a j_v(xt, q^2)j_v(yt, q^2)t^{2v+1} q^{2v+1} dt.
\]
The function \( \psi_i \) is an eigenfunction of \( k_a^v \)
\[
k_a^v \psi_i = \lambda_i^2 \psi_i.
\]

**Lemma 1** The function \( \psi_i \) initially defined on \( \mathbb{R}_q \) can be extended as an analytic function on \( \mathbb{C} \).

**Proof.** The result follows from the relation
\[
\psi_i(z) = \frac{1}{\lambda_i} c_{q,v} \int_0^a \psi_i(t)j_v(zt, q^2)t^{2v+1} dq_t,
\]
and the fact that \( j_v(., q^2) \) is an entire function. ■

**Proposition 4** The function \( \psi_i \) belonging to the Paley-Wiener space \( PW_{q,a}^v \)
Proof. Let
\[ \phi_i(x) = \frac{1}{\lambda_i} \psi_i(x) \chi_{[0,a]}(x), \]
then
\[ \mathcal{F}_{q,v} \phi_i(x) = c_{q,v} \int_0^\infty \phi_i(t) j_v(xt, q^2 t^{2v+1} d_q t) \]
\[ = \frac{c_{q,v}}{\lambda_i} \int_0^a \psi_i(t) j_v(xt, q^2 t^{2v+1} d_q t = \psi_i(x), \]
which implies that \( \psi_i \in PW_{q,a}^v \).

In the following we assume that
\[ \| \psi_i \|^2_{q,2,v} = \langle \psi_i, \psi_i \rangle = 1. \]

**Proposition 5** The sequence \( \{\psi_i\}_{i \in \mathbb{N}} \) form an orthonormal basis of \( PW_{q,a}^v \).

Proof. The \( q \)-Bessel Fourier transform
\[ \mathcal{F}_{q,v} : L_{q,a}^v \rightarrow PW_{q,a}^v, \]
define an isomorphism, and the sequence \( \{\phi_i\}_{i \in \mathbb{N}} \) form an orthogonal basis of the Hilbert space \( L_{q,a}^v \), which lead to the result. \( \blacksquare \)

**Proposition 6** Let
\[ k_x : y \mapsto k(x, y), \]
then
\[ f \in PW_{q,a}^v \iff f(x) = \langle f, k_x \rangle, \quad \forall x \in \mathbb{R}_q. \]

Proof. Let
\[ \sigma_a(y) = \mathcal{F}_{q,v} \left( \chi_{[0,a]} \right)(x) = c_{q,v} \int_0^a j_v(ty, q^2 t^{2v+1} d_q t, \]
therefore
\[ T_{q,v} \sigma_a(y) = c_{q,v} \int_0^a j_v(tx, q^2 t^{2v+1} d_q t = \frac{1}{c_{q,v}} k(x, y), \]
and then
\[ f \in PW_{q,a}^v \iff \mathcal{F}_{q,v} f(x) = \mathcal{F}_{q,v} f(x) \chi_{[0,a]}(x) = \mathcal{F}_{q,v} f(x) \mathcal{F}_{q,v} \sigma_a(x) \]
\[ \iff f(x) = f * q \sigma_a(x) = c_{q,v} \langle f, T_{q,v} \sigma_a \rangle = \langle f, k_x \rangle. \]
This finish the proof \( \blacksquare \)
Corollary 1 We have

\[ k(x, y) = \sum_{i=0}^{\infty} \psi_i(x)\psi_i(y), \quad \forall x, y \in \mathbb{R}. \]

**Proof.** In fact \( k_x \in PW_{q,a}^v \). Then

\[ k_x(y) = \sum_{i=0}^{\infty} \langle k_x, \psi_i \rangle \psi_i(y). \]

On the other hand

\[ \psi_i \in PW_{q,a}^v \iff \langle \psi_i, k_x \rangle = \psi_i(x), \]

which prove the result. ■

Lemma 2 For \( i, j \in \mathbb{N} \)

\[ \int_0^a \psi_i(x)\psi_j(x)x^{2v+1}d_qx = \lambda_i\lambda_j\delta_{ij}. \]

**Proof.** In fact

\[ \langle \phi_i, \phi_j \rangle = \langle F_{q,v}\phi_i, F_{q,v}\phi_j \rangle = \langle \psi_i, \psi_j \rangle, \]

and

\[ \langle \phi_i, \phi_j \rangle = \frac{1}{\lambda_i\lambda_j} \int_0^a \psi_i(x)\psi_j(x)x^{2v+1}d_qx. \]

On the other hand, if \( i \neq j \) then

\[ \langle \phi_i, \phi_j \rangle = \int_0^a \phi_i(t)\phi_j(t)t^{2v+1}d_qt = 0. \]

Moreover, \( \|\phi_i\|_{2,v,q} = \|\psi_i\|_{2,v,q} = 1 \) which prove that \( \langle \phi_i, \phi_j \rangle = \delta_{ij} \). This leads to the result. ■

In order to be more precise about what it means for the energy of a \( q \)-bandlimited single \( f \in PW_{q,a}^v \) to be mainly concentrated on the interval \([0, a]_q\), we consider the concentration index:

\[ \theta_v^a f = \frac{\int_0^a f(x)^2x^{2v+1}d_qx}{\|f\|_{q,v,2}^2}, \]

whose values range from 0 to 1.
Proposition 7 The maximum value of $\theta_v^w f$ is attained for $f = \psi_0$ and

$$\theta_v^w f = \frac{\sum_{i=0}^{n} \lambda_i^2 (f, \psi_i)^2}{\sum_{i=0}^{n} (f, \psi_i)^2} \geq \lambda_n^2, \quad \text{if} \quad f \in \text{span}\{\psi_0, \ldots, \psi_n\},$$

$$\theta_v^w f = \frac{\sum_{i=n+1}^{\infty} \lambda_i^2 (f, \psi_i)^2}{\sum_{i=n+1}^{\infty} (f, \psi_i)^2} \leq \lambda_{n+1}^2, \quad \text{if} \quad f \in \text{span}\{\psi_0, \ldots, \psi_n\}^\perp.$$  

Proof. With the Parseval equality

$$\int_0^\alpha f(x)^2 x^{2v+1} d_q x = \sum_{i=0}^{\infty} (f, \phi_i)^2,$$

and the fact that

$$\sum_{i=0}^{\infty} (f, \phi_i)^2 = \sum_{i=0}^{\infty} (F_{q,v} f, \psi_i)^2$$

$$= \sum_{i=0}^{\infty} \lambda_i^2 (F_{q,v} f, \phi_i)^2 = \sum_{i=0}^{\infty} \lambda_i^2 (f, \psi_i)^2,$$

$$\|f\|_{q,v,2}^2 = \sum_{i=0}^{\infty} (f, \psi_i)^2,$$

We get

$$\theta_v^w f = \frac{\sum_{i=0}^{\infty} \lambda_i^2 (f, \psi_i)^2}{\sum_{i=0}^{\infty} (f, \psi_i)^2} \leq \lambda_0^2 = \theta_v^w \psi_0,$$

which leads to the result. \(\blacksquare\)

Remark 2 If $b > a$ then

$$PW_{q,a}^v \subset PW_{q,b}^v.$$

Now let $\{\mu_n\}_{n\in\mathbb{Z}}$ the sequence of eigenvalues of the operator $T_b^v$ then we have

$$\lambda_0^2 = \theta_v^w \psi_0 \leq \theta_b^v \psi_0 \leq \mu_0^2.$$

Proposition 8 The $q$-Paley-Wiener space $PW_{q,a}^v$ is a closed subspace of $L_{q,2,v}$.  

**Proof.** First we show that $PW_{q,a}^v$ is a subspace of $L_{q,2,v}$. In fact let
\[ f \in PW_{q,a}^v \]
then there exist $u \in L_{q,a}^v$ such that
\[ f(x) = c_{q,v} \int_0^a u(t) j_v(xt, q^2) t^{2v+1} d_q t = F_{q,v}(u)(x). \]
As $L_{q,a}^v \subset L_{q,2,v}$ and from the Theorem [3] we show that $F_{q,v}(u) \in L_{q,2,v}$ which implies
\[ PW_{q,a}^v \subset L_{q,2,v}. \]
Now, given $f \in L_{q,2,v}$ and let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of element of $PW_{q,a}^v$ which converge to $f$ in $L^2$-norm. For $n \in \mathbb{N}$, there exist $u_n \in L_{q,a}^v$ such that
\[ f_n(x) = c_{q,v} \int_0^a u_n(t) j_v(xt, q^2) t^{2v+1} d_q t. \]
Moreover
\[ \lim_{n \to \infty} \|f_n - f\|_{q,2,v} = 0, \]
this give
\[ \lim_{n \to \infty} \|F_{q,v} f_n - F_{q,v} f\|_{q,2,v} = 0, \]
and then
\[ \int_0^a |F_{q,v} f_n(x) - F_{q,v} f(x)|^2 x^{2v+1} d_q x + \int_a^\infty |F_{q,v} f(x)|^2 x^{2v+1} d_q x \to 0, \]
which implies $F_{q,v} f(x) = 0$ if $x \in \mathbb{R}_q$ and $x > a$ and then $f \in PW_{q,a}^v$. ■

**Theorem 2** For any function $f \in PW_{q,a}^v$ we have
\[ f(z) = (1 - q) \sum_{k \in \mathbb{Z}} q^{2k(v+1)} f(q^k) k_z(q^k), \quad \forall z \in \mathbb{C}. \quad (1) \]

**Proof.** In fact $f$ is an analytic function, and from Proposition [3]
\[ f(x) = \langle f, k_x \rangle, \quad \forall x \in \mathbb{R}_q. \]
We have
\[ \langle f, k_x \rangle = \langle F_{q,v} f, F_{q,v} k_x \rangle = c_{q,v} \langle F_{q,v} f, j_v(x \cdot q^2) \chi_{[0,a]} \rangle \]
\[ = c_{q,v} \int_0^a F_{q,v} f(t) j_v(xt, q^2) t^{2v+1} d_q t. \]
which prove that
\[ z \mapsto \langle f, k_z \rangle, \]
is an analytic function. On the other hand
\[ \langle f, k_z \rangle = (1 - q) \sum_{k \in \mathbb{Z}} q^{2k(v+1)} f(q^k) k_z(q^k), \]
and
\[ \langle f, q^n \rangle = f(q^n), \quad \forall k \in \mathbb{Z}. \]
As \( \{0\} \) is an accumulation point of the following set
\[ \{q^k, \quad k \in \mathbb{Z}\}, \]
we conclude that \( \langle f, k_z \rangle = f(z), \quad \forall z \in \mathbb{C}. \)

**Remark 3** In many fields, telecommunication in particular, the Whittaker-Shannon-Kotel’nikov sampling theorem plays a central role. It is known that sampling is the process of converting a signal (e.g., a function of continuous time or space) into a numeric sequence (a function of discrete time or space). Namely this theorem says that every function in the cosine Paley-Wiener space:
\[ PW_{a}^{\frac{1}{2}} = \left\{ f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{a} u(t) \cos(\pi t) dt, \quad u \in L^2[0,a] \right\}, \]
can be written as
\[ f(x) = \sqrt{\frac{2}{\pi}} \sum_{n \in \mathbb{Z}} f\left(\frac{\pi}{a} n\right) \sin\left(\pi x - \pi n\right). \]
Then the above theorem can be viewed as a sampling formula where the sampling points are \( q^n \) independent of \( a \). By the use of Proposition 1 we get
\[ k_z(q^n) = \frac{(1 - q)c_{q,v}^{2} q^{2n+2} q^{2n+1}(aq^n, q^2) j_{v}(aq^{-1}z, q^2) - z^2 j_{v+1}(az, q^2) j_{v}(aq^{-1+n}, q^2)}{q^{2n} - z^2}. \]

**Proposition 9** Given a function \( f \in L_{q,2,v} \) and let
\[ f_a(x) = \langle f, k_x \rangle, \]
then
\[ f_a \in PW_{q,a}^{v}, \]
and for all \( \delta > 0 \) we have
\[ \lim_{a \to \infty} \sup_{x \in \delta, x \in \mathbb{R}_q} |f(x) - f_a(x)| = 0. \]
Proof. First
\[ |f_a(x)| \leq \|f\|_{q,v,2}\|k_x\|_{q,v,2} < \infty. \]

Now we can write
\[
\begin{aligned}
f_a(x) &= \langle f, k_x \rangle = \langle \mathcal{F}_{q,v}f, \mathcal{F}_{q,v}k_x \rangle = c_{q,v} \langle \mathcal{F}_{q,v}f, j_v(x.,q^2)\chi_{(0,a]} \rangle \\
&= c_{q,v} \int_0^a \mathcal{F}_{q,v}f(t)j_v(x,t,q^2)t^{2v+1}dq_t.
\end{aligned}
\]
which prove that \( f_a \in PW_{q,a}^v \). On the other hand
\[
\begin{aligned}
f(x) &= c_{q,v} \langle \mathcal{F}_{q,v}f, j_v(x.,q^2) \rangle,
\end{aligned}
\]
and therefore
\[
\begin{aligned}
|f(x) - f_a(x)|^2 &= c_{q,v}^2 \left| \int_a^\infty \mathcal{F}_{q,v}f(t)j_v(x,t,q^2)t^{2v+1}dq_t \right|^2 \\
&\leq c_{q,v}^2 \left( \int_a^\infty |\mathcal{F}_{q,v}f(t)||j_v(x,t,q^2)|t^{2v+1}dq_t \right)^2 \\
&\leq c_{q,v}^2 \int_a^\infty |\mathcal{F}_{q,v}f(t)|^2t^{2v+1}dq_t \int_a^\infty |j_v(x,t,q^2)|^2t^{2v+1}dq_t \\
&\leq \frac{c_{q,v}^2}{x^{2v+2}} \int_a^\infty |\mathcal{F}_{q,v}f(t)|^2t^{2v+1}dq_t \int_a^\infty |j_v(t,q^2)|^2t^{2v+1}dq_t \\
&\leq \frac{c_{q,v}^2\|j_v(.q^2)\|_{q,v,2}^2}{x^{2v+2}} \int_a^\infty |\mathcal{F}_{q,v}f(t)|^2t^{2v+1}dq_t.
\end{aligned}
\]
Using the fact that
\[
\int_0^\infty |\mathcal{F}_{q,v}f(t)|^2t^{2v+1}dq_t = \|\mathcal{F}_{q,v}f\|_{q,v,2}^2 = \|f\|_{q,v,2}^2 < \infty,
\]
we finish the proof. ■

4 Application

In this section we tuck \( v = -1/2 \) and \( q = 0.5 \) and we put
\[
f(x) = \frac{1}{1 + x^2},
\]
an even function belong to the space \( L_{q,2,v} \). Using the sampling formula (1) for the function \( f_a(x) = \langle f, k_x \rangle \) respectively for \( a = 1 \), \( a = 1/q \) and \( a = 1/q^2 \) with sampling point
\[
q^n, \quad n = -1 \ldots 10
\]
we obtain

![Graphs showing data points and lines]

References


