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ABSTRACT. In this paper we study existence and uniqueness of renormalized solution to the following problem

$$\begin{cases} \lambda(x,u) - \operatorname{div}(\mathbf{a}(x,Du) + \Phi(x,u)) = f & \text{in } \Omega, \\ \mathbf{a}(x,Du) + \Phi(x,u) \cdot \mathbf{n} = 0 & \text{on } \Gamma_n, \\ u = 0 & \text{on } \Gamma_d. \end{cases}$$

The main difficulty in this task is that in general the operator entering in the above equation is not coercive in a Sobolev space. Moreover, the possible degenerate character of λ with respect to u renders more complex the proof of uniqueness for integrable data f.

1. Introduction

In the present paper we study the class of nonlinear equations of the type

$$\lambda(x,u) - \operatorname{div}(\mathbf{a}(x,Du) + \Phi(x,u)) = f \quad \text{in } \Omega, \tag{1.1}$$

$$\big(\mathbf{a}(x,Du) + \Phi(x,u)\big) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_n, \tag{1.2}$$

$$u = 0 \quad \text{on } \Gamma_d, \tag{1.3}$$

where Ω is a bounded connected and open subset of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$, Γ_n and Γ_d are such that $\Gamma_n \cup \Gamma_d = \partial\Omega$, $\Gamma_n \cap \Gamma_d = \emptyset$ and $\sigma(\Gamma_d) > 0$ (where σ denotes the $N-1$ dimensional Lebesgue–measure on $\partial\Omega$). The vector $\mathbf{n}$ is the outer unit normal to $\partial\Omega$ and the data f is assumed to belong to $L^1(\Omega)$. The operator $u \mapsto -\operatorname{div}(\mathbf{a}(x,Du))$ is monotone (but not necessarily strictly monotone) from the Sobolev space $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ with $1 < p \leq N$ $(p' = p/(p-1))$. The functions $\lambda : \Omega \times \mathbb{R} \longmapsto \mathbb{R}$ and $\Phi : \Omega \times \mathbb{R} \longmapsto \mathbb{R}^N$ are Carathéodory functions such that $\lambda(x,r)r \geq 0$ for any $r \in \mathbb{R}$, almost everywhere in Ω and such that $|\Phi(x,r)| \leq b(x)(1+|r|)^{p-1}$ for any $r \in \mathbb{R}$, almost everywhere in Ω with b satisfying some appropriate summability hypotheses that depend on p and N (see condition (2.7) below).

Problem (1.1)–(1.3) is motivated by the homogenization in the particular case where $\mathbf{a}(x,\xi) = A(x)\xi$ and where Ω is a perforated domain with Neumann condition on the boundary of the holes and Dirichlet condition on the outside boundary of Ω (see [2] and [3]).

Key words and phrases. existence, uniqueness, non-coercive problems, integrable data.

The main difficulty in dealing with the existence of a solution of (1.1)–(1.3) is the lack of coercivity due to the term $-\operatorname{div}(\Phi(x,u))$. As an example, consider the pure Dirichlet case (i.e. $\Gamma_n = \emptyset$), the operator $\mathbf{a}(x, Du) + \Phi(x, u) = |Du|^{p-2} Du + b(x) |u|^{p-2} u$ with $b \in L^{N/(p-1)}(\Omega)$. Then thanks to Sobolev's embedding theorem, the operator $u \longmapsto -\operatorname{div}(\mathbf{a}(x, Du) + \Phi(x, u))$ is well defined from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$ but it is not coercive in general except if $\|b\|_{L^{N/(p-1)}(\Omega)}$ is small enough.

Existence results for some similar non-coercive problems (with in addition lower order terms) are proved in [16] when $f \in W^{-1,p'}(\Omega)$ and in [15] and [21] when f is a Radon measure with bounded total variation in Ω (solutions in the sense of distributions are then used in this case). A non-coercive linear case is studied in [18]. In [19] the author gives local and global estimates for nonlinear non-coercive equations with measure data (with a stronger assumption of type (2.6) below than the one used in the present paper, see (2.7), in the case $p = N$). Entropy solutions to similar equations are considered in [9].

For integrable data f we give in the present paper an existence result (see Theorem 3.1 in Section 3) using the framework of renormalized solution. This notion has been introduced by R. J. DiPerna and P.-L. Lions in [17] for first order equations and has been developed for elliptic problems with L^1 data in [24] (see also [23]). In [14] the authors give a definition of a renormalized solution for elliptic problems with general measure data and prove the existence of such a solution (a class of nonlinear elliptic equations with lower-order terms which are not coercive and right-hand side measure is also studied in [6]).

Another interesting question related to problem (1.1)–(1.3) deals with the uniqueness of a solution. In [24] F. Murat proves that the renormalized solution of

$$\begin{aligned} \lambda u - \operatorname{div}(A(x)Du + \phi(u)) &= f \quad \text{in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where $f \in L^1(\Omega)$ and $\lambda > 0$ is unique as soon as ϕ is a locally Lipschitz continuous vector field. In this result it is important to assume that ϕ does not depend on x together with pure homogeneous Dirichlet boundary conditions (see also [23] and [26] for more general operators and [8] in the parabolic case). When $\lambda(x, s)$ is strictly monotone we prove in Theorem 4.1 that the renormalized solution of (1.1)–(1.3) is unique if $\Phi(x, s)$ is locally Lipschitz continuous with respect to the second variable.

As far as the case $\lambda(x, u) \equiv 0$ is concerned, gathering the result of [1], [11] and [13], let us recall that when $1 < p \le 2$ and $f \in W^{-1,p'}(\Omega)$, the uniqueness of the variational solution of

$$\begin{cases} -\operatorname{div}(\mathbf{a}(x, u, Du) + \phi(u)) = f \quad \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

is obtained under strongly monotonicity assumption on the operator $\mathbf{a}(x,s,\xi)$ and under global Lipschitz conditions on the functions $\mathbf{a}(x,s,\xi)$ and $\phi(s)$ with respect to the variable s (or a strong control of the modulus of continuity). Moreover uniqueness may fail if $2<p<\infty$ (see [11]). In the quasi-linear case (i.e. $a(x,s,\xi)=A(x,s)\xi$) and for integrable data uniqueness results have been obtained in [25] under a very general condition on the matrix field A and the function ϕ (the author uses strongly the quasi-linear character of the problem). When $\lambda(x,s)\equiv 0$ we investigate in the present paper the uniqueness question in the nonlinear case for $1<p\le 2$ and integrable data; global conditions on $\mathbf{a}$ and Φ which insure uniqueness of the renormalized solution are given in Theorem 4.2.

The content of the paper is as follows. In section 2 we precise the assumptions on the data and we give the definition of a renormalized solution of problem (1.1)–(1.3). This section is completed by giving a few properties on the renormalized solutions of (1.1)–(1.3). Section 3 is devoted to the existence result. At last, in Section 4 we prove two uniqueness results. The results of the present paper were announced in [4] and here we improve the uniqueness results.

2. Assumptions and definitions

2.1. **Notations and hypotheses.** In the whole paper, for $q\in[1,\infty[$ we denote by $W^{1,q}_{\Gamma_d}(\Omega)$ the space of functions belonging to $W^{1,q}(\Omega)$ which have a null trace on Γ_d. Since Ω is a bounded and connected open subset of $\mathbb{R}^N$ with Lipschitz boundary and since $\sigma(\Gamma_d)>0$, the space $W^{1,q}_{\Gamma_d}(\Omega)$ is provided by the norm $\|v\|_{W^{1,q}_{\Gamma_d}(\Omega)}=\|Dv\|_{L^q(\Omega)}$ (see e.g. [27]).

We assume that $\mathbf{a}:\Omega\times\mathbb{R}\longmapsto\mathbb{R}^N$, $\lambda:\Omega\times\mathbb{R}\longmapsto\mathbb{R}$ and $\Phi:\Omega\times\mathbb{R}\longmapsto\mathbb{R}^N$ are Carathéodory functions such that for $1<p\le N$ we have :

$$\exists\alpha>0,\quad \mathbf{a}(x,\xi)\cdot\xi\ge\alpha\,|\xi|^p\,,\ \forall\xi\in\mathbb{R}^N,\quad\text{a.e. } x \text{ in } \Omega; \tag{2.1}$$

$$\big(\mathbf{a}(x,\xi)-\mathbf{a}(x,\xi')\big)\cdot(\xi-\xi')\ge 0\ \ \forall\xi,\xi'\in\mathbb{R}^N,\quad\text{a.e. } x \text{ in } \Omega; \tag{2.2}$$

$$|\mathbf{a}(x,\xi)|\le\beta(|d(x)|+|\xi|^{p-1})\ \forall\xi\in\mathbb{R}^N,\quad\text{a.e. } x \text{ in } \Omega,\ \text{with } d\in L^{p'}(\Omega); \tag{2.3}$$

$$\lambda(x,s)s\ge 0\ \ \forall s\in\mathbb{R},\quad\text{a.e. } x \text{ in } \Omega; \tag{2.4}$$

$$\forall k>0,\ \exists c_k>0 \text{ such that } |\lambda(x,s)|\le c_k\ \ \forall\,|s|\le k,\quad\text{a.e. } x \text{ in } \Omega, \tag{2.5}$$

$$|\Phi(x,s)|\le|b(x)|\,(1+|s|)^{p-1}\ \ \forall s\in\mathbb{R},\quad\text{a.e. } x \text{ in } \Omega, \tag{2.6}$$

with

$$\begin{cases} b\in L^{\frac{N}{p-1}}(\Omega) \text{ if } p<N,\\ \displaystyle\int_\Omega(1+|b|)^{\frac{N}{N-1}}(\ln(1+|b|))^{N-1}dx<\infty\ \text{ if } p=N; \end{cases} \tag{2.7}$$

$$f\in L^1(\Omega). \tag{2.8}$$

For any $k \geq 0$, the truncation function at height $\pm k$ is defined by $T_k(s) = \max(-k, \min(s, k))$. For any integer $n \geq 1$, let us define the bounded positive function

$$h_n(s) = 1 - \frac{|T_{2n}(s) - T_n(s)|}{n}. \tag{2.9}$$

For any measurable subset E of Ω, $\mathbb{1}_E$ denotes the characteristic function of the subset E.

2.2. **Definition of a renormalized solution of (1.1)–(1.3).** Following [5] let us recall the definition of the gradient of functions whose truncates belong to $W^{1,p}_{\Gamma_d}(\Omega)$.

Definition 2.1. Let u be a measurable function defined on Ω which is finite almost everywhere such that $T_k(u) \in W^{1,p}_{\Gamma_d}(\Omega)$ for every $k > 0$. Then there exists a unique measurable function $v : \Omega \longrightarrow \mathbb{R}^N$ such that

$$DT_k(u) = \mathbb{1}_{\{|u|<k\}} v \quad \text{a.e. in } \Omega, \ \forall k > 0.$$

This function v is called the gradient of u and is denoted by Du.

We now give the definition of a renormalized solution of problem (1.1)–(1.3).

Definition 2.2. A measurable function u defined on Ω and finite almost everywhere on Ω is called a renormalized solution of (1.1)–(1.3) if

$$T_k(u) \in W^{1,p}_{\Gamma_d}(\Omega), \quad \forall k > 0; \tag{2.10}$$

$$\lim_{n \to \infty} \frac{1}{n} \int_{\{|u|<n\}} \mathbf{a}(x, Du) \cdot Du dx = 0; \tag{2.11}$$

and if for every $h \in W^{1,\infty}(\mathbb{R})$, with compact support and any $\varphi \in W^{1,p}_{\Gamma_d}(\Omega) \cap L^\infty(\Omega)$,

$$\begin{aligned} &\int_\Omega \lambda(x, u)\varphi h(u) dx + \int_\Omega h(u)\mathbf{a}(x, Du) \cdot D\varphi dx + \int_\Omega h(u)\Phi(x, u) \cdot D\varphi dx \\ &\quad + \int_\Omega \varphi h'(u)\mathbf{a}(x, Du) \cdot Du dx + \int_\Omega \varphi h'(u)\Phi(x, u) \cdot Du dx = \int_\Omega f\varphi h(u) dx. \end{aligned} \tag{2.12}$$

Remark 2.3. Condition (2.10) and Definition 2.1 allow to define Du almost everywhere in Ω. Condition (2.11) which is crucial to obtain uniqueness results is standard in the context of renormalized solution and gives additional information on Du for large value of $|u|$. Equality (2.12) is formally obtained by using in (1.1) the test function $\varphi h(u)$ and taking into account the boundary conditions (1.2) and (1.3).

Every term in (2.12) is well defined. Indeed let $k > 0$ such that $\operatorname{supp}(h) \subset [-k, k]$. From assumption (2.5) we have $|\lambda(x, u)\varphi h(u)| \leq c_k \|\varphi h\|_{L^\infty(\Omega)}$ a.e. in Ω and then $\lambda(x, u)\varphi h(u)$ lies in $L^1(\Omega)$. Since $h(u)(\mathbf{a}(x, Du) + \Phi(x, u)) = h(u)(\mathbf{a}(x, DT_k(u)) +$

$\Phi(x, T_k(u)))$ a.e. in Ω, from (2.3), (2.6) and (2.10) it follows that $h(u)(\mathbf{a}(x, Du) + \Phi(x, u))$ belongs to $\big(L^{p'}(\Omega)\big)^N$. Thus $h(u)(\mathbf{a}(x, Du) + \Phi(x, u)) \cdot D\varphi$ is integrable on Ω. The same arguments imply that $\varphi h'(u)\mathbf{a}(x, Du) \cdot Du = \varphi h'(u)\mathbf{a}(x, DT_k(u)) \cdot DT_k(u)$ and $\varphi h'(u)\Phi(x, u) \cdot Du = \varphi h'(u)\Phi(x, T_k(u)) \cdot DT_k(u)$ lie in $L^1(\Omega)$. At last it is clear that $f\varphi h(u) \in L^1(\Omega)$.

2.3. **Properties of a renormalized solution of (1.1)–(1.3).** It is well known (see [14]) that if $f \in L^1(\Omega)$, any renormalized solution of the equation $-\operatorname{div}(\mathbf{a}(x, Du)) = f$ in Ω, $u = 0$ on $\partial\Omega$ is also a solution in the sense of distribution. We establish a similar result in Proposition 2.8 namely that any function $\psi \in W^{1,q}_{\Gamma_d}(\Omega)$, with $q > N$, is an admissible test function in (1.1)–(1.3). We first give two technical lemmas which will be used in the limit case $p = N$.

Lemma 2.4. $\forall \omega > 0$, $\exists \eta > 0$ *such*

$$\forall v \in W^{1,N}_{\Gamma_d}(\Omega), \quad \int_\Omega \exp\left(\Big(\frac{v}{\eta \,\|Dv\|_{(L^N(\Omega))^N}}\Big)^{\frac{N}{N-1}} - 1\right) dx \le \omega. \tag{2.13}$$

Lemma 2.5. $\exists c(N) > 0$ *such that*

$$\forall \theta > 0,\ \forall x, y \in \mathbb{R}, \quad |xy| \le \frac{c(N)}{\theta}\Big((1+|x|)\big(\ln(1+|x|)\big)^{N-1} + e^{|\theta y|^{\frac{1}{N-1}}} - 1\Big). \tag{2.14}$$

Remark 2.6. Property (2.13) is a consequence of the limit-case of Sobolev's embedding theorem (see [20]). Inequality (2.14) can be easily derived by induction using the standard inequality $|xy| \le (1 + |x|)\ln(1 + |x|) + e^{|y|} - 1$, $\forall x, y \in \mathbb{R}$. We leave the details of the proofs to the reader.

In the following lemma we give some regularity results of a renormalized solution of (1.1)–(1.3).

Lemma 2.7. *Assume that (2.1)–(2.8) hold true. If u is a renormalized solution of (1.1)–(1.3) then*

$$\lambda(x, u) \in L^1(\Omega); \tag{2.15}$$

$$\lim_{n\to\infty} \frac{1}{n} \int_{\{|u|<n\}} \big|\Phi(x, u)\big| \cdot \big|Du\big| dx = 0; \tag{2.16}$$

$$|Du|^{p-1} \in L^q(\Omega), \quad \forall\, 1 < q < \frac{N}{N-1}; \tag{2.17}$$

$$|u|^{p-1} \in L^q(\Omega), \quad \forall\, 1 < q < \frac{N}{N-p}; \tag{2.18}$$

$$\frac{\big|Du\big|^p}{\big(1 + |u|\big)^{1+m}} \in L^1(\Omega), \quad \forall m > 0. \tag{2.19}$$

Sketch of the proof of Lemma 2.7. Regularities (2.17), (2.18) and (2.19) are easy consequences of the estimate techniques of L. Boccardo and T. Gallouët developed in [10] (see also [5] and [6]). Indeed (2.1), (2.10) and (2.11) yield that $\int_\Omega |DT_k(u)|^p\,dx \le ck + L\ \forall k > 0$.

Let us prove (2.16). Assumption (2.6) and Hölder's inequality lead to

$$\frac{1}{n}\int_{\{|u|<n\}} \big|\Phi(x,u)\big|\cdot\big|Du\big|dx \le \Big(\frac{1}{n}\int_{\{|u|<n\}} |b|^{p'}\big(1+|u|\big)^p dx\Big)^{\frac{1}{p'}}\Big(\frac{1}{n}\int_{\{|u|<n\}} |Du|^p\,dx\Big)^{\frac{1}{p}}. \tag{2.20}$$

If $p < N$, using Sobolev's embedding theorem we have

$$\begin{aligned}\Big(\int_{\{|u|<n\}} |b|^{p'}\big(1+|u|\big)^p dx\Big)^{\frac{1}{p'}} &\le \|b\|_{L^{\frac{N}{p-1}}(\Omega)}\Big(\int_\Omega \big(1+|T_n(u)|\big)^{\frac{pN}{N-p}}dx\Big)^{\frac{(p-1)(N-p)}{pN}}\\ &\le c\,\|b\|_{L^{\frac{N}{p-1}}(\Omega)}\Big(1+\|DT_n(u)\|^{p-1}_{(L^p(\Omega))^N}\Big)\end{aligned}$$

and with (2.20) we obtain

$$\frac{1}{n}\int_{\{|u|<n\}} |\Phi(x,u)\cdot Du|\,dx \le \frac{c}{n}\|b\|_{L^{\frac{N}{p-1}}(\Omega)}\,(1+\|DT_n(u)\|^p_{(L^p(\Omega))^N}), \tag{2.21}$$

where c is a constant independent on n.

If $p = N$, using Lemma 2.4 and Lemma 2.5 we obtain after a few computations, for $\eta > 0$,

$$\begin{aligned}&\int_{\{|u|<n\}} |b|^{N'}(1+|u|)^N dx\\ &\le c\,\|b\|^{N'}_{L^{\frac{N}{N-1}}(\Omega)} + c\int_\Omega |b|^{N'}|T_n(u)|^N\,dx\\ &\le c\,\|b\|^{N'}_{L^{\frac{N}{N-1}}(\Omega)} + cc(N)\,\|DT_n(u)\|^N_{(L^N(\Omega))^N}\int_\Omega \big(1+\eta^N|b|^{N'}\big)\big(\ln(1+\eta^N|b|^{N'})\big)^{N-1}dx\\ &\quad + cc(N)\,\|DT_n(u)\|^N_{(L^N(\Omega))^N}\int_\Omega\Big(\exp\Big(\frac{|T_n(u)|}{\eta\,\|DT_n(u)\|_{(L^N(\Omega))^N}}\Big)^{N'}-1\Big)dx,\end{aligned}$$

where c and $c(N)$ are positive constants independent of n. Choosing $\omega = 1$ in Lemma 2.4 (then η is fixed), we get

$$\begin{aligned}\int_{\{|u|<n\}} |b|^{N'}(1+|u|)^N dx &\le c\,\|b\|^{N'}_{L^{\frac{N}{N-1}}(\Omega)} + cc(N)\,\|DT_n(u)\|^N_{(L^N(\Omega))^N}\\ &\quad + cc(N)\,\|DT_n(u)\|^N_{(L^N(\Omega))^N}\int_\Omega(1+|b|)^{N'}(\ln(1+|b|))^{N-1}dx\\ &\le c\big(1+\|DT_n(u)\|^N_{(L^N(\Omega))^N}\big),\end{aligned} \tag{2.22}$$

where c is a positive constant depending on N, η, b and Ω.

From (2.20) together with (2.21) and (2.22) it follows that in both cases $p < N$ and $N = p$

$$\frac{1}{n}\int_{\{|u|<n\}} |\Phi(x,u)\cdot Du|\,dx \le \frac{c}{n}(1+\|DT_n(u)\|^p_{(L^p(\Omega))^N}) \quad \forall n \ge 1, \tag{2.23}$$

where c is a positive constant which depends on N, p, b and Ω. Assumption (2.1), condition (2.11) and (2.23) lead to (2.16).

We are now in a position to obtain (2.15). For any $n > 0$, the function h_n (see (2.9)) is Lipschitz continuous with compact support, so that (2.12) yields with $\varphi = T_1(u) \in W^{1,p}_{\Gamma_d}(\Omega) \cap L^\infty(\Omega)$

$$\begin{aligned}\int_\Omega \lambda(x,u)T_1(u)h_n(u)dx + \int_\Omega h_n(u)\mathbf{a}(x,Du)\cdot DT_1(u)dx \\ + \int_\Omega h_n(u)\Phi(x,u)\cdot DT_1(u)dx + \int_\Omega T_1(u)h_n'(u)\mathbf{a}(x,Du)\cdot Dudx \\ + \int_\Omega T_1(u)h_n'(u)\Phi(x,u)\cdot Dudx = \int_\Omega fT_1(u)h_n(u)dx.\end{aligned}$$

Because $0 \le h_n(u) \le 1$ and $\mathbf{a}(x,Du)\cdot DT_1(u) \ge 0$ almost everywhere in Ω, we deduce that

$$\begin{aligned}\int_\Omega \lambda(x,u)T_1(u)h_n(u)dx \le \|f\|_{L^1(\Omega)} + \int_\Omega |\Phi(x,u)\cdot DT_1(u)|\,dx \\ + \frac{1}{n}\int_{\{|u|<n\}} (\mathbf{a}(x,Du)\cdot Du + |\Phi(x,u)\cdot Du|)dx.\end{aligned}$$

Since u is finite almost everywhere in Ω, $h_n(u)$ converges to 1 almost everywhere in Ω and it is bounded by 1. Assumption (2.4) and Fatou lemma together with (2.11) and (2.16) allow to conclude that

$$\int_\Omega \lambda(x,u)T_1(u)dx \le \|f\|_{L^1(\Omega)} + \int_\Omega |\Phi(x,u)\cdot DT_1(u)|\,dx. \tag{2.24}$$

At last writing $|\lambda(x,u)| \le \lambda(x,u)T_1(u) + |\lambda(x,u)|\,\mathbb{1}_{\{|u|\le 1\}}$ a.e. in Ω, and using (2.5), (2.21) and (2.24) lead to (2.15). □

Proposition 2.8. *Assume that (2.1)–(2.8) hold true. If u is a renormalized solution of (1.1)–(1.3) then for any $\psi \in \bigcup_{r>N} W^{1,r}_{\Gamma_d}(\Omega)$ we have*

$$\int_\Omega \lambda(x,u)\psi dx + \int_\Omega \mathbf{a}(x,Du)\cdot D\psi dx + \int_\Omega \Phi(x,u)\cdot D\psi dx = \int_\Omega f\psi dx. \tag{2.25}$$

Sketch of proof. Let $\psi \in W^{1,r}_{\Gamma_d}(\Omega)$ with $r > N$. Sobolev's embedding theorem implies that $\varphi \in W^{1,r}_{\Gamma_d}(\Omega) \cap L^\infty(\Omega)$ and (2.12) with $h = h_n$ leads to

$$\int_\Omega \lambda(x,u)\psi h_n(u)dx + \int_\Omega h_n(u)\mathbf{a}(x,Du)\cdot D\psi dx + \int_\Omega h_n(u)\Phi(x,u)\cdot D\psi dx + \int_\Omega \psi h_n'(u)\mathbf{a}(x,Du)\cdot Dudx + \int_\Omega \psi h_n'(u)\Phi(x,u)\cdot Dudx = \int_\Omega f\psi h_n(u)dx. \tag{2.26}$$

Assumptions (2.3) and (2.17) lead to

$$\mathbf{a}(x,Du) \in (L^q(\Omega))^N \quad \forall 1 \le q < \frac{N}{N-1},$$

and then $\mathbf{a}(x,Du)\cdot D\psi \in L^1(\Omega)$. Similarly (2.6) and (2.18) give that $\Phi(x,u)\cdot D\psi \in L^1(\Omega)$. Since $h_n(u)$ converges to 1 almost everywhere in Ω and it is bounded by 1 and recalling that $|h_n'(s)| = 1/n\mathbb{1}_{\{n<|s|<2n\}}(s)$ a.e. on $\mathbb{R}$, it is then a straightforward task to pass the limit in (2.26) using Lebesgue's dominated convergence theorem, (2.11), (2.15) and (2.16). Such a limit process leads to (2.25). □

3. Existence of a renormalized solution

Theorem 3.1. *Under assumptions (2.1)–(2.8) there exists a renormalized solution of equation (1.1)–(1.3).*

Proof. The proof relies on passing to the limit in an approximate problem.

Step 1. For $\varepsilon > 0$, let us define

$$\lambda_\varepsilon(x,s) = \varepsilon\,|s|^{p-2}\,s + \lambda\big(x, T_{1/\varepsilon}(s)\big), \tag{3.1}$$

$$\Phi_\varepsilon(x,s) = \Phi\big(x, T_{1/\varepsilon}(s)\big) \tag{3.2}$$

and $f^\varepsilon \in L^{p'}(\Omega)$ such that

$$f^\varepsilon \xrightarrow{\varepsilon\to 0} f \qquad \text{strongly in } L^1(\Omega). \tag{3.3}$$

From the classical results of Leray–Lions [22], an application of the Leray-Schauder fixed point theorem allows to show that for any $\varepsilon > 0$ there exists $u^\varepsilon \in W^{1,p}_{\Gamma_d}(\Omega)$ such that $\forall\psi \in W^{1,p}_{\Gamma_d}(\Omega)$

$$\int_\Omega \lambda_\varepsilon(x,u^\varepsilon)\psi dx + \int_\Omega \mathbf{a}(x,Du^\varepsilon)\cdot D\psi dx + \int_\Omega \Phi_\varepsilon(x,u^\varepsilon)\cdot D\psi dx = \int_\Omega f^\varepsilon\psi dx. \tag{3.4}$$

We now derive some estimates on u^ε. Using $\psi = T_k(u^\varepsilon)$ for $k > 0$ in (3.4) we obtain

$$\int_\Omega \lambda_\varepsilon(x,u^\varepsilon)T_k(u^\varepsilon)dx + \int_\Omega \mathbf{a}(x, DT_k(u^\varepsilon)) \cdot DT_k(u^\varepsilon)dx \\ + \int_\Omega \Phi_\varepsilon(x, T_k(u^\varepsilon)) \cdot DT_k(u^\varepsilon)dx = \int_\Omega f^\varepsilon T_k(u^\varepsilon)dx.$$

From the coercivity of the operator $\mathbf{a}$, the positivity of the function λ and (2.6) it follows that

$$\int_\Omega \lambda(x,u^\varepsilon)T_k(u^\varepsilon)dx + \alpha \int_\Omega |DT_k(u^\varepsilon)|^p \, dx \\ \leq \int_\Omega f^\varepsilon T_k(u^\varepsilon)dx + \int_\Omega b(x)(1+|T_k(u^\varepsilon)|)^{p-1} |DT_k(u^\varepsilon)| \, dx$$

and then Young's inequality gives

$$\int_\Omega \lambda(x,u^\varepsilon)T_k(u^\varepsilon)dx + \int_\Omega |DT_k(u^\varepsilon)|^p \, dx \leq (M+1)(k+k^p) \tag{3.5}$$

where M is a generic constant independent of k and ε. Inequality (3.5) implies that $\forall k > 0$

$$T_k(u^\varepsilon) \text{ is bounded in } W^{1,p}_{\Gamma_d}(\Omega) \tag{3.6}$$

and due to (2.4) and (2.5)

$$\lambda(x,u^\varepsilon) \quad \text{is bounded in } L^1(\Omega). \tag{3.7}$$

As a consequence of (2.3), (3.6) and (3.7) there exists a subsequence (still denoted by ε) and a measurable function $u : \Omega \longrightarrow \overline{\mathbb{R}}$ such that $\lambda(x,u) \in L^1(\Omega)$ and such that

$$u^\varepsilon \xrightarrow{\varepsilon\to 0} u \;\text{ almost everywhere in } \Omega, \tag{3.8}$$

$$\forall k > 0 \qquad T_k(u^\varepsilon) \overset{\varepsilon\to 0}{\rightharpoonup} T_k(u) \text{ weakly in } W^{1,p}_{\Gamma_d}(\Omega), \tag{3.9}$$

$$\forall k > 0, \quad \mathbf{a}(x, DT_k(u^\varepsilon)) \overset{\varepsilon\to 0}{\rightharpoonup} \sigma_k \text{ weakly in } (L^{p'}(\Omega))^N, \tag{3.10}$$

where $\sigma_k \in \left(L^{p'}(\Omega)\right)^N$.

We claim that u is finite almost everywhere in Ω (remark that if $\lambda(x,s) = \lambda\,|s|^{p-2}\,s$, with $\lambda > 0$, it is obvious since $\lambda(x,u) \in L^1(\Omega)$) through a "log–type" estimate on u^ε (such a "log–type" estimate is also performed in [9], see also [12, 18, 19]). Let us consider the real valued function

$$\psi_p(r) = \int_0^r \frac{ds}{(1+|s|)^p} \quad \forall r \in \mathbb{R}.$$

Since $p > 1$, $\psi_p(u) \in W^{1,p}_{\Gamma_d}(\Omega) \cap L^\infty(\Omega)$ with $\|\psi_p(u)\|_{L^\infty(\Omega)} \le \frac{1}{p-1}$ and $D\psi_p(u) = \frac{Du}{(1+|u|)^p}$ almost everywhere in Ω, $\psi_p(u^\varepsilon)$ is an admissible test function in (3.4). It follows that

$$\int_\Omega \lambda_\varepsilon(x,u^\varepsilon)\psi_p(u^\varepsilon)dx + \int_\Omega \mathbf{a}(x,u^\varepsilon)\cdot D\psi_p(u^\varepsilon)dx + \int_\Omega \Phi_\varepsilon(x,u^\varepsilon)\cdot D\psi_p(u^\varepsilon)dx \le \frac{M}{p-1}$$

and due to the definition of λ_ε together with (2.1), (2.4) and (2.6) we have

$$\alpha\int_\Omega \frac{|Du^\varepsilon|^p}{(1+|u^\varepsilon|)^p}dx \le \frac{M}{p-1} + \int_\Omega b(x)\frac{|Du^\varepsilon|}{(1+|u^\varepsilon|)}dx$$

where M is a generic constant independent of ε. Young's inequality leads to

$$\int_\Omega \frac{|Du^\varepsilon|^p}{(1+|u^\varepsilon|)^p}dx \le M\Big(1+\int_\Omega |b(x)|^{\frac{p}{p-1}}\,dx\Big).$$

Since $p \le N$ and $u^\varepsilon \in W^{1,p}_{\Gamma_d}(\Omega)$, the regularity of the function b (see (2.7)) implies that the field $\ln(1+|u^\varepsilon|)$ is bounded in $W^{1,p}_{\Gamma_d}(\Omega)$ uniformly with respect to ε. From (3.8) and (3.9) it follows that $\ln(1+|u|)$ belongs to $W^{1,p}_{\Gamma_d}(\Omega)$ and then u is finite almost everywhere in Ω.

Step 2. We prove the following lemma.

Lemma 3.2.

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\int_{\{|u^\varepsilon|<n\}}|Du^\varepsilon|^p\,dx = 0. \tag{3.11}$$

Proof. Taking the admissible test function $T_n(u^\varepsilon)/n$ (which belongs to $W^{1,p}_{\Gamma_d}(\Omega) \cap L^\infty(\Omega)$) in (3.4) yields that

$$\frac{1}{n}\int_\Omega \lambda_\varepsilon(x,u^\varepsilon)T_n(u^\varepsilon)dx + \frac{1}{n}\int_\Omega \mathbf{a}(x,Du^\varepsilon)\cdot DT_n(u^\varepsilon)dx + \frac{1}{n}\int_\Omega \Phi_\varepsilon(x,u^\varepsilon)\cdot DT_n(u^\varepsilon)dx = \frac{1}{n}\int_\Omega f^\varepsilon T_n(u^\varepsilon)dx.$$

Since $\lambda_\varepsilon(x,u^\varepsilon)T_n(u^\varepsilon) \ge 0$ almost everywhere in Ω, using (2.1) and (2.6) we get

$$\frac{\alpha}{n}\int_\Omega |DT_n(u^\varepsilon)|^p\,dx \le \frac{1}{n}\Big(\int_\Omega |b(x)|\,(1+|T_n(u^\varepsilon)|)^{p-1}\,|DT_n(u^\varepsilon)|\,dx + \int_\Omega f^\varepsilon T_n(u^\varepsilon)dx\Big). \tag{3.12}$$

As a consequence of (3.3) and (3.8) and using the fact that u is finite almost everywhere in Ω, Lebesgue's convergence theorem leads to

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{n}\int_\Omega |f^\varepsilon T_n(u^\varepsilon)|\,dx = \lim_{n\to\infty}\frac{1}{n}\int_\Omega |fT_n(u)|\,dx = 0.$$

Due to (3.2), (3.8) and (3.9) we have

$$\limsup_{\varepsilon\to 0}\frac{1}{n}\int_\Omega |b(x)|\,(1+|T_n(u^\varepsilon)|)^{p-1}\,|DT_n(u^\varepsilon)|\,dx = \frac{1}{n}\int_\Omega |b(x)|\,(1+|T_n(u)|)^{p-1}\,|DT_n(u)|\,dx. \tag{3.13}$$

We will prove in the sequel by splitting techniques that

$$\forall\eta>0,\qquad \frac{1}{n}\int_\Omega |b(x)|\,(1+|T_n(u)|)^{p-1}\,|DT_n(u)|\,dx \le \omega_\eta(n)+\frac{\eta}{n}\int_\Omega |DT_n(u)|^p\,dx, \tag{3.14}$$

where $\omega_\eta(n)$ goes to zero as n goes to infinity. Since $T_n(u^\varepsilon)$ converges to $T_n(u)$ weakly in $W^{1,p}_{\Gamma_d}(\Omega)$, choosing η small enough in (3.14) together with (3.12) give (3.11).

In order to complete the proof, it remains to prove (3.14). Let $\eta>0$ and $R>0$ (R will be fixed later). By denoting E_R the measurable set $E_R=\{x\in\Omega\,;\,|u(x)|>R\}$, we write

$$\frac{1}{n}\int_\Omega |b(x)|\,(1+|T_n(u)|)^{p-1}\,|DT_n(u)|\,dx = I_1(n,R)+I_2(n,R) \tag{3.15}$$

with

$$I_1(n,R)=\frac{1}{n}\int_{\Omega\setminus E_R}|b(x)|\,(1+|T_n(u)|)^{p-1}\,|DT_R(u)|\,dx$$

and

$$I_2(n,R)=\frac{1}{n}\int_{E_R}|b(x)|\,(1+|T_n(u)|)^{p-1}\,|DT_n(u)|\,dx.$$

Since $T_R(u)\in W^{1,p}_{\Gamma_d}(\Omega)$ we have $\lim_{n\to\infty} I_1(n,R)=0$, $\forall R>0$.

We now deals with $I_2(n,R)$ by distinguishing the cases $p<N$ and $p=N$.

First case. Assuming that $p<N$, Hölder's inequality and Sobolev's embedding theorem lead to, $\forall n\ge 1$,

$$\begin{aligned} I_2(n,R) &\le \frac{1}{n}\|b\|_{L^{\frac{N}{p-1}}(E_R)}\,\big\|1+|T_n(u)|\big\|^{p-1}_{L^{\frac{Np}{N-p}}(\Omega)}\,\|DT_n(u)\|_{(L^p(\Omega))^N} \\ &\le \frac{M}{n}\|b\|_{L^{\frac{N}{p-1}}(E_R)}\big(1+\|DT_n(u)\|^p_{(L^p(\Omega))^N}\big), \end{aligned} \tag{3.16}$$

where M is a generic constant not depending on n and R. Since u is finite almost everywhere in Ω and since $b\in L^{\frac{N}{p-1}}(\Omega)$, let $R>0$ such that $M\|b\|_{L^{\frac{N}{p-1}}(E_R)}<\eta$. Due to (3.15) and (3.16) we obtain (3.14).

Second case. We assume that $p=N$. Let us define $A_n=\|DT_n(u)\|_{(L^N(\Omega))^N}$ and let $\rho>0$ (ρ will be fixed in the sequel). A few computations and Lemma 2.5 (with

$\theta = A_n^{-N}$) give, $\forall n \geq 1$,

$$\begin{aligned}
I_2(n,R) &\leq \frac{2^{N-2}}{n}\Big[\int_{E_R} |b(x)|\,|DT_n(u)|\,dx + \int_{E_R} |b(x)|\,|T_n(u)|^{p-1}\,|DT_n(u)|\,dx\Big]\\
&\leq \frac{2^{N-2}}{n} A_n \Bigg[\|b\|_{L^{\frac{N}{N-1}}(\Omega)} + \Big(\int_{E_R} \big(\rho^{N-1}\,|b(x)|\,\big)^{\frac{N}{N-1}} \Big(\frac{|T_n(u)|}{\rho}\Big)^N dx\Big)^{(N-1)/N}\Bigg]\\
&\leq \frac{2^{N-2}}{n} A_n \Bigg[\|b\|_{L^{N'}(\Omega)} + \Bigg(A_n^N C(N)\Big[\int_{E_R}\Big\{\exp\Big[\Big(\frac{|T_n(u)|}{\rho A_n}\Big)^{\frac{N}{N-1}} - 1\Big]\Big\}dx\\
&\qquad + \int_{E_R}\Big(1+\rho^N\,|b(x)|^{\frac{N}{N-1}}\Big)\ln\Big(1+\rho^N\,|b(x)|^{\frac{N}{N-1}}\Big)^{N-1}dx\Big]\Bigg)^{(N-1)/N}\Bigg]\\
&\leq \frac{2^{N-2}}{n} A_n\|b\|_{L^{N'}(\Omega)} + \frac{2^{N-2}C(N)}{n} A_n^N\Big[\int_{\Omega}\Big\{\exp\Big[\Big(\frac{|T_n(u)|}{\rho A_n}\Big)^{\frac{N}{N-1}} - 1\Big]\Big\}dx\Big]^{(N-1)/N}
\end{aligned}$$

$$+\frac{2^{N-2}C(N)}{n} M(\rho,N) A_n^N \Big[\int_{E_R}\big(1+|b(x)|\,\big)^{\frac{N}{N-1}}\ln\big(1+|b(x)|\,\big)^{N-1}dx\Big]^{(N-1)/N}, \tag{3.17}$$

where $C(N) > 0$ is a constant only depending on N (from Lemma 2.5) and $M(\rho,N)$ only depends on ρ and N. Using Lemma 2.4 and since $T_n(u)$ lies in $W^{1,N}_{\Gamma_d}(\Omega)$ we can choose firstly $\rho > 0$ such that the quantity

$$2^{N-2}C(N)\Big[\int_{\Omega}\Big\{\exp\Big[\Big(\frac{|T_n(u)|}{\rho A_n}\Big)^{\frac{N}{N-1}} - 1\Big]\Big\}dx\Big]^{\frac{N-1}{N}}$$

is small enough independently of n. Secondly since u is finite almost everywhere in Ω (i.e. $\lim_{R\to\infty}\operatorname{meas}(E_R) = 0$) we can choose $R > 0$ such that the quantity

$$2^{N-2}C(N)M(\rho,N)\Big[\int_{E_R}\big(1+|b(x)|\,\big)^{\frac{N}{N-1}}\ln\big(1+|b(x)|\,\big)^{N-1}dx\Big]^{\frac{N-1}{N}}$$

is small enough (notice that it is crucial to choose ρ before choosing R). At last we deduce from (3.17) that there exists $R > 0$ such that $\forall n \geq 1$

$$\begin{aligned}
I_2(n,R) &\leq \frac{2^{N-2}}{n}\,\|b\|_{L^{N'}(\Omega)}\,\|DT_n(u)\|_{(L^N(\Omega))^N} + \frac{\eta}{2n}\,\|DT_n(u)\|^N_{(L^N(\Omega))^N}\\
&\leq \frac{2^{N+1}\,\|b\|^{N'}_{L^{N'}(\Omega)}}{n\eta^{1/(N-1)}} + \frac{\eta}{n}\,\|DT_n(u)\|^N_{(L^N(\Omega))^N}.
\end{aligned}$$

From (3.15) and the behavior of $I_1(n,R)$ as n goes to infinity it follows that (3.14) holds true. □

Step 3. We are now in a position to prove the following lemma.

Lemma 3.3. *For any $k > 0$,*

$$\lim_{\varepsilon\to 0}\int_\Omega \big(\mathbf{a}(x, DT_k(u^\varepsilon)) - \mathbf{a}(x, DT_k(u))\big)\cdot\big(DT_k(u^\varepsilon) - DT_k(u)\big)dx = 0.$$

Proof of Lemma 3.3. The proof relies on similar techniques developed in [7].

Let k be a positive real number and let $n \geq 1$. Using the test function $(T_k(u^\varepsilon) - T_k(u))h_n(u^\varepsilon)$ which belongs to $W^{1,p}_{\Gamma_d}(\Omega) \cap L^\infty(\Omega)$ yields that

$$\begin{aligned}(3.18)\quad &\int_\Omega \lambda_\varepsilon(x, u^\varepsilon)(T_k(u^\varepsilon) - T_k(u))h_n(u^\varepsilon)dx \\ &+\int_\Omega h_n(u^\varepsilon)\mathbf{a}(x, Du^\varepsilon)\cdot D(T_k(u^\varepsilon)-T_k(u))dx+\int_\Omega (T_k(u^\varepsilon)-T_k(u))h'_n(u^\varepsilon)\mathbf{a}(x, Du^\varepsilon)\cdot Du^\varepsilon dx \\ &+\int_\Omega h_n(u^\varepsilon)\Phi_\varepsilon(x, u^\varepsilon)\cdot D(T_k(u^\varepsilon)-T_k(u))dx+\int_\Omega (T_k(u^\varepsilon)-T_k(u))h'_n(u^\varepsilon)\Phi_\varepsilon(x, u^\varepsilon)\cdot Du^\varepsilon dx \\ &= \int_\Omega f^\varepsilon(T_k(u^\varepsilon) - T_k(u))h_n(u^\varepsilon)dx\end{aligned}$$

We study in the sequel the behavior of each term of (3.18) as $\varepsilon \to 0$ and $n \to \infty$.

Since h_n has a compact support, condition (2.5) implies that the field $h_n(u^\varepsilon)\lambda_\varepsilon(x, u^\varepsilon)$ is bounded in $L^\infty(\Omega)$. Moreover $T_k(u^\varepsilon) - T_k(u)$ converges to 0 almost everywhere in Ω and in $L^\infty(\Omega)$ weak–$*$ as ε goes to zero. Therefore we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega \lambda_\varepsilon(x, u^\varepsilon)(T_k(u^\varepsilon) - T_k(u))h_n(u^\varepsilon)dx = 0, \tag{3.19}$$

and similarly one has

$$\lim_{\varepsilon\to 0}\int_\Omega f^\varepsilon(T_k(u^\varepsilon) - T_k(u))h_n(u^\varepsilon)dx = 0.$$

Recalling that $h'_n(r) = -1\!\!1_{\{n<|r|<2n\}}\mathrm{sign}(r)/n$ a.e. on $\mathbb{R}$, from assumption (2.3) it follows that

$$\left|\int_\Omega (T_k(u^\varepsilon) - T_k(u))h'_n(u^\varepsilon)\mathbf{a}(x, Du^\varepsilon)\cdot Du^\varepsilon dx\right| \leq \frac{M}{n}\Big(\int_{\{n<|u^\varepsilon|<2n\}} |Du^\varepsilon|^p\,dx + \int_\Omega |d(x)|^{p'}\,dx\Big),$$

with $M > 0$ not depending on ε and n. Lemma 3.2 and the regularity of d allow us to conclude that

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\left|\int_\Omega (T_k(u^\varepsilon) - T_k(u))h'_n(u^\varepsilon)\mathbf{a}(x, DT_n(u^\varepsilon))\cdot Du^\varepsilon dx\right| = 0. \tag{3.20}$$

In view of (2.6) and since h_n has a compact support we get $|h_n(u^\varepsilon)\Phi_\varepsilon(x, u^\varepsilon)| \leq Mb(x)$ almost everywhere in Ω where M is a constant independent of ε. Moreover the pointwise convergence of u^ε and the definition of Φ_ε give that

$$h_n(u^\varepsilon)\Phi_\varepsilon(x,u^\varepsilon) \xrightarrow{\varepsilon\to 0} h_n(u)\Phi(x,u) \text{ a.e. in } \Omega.$$

Thus the regularity of b and Lebesgue's convergence theorem imply that $h_n(u^\varepsilon)\Phi_\varepsilon(x,u^\varepsilon)$ converges to $h_n(u)\Phi(x,u)$ strongly in $L^{\frac{p}{p-1}}(\Omega)$. Due to (3.9) we conclude that

$$\lim_{\varepsilon\to 0}\int_\Omega h_n(u^\varepsilon)\Phi_\varepsilon(x,u^\varepsilon)\cdot D(T_k(u^\varepsilon)-T_k(u))dx = 0 \tag{3.21}$$

and similar arguments lead to

$$\lim_{\varepsilon\to 0}\int_\Omega (T_k(u^\varepsilon)-T_k(u))h_n'(u^\varepsilon)\Phi_\varepsilon(x,u^\varepsilon)\cdot Du^\varepsilon dx = 0. \tag{3.22}$$

From (3.18) together with (3.19)–(3.22) it follows that

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\int_\Omega h_n(u^\varepsilon)\mathbf{a}(x,Du^\varepsilon)\cdot D(T_k(u^\varepsilon)-T_k(u))dx = 0. \tag{3.23}$$

Since $\mathbf{a}(x,0)=0$ almost everywhere in Ω we have for $k'>k$

$$\mathbf{a}(x,DT_k(u^\varepsilon)) = \mathbb{1}_{\{|u^\varepsilon|<k\}}\mathbf{a}(x,DT_{k'}(u^\varepsilon)) \quad \text{almost everywhere in } \Omega$$

and due to (3.8) and (3.10) we get

$$\sigma_k = \mathbb{1}_{\{|u|<k\}}\sigma_{k'} \text{ a.e. on } \Omega\setminus\{|u|=k\}.$$

Since $DT_k(u)=0$ a.e. on $\{|u|=k\}$ we obtain that

$$\sigma_k\cdot DT_k(u) = \sigma_{k'}\cdot DT_k(u) \text{ a.e. on } \Omega, \tag{3.24}$$

and then if $n\geq k$

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega h_n(u^\varepsilon)\mathbf{a}(x,Du^\varepsilon)\cdot DT_k(u)dx &= \lim_{\varepsilon\to 0}\int_\Omega h_n(u^\varepsilon)\mathbf{a}(x,DT_{2n}(u^\varepsilon))\cdot DT_k(u)dx\\ &= \int_\Omega h_n(u)\sigma_{2n}\cdot DT_k(u)dx = \int_\Omega \sigma_k\cdot DT_k(u)dx.\end{aligned}$$

From (3.23) and (3.24) we get

$$\limsup_{\varepsilon\to 0}\int_\Omega h_n(u^\varepsilon)\mathbf{a}(x,Du^\varepsilon)\cdot DT_k(u^\varepsilon)dx \leq \int_\Omega \sigma_k\cdot DT_k(u)dx. \tag{3.25}$$

At last, writing

$$\begin{aligned}&\int_\Omega (\mathbf{a}(x,DT_k(u^\varepsilon))-\mathbf{a}(x,DT_k(u)))\cdot(DT_k(u^\varepsilon)-DT_k(u))dx\\ &\quad= \int_\Omega \mathbf{a}(x,DT_k(u^\varepsilon))\cdot DT_k(u^\varepsilon)dx - \int_\Omega \mathbf{a}(x,DT_k(u^\varepsilon))\cdot DT_k(u)dx\\ &\qquad - \int_\Omega \mathbf{a}(x,DT_k(u))\cdot(DT_k(u^\varepsilon)-DT_k(u))dx,\end{aligned}$$

using (3.9), (3.10), (3.25) and the monotone character of the operator **a** allow to conclude the proof of Lemma 3.3. $\square$

From Lemma 3.3 we deduce that $\forall k > 0$

$$\lim_{\varepsilon \to 0} \int_\Omega \mathbf{a}(x, DT_k(u^\varepsilon)) \cdot DT_k(u^\varepsilon) dx = \int_\Omega \sigma_k \cdot DT_k(u) dx,$$

which gives thanks to a Minty argument

$$\forall k > 0, \qquad \sigma_k = \mathbf{a}(x, DT_k(u)) \text{ a.e. in } \Omega. \tag{3.26}$$

Using again Lemma 3.3 together with (3.9) and (3.26) we conclude that

$$\forall k > 0, \qquad \mathbf{a}(x, DT_k(u^\varepsilon)) \cdot DT_k(u^\varepsilon) \overset{\varepsilon \to 0}{\rightharpoonup} \mathbf{a}(x, DT_k(u)) \cdot DT_k(u) \text{ in } L^1(\Omega)\text{–weak.}$$

Step 4. We now pass to the limit in the approximate problem.

Let h be an element of $W^{1,\infty}(\mathbb{R})$ with compact support, let $k > 0$ such that $\operatorname{supp}(h) \subset [-k, k]$ and let φ be an element of $W^{1,p}_{\Gamma_d}(\Omega) \cap L^\infty(\Omega)$. Plugging the test function $\varphi h(u^\varepsilon)$ in (3.4) yields

$$\begin{aligned} &\int_\Omega \lambda_\varepsilon(x, u^\varepsilon)\varphi h(u^\varepsilon) dx + \int_\Omega h(u^\varepsilon)\mathbf{a}(x, Du^\varepsilon) \cdot D\varphi dx \\ &\quad + \int_\Omega \varphi h'(u^\varepsilon)\mathbf{a}(x, Du^\varepsilon) \cdot Du^\varepsilon dx + \int_\Omega h(u^\varepsilon)\Phi_\varepsilon(x, u^\varepsilon) \cdot D\varphi dx \\ &\qquad + \int_\Omega \varphi h'(u^\varepsilon)\Phi_\varepsilon(x, u^\varepsilon) \cdot Du^\varepsilon dx = \int_\Omega f^\varepsilon \varphi h(u^\varepsilon) dx. \end{aligned} \tag{3.27}$$

Let us pass to the limit in (3.27) as ε goes to zero. Since h has a compact support, assumption (2.5) and the pointwise convergence of u^ε give that the field $\lambda_\varepsilon(x, u^\varepsilon)\varphi h(u^\varepsilon)$ converges to $\lambda(x, u)\varphi h(u)$ a.e. in Ω and in $L^\infty(\Omega)$ weak–$*$. From (3.3) and (3.8) it follows that $f^\varepsilon \varphi h(u^\varepsilon)$ converges strongly to $f \varphi h(u)$ in $L^1(\Omega)$. Using assumption (2.6) together with (3.8) (and since $\operatorname{supp}(h)$ is compact) and Lebesgue's convergence theorem we obtain that $h(u^\varepsilon)\Phi_\varepsilon(x, u^\varepsilon)$ converges strongly to $h(u)\Phi(x, u)$ in $L^{N/(p-1)}(\Omega)$. Recalling that $N \geq p$ leads to

$$\lim_{\varepsilon \to 0} \int_\Omega h(u^\varepsilon)\Phi_\varepsilon(x, u^\varepsilon) \cdot D\varphi dx = \int_\Omega h(u)\Phi(x, u) \cdot D\varphi dx. \tag{3.28}$$

Similarly the weak convergence of $T_k(u^\varepsilon)$ yields that

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi h'(u^\varepsilon)\Phi_\varepsilon(x, u^\varepsilon) \cdot Du^\varepsilon dx = \int_\Omega \varphi h'(u)\Phi(x, u) \cdot DT_k(u) dx, \tag{3.29}$$

and from (3.10) and (3.26) it follows that

$$\lim_{\varepsilon \to 0} \int_\Omega h(u^\varepsilon)\mathbf{a}(x, DT_k(u^\varepsilon)) \cdot D\varphi dx = \int_\Omega h(u)\mathbf{a}(x, DT_k(u)) \cdot D\varphi dx. \tag{3.30}$$

Finally due to Lemma 3.3 we have

(3.31)

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi h'(u^\varepsilon)\mathbf{a}(x,Du^\varepsilon)\cdot Du^\varepsilon dx = \lim_{\varepsilon\to 0}\int_\Omega \varphi h'(u^\varepsilon)\mathbf{a}(x,DT_k(u^\varepsilon))\cdot DT_k(u^\varepsilon)dx$$
$$= \int_\Omega \varphi h'(u)\mathbf{a}(x,DT_k(u))\cdot DT_k(u)dx.$$

Due to (3.27)–(3.31) we obtain that the field u verifies condition (2.12) of Definition 2.2. Condition (2.16) is a consequence of (2.3), (3.9) and Lemma (3.2).

The proof of Theorem 3.1 is now complete. ☐

4. Uniqueness results

As mentioned in the introduction, we give in this section two uniqueness results. In Theorem 4.1 below we establish that if $\lambda(x,s)$ is strictly monotone with respect to s then the renormalized solution of (1.1)–(1.3) is unique under a local Lipschitz condition on $\Phi(x,s)$ with respect to s.

Theorem 4.1. *Assume that (2.1)–(2.8) hold true. Moreover assume that*

$$(\lambda(x,r)-\lambda(x,r'))(r-r')>0 \qquad \forall r,r'\in\mathbb{R},\ r\neq r' \quad \text{a.e. } x\in\Omega; \tag{4.1}$$

for any compact $C\subset\mathbb{R}$ there exists $L_C>0$ such that

$$|\Phi(x,r)-\Phi(x,r')|\le L_C\,|b(x)|\,|r-r'| \qquad \forall r,r'\in C. \tag{4.2}$$

Then the renormalized solution of equation (1.1)–(1.3) is unique.

When $\lambda(x,\cdot)$ is assumed to be monotone and when $1<p\le 2$ we must replace condition (4.2) by a global condition and the strong monotonicity of the operator $\mathbf{a}(x,\cdot)$ is needed.

Theorem 4.2. *Assume that (2.1)–(2.8) hold true. Moreover assume that*

$$1<p\le 2<N; \tag{4.3}$$

$$(\lambda(x,r)-\lambda(x,r'))(r-r')\ge 0 \qquad \forall r,r'\in\mathbb{R}, \quad \text{a.e. } x\in\Omega; \tag{4.4}$$

(4.5)

$$(\mathbf{a}(x,\xi)-\mathbf{a}(x,\xi'))\cdot(\xi-\xi')\ge\alpha\frac{|\xi-\xi'|^2}{(|\xi|+|\xi'|)^{2-p}} \qquad \forall \xi,\xi'\in\mathbb{R}^N, \quad \text{a.e. } x\in\Omega;$$

(4.6) *there exist $L>0$ and $\gamma<p-\dfrac{3}{2}$ such that*

$$|\Phi(x,r)-\Phi(x,r')|\le L\,|r-r'|\,|b(x)|\,(|r|+|r'|+1)^\gamma \quad \forall r,r'\in\mathbb{R}, \quad \text{a.e. } x\in\Omega.$$

Then the renormalized solution of equation (1.1)–(1.3) is unique.

Remark 4.3. An example of function Φ verifying growth condition (2.6)–(2.7) and (4.6) is $b(x)(1+|r|)^{p-1}$ with b satisfying regularity assumption (2.7). Roughly speaking, condition (4.6) implies that $\left|\frac{\partial\Phi(x,r)}{\partial r}\right| \leq |b(x)|(1+|r|)^{\gamma}$. When $p > 3/2$ a global Lipschitz condition on $\Phi(x,r)$ with respect to r is allowed (or a strong control of the modulus of continuity). If $p \leq 3/2$ it follows that $\gamma < 0$ and then $\left|\frac{\partial\Phi(x,r)}{\partial r}\right|$ goes to zero as $|r|$ tends to ∞ almost everywhere in Ω.

Proof of Theorem 4.1. Let u and v be two renormalized solutions of Problem (1.1)–(1.3). Our goal is to prove that $\int_\Omega |\lambda(x,u) - \lambda(x,v)|\, dx = 0$.

Let $q > 0$, $\sigma > 0$ and $n \geq 1$. Using $T_\sigma(T_q(u) - T_q(v))h_n(u)$ which belongs to $W^{1,p}_{\Gamma_d}(\Omega) \cap L^\infty(\Omega)$ in $(2.12)_u$ gives

(4.7)
$$\int_\Omega \lambda(x,u)T_\sigma(T_q(u) - T_q(v))h_n(u)dx + \int_\Omega h_n(u)\mathbf{a}(x,Du)\cdot DT_\sigma(T_q(u) - T_q(v))dx$$
$$+\int_\Omega h_n'(u)T_\sigma(T_q(u)-T_q(v))\mathbf{a}(x,Du)\cdot Dudx+\int_\Omega h_n(u)\Phi(x,u)\cdot DT_\sigma(T_q(u)-T_q(v))dx$$
$$+\int_\Omega h_n'(u)T_\sigma(T_q(u) - T_q(v))\Phi(x,u)\cdot Dudx = \int_\Omega fT_\sigma(T_q(u) - T_q(v))h_n(u)dx.$$

It is then easy to pass to the limit as q tends to $+\infty$ for fixed $\sigma > 0$ and $n \geq 1$. Indeed, since $\mathrm{supp}(h_n) \subset [-2n, 2n]$ one has

$$h_n(u)DT_\sigma(T_q(u) - T_q(v)) = h_n(u)DT_k(u - v) \quad \text{a.e. } x \text{ in } \Omega$$

as soon as $q > 2N + \sigma$. Moreover $T_\sigma(T_q(u) - T_q(v))$ converges to $T_\sigma(u - v)$ a.e. in Ω and in $L^\infty(\Omega)$ weak–$*$ as q goes to $+\infty$. Using such a process in (1.1)–(1.3) written for v gives by subtraction

$$\begin{aligned}
&\frac{1}{\sigma}\int_\Omega (\lambda(x,u)h_n(u) - \lambda(x,v)h_n(v))T_\sigma(u-v)dx \\
&+\frac{1}{\sigma}\int_\Omega (h_n(u)\mathbf{a}(x,Du) - h_n(v)\mathbf{a}(x,Dv))\cdot DT_\sigma(u-v)dx \\
&+\frac{1}{\sigma}\int_\Omega (h_n(u)\Phi(x,u) - h_n(v)\Phi(x,v))\cdot DT_\sigma(u-v)dx \\
&+\frac{1}{\sigma}\int_\Omega T_\sigma(u-v)\Big(h_n'(u)\mathbf{a}(x,Du)\cdot Du - h_n'(v)\mathbf{a}(x,Dv)\cdot Dv\Big)dx \\
&+\frac{1}{\sigma}\int_\Omega T_\sigma(u-v)\Big(h_n'(u)\Phi(x,u)\cdot Du - h_n'(v)\Phi(x,v)\cdot Dv\Big)dx \\
&= \frac{1}{\sigma}\int_\Omega fT_\sigma(u-v)(h_n(u) - h_n(v))dx.
\end{aligned} \tag{4.8}$$

We now pass to the limit successively as $\sigma \to 0$ and as $n \to +\infty$ in (4.8). Since

$$\frac{T_\sigma(u-v)}{\sigma} \xrightarrow{\sigma\to 0} \operatorname{sign}(u-v)\mathbb{1}_{\{u\neq v\}} \text{ a.e. in } \Omega \text{ and in } L^\infty(\Omega) \text{ weak}-*, \tag{4.9}$$

and because both functions $h_n(u)$ and $h_n(v)$ converge to 1 almost everywhere in Ω and in $L^\infty(\Omega)$ weak–$*$, we obtain thanks to Lebesgue's convergence theorem

$$\lim_{n\to\infty}\lim_{\sigma\to 0}\frac{1}{\sigma}\int_\Omega (\lambda(x,u)h_n(u)-\lambda(x,v)h_n(v))T_\sigma(u-v)dx = \int_\Omega(\lambda(x,u)-\lambda(x,v))\operatorname{sign}(u-v)dx \tag{4.10}$$

and

$$\lim_{n\to\infty}\lim_{\sigma\to 0}\frac{1}{\sigma}\int_\Omega fT_\sigma(u-v)(h_n(u)-h_n(v))dx = 0. \tag{4.11}$$

One has for any $\sigma > 0$ and any $n \geq 1$,

$$\left|\frac{1}{\sigma}\int_\Omega h_n'(u)T_\sigma(u-v)\mathbf{a}(x,Du)\cdot Dudx\right| \leq \frac{1}{n}\int_{\{|u|<2n\}}\mathbf{a}(x,Du)\cdot Dudx$$

and

$$\left|\frac{1}{\sigma}\int_\Omega h_n'(u)T_\sigma(u-v)\Phi(x,u)\cdot Dudx\right| \leq \frac{1}{n}\int_{\{|u|<2n\}}|\Phi(x,u)\cdot Du|\,dx.$$

Therefore (2.11) and (2.16) imply that

$$\lim_{n\to\infty}\limsup_{\sigma\to 0}\left|\frac{1}{\sigma}\int_\Omega h_n'(u)T_\sigma(u-v)\mathbf{a}(x,Du)\cdot Dudx\right| = 0, \tag{4.12}$$

$$\lim_{n\to\infty}\limsup_{\sigma\to 0}\left|\frac{1}{\sigma}\int_\Omega h_n'(u)T_\sigma(u-v)\Phi(x,u)\cdot Dudx\right| = 0. \tag{4.13}$$

By the same way we obtain (4.12) and (4.13) for v and then the forth and fifth terms of (4.8) tend to zero.

To study the behavior of the second term of (4.8), we split it as follows

$$\begin{aligned}\frac{1}{\sigma}\int_\Omega \big(h_n(u)\mathbf{a}(x,Du)-h_n(v)\mathbf{a}(x,Dv)\big)\cdot DT_\sigma(u-v)dx \\ = \frac{1}{\sigma}\int_\Omega h_n(u)\big(\mathbf{a}(x,Du)-\mathbf{a}(x,Dv)\big)\cdot DT_\sigma(u-v)dx \\ + \frac{1}{\sigma}\int_\Omega\big(h_n(u)-h_n(v)\big)\mathbf{a}(x,Dv)\cdot DT_\sigma(u-v)dx.\end{aligned} \tag{4.14}$$

Since h_n is a Lipschitz continuous function we have for $0 < \sigma \leq 1$

$$\frac{1}{\sigma}\Big|\int_\Omega (h_n(u) - h_n(v))\mathbf{a}(x, Dv) \cdot DT_\sigma(u - v)dx\Big| \\ \leq \frac{1}{n}\int_{\{0<|u-v|<\sigma\}} |\mathbf{a}(x, DT_{2n+1}(u))|\,|DT_{2n+1}(v) - DT_{2n+1}(v)|\,dx$$

which gives using Lebesgue's convergence theorem

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\Big|\int_\Omega (h_n(u) - h_n(v))\mathbf{a}(x, Dv) \cdot DT_\sigma(u - v)dx\Big| = 0.$$

Since $h_n(u)$ is non negative the monotone character of the operator $\mathbf{a}$ and (4.14) lead to $\forall n \geq 1$,

$$\limsup_{\sigma\to 0}\frac{1}{\sigma}\int_\Omega (h_n(u)\mathbf{a}(x, Du) - h_n(v)\mathbf{a}(x, Dv)) \cdot DT_\sigma(u - v)dx \geq 0. \tag{4.15}$$

We now deal with the third term of (4.8). We have

$$\frac{1}{\sigma}\int_\Omega (h_n(u)\Phi(x, u) - h_n(v)\Phi(x, v)) \cdot DT_\sigma(u - v)dx \\ = \frac{1}{\sigma}\int_{\{0<|u-v|<\sigma\}} h_n(u)(\Phi(x, u) - \Phi(x, v)) \cdot DT_\sigma(u - v)dx \\ + \frac{1}{\sigma}\int_{\{0<|u-v|<\sigma\}} (h_n(u) - h_n(v))\Phi(x, v) \cdot DT_\sigma(u - v)dx. \tag{4.16}$$

Since $\operatorname{supp}(h_n) = [-2n, 2n]$ we get for $0 < \sigma \leq 1$

$$\frac{1}{\sigma}\left|\int_{\{0<|u-v|<\sigma\}} h_n(u)(\Phi(x, u) - \Phi(x, v)) \cdot DT_\sigma(u - v)dx\right| \\ \leq \frac{1}{\sigma}\int_{\{0<|u-v|<\sigma, |u|<2n, |v|<2n+1\}} h_n(u)\,|\Phi(x, u) - \Phi(x, v)|\,|Du - Dv|\,dx,$$

which gives thanks to assumption (4.2)

$$\frac{1}{\sigma}\left|\int_{\{0<|u-v|<\sigma\}} h_n(u)(\Phi(x, u) - \Phi(x, v)) \cdot DT_\sigma(u - v)dx\right| \\ \leq L\int_{\{0<|u-v|<\sigma\}} |b(x)|\,\big(|DT_{2n+1}(u)| + |DT_{2n+1}(v)|\big)dx \tag{4.17}$$

where L does not depend on σ.

Using again the fact that h_n is Lipschitz continuous together with assumption (2.6) we have for $0 < \sigma \leq 1$

$$\begin{aligned}\left|\frac{1}{\sigma}\int_{\{0<|u-v|<\sigma\}}(h_n(u)-h_n(v))\Phi(x,v)\cdot DT_\sigma(u-v)dx\right|& \\ \leq \frac{1}{n}\int_{\{0<|u-v|<\sigma,|u|<2n,|v|<2n+1\}}&\left|\Phi(x,v)\cdot DT_\sigma(u-v)\right|dx \\ \leq M\int_{\{0<|u-v|<\sigma\}}&|b(x)|\,(|DT_{2n+1}(u)|+|DT_{2n+1}(v)|)dx,\end{aligned}$$

with $M > 0$ not depending on σ. From (4.16) and (4.17) it follows that for $0 < \sigma \leq 1$

$$\begin{aligned}&\left|\frac{1}{\sigma}\int_\Omega(h_n(u)\Phi(x,u)-h_n(v)\Phi(x,v))\cdot DT_\sigma(u-v)dx\right| \\ &\qquad\leq (L+M)\int_{\{0<|u-v|<\sigma\}}|b(x)|\,\big(\,|DT_{2n+1}(u)|+|DT_{2n+1}(v)|\,\big)dx.\end{aligned}$$

Since $|b(x)|\,(|DT_{2n+1}(u)|+|DT_{2n+1}(v)|)$ lies in $L^1(\Omega)$, Lebesgue's convergence theorem implies that

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\int_\Omega(h_n(u)\Phi(x,u)-h_n(v)\Phi(x,v))\cdot DT_\sigma(u-v)dx=0. \tag{4.18}$$

Gathering (4.8), (4.10)–(4.15) and (4.18) yields

$$\int_\Omega(\lambda(x,u)-\lambda(x,v))\operatorname{sign}(u-v)dx\leq 0. \tag{4.19}$$

The strict monotonicity of $\lambda(x,\cdot)$ allows to conclude that $\int_\Omega|\lambda(x,u)-\lambda(x,v)|\,dx = 0$ and then $u = v$ almost everywhere in Ω. □

Remark 4.4. In the pure Dirichlet case (i.e. $\Gamma_n = \emptyset$) if $\phi : \mathbb{R} \longmapsto \mathbb{R}^N$ is a continuous function without any growth assumption then there exists a renormalized solution of the problem

$$\lambda|u|^{p-2}u-\operatorname{div}(\mathbf{a}(x,Du)+\phi(u))=f-\operatorname{div}(g)\quad\text{in }\Omega, \tag{4.20}$$

$$u=0\quad\text{on }\partial\Omega, \tag{4.21}$$

with $f \in L^1(\Omega)$ and $g \in L^{p'}(\Omega)$ (see [23] and [24] in the linear case when $g \equiv 0$; notice that $\Gamma_n = \emptyset$ is crucial for this existence result). When $\lambda > 0$ and under a local Lipschitz hypothesis on ϕ, the method used in the proof of Theorem 4.1 and the property (see [8])

$$\operatorname{div}(h_n(u)\phi(u))-h_n'(u)\phi(u)\cdot Du=\operatorname{div}(\Psi_n(u))\quad\text{in }\mathcal{D}'(\Omega),$$

where $\Psi_n(r) = \int_0^r h_n(s)\phi'(s)ds$ allow to obtain that the renormalized solution of (4.20)–(4.21) is unique.

Proof of Theorem 4.2. Let u and v be two renormalized solutions of problem (1.1)–(1.3).

Let σ be a positive real number and $n \geq 1$. Using $h = h_n$ and $\psi = h_n(v)T_\sigma(u-v)$ in (2.12) written in u together with similar arguments already used in the proof of Theorem 4.1 yield (by subtraction with the equivalent equality written in v)

$$\begin{aligned}
&\int_\Omega h_n(u)h_n(v)(\lambda(x,u) - \lambda(x,v))T_\sigma(u-v)dx \\
&+ \int_\Omega h_n(u)h_n(v)(\mathbf{a}(x,Du) - \mathbf{a}(x,Dv)) \cdot DT_\sigma(u-v)dx \\
&+ \int_\Omega h_n(v)h_n(u)(\Phi(x,u) - \Phi(x,v)) \cdot DT_\sigma(u-v)dx \\
&+ \int_\Omega h_n'(u)h_n(v)T_\sigma(u-v)(\mathbf{a}(x,Du) + \Phi(x,u) - \mathbf{a}(x,Dv) - \Phi(x,v)) \cdot Dudx \\
&+ \int_\Omega h_n'(v)h_n(u)T_\sigma(u-v)(\mathbf{a}(x,Du) + \Phi(x,u) - \mathbf{a}(x,Dv) - \Phi(x,v)) \cdot Dvdx = 0.
\end{aligned} \tag{4.22}$$

We now pass to the limit as n goes to $+\infty$ first and then as σ goes to 0. It is worth noting that the reverse is performed in the proof of Theorem 4.1. Indeed passing to the limit as $\sigma \to 0$ first and then $n \to \infty$ leads to uniqueness of the solution only in the case where $\lambda(x,\cdot)$ is strictly monotone (assumption (4.1) in Theorem 4.1). In the case of Theorem 4.2, the zero order term namely $\lambda(x,\cdot)$, is monotone (see assumption (4.4)), and the uniqueness proof program of Theorem 4.1 yields $\int_\Omega |\lambda(x,u) - \lambda(x,v)|\, dx = 0$ which is not sufficient to ensure uniqueness. It follows that the second term of (4.22) leads us to the uniqueness of the field u letting first $n \to \infty$ and then $\sigma \to 0$. This explains why global condition and strong monotonicity are assumed.

Since $h_n \geq 0$ the monotone character of $\lambda(x,\cdot)$ implies that $\forall n \geq 1$

$$\int_\Omega h_n(v)h_n(u)(\lambda(x,u) - \lambda(x,v))T_\sigma(u-v)dx \geq 0. \tag{4.23}$$

We claim that $\forall \sigma > 0$,

$$\lim_{n\to\infty} \int_\Omega h_n'(u)h_n(v)T_\sigma(u-v)(\mathbf{a}(x,Du) + \Phi(x,u) - \mathbf{a}(x,Dv) - \Phi(x,v)) \cdot Dudx = 0. \tag{4.24}$$

Thanks to (2.11) of Definition 2.2 and (2.16) of Lemma 2.7 we have

$$\lim_{n\to\infty} \int_\Omega h_n'(u)h_n(v)T_\sigma(u-v)\mathbf{a}(x,Du) \cdot Dudx = 0 \tag{4.25}$$

and

$$\lim_{n\to\infty} \int_\Omega h_n'(u)h_n(v)T_\sigma(u-v)\Phi(x,u) \cdot Dudx = 0. \tag{4.26}$$

Assumption (2.3) gives with Hölder's inequality

$$\begin{aligned}
&\left|\int_\Omega h_n'(u)h_n(v)T_\sigma(u-v)\mathbf{a}(x,Dv)\cdot Du dx\right| \\
&\leq \frac{\beta\sigma}{n}\int_{\{|u|<2n,|v|<2n\}}(d(x)+|Dv|^{p-1})\,|Du|\,dx \\
&\leq \frac{\beta\sigma}{n}\Big(\int_{\{|v|<2n\}}(|d(x)|+|Dv|^{p-1})^{\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}}\Big(\int_{\{|u|<2n\}}|Du|^p\Big)^{\frac{1}{p}} \\
&\leq \frac{\beta\sigma}{n}\Big[\Big(\int_\Omega |d(x)|^{\frac{p}{p-1}}\,dx\Big)^{\frac{p-1}{p}}+\Big(\int_{\{|v|<2n\}}|Dv|^p\,dx\Big)^{\frac{p-1}{p}}\Big]\Big(\int_{\{|u|<2n\}}|Du|^p\,dx\Big)^{\frac{1}{p}}.
\end{aligned}$$

Using again (2.11) of Definition 2.2 together with assumption (2.1) leads to

$$\lim_{n\to\infty}\frac{1}{n}\int_{\{|u|<n\}}|Du|^p\,dx=0,$$

and since d lies in $L^{p'}(\Omega)$ we deduce that

$$\lim_{n\to\infty}\left|\int_\Omega h_n'(u)h_n(v)T_\sigma(u-v)\mathbf{a}(x,Dv)\cdot Du dx\right|=0. \tag{4.27}$$

From assumption (2.6) it follows that

$$\begin{aligned}
&\left|\int_\Omega h_n'(u)h_n(v)T_\sigma(u-v)\Phi(x,v)\cdot Du dx\right| \\
&\qquad\leq \sigma\Big(\frac{1}{n}\int_{\{|v|<2n\}}|b(x)|^{\frac{p}{p-1}}\big(1+|v|\big)^p dx\Big)^{\frac{p-1}{p}}\Big(\frac{1}{n}\int_{\{|u|<2n\}}|Du|^p\,dx\Big)^{\frac{1}{p}}.
\end{aligned}$$

Therefore using similar arguments to the ones used in the proof of Lemma 2.7 we deduce that

$$\lim_{n\to\infty}\left|\int_\Omega h_n'(u)h_n(v)T_\sigma(u-v)\Phi(x,v)\cdot Du dx\right|=0, \tag{4.28}$$

and then (4.24) is proved.

In view of (4.5) and (4.6), gathering (4.22), (4.23) and (4.24) leads to

$$\begin{aligned}
&\alpha\int_\Omega h_n(u)h_n(v)\frac{|DT_\sigma(u-v)|^2}{\big(|Du|+|Dv|\big)^{2-p}}dx \\
&\qquad\leq \omega_\sigma(n)+\int_\Omega h_n(u)h_n(v)\,|u-v|\,|b(x)|\,\big(|u|+|v|+1\big)^\gamma\,|DT_\sigma(u-v)|\,dx,
\end{aligned} \tag{4.29}$$

where $\omega_\sigma(n) \overset{n\to\infty}{\longrightarrow} 0$. Young's inequality yields

$$\alpha \int_\Omega h_n(u)h_n(v)\frac{|DT_\sigma(u-v)|^2}{\big(|Du|+|Dv|\big)^{2-p}}dx \le 2\omega_\sigma(n) \tag{4.30}$$
$$+\frac{2\sigma^2}{\alpha}\int_{\{|u-v|<\sigma\}\cap\{u\neq v\}} h_n(u)h_n(v)\,|b(x)|^2\,\big(|u|+|v|+1\big)^{2\gamma}\big(|Du|+|Dv|\big)^{2-p}dx.$$

Our goal is now to prove that $|b(x)|^2\,\big(|u|+|v|+1\big)^{2\gamma}\big(|Du|+|Dv|\big)^{2-p} \in L^1(\Omega)$ and then to pass to the limit as $n\to\infty$ in (4.30).

If $p=2$, we have (recalling that $\mathrm{supp}(h_n)=[-2n,2n]$)

$$\int_\Omega h_n(u)h_n(v)\,|b(x)|^2\,\big(|u|+|v|+1\big)^{2\gamma}dx$$
$$\le \Big(\int_\Omega |b|^N\,dx\Big)^{\frac{2}{N}}\Big(\int_{\{|u|<2n,|v|<2n\}}\big(|u|+|v|+1\big)^{\frac{2N\gamma}{N-2}}dx\Big)^{\frac{N-2}{N}}.$$

Since $\gamma < p-\frac{3}{2}=\frac{1}{2}$, we get $\frac{2\gamma N}{N-2}<\frac{N}{N-2}$ which implies, thanks to Lemma 2.7

$$\int_\Omega |u|^{\frac{2N\gamma}{N-2}}\,dx<\infty \quad\text{and}\quad \int_\Omega |v|^{\frac{2N\gamma}{N-2}}\,dx<\infty.$$

It follows that $|b(x)|^2\,\big(|u|+|v|+1\big)^{2\gamma}\in L^1(\Omega)$.

If $1<p<2$, making use of Hölder's inequality we have

$$\int_\Omega h_n(u)h_n(v)\,|b(x)|^2\,\big(|u|+|v|+1\big)^{2\gamma}\big(|Du|+|Dv|\big)^{2-p}dx \tag{4.31}$$
$$\le \Big(\int_\Omega h_n(u)h_n(v)\,|b|^{\frac{N}{p-1}}\,dx\Big)^{\frac{2(p-1)}{N}}$$
$$\times\Big(\int_{\{|u|<2n,|v|<2n\}} h_n(u)h_n(v)\big(|u|+|v|+1\big)^{\frac{2N\gamma}{N-2(p-1)}}\big(|Du|+|Dv|\big)^{\frac{(2-p)N}{N-2(p-1)}}dx\Big)^{\frac{N-2(p-1)}{N}}$$

Let m be a positive real number which be fixed in the sequel. Using Hölder's inequality we get

$$\int_{\{|u|<2n,|v|<2n\}} h_n(u)h_n(v)\big(|u|+|v|+1\big)^{\frac{2N\gamma}{N-2(p-1)}}\big(|Du|+|Dv|\big)^{\frac{(2-p)N}{N-2(p-1)}}dx \tag{4.32}$$
$$\le\Big(\int_{\{|u|<2n,|v|<2n\}} h_n(u)h_n(v)\big(|u|+|v|+1\big)^{\nu}dx\Big)^{\frac{2(p-1)(N-p)}{p(N-2(p-1))}}$$
$$\times\Big(\int_\Omega h_n(u)h_n(v)\frac{\big(|Du|+|Dv|\big)^p}{\big(|u|+|v|+1\big)^{1+m}}dx\Big)^{\frac{N(2-p)}{p(N-2(p-1))}}$$

with $\nu = \frac{N(2\gamma p+(1+m)(2-p))}{2(p-1)(N-p)}$. Because $\gamma < p - \frac{3}{2}$ we can choose $m > 0$ such that

$$\frac{N(2\gamma p + (1+m)(2-p))}{2(p-1)(N-p)} < \frac{N(2p^2-4p+2)}{2(p-1)(N-p)} < \frac{N(p-1)}{N-p}$$

which gives, using Lemma 2.7, $\big(|u|+|v|+1\big)^{\nu} \in L^1(\Omega)$. Since $m > 0$ from (2.19) we obtain

$$\frac{|Du|^p}{\big(1+|u|\big)^{1+m}} \in L^1(\Omega), \qquad \frac{|Dv|^p}{\big(1+|v|\big)^{1+m}} \in L^1(\Omega)$$

and then

$$\frac{\big(|Du|+|Dv|\big)^p}{\big(|u|+|v|+1\big)^{1+m}} \in L^1(\Omega).$$

Since $h_n(u)h_n(v) \overset{n\to\infty}{\longrightarrow} 1$ almost everywhere in Ω, Fatou's lemma, (4.31) and (4.32) imply that

$$|b(x)|^2\,\big(|u|+|v|+1\big)^{2\gamma}\big(|Du|+|Dv|\big)^{2-p} \in L^1(\Omega).$$

We are now in a position to pass to the limit as $n \to \infty$ in equation (4.30). Fatou's lemma yields that

$$\alpha\int_\Omega \frac{|DT_\sigma(u-v)|^2}{\big(|Du|+|Dv|\big)^{2-p}}dx \le \frac{2\sigma^2}{\alpha}\int_{\{|u-v|<\sigma\}\cap\{u\neq v\}} |b(x)|^2\,\big(|u|+|v|+1\big)^{2\gamma}\big(|Du|+|Dv|\big)^{2-p}dx.$$

Dividing the above inequality by σ^2 and taking the limit as $\sigma \to 0$ gives thanks to Lebesgue's convergence theorem

$$\lim_{\sigma\to 0}\frac{1}{\sigma^2}\int_\Omega \frac{|DT_\sigma(u-v)|^2}{\big(|Du|+|Dv|\big)^{2-p}}dx = 0. \tag{4.33}$$

Let us consider the field $\frac{h_n(u)T_\sigma(u-v)}{\sigma}$ which belongs to $W^{1,p}_{\Gamma_d}(\Omega)\cap L^\infty(\Omega)$. From Poincaré's inequality we have

$$\begin{aligned}
\int_\Omega \frac{h_n(u)\,|T_\sigma(u-v)|}{\sigma}dx &\le c\int_\Omega \Big|D\Big(\frac{h_n(u)T_\sigma(u-v)}{\sigma}\Big)\Big|\,dx\\
&\le c\bigg(\int_\Omega\Big|\frac{T_\sigma(u-v)}{\sigma}\Big|\,\big|h_n'(u)\big|\,|Du|\,dx + \int_\Omega \frac{h_n(u)\,|DT_\sigma(u-v)|}{\sigma}dx\bigg)\\
&\le c\bigg(\frac{1}{n}\int_{\{|u|<2n\}}|Du|\,dx + \Big(\int_{\{|u|<2n,|v|<2n+\sigma\}}(|Du|+|Dv|)^{2-p}dx\Big)^{1/2}\\
&\qquad\qquad \times\Big(\frac{1}{\sigma^2}\int_\Omega\frac{|DT_\sigma(u-v)|^2}{\big(|Du|+|Dv|\big)^{2-p}}dx\Big)^{1/2}\bigg)
\end{aligned}$$

which is licit since $2-p<p$ and (2.10) imply that both fields $\mathbb{1}_{\{|u|<2n\}}Du$ and $\mathbb{1}_{\{|v|<2n+\sigma\}}Dv$ lie in $L^{2-p}(\Omega)$.

Letting first $\sigma \to 0$ and (4.33) give

$$\int_\Omega \mathbb{1}_{\{u \neq v\}} h_n(u) dx \leq \frac{c}{n} \int_{\{|u|<2n\}} |Du| \, dx.$$

Taking the limit as n goes to infinity and using (2.11) we conclude that

$$\int_\Omega \mathbb{1}_{\{u \neq v\}} dx = 0$$

and then $u = v$ almost everywhere in Ω. □

References

[1] M. Artola. Sur une classe de problèmes paraboliques quasi-linéaires. *Boll. Un. Mat. Ital. B (6)*, 5(1):51–70, 1986.

[2] M. Ben Cheikh Ali. Homogenization of renormalized solution of elliptic equations in perforated domain. (in preparation).

[3] M. Ben Cheikh Ali. Homogénéisation des solutions renormalisées dans des domaines perforés. Thèse à l' Université de Rouen, 2001.

[4] M. Ben Cheikh Ali and O. Guibé. Résultats d'existence et d'unicité pour une classe de problèmes non linéaires et non coercifs. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(11):967–972, 1999.

[5] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J.-L. Vazquez. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa*, 22:241–273, 1995.

[6] M. F. Betta, A. Mercaldo, F. Murat, and M. M. Porzio. Existence of renormalized solutions to nonlinear elliptic equations with lower-order terms and right-hand side measure. *J. Math. Pures Appl. (9)*, 81(6):533–566, 2002.

[7] D. Blanchard. Truncations and monotonicity methods for parabolic equations. *Nonlinear Anal.*, 21(10):725–743, 1993.

[8] D. Blanchard and H. Redwane. Renormalized solutions for a class of nonlinear evolution problem. *J. Math. Pures et Appl*, 77:117–151, 1998.

[9] L. Boccardo. Some dirichlet problems with lower order terms in divergence form. Preprint.

[10] L. Boccardo and T. Gallouët. On some nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, 87:149–169, 1989.

[11] L. Boccardo, T. Gallouët, and F. Murat. Unicité de la solution de certaines équations elliptiques non linéaires. *C. R. Acad. Sci. Paris*, 1992.

[12] L. Boccardo, L. Orsina, and A. Porretta. Some noncoercive parabolic equations with lower order terms in divergence form. *J. Evol. Equ.*, 3(3):407–418, 2003.

[13] M. Chipot and G. Michaille. Uniqueness results and monotonicity properties for strongly nonlinear elliptic variational inequalities. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 16(1):137–166, 1989.

[14] G. Dal Maso, F. Murat, L. Orsina, and A. Prignet. Renormalized solutions of elliptic equations with general measure data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(4):741–808, 1999.

[15] T. Del Vecchio and M. R. Posteraro. Existence and regularity results for nonlinear elliptic equations with measure data. *Adv. Differential Equations*, 1(5):899–917, 1996.

[16] T. Del Vecchio and M. R. Posteraro. An existence result for nonlinear and noncoercive problems. *Nonlinear Anal.*, 31(1-2):191–206, 1998.

[17] R. J. DiPerna and P.-L. Lions. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2)*, 130(2):321–366, 1989.

[18] J. Droniou. Non-coercive linear elliptic problems. *Potential Anal.*, 17(2):181–203, 2002.

[19] J. Droniou. Global and local estimates for nonlinear noncoercive elliptic equations with measure data. *Comm. Partial Differential Equations*, 28(1-2):129–153, 2003.

[20] D. Gilbarg and N. Trudinger. *Elliptic partial differential equations of second order.* Springer, 1977.

[21] J. Liang. Weakly-coercive quasilinear elliptic equations with inhomogenous measure data. In *Progress in partial differential equations: elliptic and parabolic problems.* Pitman Research Notes in Math, (266) 1992.

[22] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod et Gauthier-Villars, Paris., 1969.

[23] P.-L. Lions and F. Murat. Solutions renormalisées d'équations elliptiques. (in preparation).

[24] F. Murat. Soluciones renormalizadas de EDP elipticas non lineales. Technical Report R93023, Laboratoire d'Analyse Numérique, Paris VI, 1993. Cours à l'Université de Séville.

[25] A. Porretta. Uniqueness of solutions for nonlinear elliptic Dirichlet problems with L^1 data. *NoDEA Nonlinear Differential Equations Appl.*, 11(4):407–430, 2004.

[26] J.-M. Rakotoson. Uniqueness of renormalized solutions in a T-set for the L^1-data problem and the link between various formulations. *Indiana Univ. Math. J.*, 43(2):685–702, 1994.

[27] W. P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.

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