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Abstract. We prove a few existence results of a solution for a static system with a coupling of thermoviscoelastic type. As this system involves L^1 coupling terms we use the techniques of renormalized solutions for elliptic equations with L^1 data. We also prove partial uniqueness results.

1. Introduction

In the present paper we consider the following nonlinear coupled system:

$$\lambda u - \operatorname{div}\big(\mathbf{A}(x)Du - f(\theta)\big) = g \quad \text{in } \Omega, \tag{1}$$

$$\mu\theta - \operatorname{div}\big(\mathbf{a}(x, D\theta)\big) = \big(\mathbf{A}(x)Du - f(\theta)\big)\cdot Du \quad \text{in } \Omega, \tag{2}$$

$$u = 0 \qquad \theta = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, $\lambda, \mu > 0$, $f : \mathbb{R} \mapsto \mathbb{R}^N$ is a continuous function, $g \in L^2(\Omega)$, $\mathbf{A}(x)$ is a coercive matrix with L^∞–coefficients and $v \mapsto -\operatorname{div}\big(\mathbf{a}(x, Dv)\big)$ is a monotone operator defined from $H^1_0(\Omega)$ into $H^{-1}(\Omega)$.

Problem (1)–(3) is a static version (or time discretized-version) of a time dependent class of systems in thermoviscoelasticity. Under stronger assumptions than in the present paper, existence of a solution for these evolution systems is established in [4] (see also [3]). Moreover, for (1)–(3) uniqueness results were also proved.

The main difficulties in dealing with existence of solution of system (1)–(3) are due to equation (2) and the coupling. Indeed if u is a variational solution of (1) (i.e. $u \in H^1_0(\Omega)$) then the right–hand side of (2) belongs to $L^1(\Omega)$. It follows from L. Boccardo and T. Gallouët [6] (see also [2] and [20]) that θ is expected in $L^q(\Omega)$ for $q < N/(N-2)$ if $N \geq 3$ and $q < \infty$ if $N = 2$. With the aim of solving (1) with $f(\theta) \in L^2(\Omega)$ we are then led to assume that f satisfies the growth assumption

$$\forall r \in \mathbb{R} \quad \big|f(r)\big| \leq a + M|r|^\alpha,$$

with $a > 0$, $M > 0$ and $\alpha < N/\big(2(N-2)\big)$ if $N \leq 3$ and $\alpha < \infty$ if $N = 2$. Under this hypothesis on f, the coupling between Equations (1) and (2) together with the L. Boccardo

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and T. Gallouët estimates techniques (see [6] and Remark 5.5 of the present paper) lead to the following a priori estimate on θ,

$$\forall q \in [1, \frac{N}{N-2}[\qquad \|\theta\|_{L^q(\Omega)} \leq C\big(1 + \|f(\theta)\|^2_{L^2(\Omega)}\big) \leq C'\Big(1 + \int_\Omega |\theta|^{2\alpha}\,\mathrm{d}x\Big),$$

where C and C' are real positive constant independ of u and θ. This implies that if $2\alpha \geq 1$ the estimate above is not sufficient in general settings to obtain the existence of a solution of (1)–(3) using a fixed–point or approximation method.

As the right-hand side of (2) belongs to L^1, we use in the present paper the convenient framework of renormalized solutions that insures uniqueness and stability results for equations with L^1 data. Renormalized solutions have been introduced by R.J. DiPerna and P.-L. Lions in [11] and [12] for first order equations and have been adapted for elliptic equations in [5], [18], [19] and for elliptic equations with general measure data in [9] (see also [8]). Other frameworks as entropy solutions [2] or SOLA [10] may be used for equation (2) with L^1 data.

Another interesting question related to problem (1)–(3) deals with the uniqueness of a solution, that is an open problem in general settings due to lack of regularity of θ and the right–hand side of (2). We investigate in the present paper uniqueness of a small solution (u, θ) such that $\theta \geq 0$ almost everywhere in Ω and under additional assumptions on the data for $N = 2$ and $N = 3$.

Elliptic systems involving L^1 coupling terms are also studied in [13], [7] and [17] and use a convenient formulation for equation with the L^1 term.

The plan of this paper is as follows. In Section 2 we recall the definition of a renormalized solution and we define a weak-renormalized solution for system (1)–(3). In Section 3 we give a few useful properties on renormalized solutions. Section 4 and Section 5 are devoted to existence results for two restricted case: the first case deals with small data, the second case contains existence results under more restrictive conditions on f but for general data. Section 6 contains a partial uniqueness result of a small solution (u, θ) such that $\theta \geq 0$ almost everywhere in Ω and under additional assumptions on the data.

2. Assumptions and definitions

Let Ω be an open and bounded subset of $\mathbb{R}^N$ ($N \geq 2$). The following assumptions are made on the data:

(A1) $\mathbf{A}(x)$ is a coercive matrix field with coefficients lying in $L^\infty(\Omega)$ i.e. $\mathbf{A}(x) = \big(a_{i,j}(x)\big)_{1\leq i,j\leq N}$ with
- $a_{i,j}(x) \in L^\infty(\Omega)$
- $\exists \gamma > 0$ such that $\forall \xi \in \mathbb{R}^N$ $\mathbf{A}(x)\xi \cdot \xi \geq \gamma \|\xi\|^2_{\mathbb{R}^N}$ for almost every $x \in \Omega$;

(A2) the function $\mathbf{a} : \Omega \times \mathbb{R}^N \longmapsto \mathbb{R}^N$ is a Caratheodory function (continuous in ξ for almost every $x \in \Omega$ and measurable in x for every $\xi \in \mathbb{R}^N$) and there exists $\delta > 0$ such that

$$\forall \xi \in \mathbb{R}^N \quad \mathbf{a}(x, \xi) \cdot \xi \geq \delta \|\xi\|^2_{\mathbb{R}^N} \quad \text{for almost every } x \in \Omega;$$

(A3) for every ξ and ξ' in $\mathbb{R}^N$, and almost everywhere in Ω

$$(\mathbf{a}(x, \xi) - \mathbf{a}(x, \xi')) \cdot (\xi - \xi') \geq 0;$$

(A4) there exists $\beta > 0$ such that

$$|\mathbf{a}(x, \xi)| \leq \beta(b(x) + |\xi|)$$

holds for every $\xi \in \mathbb{R}^N$ and for almost every in $x \in \Omega$ with $b \in L^2(\Omega)$;

(A5) $\lambda > 0$, $\mu > 0$;

(A6) f is a continuous function defined on $\mathbb{R}$ with values in $\mathbb{R}^N$;

(A7) g is an element of $L^2(\Omega)$.

Throughout this paper and for any non negative real number K we denote by $T_K(r)$ the truncation function at height $\pm K$, i.e. $T_K(r) = \min\big(K, \max(r, -K)\big)$. For a measurable set E of Ω, we denote by $\mathbb{1}_E$ the characteristic function of E.

Following [18] (and [19]) we recall the definition of a renormalized solution for nonlinear equations of type (2) with L^1 right–hand side.

Definition 2.1. Let F be an element of $L^1(\Omega)$. A measurable function θ defined on Ω is called a renormalized solution of the problem

$$P(F)\left\{\begin{array}{l} \mu\theta - \operatorname{div}(\mathbf{a}(x, D\theta)) = F \quad \text{in } \Omega, \\ \theta = 0 \quad \text{on } \partial\Omega, \end{array}\right.$$

if

$$\theta \in L^1(\Omega),\ \forall K > 0\ \ T_K(\theta) \in H_0^1(\Omega); \tag{4}$$

for every function $h \in W^{1,\infty}(\mathbb{R})$ such that h has a compact support,

$$\mu\theta h(\theta) - \operatorname{div}\big(h(\theta)\mathbf{a}(x, D\theta)\big) + h'(\theta)\mathbf{a}(x, D\theta)\cdot D\theta = F h(\theta) \text{ in } \mathcal{D}'(\Omega); \tag{5}$$

$$\lim_{n\to\infty} \frac{1}{n}\int_{\{n<|\theta|<2n\}} |D\theta|^2 \,\mathrm{d}x = 0. \tag{6}$$

Under assumptions (A2)–(A4) and with $\mu > 0$, using the techniques developped in [18] (see also [9] and [19]), there exists a unique renormalized solution of $P(F)$ for any F in $L^1(\Omega)$.

We now use renormalized solutions to define a so called weak-renormalized solution of Problem (1)–(3).

Definition 2.2. A couple of functions (u, θ) defined on Ω is called a weak-renormalized solution of (1)–(3) if u and θ satisfy

$$u \in H_0^1(\Omega), \tag{7}$$

$$f(\theta) \in \big(L^2(\theta)\big)^N, \tag{8}$$

$$\lambda u - \operatorname{div}\big(\mathbf{A}(x)Du - f(\theta)\big) = g \quad \text{in } \mathcal{D}'(\Omega), \tag{9}$$

$$\theta \text{ is a renormalized solution of (2)–(3).} \tag{10}$$

Under regularities (7)–(8), the right–hand side of (2), $(\mathbf{A}(x)Du - f(\theta))\cdot Du$, belongs to $L^1(\theta)$. So we are in the framework of renormalized solution for equation (2).

3. Useful properties of renormalized solutions

We recall the following propositions on renormalized solutions of elliptic equations for L^1 data, that can be shown using the techniques developped in [9], [18] and [19].

Proposition 3.1 (Existence and uniqueness of the renormalized solution)**.** *Assume that (A2)–(A4) hold true and $\mu > 0$. Then for any F belonging to $L^1(\Omega)$, there exists a unique* *renormalized solution of Problem $P(F)$. Moreover for any function $w \in L^\infty(\Omega) \cap H_0^1(\Omega)$, if there exists $K > 0$ such that $Dw = 0$ almost everywhere in $\{x : |\theta(x)| \geq K\}$ then*

$$\mu \int_\Omega \theta w \,\mathrm{d}x + \int_\Omega \mathbf{a}(x, D\theta) \cdot Dw \,\mathrm{d}x = \int_\Omega F w \,\mathrm{d}x. \tag{11}$$

Remark 3.2. Equality (11) which is proved in [9] in the context of general measure data, is formally obtained through using the test function w in the equation of $P(F)$.

Proposition 3.3. *Assume that (A2)–(A4) hold true and $\mu > 0$. Let F_1, F_2 be two elements of $L^1(\Omega)$, and denote by θ_i the unique renormalized solution of $P(F_i)$ $(i = 1, 2)$.*

Then for any $K > 0$

$$\begin{aligned}\mu \int_\Omega (\theta_1 - \theta_2) T_K(\theta_1 - \theta_2) \,\mathrm{d}x + \int_{\{|\theta_1 - \theta_2| < K\}} \big(\mathbf{a}(x, D\theta_1) - \mathbf{a}(x, D\theta_2)\big) \cdot (D\theta_1 - D\theta_2) \,\mathrm{d}x \\ \leq \int_\Omega (F_1 - F_2) T_K(\theta_1 - \theta_2) \,\mathrm{d}x.\end{aligned} \tag{12}$$

Remark 3.4. Inequality (12) is obtained by plugging the admissible test function $h_n(\theta_1)h_n(\theta_2)T_K(\theta_1 - \theta_2)$ in the difference of the equations $P(F_1)$ and $P(F_2)$ (that is licit in view of Proposition 3.1) where h_n is a sequence of functions in $W^{1,\infty}(\mathbb{R})$ such that $h_n(r) \to 1$ as n tends to ∞ and with compact support.

Due to Proposition 3.3 we deduce that

$$\|\theta_1 - \theta_2\|_{L^1(\Omega)} \leq \frac{1}{\mu} \|F_1 - F_2\|_{L^1(\Omega)}$$

and the continuity of the renormalized solution of $P(F)$ with respect to the datum F.

We recall the following lemma that can be proved by means of the estimates techniques of L. Boccardo and T. Gallouët [6] (see also [2]).

Lemma 3.5. *Let θ be a measurable function defined on Ω, that is finite almost everywhere in Ω, and $M > 0$ such that*

$$\forall K > 0, \quad T_K(\theta) \in H_0^1(\Omega) \quad \text{and}$$
$$\int_\Omega |DT_K(\theta)|^2 \,\mathrm{d}x \leq KM.$$

Then $\theta \in W_0^{1,p}(\Omega)$ for any $1 \leq p < N/(N-1)$ and there exists a constant C (depending on Ω and p) such that

$$\|\theta\|_{W_0^{1,p}(\Omega)} \leq CM.$$

Gathering Proposition 3.1 and Lemma 3.5 we deduce the following corollary.

Corollary 3.6. *Assume that (A2)–(A4) hold true and $\mu > 0$. Let F be an element of $L^1(\Omega)$ and θ the renormalized solution of $P(F)$. Then for any $1 \leq p < N/(N-1)$, $\theta \in W_0^{1,p}(\Omega)$ and*

$$\|\theta\|_{W_0^{1,p}(\Omega)} \leq C \|F\|_{L^1(\Omega)} \tag{13}$$

where C is a constant only depending upon Ω, p and $\mathbf{a}$.

4. Existence of small solutions of (1)–(3) for small data

In this section we assume that the continuous function f satisfies the following growth assumption

$$\exists a \geq 0, \exists M > 0, \quad \forall r \in \mathbb{R} \quad \big|f(r)\big| \leq a + M|r|^{\alpha} \tag{14}$$

with $1/2 < \alpha < N/(2(N-2))$ if $N \geq 3$ and $1/2 < \alpha < +\infty$ if $N = 2$.

Under this additional assumption, Theorem 4.1 insures the existence of at least a solution of Problem (1)–(3) for small data enough. Notice that on the one hand the upper bound of α in (14) is motivated in Introduction, on the other hand the lower bound permits us to exploit the small character on the data.

Theorem 4.1. *Assume that (A1)–(A7) and (14) hold true. There exists a real positive number η such that if $a + \|g\|_{L^2(\Omega)} < \eta$, then there exists at least a weak-renormalized solution of (1)–(3) such that*

$$\big\|u\big\|_{H_0^1(\Omega)} + \big\|\theta\big\|_{L^{2\alpha}(\Omega)} \leq \omega(\eta)$$

where $\omega(\eta)$ tends to zero as η tends to zero.

Proof. The proof is divided into 2 steps. Step 1 is devoted to the construction of a fixed–point operator. In Step 2 we give a sufficient condition on the data in order to apply the Schauder fixed–point Theorem.

Step 1. Since the function f is continuous and verifies growth assumption (14), under assumptions (A1), (A5) and (A7) the mapping

$$L^{2\alpha}(\Omega) \longmapsto H_0^1(\Omega)$$

$$\hat{\theta} \longmapsto \hat{u}, \quad \text{where } \hat{u} \text{ is the unique solution of the problem}$$

$$\begin{cases} \lambda \hat{u} - \operatorname{div}\big(\mathbf{A}(x)D\hat{u} - f(\hat{\theta})\big) = g & \text{in } \Omega \\ \hat{u} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{15}$$

is continuous and the coercivity of $\mathbf{A}$ implies that

$$\int_{\Omega} \big|D\hat{u}\big|^2 \, \mathrm{d}x \leq C\Big(\big\|g\big\|^2_{L^2(\Omega)} + \big\|f(\hat{\theta})\big\|^2_{(L^2(\Omega)^N)}\Big) \tag{16}$$

where C is a generic constant independent of $\hat{\theta}$.

Let $\hat{\theta}$ be an element of $L^{2\alpha}(\Omega)$ and $\hat{u}$ the unique element of $H_0^1(\Omega)$ solution of (15). Due to growth assumption (14) on f and the regularity of $\hat{u}$, the field $(\mathbf{A}(x)D\hat{u} - f(\hat{\theta})) \cdot D\hat{u}$ belongs to $L^1(\Omega)$, and by Proposition 3.1, let θ be the unique renormalized solution of the problem:

$$\mu\theta - \operatorname{div}\big(\mathbf{a}(x, D\theta)\big) = \big(\mathbf{A}(x)D\hat{u} - f(\hat{\theta})\big) \cdot D\hat{u} \qquad \text{in } \Omega \tag{17}$$

$$\theta = 0 \qquad \text{on } \partial\Omega. \tag{18}$$

We denote by Γ the mapping defined by $\theta = \Gamma(\hat{\theta})$.

Since $1 < 2\alpha < N/(N-2)$ (and $1 < 2\alpha < +\infty$ if $N = 2$), let q be a positive real number such that $2\alpha < q^* < N/(N-2)$ (and $2\alpha < q^* < +\infty$ if $N = 2$), where q^* denotes the Sobolev conjugate exponent ($1/q^* = 1/q - 1/N$).

Using the properties of the renormalized solutions (see Remark 3.4 and Corollary 3.6), the interpolation of $L^{2\alpha}(\Omega)$ between $L^1(\Omega)$ and $L^{q^*}(\Omega)$ and the Rellich Kondrachov Theorem we deduce that Γ is defined continuous and compact from $L^{2\alpha}(\Omega)$ into itself. Moreover inequality (16) and Corollary 3.6 imply that $\forall\hat{\theta} \in L^{2\alpha}(\Omega)$, if $\theta = \Gamma(\hat{\theta})$ then

$$\|\theta\|_{W_0^{1,q}(\Omega)} \leq C\Big(\|g\|^2_{L^2(\Omega)} + \|f(\hat{\theta})\|^2_{(L^2(\Omega))^N}\Big)$$

and growth assumption (14) on f yields

$$\|\theta\|_{W_0^{1,q}(\Omega)} \leq C\Big(\|g\|^2_{L^2(\Omega)} + a^2 + M^2\|\hat{\theta}\|^{2\alpha}_{L^{2\alpha}(\Omega)}\Big) \tag{19}$$

where C is a constant independent of $\hat{\theta}$.

Step 2. Applying the Schauder fixed–point Theorem to the mapping Γ reduces to show that there exists for instance a ball B of $L^{2\alpha}(\Omega)$ such that $\Gamma(B) \subset B$.

Let $\hat{\theta}$ be an element of $L^{2\alpha}(\Omega)$ and $\theta = \Gamma(\hat{\theta})$. Since $1 < 2\alpha < q^* < N/(N-2)$ (and $1 < 2\alpha < q^* < +\infty$ if $N = 2$), the Sobolev embedding Theorem and (19) lead to

$$\|\theta\|_{L^{2\alpha}(\Omega)} \leq C\Big(\|g\|^2_{L^2(\Omega)} + a^2 + M^2\|\hat{\theta}\|^{2\alpha}_{L^{2\alpha}(\Omega)}\Big) \tag{20}$$

where C is a constant independent of $\hat{\theta}$, g, a and M.

As $2\alpha > 1$, let $\eta > 0$ and $R(\eta) > 0$ such that

$$C\big(\eta + M^2(R(\eta))^{2\alpha}\big) < R(\eta),$$
$$R(\eta) < 2C\eta.$$

If

$$\|g\|^2_{L^2(\Omega)} + a^2 < \eta \tag{21}$$

then we have

$$\Gamma\Big(B_{L^{2\alpha}(\Omega)}\big(0, R(\eta)\big)\Big) \subset B_{L^{2\alpha}(\Omega)}\big(0, R(\eta)\big).$$

Therefore, we may apply the Schauder fixed–point Theorem so that, there exists at least a solution (u, θ) of (1)–(3) in the sense of Definition 2.2.

Moreover the choice of $R(\eta)$ and (16) imply that

$$\|u\|_{H_0^1(\Omega)} + \|\theta\|_{L^{2\alpha}(\Omega)} \leq \omega(\eta)$$

where $\omega(\eta)$ tends to zero as η tends to zero.

The proof of Theorem 4.1 is complete. □

5. Existence of a solution of (1)–(3) for more general data

In order to remove the small character on the data of the previous section, we suppose by now more restrictive hypotheses on the behavior of f, which are on $\mathbb{R}^+$

$$\begin{cases} \lim\limits_{r\to+\infty} \dfrac{|f(r)|}{r^{(N+2)/(2N)}} = 0 & \text{if } N \geq 3, \\ \forall r \in \mathbb{R}^+ \quad |f(r)| \leq a + M|r| & \text{if } N = 2 \text{ with } a \geq 0 \text{ and } M \geq 0, \end{cases} \tag{22}$$

that is more restrictive than (14) since $(N+2)/(2N) < N/(2(N-2))$), and on $\mathbb{R}^-$ a behavior of which the model case is $f = 0$ for $r < r_0 \leq 0$. (see hypotheses (23) and (24) below).

Theorem 5.1. *Assume that assumptions (A1)–(A7) and (22) hold true. Moreover assume that the continuous function f is such that*

$$\lim_{r\to-\infty} \frac{|f(r)|}{\sqrt{|r|}} = 0. \tag{23}$$

Then there exists at least a weak-renormalized solution of (1)–(3).

In the case where the function f has a zero on $\mathbb{R}^-$, the structure of equation (2) allows us to remove the growth assumption on f on $\mathbb{R}^-$ and give an additional property on θ.

Theorem 5.2. *Assume that assumptions (A1)–(A7) and (22) hold true. Moreover assume that the continuous function f satisfies:*

$$\exists r_0 \in \mathbb{R}^- \quad \text{such that} \quad f(r_0) = 0. \tag{24}$$

Then there exists at least a weak-renormalized solution (u, θ) of (1)–(3) such that $\theta \geq r_0$ almost everywhere in Ω.

Remark 5.3. The existence results of Theorems 5.1 and 5.2 were announced in [16] (see also [15]) under more restrictive hypotheses on the function f. Let us notice that when $N = 2$ linear growth on $\mathbb{R}^+$ is allowed for f.

Before to prove Theorem 5.1, we give a technical lemma.

Lemma 5.4. *Assume that (22) holds true. Let θ be a measurable function defined on Ω such that*

$$\theta \in L^1(\Omega), \tag{25}$$

$$\forall K > 0 \quad T_K(\theta) \in H_0^1(\Omega),$$

$$\exists C_1 > 0 \text{ such that } \forall K > 0 \quad \frac{1}{K}\int_\Omega \big|DT_K(\theta)\big|^2 \,\mathrm{d}x < C_1\Big(\int_{\{0\leq\theta\leq K\}} f^2(\theta)\,\mathrm{d}x + 1\Big). \tag{26}$$

Then for any $1 \leq q < N/(N-2)$ (and $1 \leq q < +\infty$ if $N = 2$), there exists a constant C', only depending upon q, Ω, $\|\theta\|_{L^1(\Omega)}$, C_1 and f such that

$$\big\|\theta\big\|_{L^q(\Omega)} \leq C'. \tag{27}$$

Sketch of the proof. The proof relies on estimate techniques of L. Boccardo and T. Gallouët [6] (see also [2]). If $N = 2$ we use the limit case of the Sobolev embedding Theorem (see [1], [14] for instance) that allows us to reach linear growth on $\mathbb{R}^+$ for the function f.

Case $N \geq 3$. Let n be an element of $\mathbb{N}$, that will fixed in the sequel, and let q be such that $1 < q < N/(N-2)$. Hypothesis (22) gives that

$$\forall r \in \big[2^n, +\infty\big[\quad |f(r)| \leq \omega(n) r^{(N+2)/2N}, \tag{28}$$

where $\omega(n)$ tends to zero as n tends to infinity.

As θ is finite almost everywhere in Ω, we have

$$\begin{aligned}\int_\Omega |\theta|^q \,\mathrm{d}x &\leq 2^{nq}|\Omega| + \sum_{k=n}^{+\infty} \int_{\{2^k<|\theta|\leq 2^{k+1}\}} |\theta|^q \,\mathrm{d}x \\ &\leq 2^{nq}|\Omega| + \sum_{k=n}^{+\infty} \left(\frac{1}{2^k}\right)^{2^*-q} \int_\Omega |T_{2^{k+1}}(\theta)|^{2^*} \,\mathrm{d}x,\end{aligned}$$

where 2^* denotes the Sobolev conjugate exponent ($1/2^* = 1/2 - 1/N$).

The Sobolev embedding Theorem and (26) (with $K = 2^{k+1}$) yield

$$\int_\Omega |\theta|^q \,\mathrm{d}x \le 2^{nq}|\Omega| + CC_1^{2^*/2} \sum_{k=n}^{+\infty} \left(\frac{1}{2^k}\right)^{2^*-q} \left(2^{k+1} \int_{\{0\le\theta\le 2^{k+1}\}} f^2(\theta)\,\mathrm{d}x + 2^{k+1}\right)^{2^*/2},$$

where C is a constant depending on Ω.

Using (28) we obtain

$$\int_\Omega |\theta|^q \,\mathrm{d}x \le 2^{nq}|\Omega| + C(2C_1)^{2^*/2} \sum_{k=n}^{+\infty} \left(\frac{1}{2^k}\right)^{2^*/2-q} \Bigg(|\Omega| \max_{r\in[0,2^n]} |f(r)|^2 + 1 + \omega(n) \int_{\{2^n\le\theta\le 2^{k+1}\}} |\theta|^{(N+2)/N} \,\mathrm{d}x\Bigg)^{2^*/2}. \tag{29}$$

On the one hand Hölders inequality gives, $\forall n \le k < +\infty$,

$$\int_{\{2^n\le\theta\le 2^{k+1}\}} |\theta|^{(N+2)/N} \,\mathrm{d}x \le \left(\|\theta\|_{L^1(\Omega)}\right)^{2/N} \left(\int_{\{2^n\le\theta\le 2^{k+1}\}} |\theta|^{N/(N-2)} \,\mathrm{d}x\right)^{2/2^*}, \tag{30}$$

on the other hand as $q < N/(N-2) = 2^*/2$, the series $\sum_{k=n}^{+\infty} \left(\frac{1}{2^k}\right)^{2^*/2-q}$ is convergent and we have

$$\sum_{k=n}^{+\infty} \left(\frac{1}{2^k}\right)^{2^*/2-q} \int_{\{2^n\le\theta\le 2^{k+1}\}} |\theta|^{N/(N-2)} \,\mathrm{d}x \le C(q) \int_\Omega |\theta|^q \,\mathrm{d}x, \tag{31}$$

where $C(q)$ is a constant only depending on q.

After a few computations, from inequality (29) together with (30) and (31) it follows that

$$\int_\Omega |\theta|^q \,\mathrm{d}x \le M_1(n,q,\Omega,f,C_1) + (\omega(n))^{2^*/2} M_2\big(\|\theta\|_{L^1(\Omega)}, q, C_1, \Omega\big) \int_\Omega |\theta|^q \,\mathrm{d}x, \tag{32}$$

where M_1 is a constant only depending on n, q, Ω, f and C_1, and M_2 is a constant only depending on $\|\theta\|_{L^1(\Omega)}$, q, C_1 and Ω.

Therefore since $\omega(n)$ tends to zero as n tends to infinity, we can choose n such that $(\omega(n))^{2^*/2} M_2\big(\|\theta\|_{L^1(\Omega)}, q, C_1, \Omega\big) < 1/2$ and then (32) yields

$$\int_\Omega |\theta|^q \,\mathrm{d}x \le 2M_1(n,q,\Omega,f,C_1),$$

that is (27).

Case $N = 2$. Let a and M be two non negative real numbers such that

$$\forall r \in \mathbb{R}^+ \quad |f(r)| \le a + M|r|. \tag{33}$$

Using similar techniques as in the previous case, we obtain

$$\begin{aligned}\int_\Omega \left|\frac{D\theta}{1+|\theta|}\right|^2 \mathrm{d}x &\le C\left(1 + \sum_{k=0}^{+\infty} \frac{1}{2^{k+1}} \int_{\{0<\theta\le 2^{k+1}\}} |\theta|^2 \,\mathrm{d}x\right) \\ &\le C\left(1 + \int_\Omega |\theta| \,\mathrm{d}x\right),\end{aligned}$$

where C is a constant only depending on Ω, a, M and C_1.

It follows that

$$\big\| \ln(1+|\theta|) \big\|_{H_0^1(\Omega)} \leq C, \tag{34}$$

where C is a constant only depending on Ω, a, M, C_1 and $\|\theta\|_{L^1(\Omega)}$.

Making use of Theorem 7.15 from [14], let C_2 and C_3 be two positive real numbers only depending on N, such that

$$\int_\Omega \exp\left[\left(\frac{\ln(1+|\theta|)}{C_2\|D(\ln(1+|\theta|))\|_{L^2(\Omega)}}\right)^2\right] \mathrm{d}x \leq C_3|\Omega|$$

Therefore from (34) we have

$$\int_\Omega \exp\left[\left(\frac{\ln(1+|\theta|)}{C_2 C}\right)^2\right] \mathrm{d}x \leq C_3|\Omega|,$$

where C, C_2 and C_3 does not depend on θ.

If follows that for any $1 \leq q < +\infty$ there exists $C' > 0$ only depending on f, Ω, C_1, C_2, C_3, $\|\theta\|_{L^1(\Omega)}$ and q such that

$$\|\theta\|_{L^q(\Omega)} \leq C'.$$

The proof of Lemma 5.4 is complete. □

We are now in a position to prove Theorem 5.1.

Proof of Theorem 5.1. The proof is divided into 3 steps. In Step 1 we consider a solution $(u^\varepsilon, \theta^\varepsilon)$ of the approximate problem (1)–(3) with $f^\varepsilon = f \circ T_{1/\varepsilon}$ ($\varepsilon > 0$) in place of f and we derive a few preliminary estimates. In Step 2, using the coupling between the unknowns u^ε and θ^ε, we establish an important equality that first implies an $L^1(\Omega)$–estimate on θ^ε. In Step 3, we make use of Lemma 5.4 to obtain an $L^2(\Omega)$–estimate on $f^\varepsilon(\theta^\varepsilon)$ and, at last, we pass to the limit in the approximate problem.

Step 1. For $\varepsilon > 0$, we consider the following system

$$\lambda u^\varepsilon - \operatorname{div}\big(\mathbf{A}(x)Du^\varepsilon - f^\varepsilon(\theta^\varepsilon)\big) = g \qquad \text{in } \Omega, \tag{35}$$

$$\mu\theta^\varepsilon - \operatorname{div}\big(\mathbf{a}(x, D\theta^\varepsilon)\big) = \big(\mathbf{A}(x)Du^\varepsilon - f^\varepsilon(\theta^\varepsilon)\big)\cdot Du^\varepsilon \qquad \text{in } \Omega, \tag{36}$$

$$u^\varepsilon = 0 \qquad \theta^\varepsilon = 0 \qquad \text{on } \partial\Omega. \tag{37}$$

Analyzing the proof of Theorem 4.1 allows us to show that there exists at least a weak-renormalized solution $(u^\varepsilon, \theta^\varepsilon)$ of (35)–(37). Indeed as the continuous function f^ε is bounded, the mapping Γ constructed in the proof of Theorem 4.1 is continuous and compact from $L^1(\Omega)$ into a bounded subset of $L^1(\Omega)$. Then the Schauder fixed–point Theorem allows us to conclude.

For $\varepsilon > 0$, let $(u^\varepsilon, \theta^\varepsilon)$ be a weak-renormalized solution of (35)–(37). It follows from (16) that

$$\int_\Omega |Du^\varepsilon|^2 \,\mathrm{d}x < C\Big(1 + \|f^\varepsilon(\theta^\varepsilon)\|^2_{(L^2(\Omega))^N}\Big), \tag{38}$$

$$\big\|(\mathbf{A}(x)Du^\varepsilon - f^\varepsilon(\theta^\varepsilon))\cdot Du^\varepsilon\big\|_{L^1(\Omega)} \leq C\Big(1 + \|f^\varepsilon(\theta^\varepsilon)\|^2_{(L^2(\Omega))^N}\Big), \tag{39}$$

and, recalling that θ^ε is a renormalized solution of (36)–(37) and using Proposition 3.1,

$$\int_\Omega \left|DT_K(\theta^\varepsilon)\right|^2 \mathrm{d}x \le CK\Big(1+\|f^\varepsilon(\theta^\varepsilon)\|^2_{(L^2(\Omega))^N}\Big) \qquad \forall K>0, \tag{40}$$

where C is a constant independent of ε and K.

Remark 5.5. From Corollary 3.6 we obtain that for any $1\le p<N/(N-1)$

$$\|\theta^\varepsilon\|_{W_0^{1,p}(\Omega)} \le C\Big(1+\|f^\varepsilon(\theta^\varepsilon)\|^2_{(L^2(\Omega))^N}\Big). \tag{41}$$

In the case where $\lim_{r\to+\infty}|f(r)|^2/r=0$, deriving an L^q–estimate for any $1\le q<N/(N-2)$ (and $1\le q<+\infty$ if $N=2$) is an easy task. But under hypothesis (22), a new estimate on θ^ε is necessary to obtain a upper bound on $\|\theta^\varepsilon\|_{L^1(\Omega)}$ and more generally on $\|\theta^\varepsilon\|_{L^q(\Omega)}$.

Step 2. For $K>0$, since $T_K(\theta^\varepsilon)\in L^\infty(\Omega)\cap H_0^1(\Omega)$ and $DT_K(\theta^\varepsilon)=0$ almost everywhere on $\{x : |\theta^\varepsilon(x)|\ge K\}$ Proposition 3.1 (with $w=T_K(\theta^\varepsilon)/K$) gives that

$$\mu\int_\Omega \frac{T_K(\theta^\varepsilon)}{K}\,\theta^\varepsilon\,\mathrm{d}x+\frac{1}{K}\int_\Omega \mathbf{a}(x,D\theta^\varepsilon)\cdot DT_K(\theta^\varepsilon)\,\mathrm{d}x=\int_\Omega \frac{T_K(\theta^\varepsilon)}{K}\left(\mathbf{A}(x)Du^\varepsilon-f^\varepsilon(\theta^\varepsilon)\right)\cdot Du^\varepsilon\,\mathrm{d}x.$$

Plugging the test function u^ε in (35) and summing the result to the previous equality yield

$$\begin{aligned}\lambda\int_\Omega (u^\varepsilon)^2\,\mathrm{d}x+\mu\int_\Omega \frac{T_K(\theta^\varepsilon)}{K}\,\theta^\varepsilon\,\mathrm{d}x+\frac{1}{K}\int_\Omega \mathbf{a}(x,D\theta^\varepsilon)\cdot DT_K(\theta^\varepsilon)\,\mathrm{d}x\\ +\int_\Omega \frac{K-T_K(\theta^\varepsilon)}{K}\,\mathbf{A}(x)Du^\varepsilon\cdot Du^\varepsilon\,\mathrm{d}x=\int_\Omega \frac{K-T_K(\theta^\varepsilon)}{K}\,f^\varepsilon(\theta^\varepsilon)\cdot Du^\varepsilon\,\mathrm{d}x+\int_\Omega g u^\varepsilon\,\mathrm{d}x.\end{aligned}$$

Since $\lambda>0$ and $K-T_K(\theta^\varepsilon)$ is a non negative function, the coercivity of $\mathbf{a}$ and $\mathbf{A}$ together with Young's inequality lead to

$$\begin{aligned}\int_\Omega (u^\varepsilon)^2\,\mathrm{d}x+\int_\Omega \frac{T_K(\theta^\varepsilon)}{K}\,\theta^\varepsilon\,\mathrm{d}x+\frac{1}{K}\int_\Omega \left|DT_K(\theta^\varepsilon)\right|^2\mathrm{d}x\\ +\int_\Omega \frac{K-T_K(\theta^\varepsilon)}{K}\,|Du^\varepsilon|^2\,\mathrm{d}x\le C\left(\int_\Omega \frac{K-T_K(\theta^\varepsilon)}{K}\left|f^\varepsilon(\theta^\varepsilon)\right|^2\mathrm{d}x+\int_\Omega g^2\,\mathrm{d}x\right),\end{aligned} \tag{42}$$

where C is a constant independent of ε and K.

As $\forall\varepsilon>0$

$$\theta^\varepsilon\frac{T_K(\theta^\varepsilon)}{K}\xrightarrow{K\to 0}|\theta^\varepsilon| \quad \text{almost everywhere in } \Omega \text{ and}$$

$$\frac{K-T_K(\theta^\varepsilon)}{K}\xrightarrow{K\to 0} 2\mathbb{1}_{\{\theta^\varepsilon<0\}}+\mathbb{1}_{\{\theta^\varepsilon=0\}} \quad \text{almost everywhere in } \Omega,$$

passing to the limit as K tends to zero in inequality (42) gives that, $\forall\varepsilon>0$,

$$\int_\Omega (u^\varepsilon)^2\,\mathrm{d}x+\int_\Omega |\theta^\varepsilon|\,\mathrm{d}x\le C\left(\int_\Omega \left|f^\varepsilon(\theta^\varepsilon)\right|^2\mathbb{1}_{\{\theta^\varepsilon\le 0\}}\,\mathrm{d}x+\int_\Omega g^2\,\mathrm{d}x\right) \tag{43}$$

where C is constant independent of ε.

Due to assumption (23) on the behavior of f on $\mathbb{R}^-$, $\forall\eta>0$, $\exists C_\eta>0$ such that, $\forall r\in\mathbb{R}^-$, $|f(r)|^2\le\eta|r|+C_\eta$. If we choose η sufficiently small, then inequality (43) implies that there exists $C_1>0$ such that, $\forall\varepsilon>0$,

$$\int_\Omega (u^\varepsilon)^2\,\mathrm{d}x+\int_\Omega |\theta^\varepsilon|\,\mathrm{d}x\le C_1. \tag{44}$$

Remark 5.6. Inequality (43) shows that θ^ε and u^ε are controlled respectively in $L^1(\Omega)$ and in $L^2(\Omega)$ if $f^\varepsilon(\theta^\varepsilon)$ is controlled in $L^2(\Omega)$ only on the subset of Ω where $\theta^\varepsilon \leq 0$. Inequality (43) is also used to prove uniqueness result (see Theorem 6.1).

Step 3. As all the terms in the left hand side of (42) are non negative, one has, $\forall \varepsilon > 0$ and $\forall K > 0$,

$$\begin{aligned}\frac{1}{K}\int_\Omega \left|DT_K(\theta^\varepsilon)\right|^2 \mathrm{d}x &\leq C\left(\int_\Omega \frac{K - T_K(\theta^\varepsilon)}{K}\left|f^\varepsilon(\theta^\varepsilon)\right|^2 \mathrm{d}x + \int_\Omega g^2 \,\mathrm{d}x\right)\\ &\leq C\left(\int_{\{\theta^\varepsilon<0\}} 2\left|f^\varepsilon(\theta^\varepsilon)\right|^2 \mathrm{d}x + \int_{\{0\leq\theta^\varepsilon<K\}} \left|f^\varepsilon(\theta^\varepsilon)\right|^2 \mathrm{d}x + \int_\Omega g^2 \,\mathrm{d}x\right),\end{aligned}$$

since $K - T_K(\theta^\varepsilon) = 0$ almost everywhere on $\{x : \theta^\varepsilon(x) \geq K\}$.

Therefore growth assumption (22) on f and estimate (44) imply that there exists $C_2 > 0$ such that $\forall \varepsilon > 0$, $\forall K > 0$

$$\frac{1}{K}\int_\Omega \left|DT_K(\theta^\varepsilon)\right|^2 \mathrm{d}x \leq C_2\left(\int_{\{0\leq\theta^\varepsilon\leq K\}} \left|f^\varepsilon(\theta^\varepsilon)\right|^2 \mathrm{d}x + 1\right).$$

Let us denote f^* the real-valued function defined by $f^*(r) = \sup_{0\leq r'\leq r}|f(r')|$, $\forall r \in \mathbb{R}^+$. The function f^* satisfies (22) and $\forall \varepsilon > 0$, $\forall K > 0$

$$\frac{1}{K}\int_\Omega \left|DT_K(\theta^\varepsilon)\right|^2 \mathrm{d}x \leq C_2\left(\int_{\{0\leq\theta^\varepsilon<K\}} \left|f^*(\theta^\varepsilon)\right|^2 \mathrm{d}x + 1\right).$$

Since C_1 and C_2 are independent of ε and K, from (43) and the above inequality we can apply Lemma 5.4 to θ^ε, $\forall \varepsilon > 0$. It follows that the sequence θ^ε is bounded in $L^q(\Omega)$ for any $1 \leq q < N/(N-2)$ (and $1 \leq q < +\infty$ if $N = 2$). In particular, growth assumptions (22) and (23) on f imply that

$$f^\varepsilon(\theta^\varepsilon) \text{ is bounded in } \left(L^2(\Omega)\right)^N, \tag{45}$$

and from (41) we obtain that for any $1 \leq p < N/(N-1)$

$$\theta^\varepsilon \text{ is bounded in } W_0^{1,p}(\Omega).$$

By the Rellich Kondrachov Theorem, let θ be a measurable function defined from Ω into $\mathbb{R}$ such that, up to a subsequence, $\forall 1 \leq q < N/(N-2)$ (and $1 < q < +\infty$ if $N = 2$)

$$\theta^\varepsilon \longrightarrow \theta \text{ in } L^q(\Omega) \text{ and almost everywhere in } \Omega, \text{ as } \varepsilon \text{ tends to zero.} \tag{46}$$

Since $(N+2)/N < N/(N-2)$, the continuity of f, growth assumptions (22) and (23) and (46) allow us to deduce, by a standard equiintegrability argument, that

$$f^\varepsilon(\theta^\varepsilon) \longrightarrow f(\theta) \quad \text{in } \left(L^2(\Omega)\right)^N \text{ as } \varepsilon \text{ tends to zero.} \tag{47}$$

Next, using the linear character of equation (35) with respect to u^ε together with (47) it is easy to show that

$$u^\varepsilon \longrightarrow u \quad \text{in } H_0^1(\Omega) \text{ as } \varepsilon \text{ tends to zero,}$$

and then, (u,θ) satisfies equation (1) in $\mathcal{D}'(\Omega)$ with $u \in H_0^1(\Omega)$ and $f(\theta) \in \left(L^2(\Omega)\right)^N$.

It follows that

$$\left(\mathbf{A}(x)Du^\varepsilon - f^\varepsilon(\theta^\varepsilon)\right)\cdot Du^\varepsilon \longrightarrow \left(\mathbf{A}(x)Du - f(\theta)\right)\cdot Du \quad \text{in } L^1(\Omega) \text{ as } \varepsilon \text{ tends to zero.}$$

As far as equation (36) is concerned, the continuity of renormalized solution with respect to the data implies that θ is a renormalized solution of (2).

The proof of Theorem 5.1 is complete. $\square$

We now prove Theorem 5.2.

Proof of Theorem 5.2. Let $\widetilde{f}$ be the function defined by

$$\widetilde{f}(r) = \begin{cases} 0 \text{ if } r \leq r_0, \\ f(r) \text{ if } r > r_0. \end{cases}$$

The function $\widetilde{f}$ is continuous and satisfies assumptions (22) and (23). Making use of Theorem 5.1, let (u, θ) be a weak-renormalized solution of system (1)–(3) with $\widetilde{f}$ in place of f.

Our aim now is to prove that

$$\theta \geq r_0 \quad \text{almost everywhere in } \Omega.$$

For $K > 0$, let H be the function defined by $H(r) = -T_K^-(r - r_0)$, $\forall r \in \mathbb{R}$. We have $H \in W^{1,\infty}(\mathbb{R})$, $H'(r) = \mathbb{1}_{\{-K+r_0<r<r_0\}}$, so H' has a compact support. Since $r_0 \leq 0$, it follows that $H(\theta) \in L^\infty(\Omega) \cap H_0^1(\Omega)$ and recalling that θ is a renormalized solution of (2), Proposition 3.1 with $w = H(\theta)$ leads to

$$\mu \int_\Omega \theta H(\theta)\,\mathrm{d}x + \int_{\{-K+r_0<\theta<r_0\}} \mathbf{a}(x, D\theta) \cdot D\theta\,\mathrm{d}x = -\int_\Omega \big(\mathbf{A}(x)Du - \widetilde{f}(\theta)\big) \cdot DuH(\theta)\,\mathrm{d}x.$$

The definitions of H and $\widetilde{f}$ imply that $\widetilde{f}(r)H(r) = 0$, $\forall r \in \mathbb{R}$, and because $H(r) \leq 0$ the coercivity of $\mathbf{a}$ and $\mathbf{A}$ gives

$$\int_\Omega |\theta|\, T_K^-(\theta - r_0)\,\mathrm{d}x \leq 0.$$

It follows that

$$\theta \geq r_0 \quad \text{almost everywhere in } \Omega,$$

and according to the definition of $\widetilde{f}$,

$$\widetilde{f}(\theta) = f(\theta) \quad \text{almost everywhere in } \Omega.$$

Hence (u, θ) is a weak-renormalized solution of (1)–(3). $\square$

6. Uniqueness results

In this section we assume that

$$f(0) = 0, \tag{48}$$

and we give the following uniqueness result of a small solution (u, θ) of (1)–(3) such that $\theta \geq 0$ almost everywhere in Ω under additional assumptions on f, $\mathbf{a}$ and N.

Theorem 6.1. *Assume that assumptions (A1)–(A7), (22) and (48) hold true. Moreover assume that*

$$N = 2 \quad \text{or} \quad N = 3, \tag{49}$$

$$\big(\mathbf{a}(x,\xi) - \mathbf{a}(x,\xi')\big) \cdot (\xi - \xi') \geq \delta|\xi - \xi'|^2 \quad \text{almost everywhere in } \Omega,\ \forall \xi, \xi' \in \mathbb{R}^N, \tag{50}$$

$$\exists L > 0 \text{ such that } \forall r, r' \in \mathbb{R}^+ \quad |f(r) - f(r')| \leq L|r - r'|. \tag{51}$$

There exists $\eta > 0$ such that if $\|g\|_{L^2(\Omega)} < \eta$, then the weak-renormalized solution (u,θ) of (1)–(3), such that $\theta \geq 0$ almost everywhere in Ω, is unique.

Proof of Theorem 6.1. From Theorem 5.2 let (u_1, θ_1) and (u_2, θ_2) be two weak-renormalized solutions of (1)–(3) such that $\theta_1 \geq 0$ and $\theta_2 \geq 0$ almost everywhere in Ω.

The aim is to prove that

$$\big\|\theta_1 - \theta_2\big\|_{L^2(\Omega)} \leq \omega\big(\|g\|_{L^2(\Omega)}\big)\big\|\theta_1 - \theta_2\big\|_{L^2(\Omega)}, \tag{52}$$

where ω is independent of θ_1 and θ_2 and is such that $\omega(r)$ tends to zero as r tends to zero.

We denote by F_i the term $\big(\mathbf{A}(x)Du_i - f(\theta_i)\big) \cdot Du_i$, for $i = 1, 2$. Proposition 3.3 and (50) give

$$\forall K > 0, \quad \delta \int_{\{|\theta_1 - \theta_2| < K\}} \big|D\theta_1 - D\theta_2\big|^2 \,\mathrm{d}x \leq K \int_\Omega \big|F_1 - F_2\big| \,\mathrm{d}x.$$

From a result of [9], it follows that $T_K(\theta_1 - \theta_2) \in H_0^1(\Omega)$ for any $K > 0$. As $N = 2$ or $N = 3$ there exists $1 < p < N/(N-1)$ such that $p^* = 2$ and so Lemma 3.5 and the above inequality imply that

$$\big\|\theta_1 - \theta_2\big\|_{L^2(\Omega)} \leq C\big\|F_1 - F_2\big\|_{L^1(\Omega)}, \tag{53}$$

where C is a generic constant independent of i and g.

A calculus leads to

$$\begin{aligned}\big\|F_1 - F_2\big\|_{L^1(\Omega)} \leq & \|A\|_{(L^\infty(\Omega))^{N\times N}} \big\|Du_1 - Du_2\big\|_{(L^2(\Omega))^N} \times \big\|Du_1 + Du_2\big\|_{(L^2(\Omega))^N} \\ & + \big\|f(\theta_1) - f(\theta_2)\big\|_{(L^2(\Omega))^N} \times \big\|Du_1\big\|_{(L^2(\Omega))^N} \\ & + \big\|f(\theta_1)\big\|_{(L^2(\Omega))^N} \times \big\|Du_1 - Du_2\big\|_{(L^2(\Omega))^N}.\end{aligned}$$

The linear character of equation (1) gives

$$\begin{gathered}\|Du_1 - Du_2\|_{(L^2(\Omega))^N} \leq C\big\|f(\theta_1) - f(\theta_2)\big\|_{(L^2(\Omega))^N}, \\ \|Du_i\|_{(L^2(\Omega))^N} \leq C\big(\|f(\theta_i)\|_{(L^2(\Omega))^N} + \|g\|_{L^2(\Omega)}\big), \quad \text{for } i = 1, 2.\end{gathered}$$

Using (51) and the above inequalities we obtain

$$\big\|F_1 - F_2\big\|_{L^1(\Omega)} \leq C\big(\|f(\theta_1)\|_{(L^2(\Omega))^N} + \|f(\theta_2)\|_{(L^2(\Omega))^N} + \|g\|_{L^2(\Omega)}\big)\,\big\|\theta_1 - \theta_2\big\|_{L^2(\Omega)}, \tag{54}$$

and therefore (52) reduces to prove that

$$\big\|f(\theta_i)\big\|_{(L^2(\Omega))^N} \leq \omega\big(\|g\|_{L^2(\Omega)}\big), \quad \text{for } i = 1, 2, \tag{55}$$

where ω is independent of i and is such that $\omega(r)$ tends to zero as r tends to zero.

Since $\theta_i \geq 0$ almost everywhere in Ω, (43) implies that

$$\big\|\theta_i\big\|_{L^1(\Omega)} \leq C\|g\|_{L^2(\Omega)} \quad \text{for } i = 1, 2, \tag{56}$$

and from (42) we obtain, for $i = 1, 2$,

$$\forall K > 0 \quad \frac{1}{K} \int_{\Omega} \left| DT_K(\theta_i) \right|^2 \mathrm{d}x \leq C \left(\int_{\{0 \leq \theta_i \leq K\}} \left| f(\theta_i) \right|^2 \mathrm{d}x + \|g\|^2_{L^2(\Omega)} \right). \tag{57}$$

So if $\|g\|_{L^2(\Omega)} \leq 1$, (56) and (57) together with Lemma 5.4 allow us to deduce that, $\forall 1 \leq q < N/(N-2)$ (and $q < +\infty$ if $N = 2$),

$$\left\| \theta_i \right\|_{L^q(\Omega)} \leq C(q), \tag{58}$$

where $C(q)$ is a constant independent of i and g.

Using (22), (48) and (51), let $M > 0$ such that

$$\forall r \in \mathbb{R}^+ \quad |f(r)| \leq M r^{(N+2)/(2N)}.$$

By interpolation between $L^1(\Omega)$ and $L^{(N+1)/(N-1)}(\Omega)$ we have, for $i = 1, 2$,

$$\left\| f(\theta_i) \right\|_{(L^2(\Omega))^N} \leq M \Big(\left\| \theta_i \right\|_{L^{\frac{N+2}{N}}(\Omega)} \Big)^{\frac{N}{2(N+2)}} \leq M \Big(\|\theta_i\|_{L^1(\Omega)} \Big)^{\frac{1}{2N}} \Big(\|\theta_i\|_{L^{\frac{N+1}{N-1}}(\Omega)} \Big)^{\frac{N+1}{2N}},$$

and using (56) and (58) (indeed $\frac{N+1}{N-1} < \frac{N}{N-2}$), if $\|g\|_{L^2(\Omega)} \leq 1$ then we have

$$\left\| f(\theta_i) \right\|_{(L^2(\Omega))^N} \leq C \|g\|^{\frac{1}{2N}}_{L^2(\Omega)}, \tag{59}$$

where C is independent of i and g.

It follows from (53), (54) and (59) that (52) is proved for $\|g\|_{L^2(\Omega)} \leq 1$. Then there exists $\eta > 0$ such that if $\|g\|_{L^2(\Omega)} < \eta$ then $\omega\big(\|g\|_{L^2(\Omega)}\big) < 1$ and (52) implies that $\theta_1 = \theta_2$ almost everywhere in Ω.

The proof of Theorem 6.1 is complete. $\square$

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References

[1] R.A. Adams. *Sobolev Spaces*. Academic Press, 1975.

[2] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J.L. Vazquez. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa*, 22:241–273, 1995.

[3] D. Blanchard and O. Guibé. Existence d'une solution pour un système non linéaire en thermoviscoélasticité. *C.R. Acad. Sci. Paris Sér. I Math.*, 325:1125–1130, 1997.

[4] D. Blanchard and O. Guibé. Existence of a solution for a nonlinear system in thermoviscoelasticity. *Adv. Differential Equations*, 5(10-12):1221–1252, 2000.

[5] L. Boccardo, J.I. Diaz, D. Giachetti, and F. Murat. Existence and regularity of renormalized solutions for some elliptic problems involving derivations of nonlinear terms. *J. Diff. Eq.*, 106:215–237, 1993.

[6] L. Boccardo and T. Gallouët. On some nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, 87:149–169, 1989.

[7] B. Climent and E. Fernández-Cara. Existence an uniqueness results for a coupled problem related to the stationary Navier-Stokes. *J. Math. Pures Appl.*, 76(4), 1997.

[8] G. Dal Maso, F. Murat, L. Orsina, and A. Prignet. Definition and existence of renormalized solutions for elliptic equations with general measure data. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(5):481–486, 1997.

[9] G. Dal Maso, F. Murat, L. Orsina, and A. Prignet. Renormalized solutions of elliptic equations with general measure data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(4):741–808, 1999.

[10] A. Dall'Aglio. Approximated solutions of equations with L^1 data. Application to the H-convergence of parabolic quasi-linear equations. *Ann. Mat. Pura Appl.(4)*, 170:207–240, 1996.

[11] R.J. DiPerna and P.-L. Lions. On the Cauchy problem for Boltzmann equations : global existence and weak stability. *Ann. of Math.*, 130(2):321–366, 1989.

[12] R.J. DiPerna and P.-L. Lions. Ordinary differential equations, Sobolev spaces and transport theory. *Invent. Math.*, 98:511–547, 1989.

[13] T. Gallouët and R. Herbin. Existence of a solution to a coupled elliptic system. *Appl. Math. Letters*, 7:49–55, 1994.

[14] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Springer-Verlag, 1977.

[15] O. Guibé. *Existence de solutions pour des systèmes non linéaires couplés elliptiques ou d'évolution.* PhD thesis, Université de Rouen, 1998.

[16] O. Guibé. Solutions entropiques et renormalisées pour un système non linéaire couplé. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(6):685–690, 1998.

[17] R. Lewandowski. *Analyse mathématique et océanographie.* Masson, Paris, 1997.

[18] P.-L. Lions and F. Murat. Solutions renormalisées d'équations elliptiques. (in preparation).

[19] F. Murat. Soluciones renormalizadas de EDP elipticas non lineales. Technical Report R93023, Laboratoire d'Analyse Numérique, Paris VI, 1993. Cours à l'Université de Séville.

[20] F. Murat. Equations elliptiques non linéaires avec second membre L^1 ou mesure. In *Compte Rendus du 26ème Congrès d'Analyse Numérique*, les Karellis, 1994.