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INFINITE PERIODIC MINIMAL SURFACE: FROM 4 DIMENSIONS TO 3

R. MOSSERI and J.F. SADOC

Laboratoire de Physique des Solides de Bellevue, CNRS, F-92195 Meudon Cedex, France
*Laboratoire de Physique des Solides, Université Paris-Sud et CNRS, F-91405 Orsay Cedex, France

Abstract: We show how the labyrinths, polyhedral approximations and special points of different Infinite Periodic Minimal Surfaces can be obtained from a single regular structure in 4 Dimensions mapped onto different 3 Dimensional planes. More specifically we focus on the P and F cases.

1 - Introduction

A minimal surface is such that, by definition, its mean curvature everywhere vanishes. The theory of minimal surfaces is based on several fundamental results from 19th century mathematicians, among which we can recall Weierstrass and Bonnet /1/. Weierstrass derives the 3D real coordinates of a minimal surface in $\mathbb{R}^3$ from a line integral of elliptic functions in the complex plane. The two spaces are related a pair of mappings, a Gauss map from the real surface to $S^2$ followed by a stereographic projection from $S^2$ to the complex plane. Bonnet had shown earlier how to generate a family of 'associate' minimal surfaces to a given one. In the Weierstrass representation, it amounts to a rotation in the complex plane. An important class of minimal surface is the so-called IPMS (Infinite Periodic Minimal Surfaces /2/), whose first example was first found by Schwarz. They occur in description of the long range organization of cubic liquid crystalline phases formed by amphiphilic molecules. In the case of liquid crystal structures it is interesting to go beyond the pure crystallographic approach and try to derive the physical principles which the geometry. Sadoc and Charvolin have shown that a major ingredient is the presence of 'frustration' /3/4/, which means that incompatible packing constraints forbid the occurrence of an optimal geometry where the energy is everywhere minimized. In the spirit of older models proposed for solving the specific frustration problems in amorphous solids /5/ these authors first define an ideal bilayer structure imbedded in the curved space $S^3$, which is further decurved by disclination 'defects'. Surfaces with topology similar to the IPMS can then be generated. Disclinations carry negative curvature, which agree with the fact that tesselated IPMS can be locally modelled by a tesselated hyperbolic plane /6/.

In the present paper we would like to introduce a complementary, and related, approach to the IPMS geometry. It is proposed that a IPMS in $\mathbb{R}^3$ can be obtained as the projection of a (2D) infinite surface embedded in a higher dimensional space. Our initial hope was to get the real coordinates as the projection of a 'simpler' surface in the hyperspace. While this goal has not yet been accomplished, it will be shown how special points of the surface and their lattice net can be simply obtained. We shall focus on the famous F and P Schwarz surfaces, analyzed here through a mapping from 4 to 3 dimensions. Furthermore, and maybe more promising, these two associates surfaces, related by a Bonnet rotation in the complex plane, are here related by a rigid displacement in 4 Dimensions. The method presented here is still tentative. The link with the frustration free solution in $S^3$ is not fully understood. Perhaps one should embed $S^3$ in $\mathbb{R}^4$ and build the surface by suitably gluing pieces of tori centered at $R^4$ lattice points, as it was suggested (with polytopes instead of tori) for complex quasiperiodic and amorphous structures /7/. Note also that we could
not find the gyroid structure, in between the F and P IPMS. It might be necessary to lift into higher dimensions (some arguments leads to the value 6). Finally we show that the so-called H-T IPMS can also be derived with the same method.

2 - F and P surface : a brief presentation.

The simplest way to image these IPMS, at least topologically, is to start from the simple cubic and diamond networks, with edges connecting the nearest neighbours. Six edges at right angles meet at each vertex of the former and four edges at $109^\circ 28'$ for the latter. Then, one changes the linear edges into cylindrical like structures and inflate the surface. This rough description stresses the role of the so-called labyrinth net associated with an IPMS. F and P surfaces divides $R^3$ into two congruent regions, each being threaded by a labyrinth interlaced with the other. The F surface is built from the minimal surface bounded by the Petrie polygon of a regular tetrahedron (figure 1-a). Six such pieces can be arranged around a vertex and form the piece of a F surface contained in a cube (figure 1-b). This piece is bounded by a 12-gon whose vertices are the mid-edges of the cube. These 12 points, as well as the cube centre, are 'flat' points of the surface (umbilic parabolic points). Also of interest are the six hyperbolic point with highest absolute value of the gaussian curvature (labelled H in figure 1-a) which are connected by a network of 'asymptotic' lines.

![Figure 1](image.png)

Figure 1 : The F surface. a) the piece bounded by the Petrie polygon of a regular tetrahedron . b) six such pieces can be arranged around a vertex and form the piece of a F surface contained in a cube.

3 - The hyperspace approach: From 4D to 3D.

This approach is inspired by previous work on amorphous /5,7/ and quasiperiodic /8/ structures. These complex structures in $R^3$ are described by maps of simpler structures (polytopes) embedded in the curved space $S^3$, or cut from higher dimensional lattices (mainly in 4,5 or 6 dimensions) like the generation of the complicated Penrose-like patterns by projecting hypercubic lattice points along suitable directions (the so-called "Cut and Projection" method /8/). Here we shall focus on the F and P IPMS and show how they are related to projections of 4-Dimensional honeycombs.

3-1 Regular honeycombs in 4 Dimensions

Let us first recall standard notations /9/. \{p\} denotes a regular polygon with p sides, \{p,q\} a regular network such that q p-gons meet at each vertex, \{p,q,r\} the network such that r \{p,q\} share each edge, etc... Therefore \{4,3\} is the ordinary cube, \{4,3,3\} the hypercube (the 'measure' or '8-cell' polytope) and \{4,3,3,4\} the hypercubic lattice $Z^4$. 
Also of interest here are the \{3,3,4\} (the 'cross' or '16-cell' polytope), the \{3,4,3\} (the '24-cell' polytope) which is a self dual polytope with no counterpart in higher dimension, and the \{3,3,4,3\} and \{3,4,3,3\} honeycombs, the former being also a lattice in \( \mathbb{R}^4 \). In terms of coordinates, the polytopes read

\[
\begin{align*}
\{4,3,3\} & : \mathcal{P}(\pm 1, \pm 1, \pm 1, \pm 1) \\
\{3,3,4\} & : \mathcal{P}(\pm 1, 0, 0, 0) \\
\{3,4,3\} & : \mathcal{P}(\pm 1, \pm 1, \pm 1, \pm 1) \text{ and } \mathcal{P}(\pm 1, 0, 0, 0) \\
\end{align*}
\]

where \( \mathcal{P}() \) means all permutations in the set. For the honeycombs

\[
\begin{align*}
\{4,3,3,4\} & : \{n_1, n_2, n_3, n_4\} \quad \text{with } n_i \in \mathbb{Z} \\
\{3,3,4,3\} & : \{n_1, n_2, n_3, n_4\} \quad \text{with } n_i \in \mathbb{Z} \text{ and } \sum n_i : \text{even} \\
\end{align*}
\]

or \( \{n_1, n_2, n_3, n_4\} \) and \( \{n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, n_3 + \frac{1}{2}, n_4 + \frac{1}{2}\} \) with \( n_i \in \mathbb{Z} \) [2]

3-2 Mapping from 4 Dimensions to 3

In the following we shall perform orthogonal mappings onto 3D-planes of \( \mathbb{R}^4 \) as well as single 3D cuts. A 3-Dimensional plane in \( \mathbb{R}^4 \) is simply defined, up to translations, by its normal 4D vector. It is interesting to write the above mappings so that one recovers the coordinates in a 3D orthogonal basis. This is easily done by using quaternions. A quaternion \( q \) is given by \( /10/ \)

\[
q = x_0 + ix_1 + jx_2 + kx_3 \quad \text{with } x_i \in \mathbb{R},
\]

\[
i^2 = j^2 = k^2 = ijk = -1 \text{ and } ij = -ji
\]

\( x_0 \) is called the scalar part and the remaining the vectorial part. A point \( (x_0, x_1, x_2, x_3) \) in \( \mathbb{R}^4 \) is represented by the above quaternion. This is equivalent to representing complex numbers by points in \( \mathbb{R}^3 \) except that here the quaternion multiplication is non commutative. Now let \( r \) be a normed quaternion. It also represents a normed 4D vector in \( \mathbb{R}^4 \). Form the quaternion \( q' = r^{-1}q \). We call this a 'screw' operation along \( r \), even though it is not equivalent to a screw in \( \mathbb{R}^3 \) (rotation plus translation. In fact, it is a combination of two rotations on two orthogonal planes, which forms a rigid displacement leaving no points invariant. The vectorial part of \( q' \) gives the mapping onto a 3D plane normal to \( r \) (in an orthogonal basis) and the scalar part the algebraic distance between the point in \( \mathbb{R}^4 \) corresponding to \( q \) and the 3D-plane through the origin. With \( r = (r_1, r_2, r_3, r_4) \) of unit norm (e.g \( \sum r_i^2 = 1 \)) this is translated in the standard matrix form

\[
M = \begin{pmatrix}
    r_1 & r_2 & r_3 & r_4 \\
    -r_2 & r_1 & r_4 & -r_3 \\
    -r_3 & -r_4 & r_1 & r_2 \\
    -r_4 & r_3 & -r_2 & r_1
\end{pmatrix}
\]

\( M \) acting on a 4D vector gives a 4D vector whose first coordinate corresponds to the scalar part and the last three to the vectorial part.

A quaternion \( q \) can also be written \( q = \rho e^{\alpha y} \) where \( \rho \) is its norm and \( y \) is a unit pure quaternion (i.e with vanishing scalar part). The standard exponential notation of complex numbers is recovered when \( y = 1 \).
3-3 Cutting and projecting the hypercubic lattice along the (1,1,1,1) direction.
Let \( r = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \). A generic point of \( \mathbb{Z}^4 \), \((n_1, n_2, n_3, n_4)\) is transformed into

\[
(h, n_2 + n_3 - h, n_3 + n_4 - h, n_2 + n_4 - h) \quad \text{where} \quad h = \sum n_i / 2.
\]

(5)

The 3-D cut corresponds to a constant \( h \). Therefore, when \( h \) runs along \( \frac{1}{2} \mathbb{Z} \), a set of fcc lattices are obtained, translated along the cube principal diagonal by the vector \((-h, -h, -h)\). Calling \( f_h \) the corresponding fcc lattice it is easy to see that

\[
f_k = f_h \quad \text{if} \quad k = h \mod 4
\]

(6)

Note that since the unit cube in \( R^3 \) has edge-length 2, successive \( f_i \) are displaced by one fourth along the cube diagonal.

4 - The F and P IPMS.

4-1 The P labyrinth: mapping the \( \{3,3,4,3\} \) along \( r = (1,0,0,0) \).

From its second set of coordinates, it is clear that this honeycomb can be viewed as two \( \mathbb{Z}^4 \) lattices displaced from each other by half of the body diagonal. Let us draw edges only between vertices belonging to the same \( \mathbb{Z}^4 \). By rotating along \( r = (1,0,0,0) \) (which amounts here to an identity) and keeping only those points whose scalar part (first coordinate) is \( 0 < x_0 < \frac{1}{2} \), one obtains two interlaced simple cubic lattices in \( R^3 \), i.e. the P labyrinth (figure 2-a).

![Figure 2: a) The P labyrinth: two interlaced simple cubic networks. b) The F labyrinth: two interlaced diamond networks. The sites number refer to Table 1 where the corresponding 4D coordinates are given.](image)

4-2 The F labyrinth: mapping the \( \{3,3,4,3\} \) along \( r = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \).

The same structure as above (in \( R^4 \)) is now rotated toward the (1,1,1,1) direction, and the vertices are kept in the same interval \( 0 \leq x_0 \leq \frac{1}{2} \). Keeping the vectorial part leads to the F labyrinth (figure 2-b): two interlaced diamond networks. Note that since the point with coordinates \( \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \) is a vertex in the \( \{3,3,4,3\} \) lattice, the set of vertices in \( R^3 \) is the same in both cases, but their pre-image in \( R^4 \) (and therefore the edge connections) are different. Therefore the P and F IPMS, which are related by a \( \frac{1}{2} \) Bonnet rotation are also related through a 4 Dimensional rigid displacement. We shall say more about this below.
Table 1: Coordinates of the 24 sites, closest to the origin in the \{3,3,4,3\} 

<table>
<thead>
<tr>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
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<td>0</td>
<td>0</td>
<td>13</td>
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<td>0.5</td>
<td>0.5</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>14</td>
<td>0.5</td>
<td>-0.5</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>15</td>
<td>0.5</td>
<td>-0.5</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>16</td>
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<td>-0.5</td>
</tr>
<tr>
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<td>0</td>
<td>1</td>
<td>0</td>
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<td>0</td>
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<td>0</td>
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<tr>
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<td>0</td>
<td>0</td>
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<td>0.5</td>
</tr>
<tr>
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</tr>
<tr>
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<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>21</td>
<td>-0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>10</td>
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<td>0.5</td>
<td>0.5</td>
<td>-0.5</td>
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<td>-0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>11</td>
<td>0.5</td>
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<td>-0.5</td>
<td>0.5</td>
<td>23</td>
<td>-0.5</td>
<td>-0.5</td>
</tr>
<tr>
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<td>-0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>24</td>
<td>-0.5</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

4 - 3 The P and F polyhedral approximations.

It is well known that some IPMS can be approximated by 3D polyhedra. We now show that they are related by the same 4D rotation as above and are generated from simple arguments. In paragraph 4-1 we decided to single out some of the \{3,3,4,3\} edges. The remaining edges connect points of high symmetry between the interlaced networks. Their mapped midpoints, suitably selected in $\mathbb{R}^4$, should therefore lie on the surface itself. This is what happen if we only keep the the vertices with $x_0 = \frac{1}{4}$. The P case gives the polyhedral approximation by cubes (figure 3-a), and the F case, the approximation by Friauf-Laves polyhedra (truncated tetrahedra) with centered hexagons (figure 3-b).

There is an alternative set in $\mathbb{R}^4$ whose selective mapping should play a role, it is the honeycomb \{3,4,3,3\}, dual of the \{3,3,4,3\}. Indeed, in $\mathbb{R}^4$, the points and edges of the former are, as we have seen, farthest away from the points and edges of the latter, some of which form the labyrinths. The \{3,4,3\} cell surrounding the origin in the \{3,4,3,3\} is obtained from the second set of coordinates given in paragraph 3-1 (rescaled by a factor $\frac{1}{\sqrt{2}}$). Note that this second set is obtained from a left multiplication of the first set by $g = \frac{1}{2\sqrt{2}}(1,1,0,0)$. The vertices of the \{3,4,3\} Voronoi cell are therefore $P(\pm \frac{1}{4}, \pm \frac{1}{2}, 0, 0)$ and fall, by mapping along $(1,0,0,0)$, onto edges of the labyrinth and cannot therefore be used. By contrast, the 24 mid-edges whose scalar part is $x_0 = \frac{1}{4}$ (with coordinates $(\frac{1}{4}, P(\pm \frac{1}{2}, \pm \frac{1}{4}, 0))$) are mapped onto the well-known polyhedral approximation of P by tetrakaidecahedra (Kelvin polyhedra, figure 4-a) glued along the squares. Note that while

Figure 3: a) The P polyhedral approximation with cubes. b) The F polyhedral approximation with truncated tetrahedra.
the P labyrinth vertices pre-image in \( R^4 \) have \( x_0 = 0 \) or \( \frac{1}{2} \), the vertices of the polyhedral approximation have \( x_0 = \frac{1}{4} \), just in-between the former in the fourth dimension.

We do the same analysis in the F case and obtain the polyhedral approximation by tetrakaidecahedra, now glued on half of their hexagons (those threaded through by the labyrinth edges (figure 4-b)).

Figure 4 : a) The P polyhedral approximation with tetrakaidecahedra. The labyrinth edges thread the squares. b) The F polyhedral approximation with tetrakaidecahedra. The labyrinth edges thread half of the hexagons.

4-4 Special points and lines in both surfaces.

The vertices of the polyhedral approximations of P and F by cubes and Laves polyhedra are flat points on the respective surfaces. So all the flat points are simple 3D cuts of a 4D lattice. Some cubes and hexagon diagonals belong to the asymptotic lines network.

5 - Comments and open questions

As noted at the beginning, the present work is preliminary. We now comment about these results and review some of the open questions that proceeds naturally from them, and are of mathematical and physical nature.

(i) Is it possible to go beyond the determination of the different skeletal networks and special points, and obtain the surface itself?

One should first 'dress' the 4D honeycomb with surface patches and then map. We have already tried to use a patch close to the flat torus of \( S^3 \). By mapping along the (1,1,1,1) direction one gets a piece of hyperbolic paraboloid bounded by the Petrie polygon of a regular tetrahedra, which is a well known approximation of the Schwarz true minimal surface. Underlying this attempt is the hope that by lifting in higher dimension one could get a minimal surface of constant negative curvature, which would be the most natural embedding for the intrinsic geometry modelled on \( H^2 \) hyperbolic tesselations /6/. Since we are dealing with 4D polytopes, we might possibly use the related minimal surfaces on \( S^3 \) /11/ as basic patches /12/. One might ask also whether it is possible to apply the present analysis to more common associate minimal surfaces, such as, for instance, the catenoid and the helicoid. It would also be interesting to try an alternative method, also inspired by quasicrystals (the so-called 'Atomic Surface' method). Instead of selecting points and surface patches inside the strip defined by the \( x_0 \) allowed values, one should look for a 3D manifold in \( R^4 \), based on the 4D honeycomb, whose cut by a 3 Dimensional plane will provide the surfaces in \( R^3 \) /13/.

(ii) Can we generate other IPMS by the same method?

In view of the 'associate' character of the P and F IPMS, it is natural to look for the intermediate gyroid surface /2/. This has not yet been achieved and one may have to embed the lattice in higher dimension (maybe 5 or 6). This is a current procedure
in the quasicrystal field in order to get new local environments and symmetries. On the other hand it is also possible to get the so-called H-T IPMS /2/ from 4 Dimensions.

We start with the same lattice as above and with the same subset of edges connecting the vertices. By mapping along the (0,1,1,1) direction (after a suitable selection on $x_0$: $-\frac{1}{\sqrt{5}} \leq x_0 \leq \frac{1}{\sqrt{5}}$), on gets the two interlaced hexagonal and triangular-like networks which form the H-T IPMS labyrinth. It is interesting to use exponental notation for quaternions. Let $y = \frac{1}{\sqrt{5}}(1+j+k)$ and $r = e^{a y}$. Then depending on the value of $\alpha$, we have the following IPMS:

$$
\begin{align*}
\alpha &= 0 \text{ the P surface,} \\
\alpha &= \frac{\pi}{3} \text{ the F surface,} \\
\alpha &= \frac{\pi}{2} \text{ the H - T surface,}
\end{align*}
$$

(8) It should also be possible (but this has not yet been done) to generate the so-called 'complementary surfaces' /2/, since they share the same straight lines network, and even some others IPMS like the CLP, the S'-S" and the I-WP /2/ (by changing $y$ for instance).

It is also known /2/ that beside the exceptional non self-intersecting IPMS, there is an countable infinite set of self-intersecting IPMS. It is very easy to provide a countable infinite set of 3 Dimensional planes in $R^4$ such that the mapping will be periodic (even though probably self-intersecting). It is the set of reticular 3D plane of the $\{3,3,4,3\}$ lattice. Even with the above value of $y = \frac{1}{\sqrt{5}}(1+j+k)$ we can proceed as follow: draw a 2D rectangular lattice with edges 1 and $\sqrt{3}$. Each point (p,q), visible from the origin (p prime with q) defines an angle $\alpha$ such that the corresponding $r$ is perpendicular to a reticular 3-plane of the lattice. The orientations, irrational with the lattice, will define quasiperiodic sets. It would be interesting to know whether there are non self-intersecting infinite quasiperiodic minimal surfaces associated with these sets.

(iii) What is the relation with the Weierstrass and Bonnet representations?

We do not know the answer. There exist a relation between 4D geometry and the complex plane, associated with the celebrated Hopf fibration and map, which might be a possible track to follow. A preliminary inspection of the effect of the 4D rigid displacement seems to show that lines of curvature (resp. asymptotic lines) transform into lines of curvature (resp. asymptotic lines) which suggest a relation to the Goursat transformation /1/. It may be that, using true rotations in $R^4$, instead of screws, will interchange these networks.

(iv) A very different issue concerns the 3 Dimensional nets and polyhedra which are obtained upon mapping. They belong to very large families as described by A.F. Wells /14/. It is clear that rather different sets are just separate facets of a single lattice in higher dimension. This could be generalized to other sets and provide new classification schemes.

(v) At a physical level, our approach might be useful to study the possible topological defects that can occur. Indeed it is possible to define, in the context of the Cut-and-Projection method, standard defects like dislocations as well as new ones like phasons (which correspond translations of the strip in the fourth dimension). In addition new paths for geometrical transformation near phase transitions could be studied. At the level of diffraction studies, the several structure factors in $R^3$ for the different IPMS should be obtained as simple cuts from a unique 4 Dimensional structure factor.

Finally a related field (from the geometrical point of view) is the Blue phases /15/. It is possible to build a frustrated director field on $S^3$ for an ideal blue phase /16/. In
the spirit of the present work, one then ask if there exists a director field in $R^4$ whose cut along different direction would lead to the different blue phase structures. Note that all the proposed structures have disclinations where the field cannot be defined. It could correspond to the locus where the director field in $R^4$ is perpendicular to the 3D plane (providing an 'escape in the 4th dimension' for the disclination).

References

[12] This follows from a discussion with J. Charvolin.
[13] After this conference, S. Hyde has provided us with unpublished work by D.W. Brisson (1976) who also considered 3D IPMS obtained as cuts from a 3 Dimensional hypersurface (the so-called 'Hyper-Schwarz-Surface'). His work seems to correspond to what we have done in the (1,0,0,0) direction. He obtains the $P$ surface as well as a polyhedral approximation of the constant mean curvature surface which shares the $P$ labyrinth. However he does not consider the kind of rotations in 4 Dimension which allows us to get the $F$ surface geometry.