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COUPLED NONLINEAR SCHRODINGER EQUATIONS ARISING IN FIBRE OPTICS

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Abstract - For systems with weak dispersion and cubic nonlinearity, the equation governing fluctuations in amplitude of a signal is usually taken to be the cubic Schrödinger equation. Consequently, this equation appears in the fibre optics literature. However, in axially symmetric waveguides, modes of propagation occur in pairs which have identical dispersion relations. Analysis of a 'single-mode' optical fibre shows that a general quasi-monochromatic signal is governed by two coupled cubic Schrödinger equations describing the interaction between the two independent complex amplitudes. Some exact solutions of this coupled system are discussed, including single-soliton solutions describing either linearly-polarized or circularly-polarized signals. Numerical results showing interaction between linearly-polarized solitons of differing polarization angles are also presented, indicating that after the collision the signal separates into two self-similar solutions which are more general than the usual soliton.

1. Introduction.

Optical fibres are cylindrical glass waveguides capable of transmitting short pulses of quasi-monochromatic light over large distances with high intensity and negligible attenuation. The modes predicted by linear theory are dispersive and, since the leading nonlinear effects are cubic, the standard treatment of nonlinearity in fibre-optics uses the nonlinear Schrödinger (NLS) equation. However, optical fibres, in common with other guiding structures which have cylindrical symmetry, allow both axially symmetric modes and modes which (in linear theory) involve azimuthal dependence \( \cos \ell \theta \) or \( \sin \ell \theta \), for any integer \( \ell \). For each value of \( \ell \) there is a 'dispersion curve' relating the angular frequency \( \omega \) to the wavenumber \( k \). Consequently, at a typical frequency, signals may propagate with various wavenumbers, each having its distinct phase speed and group speed. The designer, wishing to avoid this complication, arranges that all except one of the dispersion curves have 'cut-off frequency' above the operating frequency of the laser source. Such a fibre is called a 'mono-mode' fibre.

However, it is well established (Marcuse /1/, Snyder and Love /2/) that the corresponding mode is not axisymmetric. It is defined by \( \ell = 1 \) and has a real dispersion curve for all real \( \omega \). Consequently, a 'mono-mode' fibre transmits signals which are linear combinations of modes polarized in each of two reference directions perpendicular to the fibre axis. To describe a typical signal, two independent complex amplitudes are required. The variation of these amplitudes over scales much longer than the carrier wavelength or wave period describes the optical signal. The equations governing evolution of these two complex amplitudes form the subject of this paper.

An asymptotic procedure for deriving the two coupled NLS equations from the electromagnetic
field equations is outlined. Some of the effects embodied in these equations are illustrated by some special solutions. Firstly, analysis of general uniform wavetrains confirms how both dispersion and nonlinearity affect the phase speed of linearly polarized signals. Similar observations apply for circularly polarized signals. More generally, an elliptically polarized signal exhibits gradual rotation of the field pattern about the fibre axis - a nonlinearly-induced 'double circular refraction'. Next, it is observed that every linearly polarized signal is governed by a single NLS equation, as is every circularly polarized signal. Consequently, all signals in either of these two cases possess all the soliton properties. This raises the question whether 'perfect' collisions between solitons of two distinct types are possible. Analysis using Hirota's procedure suggests that such collisions are imperfect. To investigate such situations, numerical computations have been performed. These indicate that collisions between two 'sech-profile' pulses of different polarizations yield two localized pulses, but that these are not linearly polarized solitons but appear to be signals of permanent form akin to those termed 'vector solitons' by Christodoulides and Joseph.

Finally, some results concerning other types of similarity solution are summarized.

2. Derivation of the Equations.

The essential feature of an optical fibre is its core, which has radius \( \approx 5 \mu m \) and has speed of light fractionally below that in the surrounding cladding of outer radius \( \approx 50 \mu m \). Since the fields associated with each guided mode decay rapidly outside the core, it is usual to analyse Maxwell's equations in an unbounded region, with dielectric properties depending on the radius \( r \), where \((r, \theta, z)\) are cylindrical polar coordinates. Thus the governing equations are

\[
\nabla \times E = -\mu_0 \frac{\partial H}{\partial t} \quad , \quad \nabla \times H = -\varepsilon_0 \frac{\partial E}{\partial t} \quad , \quad \mu_0 \nabla \cdot H = 0 \quad , \quad \nabla \cdot D = 0 ,
\]

where the fields \( E \), \( D \) and \( H \) are \( o(r^{-1}) \) as \( r \to \infty \) and finite at \( r = 0 \). There is no material magnetization and the electric displacement is taken as

\[
D = \varepsilon_0 E + P = \varepsilon E(f(r) + N(E, r))
\]

Here, \( \varepsilon(r) \) is the permittivity at radius \( r \) and the function \( N \) describes nonlinear effects, typically of order \( \varepsilon(r^2) \). For simplicity, material dispersion is omitted and the material is treated as isotropic.

Linear theory allows both left- and right-handed circularly polarized modes of propagation

\[
E = E^\ell(r) e^{i(\ell \theta + k z - \omega t)} \quad , \quad \ell = 0, 1, 2, \ldots
\]

\[
H = H^\ell(r) e^{i(\ell \theta + k z - \omega t)}
\]

of the form

\[
E^\ell = \bar{E}_1 e_{r} + \bar{E}_\theta e_{\theta} + \bar{E}_z e_{z} \quad , \quad H^\ell = \bar{H}_1 e_{r} + \bar{H}_\theta e_{\theta} + \bar{H}_z e_{z} ,
\]

in which the real functions \( \bar{E}_1 = \bar{E}_1(r) \), \( \bar{H}_1 = \bar{H}_1(r) \) satisfy the differential equations

\[
\frac{\partial \bar{E}_1}{\partial r} - kr \bar{E}_2 - \mu_0 \omega r \bar{H}_1 = 0 \quad , \quad \frac{\partial \bar{H}_1}{\partial r} - kr \bar{H}_2 + \omega r f \bar{E}_1 = 0 ,
\]

\[
\frac{\partial \bar{E}_2}{\partial r} + k \bar{E}_1 - \mu_0 \omega \bar{H}_2 = 0 \quad , \quad \frac{\partial \bar{H}_2}{\partial r} - k \bar{H}_1 - \omega r f \bar{E}_2 = 0 ,
\]

\[
(r \bar{E}_2)' + \phi \bar{E}_1 + \mu_0 \omega r \bar{H}_3 = 0 \quad , \quad (r \bar{H}_2)' - \phi \bar{H}_1 + \omega r f \bar{E}_3 = 0 .
\]

(The equations resulting from the expressions \( \nabla \cdot H \) and \( \nabla \cdot D \) are redundant, since they are consequences of the other equations.)

The condition that the system (4) has nontrivial solutions \( \bar{E}_1, \bar{H}_1 \) satisfying the conditions at \( r = 0 \) and as \( r \to \infty \) is the dispersion relation \( \omega = \omega(k) \) connecting \( \omega \) to \( k \) for specified integer \( \ell \). Since we are considering 'weakly-guiding' mono-mode fibres, we henceforth set \( \ell = 1 \). Then, for \( \omega = \omega(k) \), solutions \( \bar{E}_1(r), \bar{H}_1(r) \) to (4) depend also on the parameter \( k \).

When solutions to (1), (2) are expressed as power series in an amplitude parameter \( \nu \) in the form

\[
E = \nu E^1(r) + \nu^2 E^2(r) + \ldots \quad , \quad H = \nu H^1(r) + \nu^2 H^2(r) + \ldots ,
\]

they have leading order expressions

\[
E^{1} = A^+ E^1(r) e^{i(\theta + \psi)} + A^- E^{-1}(r) e^{i(-\theta + \psi)} + c.c. \quad , \quad \psi = k z - \omega t ,
\]
with a similar expression for $H^1$. Also, $D^{(1)} = \imath\sigma_f(1)$. Here, $\psi$ is a phase variable, c.c. denotes a complex conjugate and the two amplitudes $A^+$ and $A^-$ are complex.

Since the nonlinear term $\mathcal{E}N(\mathcal{E}, r)$ in (2), has magnitude $O(\nu^3)$, any fluctuations in $A^+$ and $A^-$ which take place on the scales of $\nu z$ and $\nu t$ are governed by linear theory. Specifically, it may be shown that

$$\frac{\partial A^\pm}{\partial t} + \omega'(k) \frac{\partial A^\pm}{\partial z} = 0(\nu^2),$$

where $\omega'(k)$ is the group speed of the $l = 1$ modes of wavenumber $k$. This motivates us to introduce scaled variables

$$\chi = \nu(z - \omega'(k)t), \quad Z = \nu^2 z,$$

so allowing us to write $A^\pm$ as

$$A^\pm = A^\pm(\chi, Z),$$

where the functions have bounded derivatives as $\nu \to 0$. Henceforth, fields will be treated as functions of the coordinates $r, \theta, \psi, \chi$ and $Z$, which are strictly $2\pi$-periodic in both $\theta$ and $\psi$. Derivatives are then replaced by

$$\frac{\partial}{\partial z} = -k \frac{\partial}{\partial \psi} + \nu \frac{\partial}{\partial \chi} + \nu^2 \frac{\partial}{\partial Z}, \quad \frac{\partial}{\partial t} = -\omega \frac{\partial}{\partial \psi} - \nu \omega'(k) \frac{\partial}{\partial \chi}. $$

After splitting the fields into their leading order terms and corrections as

$$\mathcal{E} = \nu \mathcal{E}^{(1)} + \nu^2 \mathcal{E}, \quad \mathcal{H} = \nu \mathcal{H}^{(1)} + \nu^2 \mathcal{H}, \quad D = \nu D^{(1)} + \nu^2 D,$$

substitution into equations (1) gives six independent scalar equations, of which a typical example (obtained from the $E_r$ component of (1),) is

$$-\nu^2 \left[ \frac{\partial A^+}{\partial \chi} \frac{\partial E^2}{\partial \psi} + \nu \frac{\partial A^+}{\partial \psi} \frac{\partial E^2}{\partial \chi} + \mu_o \omega (A^+ \frac{\partial A^+}{\partial \chi}) \right] e^{i(\theta + \psi)}$$

$$+ \nu^2 \left[ \frac{\partial A^-}{\partial \chi} \frac{\partial E^2}{\partial \psi} + \nu \frac{\partial A^-}{\partial \psi} \frac{\partial E^2}{\partial \chi} + \mu_o \omega (A^- \frac{\partial A^-}{\partial \chi}) \right] e^{i(-\theta + \psi)} + c.c.$$

$$+ \nu^2 \left[ 1 \frac{\partial E^2}{\partial \psi} - k \frac{\partial E^2}{\partial \psi} - \mu_o \omega \frac{\partial E^2}{\partial \chi} + \nu \frac{\partial E^2}{\partial \chi} - \nu^2 \frac{\partial E^2}{\partial Z} - \nu \omega'(k) \frac{\partial E^2}{\partial \chi} \right] = 0 \quad (6)$$

Then, using $\tilde{D} = \sigma_f(r) \tilde{E} + O(\nu)$, which follows from equation (2) when $N(\nu \mathcal{E}^{(1)}) = O(\nu^2)$, it is seen that the system has the form

$$\nabla' \times \hat{E} - \mu_o \omega \frac{\partial \mathcal{H}}{\partial \psi} - \mathcal{E} = \left( \mathcal{E}^+ \times \mathcal{E}^+ + \mu_o \omega' \mathcal{H}^+ \right) \frac{\partial A^+}{\partial \chi} e^{i(\theta + \psi)}$$

$$+ \left( \mathcal{E}^- \times \mathcal{E}^- + \mu_o \omega' \mathcal{H}^- \right) \frac{\partial A^-}{\partial \chi} e^{i(-\theta + \psi)} + c.c. + O(\nu) \quad , \quad (7)$$

$$\nabla' \times \hat{H} + \omega \sigma_f(r) \frac{\partial \mathcal{E}}{\partial \psi} - \mathcal{F} = \left( \mathcal{H}^+ \times \mathcal{E}^+ - \epsilon \sigma_f(r) \omega' \mathcal{E}^+ \right) \frac{\partial A^+}{\partial \chi} e^{i(\theta + \psi)}$$

$$+ \left( \mathcal{H}^- \times \mathcal{E}^- - \epsilon \sigma_f(r) \omega' \mathcal{E}^- \right) \frac{\partial A^-}{\partial \chi} e^{i(-\theta + \psi)} + c.c. + O(\nu) \quad ,$$

where

$$\nabla' = \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} + k \frac{\partial}{\partial \psi} \quad .$$

A solution to these equations neglecting the $O(\nu)$ terms is

$$\mathcal{E} = -i \frac{\partial A^+}{\partial \chi} \frac{\partial \mathcal{E}^+}{\partial k} e^{i(\theta + \psi)} - i \frac{\partial A^-}{\partial \chi} \frac{\partial \mathcal{E}^-}{\partial k} e^{i(-\theta + \psi)} + c.c. = \mathcal{E}^{(2)}$$

with a similar expression for $\mathcal{H}$. This observation follows from substituting the fields $\mathcal{E}^\pm \exp i(\theta + \psi)$ and $\mathcal{H}^\pm \exp i(\theta + \psi)$ into the linearized form of equations (1), and (1), with $\omega = \omega(k)$ and with $\nabla$ and $\partial / \partial t$ replaced by $\nabla'$ and $-\omega \partial / \partial \psi$. Then, differentiating the resulting equations with respect to $k$ gives
and a similar equation from (I).

Omitting solutions to the homogeneous form of (7) (which correspond merely to adjustments to the amplitudes $A^+$ and $A^-$ in $E^+(t)$) the fields $E$, $H$ and $D$ are now known correctly to $O(\nu^2)$. Consequently, the representation for the exact fields may be improved as

$$
E = \nu E^{(1)} + \nu^2 E^{(2)} + \nu^3 E
$$

with a similar form for $H = \nu H^{(1)} + \nu^2 H^{(2)} + \nu^3 H$ and with $D = \epsilon f(r)(\nu E^{(1)} + \nu^2 E^{(2)}) + \nu^3 D$.

Then, repeating the process which led to equations (7) yields the system

$$
\nabla' \times \left[ \frac{dE^z}{dt} e^{i(\theta + \psi)} \right] - \mu_0 \omega \frac{\partial}{\partial r} \left[ \frac{dH^z}{dt} e^{i(\theta + \psi)} \right]
+ \epsilon \frac{\partial E^z}{\partial r} \ e^{i(\theta + \psi)} - \mu_0 \omega' (k) \ H^z \ e^{i(\theta + \psi)} = 0
$$

However, in this system $\widetilde{D}$ is not linearly related to $\widetilde{E}$. The constitutive relation (2) shows that

$$
\widetilde{D} = \epsilon f(r)\widetilde{E} + \epsilon M(E^{(1)}(r),E^{(2)}(r)) + O(\nu),
$$

where $N(\nu E^{(1)},r) = \nu^2 M(E^{(1)},r) + O(\nu^3)$. Thus equations (8) for $\widetilde{E}$ and $\widetilde{H}$ may be put into a form analogous to (7) as

$$
\nabla' \times \left[ \frac{d\tilde{E}}{dt} e^{i(\theta + \psi)} \right] - \mu_0 \omega \frac{\partial}{\partial r} \left[ \frac{d\tilde{H}}{dt} e^{i(\theta + \psi)} \right]
+ \epsilon \frac{\partial \tilde{E}}{\partial r} \ e^{i(\theta + \psi)} - \mu_0 \omega' (k) \ \tilde{H} \ e^{i(\theta + \psi)} = \tilde{F}
$$

where leading order the quantities $\tilde{F}$ and $\tilde{G}$ are known in terms of the mode distributions $E^z(r), H^z(r)$ and the amplitudes $A^\pm(\chi,z)$, including the nonlinear term $-\omega \epsilon \partial(ME^{(1)})/\partial \psi$ in $F$.

For this system, unlike (7), explicit solutions cannot readily be found. However, this is unnecessary. Equations (9) form an inhomogeneous system for $\tilde{E}$ and $\tilde{H}$, in which $\tilde{F}$ and $\tilde{G}$ are $2\pi$-periodic in both $\theta$ and $\psi$. Solutions are sought which are $2\pi$-periodic in $\theta$ and $\psi$, bounded at $r = 0$ and which decay as $r \to \infty$. This boundary value problem has a compatibility condition on the source-like terms $\tilde{F}$ and $\tilde{G}$, which is readily shown to be

$$
\int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \left( \tilde{P} \cdot \tilde{E} - \tilde{Q} \cdot \tilde{G} \right) \ r \ d\theta \ d\psi \ dr = 0
$$

where

$$
\tilde{P} = \alpha_1 E^+(r) e^{i(\theta + \psi)} + \alpha_2 E^-(r) e^{i(-\theta + \psi)} + c.c.
$$

$$
\tilde{Q} = \alpha_1 H^+(r) e^{i(\theta + \psi)} + \alpha_2 H^-(r) e^{i(-\theta + \psi)} + c.c.
$$

forms the general solution to the equivalent homogeneous problem. When condition (10) is applied with $\tilde{F}$ and $\tilde{G}$ approximated to leading order, only those contributions from which the complex exponential cancels give a non-zero contribution to the integral (10). Then, since $\alpha_1$ and $\alpha_2$ are arbitrary complex constants, it is found that
together with a similar equation for $A^-$. In the standard case, with permittivity quadratic in $|E|^2$, so that $M(E(\cdot),r) = m(r)|E|^2$, it is found that the two propagation equations have the form

$$\begin{align*}
\frac{\partial A^+}{\partial \tau} &+ \int_0^\infty (E^+ \times H^+ + E^+ \times H^+^*) \cdot \varepsilon_z \, r \, dr \\
- i \frac{\partial^2 A^+}{\partial \chi^2} &+ \int_0^\infty \left\{ E^+ \times \frac{dH^+}{dk} + \frac{dE^+}{dk} \times H^+^* \right\} \cdot \varepsilon_z - \varepsilon f \omega E^+ \times \frac{dE^+}{dk} - \mu \omega H^+^* \times \frac{dH^+}{dk} \right\} \, r \, dr \\
- \omega e \frac{2\pi}{\varepsilon} \int_0^\infty \int_0^\infty e^{-i(\theta + \psi)} \frac{\partial}{\partial \psi} (ME(\cdot)) \, d\theta \, d\psi \cdot E^+ \times \tau \, r \, dr = 0
\end{align*}$$

where the real coefficients $f_1, f_2, f_3$ and $g$ are given by

$$\begin{align*}
f_1 &= 2 \int_0^\infty \left( \tilde{H}_2 \tilde{E}_2 - \tilde{H}_1 \tilde{E}_1 \right) \, r \, dr \\
g &= \frac{f_1^2 \omega (k)}{2 \omega^2 (k)} \\
f_2 &= -\omega e \int_0^\infty \left\{ (\tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3)(\tilde{E}_3) \right\} m(r) \, r \, dr \\
f_3 &= 2\omega e \int_0^\infty \left\{ (\tilde{E}_1 + \tilde{E}_2 - \tilde{E}_3)(\tilde{E}_3) \right\} m(r) \, r \, dr
\end{align*}$$

After scaling the independent variables by defining

$$\tau = \frac{f_2}{f_1} \tau \quad , \quad x = \sqrt{\frac{f_2}{f_1}} \chi \quad , \quad h = \frac{f_2}{f_1} \, h$$

these equations may be written as

$$\frac{\partial^2 A^\pm}{\partial \tau^2} = \frac{\partial^2 A^\pm}{\partial \chi^2} + (|A^\pm|^2 + h|A^\pm|^2)A^\pm$$

a symmetric pair of coupled nonlinear Schrödinger equations for the complex amplitudes of the two circularly polarized modes.

A number of generalizations of the system (14) have appeared in the nonlinear optics literature. In particular, Blow et al. /5/ include an extra term $kA$ on the right-hand side, to account for small differences in the phase speeds of linearly polarized modes in a 'birefringent fibre'. However, until a recent analysis of the 'step-index' fibre (Newboult, Parker and Faulkner, submitted for publication) it has not been observed that in axi-symmetric fibres, two coupled equations are still needed.

Although an analysis in terms of two orthogonal linearly polarized modes is possible, the resulting equations (obtained by setting $B_1 = i(A^+ + A^-)$, $B_2 = A^+ - A^-$)

$$\begin{align*}
i \frac{\partial B_1}{\partial \tau} &= \frac{\partial^2 B_1}{\partial \chi^2} + \frac{h+1}{4} \left( iB_1^1 \, i^2 + iB_2^1 \, i^2 \right) B_1 + \frac{h-1}{4} \left( B_2^2 B_1^1 - iB_1^2 B_2^1 \right) \\
i \frac{\partial B_2}{\partial \tau} &= \frac{\partial^2 B_2}{\partial \chi^2} + \frac{h+1}{4} \left( iB_1^2 \, i^2 + iB_2^2 \, i^2 \right) B_2 + \frac{h-1}{4} \left( B_2^2 B_1^2 - iB_1^2 B_2^2 \right)
\end{align*}$$

are not as tractable as (14).

3. Interpretation of some Special Solutions.

3.1 Uniform Waves

It is readily seen that equations (14) possess solutions in which both $A^+$ and $A^-$ are periodic in $x$ and $\tau$, of the form

$$A^+ = a_1 \exp i(c_1 x + (c_1, c_1^2 - i a_1, 1^2 - h i a_1, 1^2) \tau)$$

$$A^- = a_2 \exp i(c_2 x + (c_2, c_2^2 - i a_2, 1^2 - h i a_2, 1^2) \tau)$$

where $a_1, a_2$ (complex) and $c_1, c_2$ (real) are arbitrary constants. Inserting these into (5) and using the definitions (13) of $x$ and $\tau$ shows that the solution describes two superposed circularly
polarized modes, with angular frequencies

\[ \omega_1 = \omega + \nu \omega'(k) \left( \frac{f_2}{g} \right) \frac{1}{4} c_1, \quad \omega_2 = \omega + \nu \omega'(k) \left( \frac{f_2}{g} \right) \frac{1}{4} c_2 \]

and with wavenumbers given by

\[ k_1 = k + \nu \left( \frac{f_2}{g} \right) \frac{1}{4} c_1 + \nu^2 \left( \frac{f_2}{f_1} \right)^2 \left( c_1^2 - 1 a_1^2 - h a_2^2 \right) \]

and a similar equation for \( k_2 \).

These expressions demonstrate how the phase speeds are affected by the amplitudes \( |a_1| \) and \( |a_2| \). Also it is readily seen, by Taylor expansion in \( k_1 - k \) and \( k_2 - k \), that

\[ \frac{g}{f_1} = \frac{-\omega''(k)}{2\omega'(k)} \]

an expression for the group delay dispersion which is consistent with the definitions (12) and the dispersion relation. By setting \( a_2 = 0 \), it is seen that \( f_2/f_1 \) may be regarded as the nonlinear susceptibility for circularly polarized modes. The parameter \( h = f_3/f_2 \) may be regarded as a coupling parameter between the right- and left-handed modes.

Alternatively, \( (h+1)f_2/f_1 = (f_3 + f_2)/f_1 \) is the nonlinear susceptibility for (suitably scaled) linearly polarized modes, since the case \( c_1 = c_2 = 0, a_1 = a e^{i2\alpha_1}, a_2 = a e^{i2\alpha_2} \) gives

\[ k_1 = k - (h + 1) \frac{f_2}{f_1} \nu^2 a^2 \]

corresponding to waves with radial field having the fundamental approximation

\[ r E^{(1)} \cdot g_{\theta} = -4\nu a E_\theta(r) \cos(\theta + \alpha_1 - \alpha_2) \sin(kz - \omega t + \alpha_1 + \alpha_2) \]

which has fixed polarization angle \( \alpha_2 - \alpha_1 \).

More generally, when \( a_1 = a e^{i2\alpha_1}, a_2 = b e^{i2\alpha_2} (b \neq a) \), the field pattern of a monochromatic mode \( c_1 = c_2 = 0 \) rotates as it propagates, since

\[ r E^{(1)} \cdot g_{\theta} = -2\nu (b + a) E_{\phi}(r) \cos(\theta + \rho z + \alpha_1 - \alpha_2) \sin(kz - \omega t + \alpha_1 + \alpha_2) \]

\[ + 2\nu (b - a) E_{\phi}(r) \sin(\theta + \rho z + \alpha_1 - \alpha_2) \cos(kz - \omega t + \alpha_1 + \alpha_2) \]

with

\[ k = k - \frac{1}{4} (h + 1) \nu^2 (a^2 + b^2) \frac{f_2}{f_1}, \quad \rho = \frac{1}{4} (h - 1) \nu^2 (a^2 - b^2) \frac{f_2}{f_1} \]

By analogy with elliptically polarized plane waves in which the radial component of \( E \) has the form

\[ A \cos \theta \sin(kz - \omega t) + B \sin \theta \cos(kz - \omega t) \]

with \( A \) and \( B \) arbitrary constants, these modes are called elliptically polarized. The rotation rate of the field pattern is given by \( \rho \), which is analogous to the ellipse rotation rate for plane waves discussed by Maker, Terhune and Savage [6]. Since the phenomenon is associated with differing phase speeds for left- and right-handed modes, it may be regarded as a nonlinearly-induced double circular refraction.

We emphasize here that the parameters \( f_1, f_2, f_3 \) and \( g \) are completely defined, via equations (12), in terms of the fields within the propagating modes governed by linear theory and by the coefficient \( m(r) \) in the nonlinear part of the constitutive law. The nonlinear effects are not approximated by some ad hoc averaging over the fibre cross-section. We note also that, although the designer can vary the coupling constant \( h \) by altering \( m(r) \), the identity

\[ f_3 = 2f_2 - 16\omega e \int_0^\infty \tilde{E}_2^2 \tilde{E}_3^2 m(r) r dr \]

together with the 'weak guiding' result \( \tilde{E}_3^2 << \tilde{E}_1^2 + \tilde{E}_2^2 \), shows that \( h \) is usually close to the value 2.

3.2 Circular Polarization

Whenever \( A^- = 0 \), equations (14) reduce to

\[ i \frac{\partial \xi^+}{\partial \tau} = \frac{\partial^2 \xi^+}{\partial \tau^2} + iA^1 \xi^+ + A^1 A^+ \xi^+ \quad \text{,} \]  

(15)
which is the NLS equation for which many solution procedures and many explicit solutions are known. In particular, the equation possesses soliton and multi-soliton solutions, exhibiting the familiar property that solitons emerge from collisions without change of amplitude, form or propagation speed.

The simplest of these solutions is the single soliton, which may be written as

$$A^+ = a /2 e^{-1/2} \text{sech}(x - 2\sqrt{\gamma})$$

This solution corresponds to

$$\nu \Phi(r) \cdot e^r = -2 /2r a \text{sech}(\bar{k}z - \tilde{\omega}t) \hat{E}_1(r) \sin(\theta + \bar{k}z - \tilde{\omega}t - \beta),$$

where

$$\bar{k} = \nu \left[ \frac{\bar{f}_x}{\bar{f}_y} \right]^{1/2} - 2r^2 \nu \frac{f_x}{f_y}, \quad \tilde{\omega} = \nu \left[ \frac{\bar{f}_x}{\bar{f}_y} \right]^{1/2} \omega'(k),$$

Expression (16) describes a sech-profile envelope soliton for circularly polarized signals. A similar expression for $$A^-$$, with $$A^+ = 0$$, describes right-handed envelope solitons.

### 3.3. Linear Polarization

Whenever $$A^- = A^+ e^{i2\alpha}$$, both of the equations (14) are satisfied provided that

$$\frac{\partial A^+}{\partial r} = \frac{\partial^2 A^+}{\partial x^2} + (h + 1) A^+ a^2 A^+, \quad (h + 1) A^+ a^2 A^+,$$

which is a trivial rescaling of the NLS equation (15). The single soliton solution analogous to (16) is

$$A^+ = \sqrt{\frac{2}{h + 1}} e^{-1/2} \text{sech}(x - 2\sqrt{\gamma})$$

with $$\gamma$$ given as before so that the radial field has the approximation

$$\nu \Phi(r) \cdot e^r = -4a \sqrt{\frac{2}{h + 1}} \hat{E}_1(r) \cos(\theta - \alpha) \text{sech}(\bar{k}z - \tilde{\omega}t) \sin(\bar{k}z - \tilde{\omega}t + \alpha - \beta), \quad (17)$$

where $$\bar{k}, \tilde{\omega}, \bar{\kappa}, \tilde{\omega}$$ are defined as in §3.2. This describes a sech-profile envelope soliton of a linearly-polarized field having fixed orientation angle $$\alpha$$ relative to the fibre. The formulae for the phase speed of the carrier signal and for the soliton speed $$\tilde{\omega}\bar{k}$$ are identical to those in §3.2, while the magnitude of the radial field is scaled from (16) by the factor $$2(h + 1)^{-1}$$.

Although it is clear that any number of solitons having the same polarization angle $$\alpha$$ may collide and then emerge as perfect solitons, it is not obvious what is the result of collisions between solitons of differing polarization angles. This situation is discussed in the next section.

### 4. Soliton Interactions

The possibility that solitons satisfying the coupled NLS equations (14) might collide perfectly has been investigated using the method of Hirota [13]. Solutions are sought in the form $$A^+ = p/q, \quad A^- = r/s$$, with $$p, q, r$$ and $$s$$ being suitable sums of exponentials. Analysis leads to the conclusion that exact two-soliton solutions exist only in two cases: $$r = 0$$ (or $$p = 0$$) giving circular polarization (§3.2); or $$A^- = A^+ \exp i2\alpha$$ giving linear polarization (§3.3).

Numerical integration of equations (14) has been performed with 'initial conditions' chosen at $$\tau = 0$$ to agree with

$$A^+ = \sum_{j=1, 2} \gamma_j e^{-i(\beta_j(x - x_j) - (\beta_j - \gamma_j^2)\tau)} \text{sech} \gamma_j(x - x_j - 2\beta_j \tau), \quad (18)$$

Expressions (18) describe the linear superposition of two solitons of amplitudes $$\gamma_1, \gamma_2$$, soliton speeds $$2\beta_1, 2\beta_2$$, polarization angles $$\alpha_1, \alpha_2$$ and centred 'initially' at $$x = x_1$$ and $$x = x_2$$, respectively. Although the superposition (18) does not satisfy equations (14), a choice of parameters satisfying $$\gamma_1(x_2 - x_1) > 1, \gamma_2(x_2 - x_1) > 1, \beta_1 > 0 > \beta_2$$, describes two initially well-separated solitons converging upon each other. Computations have been performed using an
Fig. 1 Collision between two linearly polarized solitons of the same amplitude and polarization angle. $|A^+| (-|A^-|)$ as a function of phase $x$ and distance $r$.

Fig. 2. Collision between two solitons of equal amplitude $\gamma_1 = \gamma_2 = 1$ but polarized with $\alpha_1 = 0, \alpha_2 = \frac{1}{4}\pi$. Note the differences between the interaction ripples of $|A^+| (\text{Fig. 2a})$ and of $|A^-|$ (Fig. 2b).

An explicit scheme with coupling parameter $h = 2.5$ and with the other parameters chosen as

$\gamma_1 = \gamma_2 = 1, \ \beta_1 = -\beta_2 = 4, \ x_1 = 10, \ x_2 = 25, \ \alpha_1 = 0$

while the second polarization angle $\alpha_2$ has been taken as $\alpha_2 = 0, \frac{1}{4}\pi, \frac{1}{4}\pi, \frac{1}{4}\pi$. Steplengths in the numerical discretization were $\Delta x = 5 \times 10^{-2}, \Delta r = 5 \times 10^{-4}$, giving results which showed good agreement when the mesh was refined to $\Delta x = 4 \times 10^{-2}, \Delta r = 2.5 \times 10^{-4}$.

Figures 1–3 show only the moduli $|A^+|$ and $|A^-|$ of the complex amplitudes $A^+$ and $A^-$.
associated with the two circularly polarized modes. Figure 1 shows the case $\alpha_1 = 0 = \alpha_2$, for which it is found that $A^+ = A^-$, for all $(x, \tau)$. This shows the perfect collision between two linearly polarized solitons having equal polarization angles (§3.3). The oscillations in $|A^\pm|$ are associated with the difference $\tau(\beta_2 - \beta_1)$ in the wavenumbers of the two solitons, as arise also in the explicit two-soliton representation of Satsuma and Yajima /7/. In the two cases $\alpha_2 = \frac{1}{4}\pi$ and $\alpha_2 = \frac{1}{2}\pi$, the behaviour is broadly identical. Figures 2a and 2b show profiles of $|A^+|$ and $|A^-|$ respectively at intervals $\tau = 0(0.125)2.5$. Perhaps surprisingly, the solitons emerge from the collision as two humps, with no apparent radiation. However, there is a breaking of symmetry. In Figure 2a it may be seen that in the faster ($\beta_1$) hump $|A^+|$ has increased, while in the slower ($\beta_2$) hump it has decreased. In Figure 2b the situation is reversed. Indeed, throughout the integration range each profile of $|A^-|$ is obtained from that of $|A^+|$ by reflection in $x = \frac{1}{2}(x_0 + x_1)$.

When $\alpha_2 = \frac{1}{4}\pi$, so that initially the solitons are polarized orthogonally, both humps emerge with $|A^+| = |A^-|$ (and within 1% of the initial value). However, careful comparison of Figures 3a and 3b shows that profiles of $|A^+|$ and $|A^-|$ are quite distinct in the interaction region.

These computations suggest that collisions between linearly polarized solitons produce two envelope pulses of permanent form, without radiation. However, these pulses are linearly polarized sech-profiles only in the cases $\alpha_2 - \alpha_1 = 0, \frac{1}{4}\pi$. The situation is analogous to that found by Cadet /8/ for multicomponent solitons in atomic lattices, where elliptically polarized solitons emerge when $\alpha_2 - \alpha_1 \neq 0, n\pi/2$.

The pulses of permanent form are special cases of similarity solutions investigated by Parker and Torrisi (unpublished, see also Parker /9/) using the procedure of Bluman and Cole /10/. This shows the following possibilities for a similarity variable $\sigma$ for the system (14):

\begin{align*}
1. & \quad \sigma = x - 2\sqrt{\tau} \\
2. & \quad \sigma = \tau \\
3. & \quad \sigma = x - \sqrt{\tau^2} \\
4. & \quad \sigma = \tau^{-\frac{1}{2}}(x - 2\sqrt{\tau}) \\
5. & \quad \sigma = \tau^{-\frac{1}{4}}x
\end{align*}
Only the possibility \( \sigma = x - 2V\tau \) is relevant here. Besides a description of uniform wavetrains (§3.1), it allows the possibility

\[
A^+ = i \, F_1(\sigma) \exp -i(\beta_1 \tau + \Delta_1 \sigma + \delta_1) \quad , \quad A^- = i \, F_2(\sigma) \exp -i(\beta_2 \tau + \Delta_2 \sigma + \delta_2) ,
\]

where the equations governing the (real) amplitudes \( F_1, F_2 \) reduce to

\[
F_1''(\sigma) + (F_1^2 + hF_2^2 + V^2 - \beta_1)F_1 = 0 ,
\]

\[
F_2''(\sigma) + (hF_1^2 + F_2^2 + V^2 - \beta_2)F_2 = 0 .
\]

Clearly the cases \( F_2 = 0 \) or \( F_1 = 0 \) describe circularly polarized solitons, while \( F_2 = F_1 \) (with \( \beta_2 = \beta_1 \)) describes linearly polarized solitons. The symmetric humps seen in Figure 2 should correspond to bounded even solutions of (19). These have been computed for \( F_2(0)/F_1(0) = 1.3413 \), corresponding to the value \( iA^+_{\text{max}}/iA^-_{\text{max}} \) found in the faster pulse emerging from the interaction in the case \( \alpha_1 = 0, \alpha_2 = \frac{3}{4} \pi \). Careful search for the parameters \( C_1 = (\beta_1 - V^2)/F_1^2(0), C_2 = (\beta_2 - V^2)/F_1^2(0) \) yields the graphs shown in Figure 4, when \( C_1 = 2.91516 \) and \( C_2 = 2.05710 \). (It may be noted here that profiles have also been found in which both \( F_1(\sigma) \) and \( F_2(\sigma) \) are even functions, but with graphs which cross the axis before decaying. Analogous solutions in which \( F_1 \) is odd but \( F_2 \) is even have been termed vector solitons by Christodoulides and Joseph /4/.)

![Fig. 4. Bounded symmetric envelope profiles for \( F_1(\sigma) \) and \( F_2(\sigma) \) satisfying equations (19) when \( F_2(0)/F_1(0) = 1.3413 \).](image)

5. Conclusions.

It is emphasized that even in perfectly axisymmetric 'mono-mode' optical fibres, signals are governed by two coupled NLS equations possessing a high degree of symmetry. These equations allow signals which are either linearly polarized or circularly polarized. In each case, signal modulation is governed by the usual NLS equation, for which the familiar soliton properties hold. Except in these cases, Hirota analysis suggests that soliton collisions are not perfect. However, computation of interactions between linearly polarized solitons having distinct polarization angles indicates that self-similar pulses emerge. Such pulses are just very special cases of solutions of the coupled NLS equations which may be investigated using similarity analysis.

REFERENCES

/8/ Cadet, S., in this issue.
/9/ Parker, D.F., to appear in Proc. 5th Meeting, Waves and Stability in Continuous Media, (Editel, Cosenza, 1988)