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Abstract

Transorthogonality for a sequence of polynomials on $\mathbb{R}^d$ has been recently introduced in order to characterize the reference probability measures, which are multivariate distributions of the natural exponential families (NEFs) having a simple cubic variance function. The present paper pursues this characterization of three various manners through exponential generating functions, transdiagonality of Bhattacharyya matrices and semigroup-Sheffer systems, respectively. The obtained results extend those well-known of simple quadratic NEFs based on the classical orthogonality of associated polynomials. The transorthogonality property is then compared to the 2-pseudo-orthogonality one which globaly characterizes the cubic NEFs. Finally, some techniques of calculation of these polynomials are presented and then illustrated on a multivariate normal inverse Gaussian Lévy process.

Key words: Bhattacharyya matrix, exponential generating function, Lévy process, natural exponential family, normal inverse Gaussian distribution, Sheffer polynomial, variance function, 2-pseudo-orthogonality.

MSC: 60G50, 62E10, 42C05.

1 Introduction

It is well-known that the orthogonality property of real polynomials $(Q_n)_{n\in\mathbb{N}}$ with respect to a probability measure μ provides six categories of polynomials: Hermite, Charlier, Laguerre, Krawchouk, Meixner and Pollaczek. The associated probability measures are also of six types: Gaussian, Poisson, gamma, binomial, negative binomial and hyperbolic cosine, respectively. Independently, the class of these probability measures μ is the set of real natural exponential families (NEFs) having

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quadratic variance functions (see Morris [22]), while the set of the orthogonal polynomials has been characterized in four different ways by several authors. Indeed, Meixner [21] has been the first one who characterized these orthogonal polynomials using the criterion of exponential generating functions. Then, Shanbhag ([34], [35]) proposed some characterizations based on the diagonality of infinite Bhattacharyya matrices. Feinsilver [6] built the orthogonal polynomials from successive derivatives of probability density. Finally, Schoutens and Teugels [33] introduced a connection to stochastic processes with some applications (see, also, Schoutens [31] and [32]). From the Casalis [4] description of the multivariate simple quadratic NEFs generalizing the Morris [22] class on $\mathbb{R}^d$, Pommeret ([24], [25] and [26]) has extended all the previous results on orthogonal polynomials in multivariate cases. However, the simple quadratic NEFs are not the only multivariate NEFs which have quadratic variance functions. In the global situation of multivariate quadratic NEFs, Pommeret proposed in its papers a notion of pseudo-orthogonality for completing the above characterizations. Note here that Lancaster [18] obtained chraracterization of marginal distributions in NEF by the use of bi-orthogonal sequences of polynomials.

In order to continue the characterization of (multivariate) NEFs with respect to their associated polynomials, the classical orthogonality may be extended. In this spirit, Hassairi and Zarai [8] introduced the 2-pseudo-orthogonality for characterizing the real NEFs with cubic variance functions which have been described by Letac and Mora [20]. The characterizations of the univariate cubic NEFs are done both in the Meixner [21] and Feinsilver [6] ways; see Hassairi and Zarai [9] for the Shanbhag ([34], [35]) one. Kokonendji ([12] and [13]) considered a notion of k-pseudo-orthogonality ($k \in \{2, 3, \cdots\}$) in order to characterize, in all the four ways (*i.e.*, Meixner [21], Feinsilver [6], Shanbhag [35] and Schoutens and Teugels [33]), the univariate NEFs having real polynomial variance functions of exact degree $2k-1$. Let us quote two remarkable subclasses of real NEFs with polynomials variance functions: Hinde-Demétrio's class described recently by Kokonendji *et al.* [14] and Tweedie's class (see, *e.g.*, Jørgensen [11]) also called power variance functions which contain all stable distributions. In higher dimensions ($d > 1$), for instance, only homogeneous and simple quadratic NEFs of Casalis [4] and simple cubic NEFs of Hassairi [7] are completely described. Hence, more recently, Hassairi and Zarai [10] provided the characterization of simple cubic NEFs by transorthogonality property, which is a bit different to the multivariate extension of 2-pseudo-orthogonality (see Definition 3.1). This result is simply presented following the only one way of Feinsilver [6]. In the other respects, Kokonendji and Pommeret [15] obtained a characterization of multivariate NEFs with lth degree polynomial variance functions ($l \in \{1, 2, 3, \cdots\}$) via another notion of l-orthogonality of some associated polynomials.

The aim of this paper is to pursue the characterization of simple cubic NEFs by transorthogonality given in [10] following the three other directions: Meixner [21], Shanbhag ([34], [35]) and Schoutens and Teugels [33]). We also compare the transorthogonality to the 2-pseudo-orthogonality which is related to the global cubic NEFs on $\mathbb{R}^d$. We finally illustrate some results on a multivariate normal inverse Gaussian NEF. Let us mention that there exist other extensions of orthogonality in the literature, like quasi-orthogonality of a certain order and another l-orthogonality (*e.g.*, [5]). Hence, this paper can be considered as a review and we organize it as follows. In Section 2, we recall some relevant materials of simple cubic NEFs on $\mathbb{R}^d$ and their associated polynomials. Section 3 presents the origin of transorthogonality and compares it to the 2-pseudo-orthogonality. Sections 4, 5 and 6 characterize simple cubic NEFs by transorthogonality in the sense of Meixner [21], Shanbhag ([34], [35]) and Schoutens and Teugels [33]), respectively. Section 7 is devoted to illustrate some results with a multivariate normal inverse Gaussian Lévy process.

2 Simple cubic exponential families

In this section, we first recall some basic properties of NEFs and their associated polynomials. We then conclude by a presentation of the simple cubic NEFs.

Througout the paper, we denote by $(e_1, \cdots, e_d)$ the canonical basis of $\mathbb{R}^d$. In order to simplify expressions, it is convenient to use some classical multidimensional notation. If $n = (n_1, \cdots, n_d) \in \mathbb{N}^d$ and $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, then $|n| = \sum_{i=1}^{d} n_i$, $n! = \prod_{i=1}^{d} (n_i!)$ and $x^n = \prod_{i=1}^{d} x_i^{n_i}$. A polynomial $P_n(x)$ in $x \in \mathbb{R}^d$ with degree $n \in \mathbb{N}^d$ is written as

$$P_n(x) = \sum_{q \in \mathbb{N}^d; |q| \leq |n|} \alpha_q x^q,$$

where at least one of the real coefficent α_q is nonnull when $|q| = |n|$.

2.1 Natural exponential families (NEFs)

Let $\mathcal{M}(\mathbb{R}^d)$ denotes the set of σ-finite positive measures μ on $\mathbb{R}^d$ (not necessarily a probability and) not concentrated on an affine subspace of $\mathbb{R}^d$, with the cumulant transform of μ given by

$$K_\mu(\theta) = \log \int_{\mathbb{R}^d} \exp(x^t\theta)\, \mu(dx) \leq \infty$$

and such that the interior $\Theta(\mu)$ of the domain $\{\theta \in \mathbb{R}^d; K_\mu(\theta) < \infty\}$ is nonempty. The NEF generated by $\mu \in \mathcal{M}(\mathbb{R}^d)$, denoted by $F = F(\mu)$, is the family of probability measures

$$P(\theta; \mu)(dx) = \exp\left\{x^t\theta - K_\mu(\theta)\right\} \mu(dx), \quad \theta \in \Theta(\mu). \qquad (2.1)$$

Any NEF can be reparametrized in terms of the mean m such that

$$m = m(\theta) = \mathbb{E}_\theta(X) = K'_\mu(\theta) = \int_{\mathbb{R}^d} x P(\theta; \mu)(dx),$$

where X is a random vector distributed according to a $P(\theta; \mu)$ in F. The mean domain $M_F = K'_\mu(\Theta(\mu))$ depends only on F, and not on the choice of the generating measure μ of F; so we can write $F = \{P(m, F)\,; m \in M_F\}$. The function

$$V_F(m) = Var_{\theta_\mu(m)}(X) = K''_\mu(\theta_\mu(m)) = \left[\theta'_\mu(m)\right]^{-1}, \quad m \in M_F$$

is called the *variance function* of the family F. Here $\theta_\mu(\cdot)$ denotes the inverse of the mapping $m(\theta) = K'_\mu(\theta)$. The pair $(V_F(\cdot), M_F)$ characterizes F within the class of all NEFs. See, for examples, Letac [19] and Kotz *et al.* [16, Chap. 54] for more details.

Let us also recall two elementary transformations which preserve any type of NEF. The first one is the affinity. Let $\varphi(x) = Ax+b$ where A is in the linear group $GL(\mathbb{R}^d)$ of $\mathbb{R}^d$ and b is in $\mathbb{R}^d$. Then, for any NEF $F = F(\mu)$ on $\mathbb{R}^d$, one has

$$\varphi(F) = F(\varphi(\mu)), \;\; M_{\varphi(F)} = \varphi(M_F) \;\text{ and }\; V_{\varphi(F)}(m) = AV_F(\varphi^{-1}(m))^t A. \tag{2.2}$$

The second one is the positive power of convolution. If μ is in $\mathcal{M}(\mathbb{R}^d)$, let us introduce the Jørgensen set

$$\Lambda = \Lambda(\mu) = \{t > 0; \exists \mu_t \in \mathcal{M}(\mathbb{R}^d) : \Theta(\mu_t) = \Theta(\mu) \text{ and } K_{\mu_t}(\cdot) = tK_\mu(\cdot)\}. \tag{2.3}$$

Denoting $F_t = F(\mu_t) = F(\mu^{*t}) = F^{*t}$ where t is in Λ and $*$ means the product of convolution, one has

$$M_{F_t} = tM_F \;\text{ and }\; V_{F_t}(m) = tV_{F_t}\left(\frac{m}{t}\right). \tag{2.4}$$

We conclude this subsection by the notion of reductibility. A NEF F on $\mathbb{R}^d$ is said to be reducible if there exist an integer $k < d$ and two NEFs F_1 on $\mathbb{R}^k$ and F_2 on $\mathbb{R}^{d-k}$ such that $F = F_1 \times F_2$. In this case, $M_F = M_{F_1} \times M_{F_2}$ and $V_F(m_1, m_2) = V_{F_1}(m_1) \otimes V_{F_2}(m_2)$.

2.2 Polynomials associated to a NEF

Let $F = \{P(m, F)\,; m \in M_F\}$ be a NEF on $\mathbb{R}^d$ and let $\mu = P(m_0, F)$ with m_0 fixed in M_F. From (2.1), the density $f_\mu(\cdot, m)$ of $P(m, F)$ with respect to $\mu(dx)$ is given by $f_\mu(x, m) = \exp\{x^t\theta_\mu(m) - K_\mu(\theta_\mu(m))\}$ with $f_\mu(\cdot, m_0)$ equals to 1. The Taylor expansion in m of the analytic function $m \mapsto f_\mu(x, m)$ in a neighborhood of m_0 is

$$f_\mu(x, m) = \exp\left\{x^t\theta_\mu(m) - K_\mu(\theta_\mu(m))\right\} = \sum_{n \in \mathbb{N}^d} \frac{(m - m_0)^n}{n!} P_n(x),$$

where for all $n \in \mathbb{N}^d$,

$$P_n(x) = \left.\frac{\partial^{|n|}}{\partial m^n} f_\mu(x, m)\right|_{m=m_0} = f_\mu^{(n)}(x, m_0)(e_1, \cdots, e_d) \tag{2.5}$$

is a polynomial in x of degree $|n|$. These polynomials expansions $(P_n)_{n\in\mathbb{N}^d}$ associated with a NEF belong to the class of Sheffer's [36] polynomials and they are such that, for all $m \in M_F$,

$$P_0(x,m) = 1 \text{ and } P_1(x,m) = \left[\theta'_\mu(m)(x-m)\right]\ ^t(e_1,\cdots,e_d).$$

More generally, we have the following proposition showing the derivative of f_μ in another basis of $\mathbb{R}^d$ (see [24, Lemma 2.2 and Theorem 2.1]).

Proposition 2.1 *Let $F = F(\mu)$ be a NEF on $\mathbb{R}^d$ and let (m_0, A) be in $M_F \times GL(\mathbb{R}^d)$. Consider the polynomials $(P_{A,n})_{n\in\mathbb{N}^d}$ defined by*

$$P_{A,n}(x) = P_{A,n}(x,m_0) = f_\mu^{(n)}(x,m_0)(Ae_1,\cdots,Ae_d), \tag{2.6}$$

i.e. the $|n|$th derivative of $m \mapsto f_\mu(x,m)$ at the mean m_0 of F in the $|n| = n_1+\cdots+n_d$ directions: Ae_1 (n_1 times), ..., Ae_d (n_d times). Then:
(i) $P_{A,n}(x) = f^{(n)}_{A^{-1}(\mu)}(A^{-1}x, A^{-1}m_0)(e_1,\cdots,e_d)$;
(ii) there exists an open ball $B(m_0,r) = \{m \in M_F \subseteq \mathbb{R}^d; |m_i - m_{0i}| < r, \forall i = 1,\cdots,d\}$ of $M_F \subseteq \mathbb{R}^d$ such that, for all $(m,x) \in B(m_0,r)\times\mathbb{R}^d$,

$$f_\mu(x,Am) = \sum_{n\in\mathbb{N}^d} \frac{(m-A^{-1}m_0)^n}{n!} P_{A,n}(x);$$

(iii) $P_{A,n}(x)$ is a polynomial in x of degree $|n|$ and $(P_{A,n})_{n\in\mathbb{N}^d}$ forms a basis of the space of all polynomials on $\mathbb{R}^d$.

Observe that if $A = I_d$ is the identity matrix in (2.6) then $P_{I_d,n} = P_n$ is clearly the polynomial defined by (2.5). In particular situation, one has explicit expressions of $P_n(x)$ (*e.g.*, [24]) or some recurrence relations of certain terms (*e.g.*, [15]). A general calculation of the sequence of polynomials $P_n(x) = P_n(x,m_0)$ can be done by mean of the Faà di Bruno formula (*e.g.*, Savits [30]) as follows:

$$P_n(x) = \sum_{\substack{1\le\lambda\le|n|\\ \Omega(n,\lambda)}} (n!) \prod_{j=1}^{q} \frac{1}{(k_j!)\,[l_j!]^{k_j}} \left(\frac{\partial^{|l_j|}}{\partial m^{l_j}} \left\{x^t\theta_\mu(m) - K_\mu(\theta_\mu(m))\right\}\Big|_{m=m_0} \right)^{k_j}, \tag{2.7}$$

where $q = q(n) = \left[\prod_{s=1}^{d}(n_s+1)\right] - 1$ and

$$\Omega(n,\lambda) = \left\{(k_1,\cdots,k_q;l_1,\cdots,l_q);(k_j,l_j)\in\mathbb{N}\times\mathbb{N}^d : \lambda = \sum_{i=1}^{q} k_i \text{ and } n = \sum_{i=1}^{q} k_i l_i\right\}.$$

2.3 Simple cubic NEFs

The cubic variance function of a NEF F on $\mathbb{R}^d$ can be defined as follows:

$$V_F(m) = M_3(m,m,m) + M_2(m,m) + M_1(m) + M_0, \tag{2.8}$$

where $M_3(m,m,m)$ ($M_2(m,m)$, $M_1(m)$ and M_0) is a real symmetric $(d\times d)$ matrix of trilinear (bilinear, linear and constant) elements in $m \in M_F \subseteq \mathbb{R}^d$. The quadratic variance function is obviously given with $M_3 = 0$ in (2.8) and the simple quadratic one is also considered from (2.8) with $M_3 = 0$ and $M_2(m,m) = \alpha\ {}^t mm$, $\alpha \in \mathbb{R}$ [4]. The *simple cubic* NEFs on $\mathbb{R}^d$ (denoted by $\mathcal{M}_3(\mathbb{R}^d)$) have been defined in Hassairi and Zarai ([8], [10]) as the NEFs on $\mathbb{R}^d$ obtained from the simple quadratic ones (denoted by $\mathcal{M}_2(\mathbb{R}^d)$) by the action of the linear group $\mathcal{G} = GL(\mathbb{R}, \mathbb{R}^d) \equiv GL(\mathbb{R}^{d+1})$ on the NEFs of $\mathbb{R}^d$. That is

$$\mathcal{M}_3(\mathbb{R}^d) = \mathcal{G}\left[\mathcal{M}_2(\mathbb{R}^d)\right]. \tag{2.9}$$

Indeed, for the simplicity, an element g of $\mathcal{G}$ is defined by its blocks $(\alpha, \beta, \gamma, \delta)$ in $\mathbb{R} \times \mathbb{R}^d_* \times \mathbb{R}^d \times \mathcal{L}(\mathbb{R}^d)$ where $\mathcal{L}(\mathbb{R}^d)$ is the space of endomorphisms on $\mathbb{R}^d$ and $\mathbb{R}^d_* \equiv \mathbb{R}^d$ denotes the dual of the linear vector space $\mathbb{R}^d$ with dimension $d < \infty$. Following Hassairi and Zarai's [10] notations, its respective actions on $\mathbb{R} \times \mathbb{R}^d$ and $\mathbb{R} \times \mathbb{R}^d_*$ are defined by

$$(x_0, x) \mapsto g(x_0, x) = \left(x_0\alpha + \beta^t x, x_0\gamma + \delta(x)\right)$$

and

$$(k, \theta) \mapsto g(k, \theta) = \left(k\alpha + \gamma^t\theta, k\beta + \delta^*(\theta)\right),$$

where δ^* is the adjoint of δ. Also, we denote

$$d_g(m) = \alpha + \beta^t m \ \text{ and } \ h_g(m) = (d_g(m))^{-1}(\gamma + \delta(m).$$

If O is an open set of $\mathbb{R}^d$ and g in $\mathcal{G}$, we write

$$O_g = \left\{m \in \mathbb{R}^d; d_g(m) > 0 \ \text{ and } \ h_g(m) \in O\right\}.$$

Hassairi [7] has shown that if m is in $\mathbb{R}^d$ such that $d_g(m) \neq 0$, then the differential $h'_g(m)$ of h_g at m is an isomorphism of $\mathbb{R}^d$.

Let $g \in \mathcal{G}$ and let O be a nonempty open set of $\mathbb{R}^d$ such that $O_g \neq \{0\}$. For $V : O \mapsto \mathcal{L}_s(\mathbb{R}^d_*, \mathbb{R}^d)$ ($\mathcal{L}_s(\mathbb{R}^d_*, \mathbb{R}^d)$ is the space of the symmetric linear maps from $\mathbb{R}^d_*$ to $\mathbb{R}^d$), we define $T_gV : O_g \mapsto \mathcal{L}_s(\mathbb{R}^d_*, \mathbb{R}^d)$ by

$$(T_gV)(m) = (d_g(m))^{-1}\left[h'_g(m)\right]^{-1} V(h_g(m))\left[h'_g(m)\right]^{*-1}.$$

When $d = 1$ the action of an element g of $\mathcal{G}$ on a real NEF F is given by

$$(T_gV_F)(m) = \frac{(\alpha + \beta m)^3}{(\alpha\delta - \beta\gamma)^2} V_F\left(\frac{\gamma + \delta m}{\alpha + \beta m}\right).$$

In particular, if $\alpha = 1$ and $\beta = 0$ then the image F_1 of F by the affinity $x \mapsto \gamma + \delta(x)$ satisfies $V_{F_1} = T_gV_F$ (2.2). Also if we have $\alpha \in \Lambda = \Lambda(F)$ (2.3), $\beta = 0 = \gamma$ and $\delta(x) = x$, then T_g corresponds to the power transformation with parameter α (2.4).

Let $\mathcal{G}_0$ be the subgroup of $\mathcal{G}$ whose the elements are such that $\beta = 0$ and $\alpha > 0$. An element g_0 of $\mathcal{G}_0$ may be written as a product of a power transformation (2.4) and an affine transformation (2.2). All the descriptions of NEFs on $\mathbb{R}^d$ are done up to affinity (2.2) and power transformation (2.4), that is up to $\mathcal{G}_0$-orbits. See Table 1 for $d = 1$ in (2.9).

Table 1 about here.

More generally, $\mathcal{M}_2(\mathbb{R}^d)$ contains $(2d+4) - \mathcal{G}_0$-orbits that one can interpret the distributions from the random variables $(X_1, \cdots, X_d)$ as follows (see Casalis [4] for more details). The $d+1$ Poisson-Gaussian $\mathcal{G}_0$-orbits are such that all X_i are independent, $X_1, \cdots, X_k$ have Poisson distributions and $X_{k+1}, \cdots, X_d$ are Gaussian variables with variance 1. The $d+1$ negative multinomial-gamma $\mathcal{G}_0$-orbits are such that the vectors $(X_1, \cdots, X_k)$ have negative multinomial distribution, the conditional variable $X_{k+1}|(X_1, \cdots, X_k)$ is gamma distributed with shape parameter $\sum_{i=1}^{k} X_i + 1$ and $(X_{k+2}, \cdots, X_d)|(X_1, \cdots, X_k)$ is a Gaussian vector with variance $diag(X_1, \cdots, X_{k+1})$. The hyperbolic $\mathcal{G}_0$-orbit is such that $(X_1, \cdots, X_{d-1})$ has a negative multinomial distribution and $X_d|(X_1, \cdots, X_{d-1})$ has an hyperbolic cosine distribution. The last $\mathcal{G}_0$-orbit of simple quadratic NEFs is composed by the classical multinomial vector. Consequently, for all $g \in \mathcal{G}$ and $F \in \mathcal{M}_2(\mathbb{R}^d)$, $(T_g V_F)(m)$ is a polynomial in m of degree less than or equal to 3; *i.e.*, $\mathcal{M}_3(\mathbb{R}^d) \supseteq \mathcal{G}\left[\mathcal{M}_2(\mathbb{R}^d)\right]$. The converse inclusion of (2.9) is given by Hassairi [7], which thus described the class of simple cubic NEFs $\mathcal{M}_3(\mathbb{R}^d)$ in $(d+3) - \mathcal{G}$-orbits.

3 Transorthogonality

In this section, we present in Theorem 3.2 below the first characterization of the multivariate simple cubic NEFs (in the Feinsilver [6] way) which allowed to introduce the transorthogonality [10]. Then, we compare the transorthogonality to the 2-pseudo-orthogonality which are two extensions of the classical orthogonality of a sequence of polynomials on $\mathbb{R}^d$. To conclude we give recurrence relations for computing these polynomials.

Let us first define the two extensions of orthogonality that we need. Then, the map $x = (x_1, \cdots, x_d) \mapsto \|x\|_+ = \max\left(-\sum x_i^-, \sum x_i^+\right)$ defines a norm on $\mathbb{R}^d$ such that for $n \in \mathbb{N}^d$, $\|n\|_+ = |n|$.

Definition 3.1 *A sequence $(Q_n)_{n\in\mathbb{N}^d}$ of polynomials on $\mathbb{R}^d$ is said to be transorthogonal (2-pseudo-orthogonal) with respect to a probability measure ν and denoted $\nu-$transorthogonal (ν − 2-pseudo-orthogonal) if, for all n and q in $\mathbb{N}^d$,*

$$\int_{\mathbb{R}^d} Q_n(x)Q_q(x)\nu(dx) = 0 \quad \textit{when } \|n-q\|_+ \geq \inf(|n|;|q|) \qquad (|n| \geq 2|q|).$$

Transorthogonality and 2-pseudo-orthogonality coincide for $d = 1$ and it has provided some characterizations of real cubic NEFs (see [8] and [9]). See Kokonendji ([12] and [13]) for several generalizations on $\mathbb{R}$ of the standard orthogonality. For multivariate cases ($d > 1$), the transorthogonality property has been characterized by Hassairi and Zarai [10] in the following sense. (See Pommeret [24] for the orthogonality and the corresponding 1-pseudo-orthogonality).

Theorem 3.2 *Let F be an irreductible NEF on $\mathbb{R}^d$ and let (m_0, A) be in $M_F \times GL(\mathbb{R}^d)$. Then: F is simple cubic with $A^{-1}V_F(m_0)^tA^{-1}$ diagonal if and only if the family of polynomials $(P_{A,n})_{n\in\mathbb{N}^d}$ defined by (2.6) is $P(m_0, F)$-transorthogonal.*

Proof. The basic case $A = I_d$ (identity matrix) is proved in [10, Theorem 3.1, pp. 76-89]. The general case follows from Proposition 2.1. □

Remark 3.3 *There exists an analog of Theorem 3.2 for cubic NEFs on $\mathbb{R}^d$ (2.8) with respect to the 2-pseudo-orthogonality. Its proof is similar and we omit it.*

The following proposition is a criterion to get the transorthogonality of polynomials on $\mathbb{R}^d$ from the 2-pseudo-orthogonality.

Proposition 3.4 *Let F be an irreductible NEF on $\mathbb{R}^d$ and let (m_0, A) be in $M_F \times GL(\mathbb{R}^d)$. Let $(P_{A,n})_{n\in\mathbb{N}^d}$ be the sequence of polynomials defined by (2.6). Then the two following statements are equivalent:*
(i) the polynomials $(P_{A,n})_{n\in\mathbb{N}^d}$ are $P(m_0, F)-2$-pseudo-orthogonal and $(P_{A,n})_{n\in\mathbb{N}^d,|n|\in\{1,2,3\}}$ are $P(m_0, F)$-transorthogonal;
(ii) the polynomials $(P_{A,n})_{n\in\mathbb{N}^d}$ are $P(m_0, F)$-transorthogonal.

Proof. $(i) \Leftarrow (ii)$ is obvious. For $(i) \Rightarrow (ii)$, according to Theorem 3.2, it suffices to show that the NEF F is simple cubic with $A^{-1}V_F(m_0)^tA^{-1}$ diagonal. This fact rises immediately from the thansorthogonality of $(P_{A,n})_{n\in\mathbb{N}^d}$ for only $|n| \in \{1, 2, 3\}$ by using the same argument in the proof of Theorem 3.2 (see also [24, Proposition 4.1] for a similarity). □

It seems not easier to point out all expressions of the transorthogonal polynomials $(P_{A,n})_{n\in\mathbb{N}^d}$ for $d > 1$ via (2.7). However, one can use the following reccurence relations which are proved in Kokonendji and Pommeret [15, Theorem 3.1 with $k = 3$].

Proposition 3.5 *Let F be a NEF on $\mathbb{R}^d$ and let m_0 be in M_F. Consider the polynomials $P_n(x) = P_n(x, m_0)$ defined by (2.5). Then the following items are equivalent:*
(i) the variance function $V_F(m) = (V_{i,j}(m))_{i,j\in\{1,\cdots,d\}}$, $m \in M_F$, has the following 3rd order form:

$$V_{ij}(m) = \sum_{q\in\mathbb{N}^d;|q|\leq 3} \alpha_{ij}(q)(m - m_0)^q,$$

for some reals α_{ij} with $\alpha_{ij}(0) = V_{ij}(m_0)$;
(i) the polynomials $P_n(x)$ satisfy

$$x_iP_n(x) = \sum_{j=1}^{d} \left\{ \alpha_{ij}(0)P_{n+e_j}(x) + \sum_{0<|q|\leq 3} \frac{n!\alpha_{ij}(q)}{(n-q)!} P_{n+e_j-q}(x) \right\} + n_iP_{n-e_i}(x) + m_{0i}P_n(x),$$

with the convention $n!/(n - q)! = 0$ if $n - q \notin \mathbb{N}^d$;

(i) For all $n, q \in \mathbb{N}^d$ such that $|q| = 2$ and $|n| = 4$, we have

$$\int P_q(x)P_n(x)P(m_0, F)(dx) = 0.$$

In this case, coefficients α_{ij} coincide in (i) and (ii).

4 Exponential generating function

This section is devoted to our first characterization of the transorthogonal polynomials on $\mathbb{R}^d$ with respect to its generating function having the exponential property, like Meixner [21] for the orthogonal real polynomials. We thus show in Corolary 4.3 below that the reference probability measures of the transorthogonality are also constituted by the simple cubic NEFs on $\mathbb{R}^d$ (2.9). Recall that similar results for other classes of NEFs can be found in [12] and in [24].

Definition 4.1 *A generating function of a sequence of polynomials $(Q_n)_{n\in\mathbb{N}^d}$ on $\mathbb{R}^d$ is said to be exponential if there exist an open ball $B(0, r)$ of $\mathbb{R}^d$ and two analytic functions $a : B(0, r) \to \mathbb{R}^d$ and $b : B(0, r) \to \mathbb{R}$ such that, for all $(z, x) \in B(0, r) \times \mathbb{R}^d$,*

$$\sum_{n\in\mathbb{N}^d} \frac{z^n}{n!} Q_n(x) = \exp\left\{x^t a(z) + b(z)\right\}.$$

Here is the main result of the section for which it is trivial to deduce the univariate case ($d = 1$) given in [12, Theorem 2] as a particular situation.

Theorem 4.2 *Let $F = F(\mu)$ be an irreductible NEF on $\mathbb{R}^d$ and let m_0 be in M_F. Let $(Q_n)_{n\in\mathbb{N}^d}$ be a sequence of $P(m_0, F) - 2$-pseudo-orthogonal polynomials on $\mathbb{R}^d$ such that Q_n is of degree $|n|$. Then the two following statements are equivalent:*
(i) the generating function of $(Q_n)_{n\in\mathbb{N}^d}$ is exponential;
(ii) there exists A in $GL(\mathbb{R}^d)$ such that, for all $n \in \mathbb{N}^d$,

$$Q_n(x) = Q_0(x)P_{A,n}(x) = Q_0(x)f_\mu^{(n)}(x, m_0)(Ae_1, \cdots, Ae_d).$$

In this case, for all $z \in B(0, r)$, we have $\sum_{n\in\mathbb{N}^d} \frac{z^n}{n!} Q_n(x) = \exp\{x^t a(z) + b(z)\}$ with $a(z) = \theta_\mu(Az + m_0)$ and $b(z) = -K_\mu(\theta_\mu(a(z)))$.

Proof. Up to consider $\widetilde{Q}_n = Q_n/Q_0$, we can assume $Q_0 = 1$.

$(i) \Leftarrow (ii)$ By Part *(ii)* of Proposition 2.1, there exists an open ball $B(0, r)$ of $\mathbb{R}^d$ such that, for all $(z, x) \in B(0, r) \times \mathbb{R}^d$, $f_{A^{-1}(\mu)}(x, z + A^{-1}m_0) = \sum_{n\in\mathbb{N}^d} \frac{z^n}{n!} Q_n(Ax)$. From Part *(i)* of Proposition 2.1, $Q_n(Ax) = f^{(n)}_{A^{-1}(\mu)}(x, A^{-1}m_0)(e_1, \cdots, e_d)$. Thus, we successively have

$$\sum_{n\in\mathbb{N}^d}\frac{z^n}{n!}Q_n(x) = f_{A^{-1}(\mu)}(A^{-1}x, z+A^{-1}m_0)$$
$$= f_\mu(x, Az+m_0)$$
$$= \exp\left\{x^t\theta_\mu(Az+m_0) - K_\mu(\theta_\mu(Az+m_0))\right\}.$$

Consequently, Part *(i)* follows with $a(z)=\theta_\mu(Az+m_0)$ and $b(z)=-K_\mu(\theta_\mu(a(z)))$.

$(i)\Rightarrow(ii)$ Let $\nu = P(m_0, F)$. From the $\mu - 2$-pseudo-orthogonality of $(Q_n)_{n\in\mathbb{N}^d}$, we get

$$\int\left(\sum_{n\in\mathbb{N}^d}\frac{z^n}{n!}Q_n(x)\right)\nu(dx) = \int\left(\sum_{n\in\mathbb{N}^d}\frac{z^n}{n!}Q_n(x)\right)Q_0(x)\nu(dx)$$
$$= \int Q_0^2(x)\nu(dx)$$
$$= 1.$$

On the other hand, the exponential generating function associated to $(Q_n)_{n\in\mathbb{N}^d}$ (Part (i)) allows to write

$$\int\left(\sum_{n\in\mathbb{N}^d}\frac{z^n}{n!}Q_n(x)\right)\nu(dx) = \int\exp\left\{x^ta(z)+b(z)\right\}\nu(dx)$$
$$= \exp\left\{K_\mu(a(z))+b(z)\right\}.$$

Hence,
$$b(z) = -K_\mu(\theta_\mu(a(z))). \tag{4.1}$$
Letting $Q(x) = {}^t(Q_{e_1}(x),\cdots,Q_{e_d}(x))$ and proceeding similarly, we obtain that for all $i\in\{1,\cdots,d\}$

$$\int\left(\sum_{n\in\mathbb{N}^d}\frac{z^n}{n!}Q_n(x)Q_{e_i}(x)\right)\nu(dx) = \sum_{\|n\|_+\le\inf(|n|;0)}\left(\int Q_n(x)Q(x)\nu(dx)\right)\frac{z^n}{n!}$$
$$= \left(\int Q_{e_i}^2(x)\nu(dx)\right)z_i \tag{4.2}$$
$$= \left(\int Q(x)^tQ(x)\nu(dx)\right)z.$$

The polynomial vector $Q(x)$ is of degree 1 in x; therefore, there exists $B\in GL(\mathbb{R}^d)$ and $c\in\mathbb{R}^d$ such that
$$Q(x) = Bx+c. \tag{4.3}$$
In fact, we have $c=\int Q(x)Q_0(x)\nu(dx) - Bm_0 = -Bm_0$ and

$$\int Q(x)^tQ(x)\nu(dx) = \int B(x-m_0)^t(x-m_0)^tB\nu(dx)$$
$$= BV_F(m_0)^tB. \tag{4.4}$$

Furthermore, it follows from (4.1), (4.2) and (4.3) that

$$\begin{aligned}\left(\int Q(x)^t Q(x)\nu(dx)\right) z &= \int \exp\left\{x^t a(z) + b(z)\right\} Q(x)\nu(dx)\\ &= B\int (x-m_0)\exp\left\{x^t a(z) - K_\mu(\theta_\mu(a(z)))\right\}\nu(dx)\\ &= B\left[K'_\mu(a(z)) - m_0\right],\end{aligned}$$

and we deduce that (4.4) can be written as

$$BV_F(m_0)^t Bz = B\left[K'_\mu(a(z)) - m_0\right].$$

Therefore, $K'_\mu(a(z)) = V_F(m_0)^t Bz + m_0$, that is $a(z) = \theta_\mu(V_F(m_0)^t Bz + m_0)$ and, finally, we obtain

$$\sum_{n\in\mathbb{N}^d} \frac{z^n}{n!} Q_n(x) = f_\mu(x, V_F(m_0)^t Bz + m_0).$$

Thus, setting $A = V_F(m_0)^t B$, we have $Q_n(x) = f_\mu^{(n)}(x, m_0)(Ae_1, \cdots, Ae_d)$. □

Corollary 4.3 *Let F be an irreductible NEF on $\mathbb{R}^d$ and let m_0 be in M_F. Then: there exists a family of $P(m_0, F)$-transorthogonal polynomials on $\mathbb{R}^d$ with an exponential generating function if and only if F is simple cubic.*

Proof. It follows from Theorem 4.2, Theorem 3.2 and Proposition 3.4. □

5 Multidimensional Bhattacharyya matrices

In this section, we introduce a notion of transdiagonality for a multidimensional Bhattacharyya matrix in order to give another characterization of the transorthogonal polynomials, which are related to the simple cubic NEFs on $\mathbb{R}^d$ (2.9). This is a new mutidimensional extension of the Shanbhag ([34], [35]) characterization for the real orthogonal polynomials, which are connected to the quadratic NEFs [22]. See [9], [12, Section 4] and [25] for other extensions of Shanbhag's results.

Let $F = F(\mu) = \{P(\theta;\mu); \theta \in \Theta(\mu)\}$ be a NEF on $\mathbb{R}^d$ generated by μ (2.1). Any $\mathcal{C}^\infty$ diffeomorphism h from an open set I of $\mathbb{R}^d$ into $\Theta(\mu)$ provides a new parametrization of F:

$$F = \{P(h(z);\mu); z \in I\},$$

where the density of $P(h(z);\mu)$ with respect to $\mu(dx)$ is given by

$$g_\mu(x,z) = \exp\{x^t h(z) - K_\mu(h(z))\} = f_\mu(x, K'_\mu(h(z))). \tag{5.1}$$

Then, for all $n \in \mathbb{N}^d$ and $A \in GL(\mathbb{R}^d)$, the function on $\mathbb{R}^d \times I$

$$S_{A,n}(x,z) = \frac{1}{g_\mu(x,z)}\ g_\mu^{(n)}(x,z)(Ae_1, \cdots, Ae_d) \tag{5.2}$$

is a polynomial in x of degree $|n|$ and independent of the choice of the generator μ of F.

For all $(z, A) \in I \times GL(\mathbb{R}^d)$, we then call *(multidimensional) Bhattacharyya matrix* the infinite matrix $B_A(z) = (B_{A;n,m}(z))_{n,m\in\mathbb{N}^d}$ where

$$B_{A;n,m}(z) = \int S_{A,n}(x,z)S_{A,m}(x,z)P(h(z);\mu)(dx). \tag{5.3}$$

For all $(k,l) \in \mathbb{N}^2$, we denote the submatrices of $B_A(z)$ by

$$B_A^{k,l}(z) = (B_{A;n,m}(z))_{n,m\in\mathbb{N}^d;|n|=k,|m|=l}\,. \tag{5.4}$$

Definition 5.1 *A d-dimensional infinite matrix $B = (B_{n,m})_{n,m\in\mathbb{N}^d}$ is said to be transdiagonal (2-pseudodiagonal) if, for all $(n,m) \in \mathbb{N}^d \times \mathbb{N}^d$ such that $\|n-m\|_+ \geq \sup(|n|,|m|)$ $\quad(|n| \geq 2|m|)$, $B_{n,m} = 0$.*

Remark 5.2 *The Bhattacharyya matrix $B_A(z) = (B_{A;n,m}(z))_{n,m\in\mathbb{N}^d}$ is transdiagonal (2-pseudodiagonal) if and only if the polynomials $(S_{A,n}(x,z))_{n\in\mathbb{N}^d}$ are $P(h(z);\mu)$-transorthogonal ($P(h(z);\mu) - 2$-pseudo-orthogonal).*

First, we show the result for 2-pseudodiagonality. To simplify, we assume $A = I_d$ (identity matrix) in (5.2), (5.3) and (5.4) and we thus denote $S_n(x,z) = S_{I_d,n}(x,z)$ and $B(z) = B_{I_d}(z)$.

Theorem 5.3 *Let $F = F(\mu)$ be an irreductible NEF on $\mathbb{R}^d$. If $h : I \to \Theta(\mu)$ is a $\mathcal{C}^\infty$ parametrization of F such that $h'(z)$ is invertible for all $z \in I$ and $B(z) = B_{I_d}(z)$ is the Bhattacharyya matrix defined by (5.3), then the following items are equivalent:*
(i) for all $z \in I$, $B(z)$ is 2-pseudodiagonal;
(ii) there exists $z \in I$ such that $B(z)$ is 2-pseudodiagonal;
(iii) for all $z \in I$, $B^{1,2}(z) = 0$ and $B^{2,4}(z) = 0$;
(iv) F is cubic and there exists $(U,v) \in GL(\mathbb{R}^d) \times \mathbb{R}^d$ such that $h(z) = \theta_\mu(Uz+v)$; i.e., $Uz+v$ is the mean of $P(h(z);\mu)$.

Proof. $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ are trivial.

$(ii) \Rightarrow (iv)$ Suppose that there exists $z_0 \in I$ such that $B(z_0)$ is 2-pseudodiagonal. Denoting $\nu = P(h(z_0);\mu)$, the polynomial $(S_n(x,z_0))_{n\in\mathbb{N}^d}$ are $\nu - 2$-pseudo-orthogonal. On the other hand, taking $\widetilde{h}(z) = h(z) - h(z_0)$, we can write

$$\begin{aligned}\sum_{n\in\mathbb{N}^d} \frac{(z-z_0)^n}{n!}\, S_n(x,z_0) &= \exp\left\{x^t\left[h(z)-h(z_0)\right] - K_\mu(h(z)) + K_\mu(h(z_0))\right\}\\ &= \exp\left\{x^t\widetilde{h}(z) - K_\mu(\widetilde{h}(z))\right\},\end{aligned}$$

which means that the polynomials $(S_n(x,z_0))_{n\in\mathbb{N}^d}$ have an exponential generating function. According to the characterization of the 2-pseudo-orthogonal polynomials with exponential generating function given in Theorem 4.2, we deduce that there exists $U \in GL(\mathbb{R}^d)$ such that

$$S_n(x, z_0) = P_{U,n}(x, m_0) = f_\nu^{(n)}(x, z_0)(Ue_1, \cdots, Ue_d)$$

with $m_0 = K_\nu'(0) = K_\mu'(h(z_0))$.

Hence, since $\widetilde{h}(z + z_0) = \theta_\nu(Uz + m_0) = \theta_\mu(Uz + m_0) - \theta_\mu(m_0)$, we have $h(z) = \theta_\mu(Uz+m_0-Uz_0) = \theta_\mu(Uz+v)$ with $v = m_0 - Uz_0 = K_\mu'(h(z_0)) - Uz_0$. Furthermore, the fact that $(P_{U,n}))_{n\in\mathbb{N}^d}$ are also $\nu - 2$-pseudo-orthogonal implies that F is cubic.

$(iv) \Rightarrow (i)$ Let z_0 be in I and let $\nu = P(h(z_0); \mu)$ and $m_0 = Uz_0 + v$. From (5.1) we have

$$f_\nu(x, Uz + v) = \frac{f_\mu(x, Uz + v)}{f_\mu(x, m_0)} = \frac{g_\mu(x, z)}{g_\mu(x, z_0)},$$

from which we deduce

$$S_n(x, z_0) = f_\nu^{(n)}(x, Uz_0 + v)(Ue_1, \cdots, Ue_d) = P_{U,n}(x, m_0).$$

Since F is cubic, the polynomials $(S_n(x, z_0))_{n\in\mathbb{N}^d}$ are $\nu - 2$-pseudo-orthogonal, *i.e.*, $B(z_0)$ is 2-pseudodiagonal. The implication is thus proved because z_0 is arbitrary.

$(iii) \Rightarrow (iv)$ Writing the polynomials $(S_n(x, z))_{n\in\mathbb{N}^d}$ as

$$S_n(x, z) = \sum_{q\in\mathbb{N}^d;|q|\le|n|} c_q(z)x^q$$

with $c_q(z) \in \mathbb{R}$, we have

$$\int S_n(x, z)S_p(x, z)P(h(z); \mu)(dx) = \sum_{q\in\mathbb{N}^d;|q|\le|n|} c_q(z)\frac{\partial^{|p|}}{\partial z^p}\int x^q P(h(z); \mu)(dx)$$

for all $(n, p) \in \mathbb{N}^d\times\mathbb{N}^d$ (see, *e.g.*, [25, Lemma 3.2]). Denoting $h(z) = {}^t(h_1(z), \cdots, h_d(z))$ and $\langle x_i\rangle = \langle x_{e_i}\rangle = \int x_i P(h(z); \mu)(dx)$, we then obtain for all $p \in \mathbb{N}^d$

$$\begin{aligned}\int S_{e_i}(x, z)S_p(x, z)P(h(z); \mu)(dx) &= \sum_{k=1}^d \frac{\partial}{\partial z_i}h_k(z)\frac{\partial^{|p|}}{\partial z^p}\langle x_k\rangle \\ &= \left(h'(z)\frac{\partial^{|p|}}{\partial z^p}\langle x\rangle\right)_i.\end{aligned}$$

In particular, for $|p| = 2$, $B^{1,2}(z) = 0$ implies that:

$$h'(z)\frac{\partial^2}{\partial z^p}\langle x\rangle = 0.$$

Since $h'(z)$ is invertible, $\langle x\rangle = \int xP(h(z); \mu)(dx)$ is a polynomial in z of degree 1 and therefore there exist a matrix U and a vector v such that

$$\langle x\rangle = K_\mu'(h(z)) = Uz + v.$$

This implies that:

$$h(z) = \theta_\mu(Uz + v),$$

where, by derivation, $U = {}^t h(z) K''_\mu(h(z))$ belongs to $GL(\mathbb{R}^d)$.

Similarly, $B^{2,4}(z) = 0$ implies for all $p \in \mathbb{N}^d$ such that $|p| = 4$:

$$\begin{aligned}\int S_{e_i+e_j}(x,z) S_p(x,z) P(h(z);\mu)(dx) &= \sum_{k,l=1}^{d} \frac{\partial}{\partial z_i} h_k(z) \frac{\partial}{\partial z_j} h_l(z) \frac{\partial^4}{\partial z^p} \langle x_k x_l \rangle \\ &= \left({}^t h'(z) \frac{\partial^4}{\partial z^p} \langle x_k x_l \rangle h'(z) \right)_{i,j} \\ &= 0,\end{aligned}$$

i.e., ${}^t h'(z) \frac{\partial^4}{\partial z^p} \langle x_k x_l \rangle h'(z) = 0$. Then $\langle x^t x \rangle = (\langle x_k x_l \rangle)_{k,l=1,\cdots,d}$ is a polynomial matrix in z of degree ≤ 3. Since

$$\langle x^t x \rangle = V_F(K'_\mu(h(z))) + \langle x \rangle^t \langle x \rangle,$$

it follows that there exists an open subset of M_F on which $V_F(m)$ is of degree ≤ 3. Therefore, F is cubic. □

Now, we come to the result concerning the transdiagonality characterization of the Bhattacharyya matrices. For this, we consider $A \in GL(\mathbb{R}^d)$ in (5.3), not necessarily the identity matrix.

Theorem 5.4 *Under the hypothesis of Theorem 5.3 and let (A, z_0) be in $GL(\mathbb{R}^d) \times I$, the three following statements are equivalent:*
(i) $B_A(z_0)$ is transdiagonal;
(ii) for all $z \in I$, $B_A^{1,2}(z) = 0$ and $B_A^{2,4}(z) = 0$ and $B_A^{1,1}(z_0)$ and $B_A^{2,3}(z_0)$ are transdiagonal;
(iii) F is simple cubic and there exists $(U, v) \in GL(\mathbb{R}^d) \times \mathbb{R}^d$ such that, for all $z \in I$, $h(z) = \theta_\mu(Uz + v)$ and $(UA)^{-1} V_F(Uz_0 + v)^t (AU)^{-1}$ is diagonal.

Proof. *(iii)* $\Rightarrow$ *(i)* It rises from Theorem 3.2 and Remark 5.2.

(i) $\Rightarrow$ *(ii)* It is easily obtained from Theorem 5.3.

(ii) $\Rightarrow$ *(iii)* We have $B_A^{1,2}(z) = B_A^{2,4} = 0$, then from Theorem 5.3 F is cubic and there exists $(U, v) \in GL(\mathbb{R}^d) \times \mathbb{R}^d$ such that

$$h(z) = \theta_\mu(Uz + v). \tag{5.5}$$

For $z_0 \in I$, let $\nu = P(h(z_0); \mu)$. Then, it is easy to see that the polynomials $(S_{A,n}(x, z_0))_{n \in \mathbb{N}^d}$ are $\nu - 2$-pseudo-orthogonal and have an exponential generating function as

$$\sum_{n \in \mathbb{N}^d} \frac{(z - A^{-1} z_0)^n}{n!} S_{A,n}(x, z_0) = \exp\left\{ x^t \widetilde{h}(Az) - K_\mu(\widetilde{h}(Az)) \right\},$$

with $\widetilde{h}(z) = h(z) - h(z_0)$. From Theorem 4.2, there exists $\widetilde{A} \in GL(\mathbb{R}^d)$ such that for all $n \in \mathbb{N}^d$

$$S_{A,n}(x, z_0) = f_\nu^{(n)}(x, m_0)(\widetilde{A}e_1, \cdots, \widetilde{A}e_d),$$

where $m_0 = K'_\mu(h(z_0)) = Uz_0 + v$ and $\widetilde{h}(A(z+z_0)) = \theta_\nu(\widetilde{A}z + m_0) = \theta_\mu(\widetilde{A}z + m_0) - \theta_\mu(m_0)$. Thus,

$$h(z) = \theta_\mu(\widetilde{A}A^{-1}(z - z_0) - m_0). \tag{5.6}$$

Since $B_A^{1,1}(z_0)$ and $B_A^{2,3}(z_0)$ are transdiagonal, the polynomials $(S_{A,n}(x, z_0))_{n\in\mathbb{N}^d, |n|\in\{1,2,3\}}$ are ν-transorthogonal. Hence, by Proposition 3.4 the polynomials $(S_{A,n}(x, z_0))_{n\in\mathbb{N}^d}$ are ν-transorthogonal and by Theorem 3.2 F is simple cubic with $\widetilde{A}^{-1}V_F(Uz_0 + v)^t\widetilde{A}^{-1}$ diagonal. Finally, by (5.5) and (5.6) we have $\widetilde{A} = UA$ and the diagonality of $(UA)^{-1}V_F(Uz_0 + v)^t(UA)^{-1}$ holds. □

6 Semigroup-Sheffer systems

This section characterizes the transorthogonality through the Sheffer [36] polynomials associated to a convolution semigroup of probability measures or NEFs, usually induced by a stochastic process with stationary and independent increments. In this way, one can refer to [13], [26], [31] and [33] for orthogonality and their other real extensions.

Let us first define the semigroup-Sheffer systems of NEFs on $\mathbb{R}^d$. If there exist an open ball $B(0, r)$ of $\mathbb{R}^d$ and two analytic functions $a : B(0, r) \to \mathbb{R}^d$ and $\bar{b} : B(0, r) \to \mathbb{R}$ such that $a(0) = 0$, $a'(0) \neq 0$ and $\bar{b}(0) \neq 0$, then the polynomials sequence $(Q_n(x))_{n\in\mathbb{N}^d}$ defined by the generating function

$$\sum_{n\in\mathbb{N}^d} \frac{m^n}{n!} Q_n(x) = \exp\left\{x^t a(m)\right\} \bar{b}(m), \quad \forall m \in B(0, r),$$

is a *Sheffer system* [36]. In the context of NEFs and following Schoutens and Teugels [33], we can introduce an additional time parameter $t \in \overline{\Lambda} \subseteq [0, \infty)$ into the above Sheffer systems as follows:

Definition 6.1 *Let $F = F(\mu)$ be a NEF on $\mathbb{R}^d$ and let $\overline{\Lambda} = \{t \geq 0; \exists \mu_t = \mu^{*t} : K_{\mu_t} = tK_\mu\}$ be the completed Jørgensen set of $F(\mu)$ with 0 as defined in (2.3). For all $(A, m_1) \in GL(\mathbb{R}^d) \times M_F$, the polynomials $(Q_{tA,n,t}(x, t))_{n\in\mathbb{N}^d; t\in\overline{\Lambda}}$ such that*

$$\sum_{n\in\mathbb{N}^d} \frac{m^n}{n!} Q_{tA,n,t}(x, t) = \exp\left\{x^t\theta_\mu(Am + m_1) - K_\mu(\theta_\mu(Am + m_1))\right\}, \quad \forall m \in B(tm_1, r),$$

are called semigroup-Sheffer systems associated to F.

Note that, for $t = 0$ in the closed additive semigroup $\overline{\Lambda} = \overline{\Lambda}(F(\mu))$ of $[0, \infty)$ with $\mathbb{N} \subseteq \overline{\Lambda} \subseteq [0, \infty)$, it is convenient to put $\mu_0 = \delta_0$ the Dirac mass at 0. For fixed $t = 1$, we have $Q_{A,n,1}(x, 1) = P_{A,n}(x, m_1) = f_\mu^{(n)}(x, m_1)(Ae_1, \cdots, Ae_d)$ as given in (2.6) with $\mu = \mu_1$.

For all $t \in \overline{\Lambda}$, we can associate the random vector X_t with distribution $\nu_t = (P(m_1, F))^{*t}$. In particular, under infinite divisibility of F (*i.e.*, $\overline{\Lambda} = [0, \infty)$), $(X_t)_{t\geq 0}$ is a Lévy process (*i.e.*, stationary process with independent increments; see [3] and [29] for more details). Thus, the polynomials $(Q_{tA,n,t}(x,t))_{n\in\mathbb{N}^d;t\geq 0}$ are known to be Lévy-Sheffer systems (*e.g.*, [26]). It follows, by the martingale property of

$$f_{\mu_t}(X_t, m) = \exp\left\{\theta_{\mu_t}(m)^t X_t - tK_\mu(\theta_{\mu_t}(m))\right\}$$

(*e.g.*, [17]), that we have the following martingale equality ([32]) as a basic application of this study:

$$\mathbb{E}\left[Q_{A,n,t}(X_t,t)|X_s\right] = Q_{A,n,s}(X_s,s), \quad 0 \leq s < t,\ n \in \mathbb{N}^d,\ A \in GL(\mathbb{R}^d). \tag{6.1}$$

Since all univariate cubic NEFs are infinitely divisible [20], it is tantalizing to say that the multivariate (simple) cubic NEFs are also infinitely divisible. For instance, this is an open problem. Let us mention that we have an analog application as (6.1) when $\overline{\Lambda} = \mathbb{N}$. This corresponds to IID-Sheffer systems (*e.g.*, [13], [31] and [32]).

Now, we can show the result of characterization only with respect to the transorthogonality. The 2-pseudo-orthogonality case is almost similar and we omit it.

Theorem 6.2 *Let F be an irreductible NEF on $\mathbb{R}^d$ and let $\overline{\Lambda} = \overline{\Lambda}_F$ be the completed Jørgensen set of F. For all $(A, m_1) \in GL(\mathbb{R}^d) \times M_F$, consider the semigroup-Sheffer systems $(Q_{tA,n,t}(x,t))_{n\in\mathbb{N}^d;t\in\overline{\Lambda}}$. Then: the transorthogonality of the semigroup-Sheffer systems occurs if and only if F is simple cubic with $A^{-1}V_F(m_1)^t A^{-1}$ diagonal.*

Proof. Let $\nu_t = (P(m_1, F))^{*t} = P(m_t, F_t)$ with $m_t = tm_1$. Assume that the polynomials $Q_{tA,n,t}(x,t)$ are ν_t-transorthogonal for all $t \in \Lambda = \overline{\Lambda}\backslash\{0\}$. Then, in particular, the polynomials $P_{A,n} = Q_{A,n,1}$ are ν_1-transorthogonal. By Theorem 3.2, we deduce that $F_1 = F$ is simple cubic with $A^{-1}V_F(m_1)^t A^{-1}$ diagonal.

Conversely, from (2.4), if $F = F_1$ is simple cubic then $F_t = F^{*t}$ is simple cubic too for all $t \in \Lambda$. Thus, it suffices to show that the polynomials $P_{A,n} = Q_{A,n,1}$ are ν_1-transorthogonal, which are also obtained by Theorem 3.2. □

7 An illustration

The most famous example of simple cubic NEFs (2.9) is the *multivariate normal inverse Gaussian* (MNIG) family. The MNIG distribution is a variance-mean mixture of a multivariate Gaussian with a univariate inverse Gaussian distribution. It can be considered as a distribution of the position of multivariate Brownian motion at a certain stopping time. See [1], [2] and [23] for more details and some interesting applications.

For instance, fix $t>0$. Consider the generating measure μ_t on $\mathbb{R}^d$ defined by

$$\mu_t(dx)=\frac{tx_1^{-(d+2)/2}}{(2\pi)^{d/2}}\exp\left\{-\frac{1}{2x_1}\left(t^2+\sum_{i=2}^{d}x_i^2\right)\right\}1_{x_1>0}dx_1\cdots dx_d.$$

It is easy to see that, for all $t>0$, the NEF $F_t=F(\mu_t)$ generated by the probability μ_t is composed by the distributions of the random variables $(X_1,\cdots,X_d)$, where X_1 is univariate inverse Gaussian distributed and $(X_2,\cdots,X_d)|X_1$ are $d-1$ real independent Gaussian variables with variance X_1. Also, we have

$$\Theta(\mu_t)=\left\{\theta\in\mathbb{R}^d; 2\theta_1+\sum_{i=2}^{d}\theta_i^2<0\right\}\quad\text{and}\quad K_{\mu_t}(\theta)=-t\left(-2\theta_1-\sum_{i=2}^{d}\theta_i^2\right)^{1/2}.$$

Thus, $M_{F_t}=(0,\infty)\times\mathbb{R}^{d-1}$, $\theta_{\mu_t}(m)=\left(-2m_1^{-2}\left[t^2+\sum_{i=2}^{d}m_i^2\right],m_1^{-1}m_2,\cdots,m_1^{-1}m_d\right)$ and

$$\begin{aligned}V_{F_t}(m)&=m^te_1\left[t^{-2}\ {}^tmm+I_d-\ {}^te_1e_1\right]\\&=\left(m_1\left[t^{-2}m_im_j+\delta_{ij}-\delta_{1i}\delta_{1j}\right]\right)_{i,j\in\{1,\cdots,d\}}\end{aligned}\tag{7.1}$$

where $\delta_{ij}=1$ for $i=j$ and 0 for $i\neq j$. It follows that $F=F_1$ is a simple cubic NEF on $\mathbb{R}^d$, namely the MNIG $\mathcal{G}$-orbit and it is from the $\mathcal{G}_0$-orbit of Gaussian (2.9). Its Jørgensen set (2.3) is $\Lambda=(0,\infty)$; *i.e.*, F is infinitely divisible. Thus, the MNIG family is associated to a Lévy process; see also [28].

Let $(Q_{n,t}(x,t))_{n\in\mathbb{N}^d;t\geq 0}$ be the associated Lévy-Sheffer systems of F as in Definition 6.1 with $A=I_d$ (identity matrix). The general expression of polynomials $Q_{n,t}(x,t)$ can be explicitly obtained by the corresponding Faà di Bruno formula (2.7) with

$$g_{x,t}(m)=x^t\theta_{\mu_t}(m)-K_{\mu_t}(\theta_{\mu_t}(m))=-\frac{x_1}{2m_1^2}\left(t^2+\sum_{i=2}^{d}m_i^2\right)+\frac{1}{m_1}\sum_{i=2}^{d}m_ix_i+\frac{t^3}{m_1^2}$$

to be derivated at any $m_0\in M_F$; see Savits [30] for some other illustrative examples. For example, in order to calculate $Q_{e_i,t}(x,t)$ and $Q_{e_i+e_j,t}(x,t)$ we first need

$$\left.\frac{\partial^{|l|}}{\partial m^l}g_{x,t}(m)\right|_{m=m_0}=\begin{cases}\frac{(-1)^{l_1}}{2m_{01}^{l_1+2}}l_1!\times\\ \left[\left(t^2+\sum_{s=2}^{d}m_{0s}\right)(1+l_1)\,x_1+2m_{01}\sum_{s=2}^{d}m_{0s}x_s\right] & \text{if } l=l_1e_1\\ \frac{(-1)^{l_1}}{m_{01}^{l_1+2}}l_1!\,(m_{01}x_j+2x_1m_{0j}) & \text{if } l=l_1e_1+e_j, j\neq 1\\ \frac{2(-1)^{l_1}}{m_{01}^{l_1+2}}l_1!x_1 & \text{if } l=l_1e_1+2e_j, j\neq 1.\end{cases}$$

Then, the Faà di Bruno formula (2.7) provides:

- $n = e_i$ then $\Omega(e_i, 1) = \{(1, e_i)\}$ and

$$Q_{e_i,t}(x,t) = (1-\delta_{1i})\left(x_i + 2x_1\frac{m_{0i}}{m_{01}}\right) - \delta_{1i}\left[\left(t^2+\sum_{s=2}^{d} m_{0s}\right)x_1 + m_{01}\sum_{s=2}^{d} m_{0s}x_s\right];$$

- $n = e_i+e_j$ with $i \neq j$ then $\Omega(e_i+e_j, 1) = \{(0,0,1,e_i,e_j,e_i+e_j)\}$ and $\Omega(e_i+e_j, 2) = \{(1,1,0,e_i,e_j,e_i+e_j)\}$, therefore

$$Q_{e_i+e_j,t}(x,t) = \begin{cases} \frac{1}{m_{01}^4}\left(m_{01}x_i + 2x_1m_{0i}\right)\left(m_{01}x_j + 2x_1m_{0j}\right) & \text{if } 1 \neq i \neq j \neq 1 \\ \frac{-1}{m_{01}^3}\left(m_{01}x_j + 2x_1m_{0j}\right)\times & \\ \left[1+\frac{1}{m_{01}^4}\left\{\left(t^2+\sum_{s=2}^{d} m_{0s}\right)x_1 + 2m_{01}\sum_{s=2}^{d} m_{0s}x_s\right\}\right] & \text{if } i=1, j\neq 1; \end{cases}$$

- $n = 2e_i$ then $\Omega(2e_i, 1) = \{(0,1,e_i,2e_i)\}$ and $\Omega(2e_i, 2) = \{(2,0,e_i,2e_i)\}$, therefore

$$\begin{aligned} Q_{2e_i,t}(x,t) =& \frac{\delta_{1i}}{m_{01}^6}\left[2\left(t^2+\sum_{s=2}^{d} m_{0s}\right)x_1 + 2m_{01}\sum_{s=2}^{d} m_{0s}x_s\right]^2 \\ &+\frac{\delta_{1i}}{m_{01}^4}\left[3\left(t^2+\sum_{s=2}^{d} m_{0s}\right)x_1 + 2m_{01}\sum_{s=2}^{d} m_{0s}x_s\right] \\ &+(1-\delta_{1i})\left[\frac{1}{m_{01}^4}\left(m_{01}x_i + 2m_{0i}x_1\right)^2 + \frac{2}{m_{01}^2}x_1\right]. \end{aligned}$$

Otherwise, by Proposition 3.5, the corresponding recurrence relations of $(Q_{n,t}(x,t))_{n\in\mathbb{N}^d;t\geq 0}$ is deduce from (7.1). Indeed, taking $m_0 = (1, 0, \cdots, 0)$ in M_F, the variance function $V_{F_t}(m) = (V_{i,j}(m))_{i,j\in\{1,\cdots,d\}}$ given by (7.1) is such that

$$\begin{aligned} V_{ij}(m) =& \sum_{q\in\mathbb{N}^d;|q|\leq 3} \alpha_{ij}(q,t)(m-m_0)^q \\ =& \begin{cases} t^{-2}\left[(m_1-1)^3 + 3(m_1-1)^2 + 3(m_1-1) + 1\right] & \text{for } i=1=j \\ t^{-2}\left[(m_1-1)^2m_j + 2(m_1-1)m_j + m_j\right] & \text{for } i=1\neq j \\ t^{-2}\left[(m_1-1)^2m_i + 2(m_1-1)m_i + m_i\right] & \text{for } i\neq 1=j \\ t^{-2}\left[(m_1-1)m_im_j + m_im_j + \delta_{ij}(m_1-1)\right] + \delta_{ij} & \text{for } i\neq 1\neq j \end{cases} \end{aligned}$$

with

$$\alpha_{ij}(q,t) = \begin{cases} t^{-2} & \text{if } \begin{cases} |q|=3; |q|=2 \text{ and } i\neq 1\neq j; |q|=1 \text{ and } i=j\neq 1; \\ |q|=1 \text{ and } i=1\neq j; |q|=1 \text{ and } i\neq 1=j; \end{cases} \\ 2t^{-2} & \text{if } |q|=2 \text{ and } i=1\neq j; |q|=2 \text{ and } i\neq 1=j; \\ 3t^{-2} & \text{if } |q|=2 \text{ and } i=1=j; |q|=1 \text{ and } i=1=j; \\ t^{-2}+d-1 & \text{if } |q|=0; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, at $m_0 = (1, 0, \cdots, 0)$ in M_F, the Lévy-Sheffer polynomials $Q_{n,t}(x,t)$ satisfy

$$x_i Q_{n,t}(x,t) = \sum_{j=1}^{d} \left\{ \alpha_{ij}(0,t) Q_{n+e_j,t}(x,t) + \sum_{0<|q|\leq 3} \frac{n! \alpha_{ij}(q,t)}{(n-q)!} Q_{n+e_j-q,t}(x,t) \right\} + n_i Q_{n-e_i,t}(x,t) + m_{0i} Q_{n,t}(x,t),$$

for all $n \in \mathbb{N}^d$ and $t \geq 0$ with the following initial conditions: $Q_{0,t}(x,t) = 1$,

$$Q_{e_i,t}(x,t) = \begin{cases} -t^2 x_1 & \text{if } i = 1 \\ x_i & \text{if } i \neq 1 \end{cases}$$

and

$$Q_{e_i+e_j,t}(x,t) = \begin{cases} x_i x_j & \text{if } 1 \neq i \neq j \neq 1 \\ -\left(1 + t^2 x_1\right) x_j & \text{if } i = 1 \neq j \\ 4t^4 x_1^2 + 3t^2 x_1 & \text{if } i = j = 1 \\ x_i^2 + 2x_1 & \text{if } i = j \neq 1. \end{cases}$$

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Table 1
The twelve $\mathcal{G}_0$-orbits of the real cubic NEFs ([20], [22]) distributed in four $\mathcal{G}$-orbits by Hassairi [7].

$\mathcal{G}$-orbit	Quadratic [22]		Cubic [20]	
1st	Gaussian		Inverse Gaussian	
	1		m^3	
2nd	Poisson	Gamma	Abel	Ressel-Kendall
	m	m^2	$m(1+m)^2$	$m^2(1+m)$
3rd	Binomial	Negative binomial	Takács ($a>0$)	
	$m(1-m)$	$m(1+m)$	$m(1+m)(1+\frac{1+a}{a}m)$	
4th	Hyperbolic		Large arcsine ($a>0$)	Strict arcsine
	$1+m^2$		$m(1+2m+\frac{1+a^2}{a^2}m^2)$	$m(1+m^2)$