QUANTUM FIELD THEORY AND STATISTICAL MECHANICS

E. Brézin

JOURNAL DE PHYSIQUE
Colloque C3, supplément au n° 12, Tome 43, décembre 1982 *page* C3-743

QUANTUM FIELD THEORY AND STATISTICAL MECHANICS

E. Brézin

DPhT, Orme des Merisiers, CEN Saclay, B.P. n° 2, 91191 Gif-sur-Yvette, France

Résumé. - Les champs de jauge sur réseau ont été étudiés de manière exhaustive par des simulations numériques. Mais dans ce rapport l'accent sera mis sur des méthodes analytiques, fréquemment inspirées de la Mécanique Statistique. Plusieurs résultats sont récents et il semble que ces méthodes soient appelées à jouer un rôle plus important pour la compréhension de la Chromodynamique quantique. Néanmoins la plupart des sujets abordés dans cet exposé sont très classiques et ce rapport n'est certainement pas destiné aux spécialistes.

Abstract. - Lattice gauge theories have been extensively studied by Monte-Carlo simulations. In this talk some analytic methods (frequently inspired by statistical mechanics) are presented. Several results are quite recent and I believe that these methods should play an increasing role in the understanding of QCD. However most of the material is completely standard and this report should not be read by specialists.

Introduction. - Connections between quantum field theory and statistical mechanics have existed for many years. The so-called second quantized formalism, Green functions techniques, Feynman diagrams etc... have been widely used in both fields. However it is only after the 70's that the two subjects became extremely similar, and it is only slightly exaggerated to say that renormalization theory is part of the theory of critical phenomena. Two basic steps led to this unification. Firstly the imaginary time (Euclidean) formalism [1] brings the quantum mechanical amplitude $\exp(iS/\hbar)$ under the form of a Boltzmann probability weight $\exp-(\beta H)$, a trivial, but in practice, very important fact. Second, the existence of a renormalized quantum field theory, i.e. a theory in which the physical observables are expressed in terms independent of the intermediate regularizing length scale (or inverse cut-off), is very similar to that of a critical point in condensed matter physics [2].

These resemblances are useful in both directions. Many quantum field theory techniques, such as the renormalization group formalism, have been particularly useful in statistical mechanics [3] but, in spite of the fact that it is a beautiful topic (it leads to qualitative and even accurate quantitative predictions) the emphasis, for this particle physics conference will be on the opposite direction : we shall discuss some of the ideas and of the techniques which emerged out of statistical mechanics and led to applications to particle physics. Thus we shall be dealing mostly with analytic approaches to lattice gauge theories. Indeed the lattice formulation in 1973 by K. Wilson [4] of gauge theories is an important and natural idea which led to many developments, in particular to remarkable numerical simulations by the Monte-Carlo techniques (reviewed by C. Rebbi in this conference). It should be stated again that the elementary length scale introduced in this formulation is a powerful technical device, which allows one to control the strong coupling limit, or to perform numerical simulations for instance. It has nothing to do with the deep question of a possible fundamental length scale in space-time. A major issue, a "sine qua non" condition for the practical usefulness of lattice gauge theories was the possibility of introducing fermions in the formalism, and in particular in a numerical simulation. This has been achieved during the last year [5] and this important break-through will, I believe, lead to many developments. (See C. Rebbi's and Martinelli's talks at this conference).

There are many important and new topics related to my subject which I should have reviewed in this talk. More than the traditional lack of time argument, I think that they are so important that they deserve a more qualified exposition than the one I could provide. It is very unfortunate for instance that A. Polyakov has not been authorized to present at this conference his new statistical theory of random surfaces [6], which leads to a completely new understanding of the old dual models, and resolves their difficulties away from 26 or 10 dimensions (see the talk by A. Neveu at this conference). The Nielsen-Ninomiya doubling theorem [7] which, at least on a lattice, implies the existence of an equal number of left and right-handed Weyl fermions, is manifestly an extremely deep and important constraint. The problem raised by Bhanot and Dashen [8] in which the universality of the continuum limit of various lattice gauge actions is questioned is important, though I believe that the observations of S. Samuel and of B. Grossman [9] are relevant. Finite temperature QCD and the issue of the deconfinement and chiral symmetry restoration temperatures is important and Monte-Carlo results are quite useful [10] but I had unfortunately to ignore all these topics.

Finally, this talk will be a survey of some of the non-perturbative, non Monte-Carlo methods in gauge theories. The first part consists simply of the traditional connection between quantum field theory and classical statistical mechanics at a critical point. In the second part we shall look at some of the strong coupling results. We will present then the mean-field ideas, and the expansion (in fact in powers of the inverse of dimension of space-time) around this picture ; some recent results on the mass spectrum will be reviewed. Finally we will discuss some new and beautiful ideas on the large N limit of an SU(N) gauge theory which may lead to future developments and may justify the use of small lattices in numerical simulations.

1. Quantum field theory and classical statistical mechanics in the neighbourhood of a critical point

This connection is so well-known that we shall very briefly recall the main features. For simplicity of the notations we shall discuss it for a self-coupled real scalar field ϕ, with the action

$$S = \int dt \int d^3x(\dot{\phi}^2-(\nabla\phi)^2-V(\phi)) \ .$$

In Feynman's path integral formulation amplitudes are given by the sum over all field configurations in space and time

$$\sum_{\{\phi\}} \exp \frac{i}{\hbar} S\{\phi\}$$

After Wick's rotation $t \to ix_4$

$$\frac{iS}{\hbar} \Rightarrow - \int d^4x(\sum_1^4 (\partial_\mu\phi)^2 + V(\phi)) \equiv -\beta H$$

and vacuum expectation values become statistical averages

$$<0|A(\phi)|0> = \frac{\sum_{\{\phi\}} A(\phi)\exp-\beta H(\phi)}{\sum_{\{\phi\}} A(\phi)}$$

This very simple reformulation is at the heart of many new techniques in quantum field theory. One can invent for instance a fictitious time, a fifth dimension in some sense, in which the configurations will evolve and approach the "equilibrium" prescribed by the Boltzmann weight $\exp-\beta H$; this is the basis of Monte-Carlo simulations [11] or of the recent stochastic approaches to quantum theory [12]. The positivity of the probability weight may be used in many ways ; it leads to

variational calculations or to mean field ideas. There is a whole dictionary of translations from one field to the other. For instance a change of vacuum for a potential $V(\phi)$ of fig. 1, from the $\phi = 0$ to the $\phi = \phi_0$ vacuum is a (first-order) phase transition. The actual time development of this change of vacuum, which was studied in particular by Callan and Coleman [13] is very similar to the theory of nucleation of a bubble of liquid in a supercooled vapor [14]. A more extended dictionary may be found in Parisi's contribution [15] to the 1980 Madison Conference.

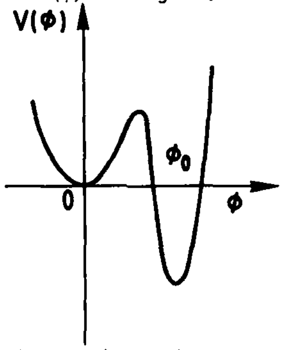


Fig.1

Regularization and renormalization

A quantum field theory cannot be constructed without regularization of the short-distance singularities. Many ways have been proposed in practice, from simple momentum cut-off Λ to a minimum distance as in a lattice regularization but they all involve some intermediate small distance a, the lattice spacing or the inverse momentum cut-off Λ^{-1}. The renormalized theory (if the theory is renormalizable) is the limit in which S-matrix elements or Green functions are expressed in terms of physical parameters, masses, coupling constants, momenta or relative distances which are held fixed when the regularizing length a is decreased to zero. Therefore the comparison between renormalized field theory and the statistical mechanics viewpoint may be expressed schematically as

Renormalized field theory	Statistical Mechanics near a critical point
masses $m \ll \Lambda$ momenta $p \ll \Lambda$	correlation length $\xi \gg a$ relative distances $r \gg a$

with $a = \Lambda^{-1}$ and ξ the characteristic correlation length is identified to the inverse of the mass. A simple example which illustrates this connection is given by the Fourier transform of the free propagator

$$\int \frac{e^{ip.x}}{p^2+m^2}\, d^4p \underset{|x|\to\infty}{\sim} \frac{e^{-m|x|}}{|x|^{3/2}} \equiv \frac{e^{-|x|/\xi}}{|x|^{3/2}}$$

Therefore in the renormalized field theory correlations extend to infinity (compared to a). This well-known and obvious fact is nevertheless somewhat surprizing. Indeed in the probability weight

$$\exp - \int d^4x[\partial_\mu\phi\partial_\mu\phi+V(\phi)]$$

(i) the potential term $[\exp-(\int V\,dx)]$ does not couple the values of ϕ at different space-time points ; if the kinetic term were not present $\phi(x)$ and $\phi(y)$ would be independent random variables and the range of correlations would be zero.

(ii) The kinetic term $[\exp-(\int(\nabla\phi)^2dx]$ is quasi-local. It couples the fields ϕ at neighbouring points. It is manifest in a lattice regularization in which $\partial_\mu\phi\partial_\mu\phi$ stands for $\left(\frac{\phi(x+ae_\mu)-\phi(x)}{a}\right)^2$ and nearest neighbours only are coupled.

Therefore the range of interactions is of order a, and in general for an arbitrary choice of the (bare) coupling constants the correlations do no extend further

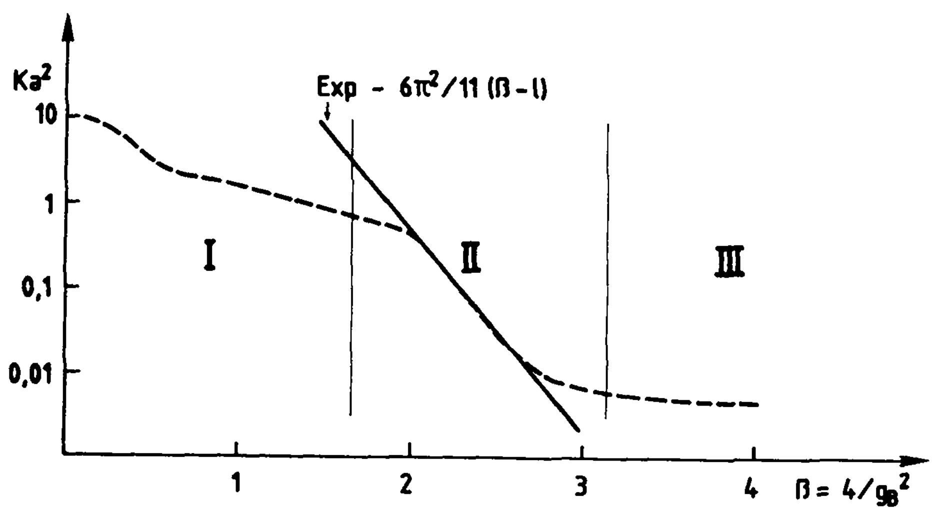
than the range of interactions. The physical requirement of renormalization $\xi \gg a$ is possible only if the coupling constants are adjusted in order to be in the vicinity of a critical point (at which a continuous phase transition takes place with a diverging correlation length).

Let us consider for example Wilson's formulation [4] of lattice gauge theories

$$\beta H = -\frac{1}{g_B^2} \sum_{\text{plaquettes}} 2\,\mathrm{Re\,Tr}(U_{x,\mu}\, U_{x+\mu,\nu}\, U^\dagger_{x+\nu,\mu}\, U^\dagger_{x,\nu})$$

One can calculate ξ by looking for instance at correlation functions of small Wilson loops (ξ would be simply the inverse of the glueball mass), and the result would be for ξ some function $\xi(a,g_B^2)$. For an arbitrary choice of g_B^2. ξ remains of the order of the interaction range a, the plaquette size. Thus there is no physics outside the critical values of g_B^2, provided they exist, at which ξ diverges. It is believed, and supported by numerical simulations, that for four-dimensional QCD ξ/a is large for small values of g_B^2 *only*. For g_B^2 small it is a consequence of asymptotic freedom that ξ/a is large. Indeed this follows from the renormalization group equations ; let us recall here simply that these equations describe trajectories of parameters of the theory which can be modified without changing the physics. For instance in the relation $\xi = \xi[g_B,a]$ we can modify the length scale $a \to a/\lambda$ and change accordingly g_B $g_B \to g_B(\lambda)$ so that ξ remains invariant : $\xi = \xi[g_B(\lambda),a/\lambda]$. The trajectory $g_B(\lambda)$ is given by $\frac{\lambda dg_B}{d\lambda} = \beta[g_B(\lambda)]$, $\beta[g_B] = -b_o g_B^3 + O(g_B^5)$. For λ large it follows that $g_B^2(\lambda) \simeq \frac{1}{2b_o \ell n\lambda}$. Therefore the ratio ξ/a is increased by a large factor λ provided the bare coupling constant goes to zero.

These well-known and elementary considerations explain the existence of a "scaling window" in any numerical simulations. Let us for instance sketch roughly the behaviour of the "measured" string tension in a Monte-Carlo simulation on a lattice L×L×L×L, as a function of the coupling constant ; such calculations initiated by M. Creutz [16] have now been repeated by many groups.



In the first region of Fig. 2 g_B is large ; ξ is of order a and the measurements display only lattice short-range effects. In the third region g_B is indeed small, but ξ is order L, the full size of the lattice (which, for obvious reasons, rarely exceeds 6 or 8a) and the calculation is showing only finite size effects. It is only in the "window" II that ξ is equal to a few lattice spacings without yet any effect of the finite total size. It is remarkable that asymptotic freedom (the straight line of

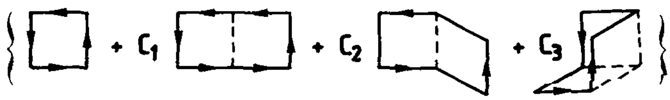
the picture) is indeed observed in this narrow window.

Universality and irrelevant variables

Various lattice actions may have the same physical content and there is a priori no reason, other than convenience to choose one rather than another.

For instance, instead of the simplest plaquette action, Wilson has found convenient to use in addition six links closed circuits [4], pictorially described as

$$\beta H = -\frac{1}{g_B^2}\sum \mathrm{Tr}\ \ \Sigma\,\mathrm{tr}$$



This action has the same (formal) continuum limit, the Yang-Mills action, plus different corrections of order a^2. Wilson has adjusted c_1, c_2 and c_3 to improve the convergence to the continuum limit. Recently Bhanot and Dashen [8] have considered

$$\beta H = -\frac{1}{g_B^2}\sum_{\mathrm{plaq.}} \mathrm{Tr}$$

and in fact questioned the common belief that this parameter c should not affect the continuum limit. Many other types of modifications of the simplest action have been proposed in the literature. Since all these actions share the same properties : gauge invariance, identical formal continuum limit, it would be embarassing if they had a different physical content in the long distance limit. The universality of this continuum limit is certainly difficult to establish rigorously ; it could be that for large deviations from the simplest action one reaches a different theory, however strong arguments have been provided by Wilson's characterization of irrelevant variables [3]. Let us choose a very simple example in order to examine this concept : consider the lattice regularization of the kinetic energy for a scalar field, and expand of for small a

$$\sum_{\mu=1}^{4}\left[\frac{\phi(x+ae_\mu)-\phi(x-ae_\mu)}{2a}\right]^2 = \sum_1^4 \partial_\mu\phi\partial_\mu\phi + \frac{a^2}{3}\sum_{\mu=1}^{4}\partial_\mu\phi\partial_\mu^3\phi + O(a^4)$$

At order a^2 appears a non-rotationaly invariant operator of dimension 6 (a scalar field has mass dimension one). Of course in the formal continuum limit $a \to 0$ this operator disappears. However the argument has to be slightly refined since a perturbative treatment of this additional dimension six operator yields a factor $1/a^2$ at each loop and the combined effect of $a^2 \times 1/a^2$ is not clear. More convincing renormalization group arguments have been introduced by Wilson who characterized the "irrelevant operators" which have no influence on the long distance limit because they scale as a/ξ raised to some positive power. In dimension four all operators of dimension larger than four are irrelevant, at least in some vicinity of the origin in coupling constant space.

Consequently

(i) In the continuum limit $a \to 0$ the full translation invariance is restored (instead of just the lattice translations) ; rotation symmetry (the Euclidean equivalent of Lorentz invariance) is also restored.

(ii) One may try to improve the lattice action, by the addition of irrelevant variables, which would accelerate the convergence to continuum by cancelling the leading irrelevant operators. This program, numerically explored initially by Wilson, is under systematic investigation by K. Symanzik and followers [17].

(iii) This picture of irrelevant operators has led to a conceptual revision of the demand of renormalizability of quantum field theories. In a renormalizable theory, since one can get rid completely of the cut-off, one could pretend that one controls the physics at all length scales down to zero. However we know very well that the physics is not understood at very small distances such as (and in fact well before)

the Planck length. Therefore one could conclude that demanding renormalizability is more than what is allowed by present day physics. However this new way of looking at continuum field theories shows that whatever theory governs physics at very short distances, renormalizable or non-renormalizable, the long distance limit with respect to these unknown short distances generates automatically an effective renormalizable theory. N. Nielsen and coworkers[18,19] S. Shenker [20] have even gone further and argued that Lorentz invariance or even gauge symmetry may result of a similar long distance limit of an unspecified short distance theory. Therefore, one can consider a renormalizable theory such as QED, or may be even QCD, as an effective "low energy" theory.

2. Strong coupling expansions

In the correspondance between field theory and statistical mechanics the (bare) coupling constant plays the role of a temperature at thermal equilibrium since $1/g_B^2$ is replaced by $\beta = 1/kT$. Therefore weak and strong coupling limits of the theory correspond respectively to low and high temperature limits of the theory. In the low temperature limit, one projects out the vicinity of the lowest energy configuration ; the "Hamiltonian"

$$H = - \sum_{Pl} \mathrm{Tr}(\square + \mathrm{h.c.})$$

is minimum if each group element attached to a link is equal to the identity (up to a gauge transformation). Expanding around this configuration (with proper gauge fixing) yields the usual perturbation theory with a lattice propagator $1/ \frac{2}{a^2} \sum_1^4 (1-\cos ap_\mu)$ instead of $1/p^2$.

The new feature opened by the lattice formulation is the control of the strong coupling limit of the theory. Indeed without lattice even if the potential energy is large, the kinetic energy can never be neglected since field configurations with rapid spatial variations of the field may exceed the potential energy ; therefore the field gradients can never be neglected unless a lattice limits all momenta to an upper-bound given by the inverse lattice spacing ($|p_\mu| \leq \pi/a$). It has been shown long ago by Wilson [4] through a beautiful and elementary argument using the high temperature expansion, that in the strong coupling limit the static potential between heavy quarks increases linearly at large distances. High temperature series are calculated simply by expanding $\exp - (\beta H)$ in powers of β. At the price of sometimes enormous combinatorial efforts any observable can be calculated in powers series of β :

$\langle O \rangle = \sum_{0}^{N_{max}} a_n \beta^n$. Free energy plaquette energies, plaquette-plaquette correlation functions etc... have been computed for various gauge groups and space dimensions [20]. The longest series have little more than 20 terms. This method has been used for many years in the area of critical phenonema ; it is believed there that the radius of convergence of the series estimated by various conventional methods (ratios of successive coefficients, Padé approximants etc...) corresponds to the critical (inverse) temperature β_c at which a phase transition takes place. A naive extrapolation of these ideas to four-dimensional non-abelian gauge theories in which no transition is expected could make one conclude that the radius of convergence is infinite. However this expectation is presumably wrong : Itzykson, Pearson and Zuber [21] have analyzed the low temperature expansion of the Ising model in three-dimensions (which is dual to the high temperature expansion of the 3-D Z_2-gauge theory) and found that the leading singularity does not correspond to the transition point. They have given strong indications that the same should be true for SU(N) gauge theories as well. Therefore one has to extrapolate the series by analytic continuation outside the circle of convergence and the numerical accuracy may be reduced by the presence of these spurious singularities.

Extensive calculations of the plaquette-plaquette correlation functions have led to numerical estimates of the glueball mass (the lowest colour singlet, pure gauge excitation). Hamiltonian high temperature series were already calculated in 1976 by Kogut, Sinclair and Susskind [22], and during the last year longer Euclidean series have been calculated by several groups [23]. Of course, since the continuum limit

corresponds to g_B^2 going to zero, these series in powers of β have to be extrapolated at $\beta = \infty$. The latest values for an SU(2) gauge group of the glueball mass [23] is $m(0^+) = (1130 \pm 350)$ MeV which is certainly compatible with the results of an extensive variational Monte-Carlo result by Berg and Billoire [24] $m(0^+) = (920 \pm 310)$ MeV.

3. Mean field methods and beyond mean field

The simplest, and most traditional approximation, to a system in which the degrees of freedom are coupled, is to replace the dynamics by that of independent degrees of freedom in some external source, chosen in the best possible way to simulate the full dynamics. For a gauge theory in which there is a group element on every link one replaces the true action

$$H = - \sum_{\text{Plaq.}} \mathrm{tr}(UUU^+U^+ + \text{h.c.})$$

by

$$H_o(X) = - \sum_{\text{links}} \mathrm{tr}(U_\ell X_\ell + \text{h.c.})$$

The arbitrary matrices X_ℓ are then fixed by a variational principle to their best possible values. These calculations carried several years ago by Balian, Drouffe and Itzykson [25] led to the conclusion that there is a phase transition between a weak coupling, deconfined phase and a strong coupling confining phase. These results are correct when the space-time dimension d is large. Indeed X_ℓ may be seen as the contribution of the 2(d-1) plaquettes adjacent to a given link and for large d the sum of these contributions has only small statistical fluctuations and may be replaced by an average self-consistent X. There is, of course, a more precise way to show that mean field becomes exact at large d [26]. The results of these calculations are in fact qualitatively correct for $d \geq 5$, since in 5-D already a first order deconfining transition is also observed in Monte-Carlo simulations [27]. However no such transition is expected in 4-D and one has to go beyond this simple picture. This was not attempted for a long time because of a deep general property of gauge theories discovered in 1975 by Elitzur [28] that mean field theory apparently violated. Elitzur's theorem simply says that no non-gauge invariant operator can acquire a non-zero expectation value. This is to be contrasted with phase transitions in a spin system for instance. Below the critical temperature if we impose some external field h (which breaks the symmetry) on a system of finite volume V, the expectation value of a spin σ has the following property

$$\lim_{V \to \infty} \lim_{h \to 0} \langle\sigma\rangle = 0 \quad ; \quad \text{however}$$

$$\lim_{h \to 0} \lim_{V \to \infty} \langle\sigma\rangle \neq 0 \quad .$$

In this second sequence of limits one discovers a broken symmetry. For a gauge theory, even if there is a transition to a deconfining phase, for all values of the parameters Elitzur proved that

$$\lim_{V \to \infty} \lim_{h \to 0} \langle U_\ell\rangle = \lim_{h \to 0} \lim_{V \to \infty} \langle U_\ell\rangle = 0$$

(in which h is an infinitesimal source breaking the gauge symmetry)

In the weak coupling phase mean field calculations lead to a non-zero expectation value for a link and contradicts Elitzur's theorem. However this leads to two puzzles (i) mean field theory is qualitatively and quantitatively exact for large d and Elitzur's theorem holds in any dimension (ii) in the weak coupling phase it is obvious that a large Wilson loop of perimeter L behaves as $\langle U\rangle^L$ within mean field theory ; this perimeter law is indeed the expected behaviour of a deconfined phase, and it is difficult to explain this perimeter law if $\langle U\rangle = 0$.

The solution to these paradoxes and the possibility of going beyond this simple mean field variational calculation lies in a reformulation of mean field theory [29,30]. For any problem there is an exact representation in which the degrees of

freedom are decoupled but they are coupled to a random source [31] (whose probability measure depends upon the interactions). Mean field picture is a simple saddle point approximation in which one keeps only the most likely external source configuration. However because of the gauge symmetry of the problem any gauge transformed of a saddle-point is also a saddle-point. Similarly in a spin system there is an orbit of saddle-points related by a global transformation, but since any infinitesimal source lifts the degeneracy, pure states correspond to one single saddle-point. Since, in a gauge theory this (larger) degeneracy is not removed by an external infinitesimal source, we do have to keep the whole orbit of saddle-points. Thus, summing over this orbit it is immediate to verify that all non-gauge invariant expectation values vanish [30]. A gauge invariant operator, such as a Wilson loop, is not affected by this averaging and takes indeed the same value as in the simple one saddle-point mean field approximation. In the weak coupling limit for instance a Wilson loop of perimeter L behaves as U^L and U is not the expectation value of a link (which vanishes).

Once mean field theory is identified as a saddle-point approximation in an integral over random external sources, it is in principle straightforward, though the calculations are often tedious, to include fluctuations of the random source around its most likely value [32]. Successive terms in this expansion are suppressed by increasing powers of $1/d$ (with possibly powers of log d). The first correction to an SU(2) pure gauge calculation eliminates the spurious transition in four dimensions and is already quite close to Monte-Carlo data. See H. Flyvbjerg's contribution to this conference.

Spectrum of gauge theories with quarks in the mean field picture

In 1981 the Brussels group [33] showed that in the strong coupling limit, large N SU(N) - (gauge + quarks) theory, large d limit, chiral symmetry is spontaneously broken, as expected from a true theory of strong interactions. This program was then examined by several groups [34,35]. In the work of H. Kluberg-Stern, A. Morel and B. Petersson [35] the assumptions of large N was relaxed and calculations were performed for SU(2) and SU(3). They started from the fact that when the bare coupling is very large, the pure gauge part of the action of quarks coupled to gluons may be neglected (it is multiplied by a factor $1/g_B^2$) and the action is simply linear in any link gauge variable which sits in between two quark fields. One can integrate out all the link variables because of the linearity of the action. The result is an effective action in terms of quark fields, involving bilinear "meson" operators

$M^{\alpha\beta}_{(r)} = \sum_{i=1}^{N} \chi_i^{\alpha}(r)\bar{\chi}_i^{\beta}(r)$, and multilinear "baryon" operators

$B^{\alpha_1\ldots\alpha_N}(r) = \sum_{i_1\ldots i_N} \varepsilon_{i_1\ldots i_N}\, \chi_{i_1}^{\alpha_1}(r)\ldots\chi_{i_N}^{\alpha_N}(r)$ (the i's are colour indices, and the α's are flavour indices). This effective action is then treated within a mean-field saddle-point approximation, which becomes exact for large d. This yields a spectrum of hadrons depending upon two parameters, the lattice spacing and the light quark mass (stranged and charmed hadrons have not been considered). Fitting the m_π and the m_ρ mass (with $a^{-1} = 440$ MeV and $m_a = 8$ MeV) the authors [35] obtain mesons and baryons masses. The precise identification with the true world is slightly obscured by the use of Susskind's fermions but they find meson states at 1010 and 1160 MeV which could correspond to the δ and A_1 mesons, and a nucleon at 1380 MeV. Corrections in the $1/d$ expansion seem to be rather small. The main problem is manifestly to move back closer to the continuum limit (which is g_B^2 small and not large), but the present results are quite encouraging.

4. Recent results concerning the large N-limit

Many attempts have been made to solve the large N limit of an SU(N) - gauge theory, and one must admit that, in spite of a large number of small advances, the problem is still unsolved. Some recent new ideas initiated by Eguchi and Kawai [37] may lead to significant improvements.

The idea of studying the large N limit goes back to statistical mechanics [38], in which a complete solution may be found (the first $1/N$ corrections may even be computed) and it is a very useful method in order to understand the structure of the

theory (non-perturbative mass generation in the 2-D non-linear σ model, for instance) [39]. Therefore there is little doubt that a complete solution of the large N-limit for gauge theories would be a tremendously important way to understand quark confinement, the mass spectrum, etc... In addition 't Hooft has argued that SU(∞) might be the only well-defined field theory (see 't Hooft's report at this conference). For an SU(N) gauge theory there are simplifications in the large N-limit : 't Hooft has shown in 1974 [40] that in this limit all non-planar Feynman diagrams (i.e. diagrams which cannot be drawn on a plane without lines crossing away from vertices) are suppressed by powers of $1/N^2$. Unfortunately, in spite of the fact that there are only few planar diagrams (μ^n instead of $n!$ at order n), it has not been possible to sum them (except in a low number of dimensions). I will shorten the very rich history of the problem, to discuss directly the 1982 attempts.

Eguchi and Kawai [37] have argued that for a lattice gauge theory in the large-N limit one can replace the infinite volume lattice by a finite periodic lattice $\ell\times\ell\times\ell\times\ell$, the simplest being $\ell = 1$. We consider, for simplicity this reduced $\ell = 1$ model. The original model

$$\beta H = -\beta N \sum_{\substack{x \\ 1\leq\mu<\nu\leq 4}} \mathrm{Tr}(U_{x,\mu}\, U_{x+\mu,\nu}\, U^{\dagger}_{x+\nu,\mu}\, U^{\dagger}_{x,\nu} + \mathrm{h.c.})$$

is replaced by the reduced model

$$\beta H_R = -\beta N \sum_{1\leq\mu<\nu\leq 4} \mathrm{Tr}(U_{\mu}\, U_{\nu}\, U^{\dagger}_{\mu}\, U^{\dagger}_{\nu} + \mathrm{h.c.})$$

in which there are only four $N \times N$ matrices. A Wilson loop on the original lattice, is mapped into the reduced model by forgetting in any link variable $U_{x,\mu}$ (starting from the point x pointing in the direction μ) the origin of the link : $U_{x,\mu} \to U_{\mu}$. The authors of ref [40] proved that for an arbitrary Wilson loop W, mapped by the previous rule into a reduced operator W_R

$$\langle W\rangle_{\text{true}} = \langle W_R\rangle_{\substack{\text{reduced}\\ \text{model}}} + O\left(\frac{1}{N^2}\right) .$$

The proof relies on an identification, in the large N limit, of the equations of motion of the reduced and full problems. Their proof is certainly valid to all orders in the strong coupling expansion. However it is assumed that the $[U(1)]^4$ symmetry of the reduced model, the very simple symmetry $U_{\mu} \to e^{i\theta}\, U_{\mu}$ which leaves H_R invariant, is unbroken in order to cancel unwanted extra-terms. It was immediately pointed out by Bhanot, Heller and Neuberger [41] that in weak coupling this $[U(1)]^4$ symmetry was certainly broken, and therefore that the reduced model, as it stands, cannot describe the physical region (which is the weak coupling limit of the theory). They proposed, in order to cover the whole range of coupling constants, to use a "quenched" version of the reduced model. [This concept of quenching is well-known in the statistical mechanics of *random media* ; consider a system of particles, for instance, an electron gas, interacting with impurities. If these impurities are mobile, they will thermalize with the electron gas and the average physical quantities are obtained by a trace over the electron gas and the impurities degrees of freedom. However if the impurities are frozen, the "quenched" case, the physical observables are obtained by calculating their value for fixed impurities and then averaging over these impurities]. Parisi [42] has presented a simple version of the quenching procedure for the reduced model of Eguchi and Kawai. He introduces a set of arbitrary phase factors and instead of periodic boundary conditions on the unit cell of the reduced model, a phase factor is associated to a unit translation :

$$\text{in the reduced model } U_{x+\mu,\nu} = U_{x,\nu}$$

$$\text{in the quenched reduced model } (U_{x+\mu,\nu})_{ab} = e^{i\theta^a_{\mu}}(U_{x,\nu})_{ab} .$$

Wilson loops are then mapped, with this new rule, into a reduced loop. Its expectation value is calculated for a fixed set of $4 \times N$ phases θ_μ^a ; finally an average is taken over these phases with a uniform distribution on the unit circle. Again this procedure is exact in the large N-limit, and Gross and Kitazawa [43] have shown that it now works for the whole range of coupling constants. This remarkable reduction is still not a solution of the problem, however it may lead to new developments (see I. Bar's contribution to this conference).

Let me conclude with a few additional remarks on this reduced problem.

(i) The reduced model is also exact for fixed N and large dimension d:indeed, in this limit, mean field theory holds and in this approximation link variables take the same value all over the lattice. Therefore the error of the reduced model, compared to the true SU(N) gauge theory, is proportional to $1/N^2 \times 1/d$, which is less than 3% for SU(3) in four dimensions.

(ii) This reduction may explain the success of Monte-Carlo simulations on rather small lattices. Indeed the arguments of Eguchi and Kawai are valid for any periodic lattice $\ell\times\ell\times\ell\times\ell$. One can show [44] that for any ℓ there is a breakdown of this (unquenched) reduced model below some critical value $g_c^2(\ell)$ of this coupling constant. A simple analysis shows that $g_c^2(\ell)$ decreases certainly rapidly with ℓ. For the present typical Monte-Carlo simulations in which ℓ is somewhere between 4 and 8, $g_c^2(\ell)$ could be so small that the breakdown of the reduced model would be invisible, and therefore Monte-Carlo data on these small lattices would be exact in the large N limit. I hope that a more quantitative estimate of $g_c^2(\ell)$ could shed some light on this question.

Conclusion : Numerical simulations remain certainly an irreplaceable tool in order to understand the spectrum of gauge theories. However analytic methods, such as strong coupling expansions, mean field expansions, large N limit, etc... may become increasingly powerful.

References

[1] - J. Schwinger, Proc. Nat. Acad. Sc. (U.S.) 44, 956 (1958)
K. Symanzik, J. Math. Phys. 7, 510 (1966)
[2] - These ideas and many others throughout this report were initiated by K. Wilson
[3] - K. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974) and e.g. E. Brézin, J.C. Le Guillou and J. Zinn-Justin in "Phase transitions and critical phenomena" edited by C. Domb and M.S. Green (Academic, New York 1976) vol VI.
[4] - K. Wilson, Cargèse 1973 (private communication), Phys. Rev. D 10, 2445 (1974)
[5] - E. Marinari, G. Parisi and C. Rebbi, Phys. Rev. Letters. 47, 1795 (1981)
H. Hamber and G. Parisi, Phys. Rev. Letters 47, 1792 (1981)
D. Weingarten, Phys. Letters B109, 57 (1981)
A. Hasenfratz and P. Hasenfratz, Phys. Lett. 104B, 489 (1981)
A. Hasenfratz, Z. Kunszt, P. Hasenfratz and C.B. Lang, Cern-Th 3220 Preprint (1981), etc...
[6] - A. Polyakov, Phys. Lett. 103B, 207 and 211 (1981)
[7] - H.B. Nielsen and M. Ninomiya, Nucl. Phys. B185 20 (1981) and B193, 173 (1981)
D. Friedan, Comm. Math. Phys. 85, 481 (1982)
[8] - G. Bhanot and R. Dashen "Do numbers extracted near crossovers represent continuum physics" I.A.S. Princeton preprint (1982)
[9] - S. Samuel "Large N lattice QCD with fundamental and adjoint action terms" CU.TP225 Columbia preprint (1982)
B. Grossman and S. Samuel "Universality in SU(2) lattice gauge theory" RU62/B/25 Rockefeller Univ. Preprint (1982)
[10] - A. Polyakov, Phys. Lett. 72B, 477 (1978)
J. Kuti, J. Polonyi and K. Szlachanyi, Phys. Lett. 98B, 199 (1981)
J. Engels, F. Karsch, I. Montvay and H. Satz, Phys. Lett. 101B, 89 (1981), Phys. Lett. 102B, 332 (1981)
J. Engels, F. Karsch and H. Satz "The high temperature behaviour of lattice QCD with fermions" BI-TP 82/8 Bielefeld preprint
[11] - References to the Monte-Carlo method should be consulted in C. Rebbi's report at this conference.

[12] - G. Parisi and Wu Yong-Shi, Scientia Sinica 24, 483 (1981)
[13] - S. Coleman, Phys. Rev. D15, 2929 (1977)
C. Callan and S. Coleman D16, 1762 (1977)
[14] - J. Langer, Ann. Phys. (N.Y.) 41, 108 (1967)
[15] - G. Parisi, Proc. of the XX International Conference on High Energy Physics, Madison, Am. Inst. of Physics (1981)
[16] - M. Creutz, Phys. Rev. Lett. 45, 313 (1980)
[17] - K. Symanzik "Improved lattice actions" DESY Preprint 81-068 (1981)
G. Martinelli, G. Parisi and R. Petronzio, CERN preprint TH 3283 (1983)
P. Weisz, DESY Preprint (1982)
[18] - H.B. Nielsen, in "Fundamentals of Quark Models", I. Barbour and A. Davies eds. (Scottish Univ. Summer School in Physics Publ. 1977)
[19] - S. Chadha and H.B. Nielsen, Niels Bohr Institute report (1977) unpublished
D. Förster, H.B. Nielsen and M. Ninomiya, Phys. Lett. 94B, 135 (1980)
[20] - S. Shenker (1980) reference unknown
[21] - C. Itzykson, R. Pearson and J.B. Zuber, Saclay Preprint
[22] - J. Kogut, D.K. Sinclair and L. Susskind, Nucl. Phys. B114, 199 (1976)
G. Münster, Nucl. Phys. B190 (FS3), 439 (1981)
[23] - N. Kimura and A. Ukawa, Preprint
J. Smit, Amsterdam ITFA-82-3 (1982) preprint
[24] - B. Berg and A. Billoire, Phys. Lett. 113B, 65 (1982) and CERN preprint TH 3267 (1982)
[25] - R. Balian, J.M. Drouffe and C. Itzykson, Phys. Rev. D10 3376 (1974) ; D11, 2098 (1975) ; D11, 2104 (1975)
[26] - This is quite standard in statistical mechanics ; see for instance ref [3]
[27] - M. Creutz, Phys. Rev. Letters 43, 553 (1979)
[28] - S. Elitzur, Phys. Rev. D12 3978 (1975)
[29] - J.M. Drouffe, Nucl. Phys. B170 (FS1), 211 (1980)
[30] - E. Brézin and J.M. Drouffe, Nucl. Phys. B 200 (FS4), 93 (1982)
[31] - E. Brézin, J.C. Le Guillou and J. Zinn-Justin, see ref [3] p. 169-170
[32] - H. Flyvbjerg, B. Lautrup and J.B. Zuber, Phys. Lett. 110B, 279 (1982)
V. Alessandrini, V. Hakim and A. Krsywicki, Nucl. Phys. B (to be published)
J.M. Alberty, H. Flyvberg and B. Lautrup, Nordita Preprint NBl-He-82-20 (1982)
V.F. Müller and W. Rühl, Kaiserslautern preprint (1982)
[33] - J.M. Blairon, R. Brout, F. Englert and J. Greensite, Nucl. Phys. B180 (FS2) 439, (1981)
[34] - N. Kawamoto and J. Smit, Nucl. Phys. B182 100 (1981)
H. Kluberg-Stern, A. Morel, O. Napoly and B. Peterson, Nucl. Phys. B190 (FS3) 504 (1981)
[35] - H. Kluberg-Stern, A. Morel and B. Peterson, Saclay Preprint DPh-T/82-21 (1982)
[36] - Since by mistake, there is no reference [36] in the text, I'll use the available space to apologize for all the omissions in this brief and incomplete list of references
[37] - T. Eguchi and H. Kawai, Phys. Rev. Letters 48, 1063 (1982)
[38] - E. Stanley, Phys. Rev. 176B 178 (1968) ,
K.G. Wilson 1972 (private communication)
[39] - A. Polyakov, Phys. Lett. B59, 79 (1975)
E. Brézin, J. Zinn-Justin, Phys. Rev. B14, 3110 (1976)
[40] - 't Hooft, Nucl. Phys. B72, 461 (1974)
[41] - G. Bhanot, U. Heller and H. Neuberger, Phys. Lett. 113B, 47 (1982)
This breakdown in weak coupling has independently been noticed by other group. See also V. Kazakov and A.A. Migdal, Phys. Lett (1982)
[42] - G. Parisi, Phys. Lett. 112B, 463 (1982)
[43] - D.J. Gross and Y. Kitazawa, Princeton Univ. preprint (1982)
[44] - This may be done (E. Brézin, J.B. Zuber unpublished) with the use of special field configurations discussed in a different context by
A. Gonzalez-Arroyo, J. Jurkiewicz and C.P. Korthals-Altes, Marseille Preprint (1982)

Discussion

M. WEINSTEIN (SLAC).- I only raise the question because you did, but the Nielsen-Ninomiya theorem is careful to exclude the SLAC (or long range derivative) to which it does not apply. The only work which bears upon this is the claim of Karsten and Smit which has recently been shown by Rabin to be trivially incorrect.