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On the Computation of the Topology of a Non-Reduced Implicit Space Curve

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ABSTRACT
An algorithm is presented for the computation of the topology of a non-reduced space curve defined as the intersection of two implicit algebraic surfaces. It computes a Piecewise Linear Structure (PLS) isotopic to the original space curve.

The algorithm is designed to provide the exact result for all inputs. It’s a symbolic-numeric algorithm based on subresultant computation. Simple algebraic criteria are given to certify the output of the algorithm.

The algorithm uses only one projection of the non-reduced space curve augmented with adjacency information around some “particular points” of the space curve.

The algorithm is implemented with the Mathemagix Computer Algebra System (CAS) using the SYNAPS library as a backend.

Categories and Subject Descriptors
I.1.4 [Symbolic and Algebraic Manipulation]: Applications; I.3.5 [Computer Methodologies]: Computer Graphics, Computational Geometry and Object Modeling-Geometric Algorithms

General Terms
Algorithms

Keywords
Algebraic Curves, Subresultants Sequence, Generic Conditions, Topology Computation, Sturm-Habicht Sequence, Exact Geometric Computation

Introduction
The problem of computing the topological graph of algebraic curves plays an important role in many applications such as plotting and sectioning in Computer Aided Geometric Design (CAD). A wide literature exists on the computation of the topology of plane curves ([8], [9], [10], [11], [12], [13]). The problem of computing the topology of space curves has been less investigated. In [1], Alcázar and Sendra give a symbolic-numeric algorithm for reduced space curves using subresultant and GCD computations of approximated polynomials. If their approach gives good practical results however it doesn’t give a rigorous proof that a sufficient precision is selected for all inputs in the computation of GCD of approximated polynomials. In [11], Owen, Rockwood and Ayn give a numerical algorithm for reduced space curve using subdivision method. Their algorithm has a good complexity but the topology around the singularities of the space curve is not certified. We also mention the work in [1], where two projections of a reduced space curve are used, and where the connection algorithm is valid under genericity conditions.

To our knowledge, the general problem of computing the topology of non-reduced space curves is not investigated in the algorithmic point of view despite its significance in the problem of computing the topology of a real algebraic surface.

We present a certified algorithm that computes the topology of non-reduced algebraic space curves. We compute the topology of a plane projection of the space curve and then we lift the computed topology on the space. The topology of the projected curve is computed using a classical sweeping algorithm (see [8], [12]). For the computation of the topology of a plane algebraic curve, we present an efficient generic test that certifies the output of the algorithm in [1].

For space curves, we introduce the notion of pseudo-generic position. A space curve is said to be in pseudo-generic position with respect to the \((x, y)-plane\) if and only if almost every point of its projection on the \((x, y)-plane\) has only one geometric inverse-image. A simple algebraic criterion is given to certify the pseudo-genericity of the position of a space curve. From a theoretical point of view, the use of the notion of curve in pseudo-generic position gives us a rational parametrization of the space curve. The use of this rational parametrization allows us to lift the topology computed after projection without any supplementary effort. From a practical point of view, the use of the rational parametrization of the space curve makes the lifting faster, avoiding numerical problems.

We need to distinguish two kinds of singularities on the projected curve. A certified algorithm is given to do so. Unlike previous approaches, our algorithm uses only one projection of the space curve and works for non-reduced space curves. We therefore avoid the cost of the second
projection used by previous approaches.
In the next section we describe the fundamental algebraic tools that we use in this paper. In Section 2, we present our contribution to certify the algorithm for computing the topology of a plane algebraic curve. Our algorithm itself is introduced in Section 3. We report on our implementation and experiments in section 4.

1. SUBRESULTS

Let \( P, Q \in \mathbb{R}[X, Y, Z] \) and \( \mathcal{C}_R := \{(x, y, z) \in \mathbb{R}^3 \mid P(x, y, z) = 0\} \) be the intersection of the vanishing sets of \( P \) and \( Q \). Our curve analysis needs to compute a plane projection of \( \mathcal{C}_R \). Subresultant sequences are a suitable tool to do it. For the reader’s convenience, we recall their definition and relevant properties. For all the results of this section, we refer to [5] for proofs.

Let \( \mathcal{A} \) be a integral domain. Let \( P = \sum_{i=0}^{p} a_i X^i \) and \( Q = \sum_{i=q} b_i X^i \) be two polynomials with coefficients in \( \mathcal{A} \). We shall always assume \( a_p \neq 0, b_q \neq 0 \) and \( p \geq q \).

Let \( \mathbb{P}_r(\mathcal{A}) \) be the set of polynomials in \( \mathcal{A}[X] \) of degree not exceeding \( r \), always, with the basis (as an \( \mathcal{A} \)-module) \( 1, X, \ldots, X^r \). If \( r < 0 \), we set \( \mathbb{P}_r(\mathcal{A}) = 0 \) by convention, and we will identify an element \( S = s_0 + \ldots + s_r X^r \) of \( \mathbb{P}_r(\mathcal{A}) \) with the row vector \((s_0, \ldots, s_r)\).

Let \( k \) be an integer such that \( 0 \leq k \leq q \), and let \( \Psi_k \colon \mathbb{P}_{q-k-1}(\mathcal{A}) \times \mathbb{P}_{p-k-1}(\mathcal{A}) \rightarrow \mathbb{P}_{p+q-k-1}(\mathcal{A}) \) be the \( \mathcal{A} \)-linear map defined by \( \Psi_k(U, V) = PU + QV \), with \( M_k(P, Q) \) the \((p + q - k) \times (p + q - k)\) matrix of \( \Psi_k \). As we write vectors as row vectors, we have

\[
M_k(P, Q) = \begin{pmatrix}
a_0 & \ldots & a_p \\
& \ldots & \\
b_0 & \ldots & b_q
\end{pmatrix}
\]

That is \( M_0(P, Q) \) is the classical Sylvester matrix associated to \( P, Q \). To be coherent with the degree of polynomials, we will attach index \( i \) to the \( i \)th column of \( M_k(P, Q) \), so the indices of the columns go from 0 to \( p + q - k - 1 \).

Definition 1 For \( j \leq p+q-k-1 \) and \( 0 \leq k \leq q \), let \( sr_{k,j} \) be the determinant of the submatrix of \( M_k(P, Q) \) formed by the last \( p+q-2k \) columns, of index \( j \) and all the \((p + q - 2k)\) rows. The polynomial \( sr_k(P, Q) = sr_{k,0} + \ldots + sr_{k,k} X^k \) is the \( k \)th sub-GCD of \( P \) and \( Q \), and its leading term \( sr_{k,k} \) (sometimes noted \( sr_k \)) is the \( k \)th subresultant of \( P \) and \( Q \). So, it follows that \( sr_0(P, Q) = sr_0 \) is the usual resultant of \( P \) and \( Q \).

Remark 1

1. For \( k < j \leq p + q - k - 1 \), we have \( sr_{k,j} = 0 \), because it is the determinant of a matrix with two equal columns.
2. If \( q < p \), we have \( sr_q = (b_q)^{p-q-1}Q \) and \( sr_q = (b_q)^{p-q} \).

The following proposition will justify the name of sub-GCD given to the polynomial \( sr_k \).

Proposition 1 Let \( d \) be the degree of the GCD of \( P \) and \( Q \) (\( d \) is defined because \( \mathcal{A} \) is an integral domain, so we may compute the GCD over the quotient field of \( \mathcal{A} \)). Let \( k \) be an integer such that \( k \leq d \).

1. The following assertions are equivalent:
   a) \( k < d \);
   b) \( sr_k = 0 \);
   c) \( sr_k = 0 \).

2. \( sr_d \neq 0 \) and \( sr_d \) is the GCD of \( P \) and \( Q \).

Theorem 1 Fundamental property of subresultants

The first polynomial \( Sr_k \) associated to \( P \) and \( Q \) with \( Sr_k \neq 0 \) is the greatest common divisor of \( P \) and \( Q \).

We will often call \( (Sr_i) \), the subresultant sequence associated to \( P \) and \( Q \) (\( sr_{i,j} \)) the sequence of their subresultants coefficients. We will denote by \( \text{lcoe}_f(x) \) the leading coefficient of the polynomial \( f \) with respect to the variable \( X \).

Theorem 2 Specialization property of subresultants

Let \( P_1, P_2 \in \mathcal{A}[Y, Z] \) and \( (Sr_i(Y, Z)) \), be their subresultant sequence with respect to \( Z \). Then for any \( \alpha \in \mathcal{A} \) with:

\[
\deg_{\mathcal{A}}(P(Y, Z)) = \deg_{\mathcal{A}}(P(\alpha, Z));
\]

\[
\deg_{\mathcal{A}}(Q(Y, Z)) = \deg_{\mathcal{A}}(Q(\alpha, Z));
\]

\[
(Sr_i(\alpha, Z)), i \]

is the subresultant sequence of the polynomials \( P(\alpha, Z) \) and \( Q(\alpha, Z) \).

2. TOPOLOGY OF A PLANE ALGEBRAIC CURVE

Let \( f \in \mathbb{Q}[X, Y] \) be a square free polynomial and

\[
\mathcal{C}(f) := \{(\alpha, \beta) \in \mathbb{R}^2 \mid f(\alpha, \beta) = 0\}
\]

be the real algebraic curve associated to \( f \). We want to compute the topology of \( \mathcal{C}(f) \).

For curves in generic position, computing its critical fibers and one regular fiber between two critical ones is sufficient to obtain the topology using a sweeping algorithm (see [5]). But for a good computational behaviour, it is essential to certify the genericity of the position of the curve.

We propose an effective test allowing to certify the computation and connection, in a deterministic way. This is an important tool in order to address the case of space curves.

Now, let us introduce the definitions of generic position, critical, singular and regular points.

Definition 2 Let \( f \in \mathbb{Q}[X, Y] \) be a square free polynomial and \( \mathcal{C}(f) = \{(\alpha, \beta) \in \mathbb{R}^2 \mid f(\alpha, \beta) = 0\} \) be the curve defined by \( f \). A point \( (\alpha, \beta) \in \mathcal{C}(f) \) is called:

- a \( x \)-critical point if \( \partial_Y f(\alpha, \beta) = 0 \),
- a singular point if \( \partial_X f(\alpha, \beta) = 0 \),
- a regular point if \( \partial_X f(\alpha, \beta) \neq 0 \) or \( \partial_Y f(\alpha, \beta) \neq 0 \).

With these definitions we can describe the generic conditions required for plane curves.
**Definition 3** Let \( f \in \mathbb{Q}[X, Y] \) be a square free polynomial and \( \mathcal{C}(f) = \{(\alpha, \beta) \in \mathbb{R}^2 : f(\alpha, \beta) = 0 \} \) be the curve defined by \( f \). Let \( N_\alpha(\alpha) := \#\{\beta \in \mathbb{R}, \text{ such that } (\alpha, \beta) \text{ is a } x\text{-critical point of } \mathcal{C}(f) \} \). \( \mathcal{C}(f) \) is in generic position for the \( x \)-direction, if:

1. \( \forall \alpha \in \mathbb{C}, N_\alpha(\alpha) \leq 1 \).
2. There is no asymptotic direction of \( \mathcal{C}(f) \) parallel to the \( y \)-axis.

This notion of genericity also appears in [2] and in a slightly more restrictive form in [1]. Previous approaches succeed if genericity conditions are satisfied, but they do not guarantee to reject the curve if they are not; i.e., it does not decide genericity. So for some input curves the computed topology might not be exact.

A change of coordinates such that \( \text{locf}(f) \in \mathbb{Q}^* \) is sufficient to place \( \mathcal{C}(f) \) in a position such that any asymptotic direction is not parallel to the \( y \)-axis. It remains to find an efficient way to verify the first condition. This follows from the next propositions. We refer to [3] for proofs.

**Proposition 2** Let \( f \in \mathbb{Q}[X, Y] \) be a square free polynomial with \( \text{locf}(f) \in \mathbb{Q}^* \) and \( d := \deg_y(f) \). We denote by \( \mathcal{S}_k(X, Y) \) the \( i \)-th subresultant polynomial of \( f \) and \( \partial Y \) and \( \mathcal{S}_{i,j}(X) \) the coefficient of \( Y^j \) in \( \mathcal{S}_i(X, Y) \). We define inductively the following polynomials:

\[
\Phi_0(X) = \frac{\mathcal{S}_{0,0}(X)}{\gcd(\mathcal{S}_{0,0}(X), \mathcal{S}'_{0,0}(X))},
\]

\[
\forall i \in \{1, \ldots, d-1\}, \Phi_i(X) = \gcd(\Phi_{i-1}(X), \mathcal{S}_{i,j}(X)) \text{ and } \Gamma_i(X) = \frac{\Phi_i(X)}{\Phi_i(X)}
\]

**Proposition 3**

1. \( \Phi_0(X) = \sum_{i=1}^{d-1} \Gamma_i(X) \) and \( \forall i, j \in \{1, \ldots, d-1\}, i \neq j \implies \gcd(\Gamma_i(X), \Gamma_j(X)) = 1 \).
2. Let \( k \in \{1, \ldots, d-1\}, \alpha \in \mathbb{C}, \Gamma_k(\alpha) = 0 \) if and only if \( \gcd(f(\alpha, Y), \partial_Y f(\alpha, Y)) = \mathcal{S}_k(\alpha, Y) \).
3. \( \{\alpha, \beta\} \in \mathbb{R}^2 : f(\alpha, \beta) = \partial_Y f(\alpha, \beta) = 0 \}= \bigcup_{k=1}^{d-1} \{\alpha, \beta\} \in \mathbb{R}^2 : \Gamma_k(\alpha) = \mathcal{S}_k(\alpha, \beta) = 0 \}.

In the following theorem, we give an effective and efficient algebraic test to certify the genericity of the position of a curve with respect to a given direction.

**Theorem 3** Let \( f \in \mathbb{Q}[X, Y] \) be a square free polynomial such that \( \deg_y(f) = d \), \( \text{locf}(f) \in \mathbb{Q}^* \). Then \( \mathcal{C}(f) \) is in generic position for the \( x \)-axis if and only if \( \forall k \in \{1, \ldots, d-1\}, \forall i \in \{0, \ldots, k-1\}, k(\alpha, \beta) \in \mathbb{R}^2 : \Gamma_k(\alpha, \beta) = \mathcal{S}_k(\alpha, \beta) = 0 \) or \( k(\alpha, \beta) \in \mathbb{R}^2 : \Gamma_k(\alpha, \beta) = \mathcal{S}_k(\alpha, \beta) = 0 \).

With the same argument used in the first part of this proof we obtain

\[
\gcd(f(\alpha, Y), \partial_Y f(\alpha, Y)) = \mathcal{S}_k(\alpha, Y)
\]

and

\[
\text{gcd}(f(\alpha, Y), \partial_y f(\alpha, Y)) = \mathcal{S}_k(\alpha, Y) = \mathcal{S}_k(\alpha)(Y - \beta)^k
\]

with

\[
\beta = \frac{\mathcal{S}_{k,k-1}^{\pm 1}(\alpha)}{k \mathcal{S}_{k,k}(\alpha)}.
\]

Then we conclude that \( \gcd(f(\alpha, Y), \partial_Y f(\alpha, Y)) \) has only one distinct root and, according to Proposition 3, \( \mathcal{C}(f) \) is in generic position.

**Remark 2** Theorem 3 shows that it is possible to check with certainty if a plane algebraic curve is in generic position or not. If not, we can put it in generic position by a basis change.

In fact, it is well known that there is only a finite number of bad changes of coordinates of the form \( X := X + \lambda Y, \quad Y := Y, \quad \lambda \in \mathbb{C} \), such that if \( \mathcal{C}(f) \) is not in generic position then the transformed curve remains in a non-generic position. This number of bad cases is bounded by \( \binom{d}{c} \), where \( c \) is the number of distinct \( x \)-critical points of \( \mathcal{C}(f) \).

### 3. Topology of Implicit Three Dimensional Algebraic Curves

#### 3.1 Description of the problem

Let \( P_1, P_2 \in \mathbb{Q}[X, Y, Z] \) and

\[
\mathcal{C}_R := \{(x, y, z) \in \mathbb{R}^3 : P_1(x, y, z) = P_2(x, y, z) = 0 \}
\]

be the intersection of the surfaces defined by \( P_1 \) and \( P_2 \). We assume that \( \gcd(P_1, P_2) = 1 \) so that \( \mathcal{C}_R \) is a space curve. Our goal is to analyze the geometry of \( \mathcal{C}_R \) in the following sense: We want to compute a piecewise linear graph of \( \mathbb{R}^3 \) isotopic to the original space curve.

Our method allows to use a new sweeping algorithm using only one projection of the space curve.

To make the lifting possible using only one projection, a new definition of generic position for space curves and an algebraic characterization of it are given. We will also need to distinguish the "apparent singularities" and the "real singularities". A certified algorithm is given to distinguish
these two kinds of singularities.
For the lifting phase, using the new notion of curve in pseudo-
generic position, we give an algorithm that computes a rational parametrization of the space curve. The use of this rational parametrization allows us to lift the topology of the projected curve without any supplementary computation.

3.2 Genericity conditions for space curves
Let \( \Pi : (x, y, z) \in \mathbb{R}^3 \mapsto (x, y) \in \mathbb{R}^2 \). We still denote \( \Pi_x = \Pi|_{\mathbb{R}^2} \). Let \( \mathcal{D} = \Pi_x(C_R) \subset \mathbb{R}^2 \) be the curve obtained by projection of \( C_R \).

We assume that \( \deg_X(P_1) = \deg(P_1) \) and \( \deg_Z(P_2) = \deg(P_2) \) (by a basis change, these conditions are always satisfied). Let \( h(X, Y) \) be the squarefree part of \( \text{Res}_{Z}(P_1, P_2) \in \mathbb{Q}[X, Y] \). With the above notation and assumptions we have the following “geometric” equality, \( \Pi_x(C_R) = \mathcal{C}(h) \).

**Definition 4** [Pseudo-generic position]
Let \( C_C := \{(x, y, z) \in \mathbb{C}^3 \mid \Pi_x(z) = P_2(x, y, z) = 0 \} \).
The curve \( C_R \) is in pseudo-generic position with respect to the \((x, y)-plane\) if and only if almost every point of \( \Pi_x(C_C) \) has only one geometric inverse-image, i.e. generically, if \((\alpha, \beta) \in \Pi_x(C_C)\), then \( \Pi_x^{-1}(\alpha, \beta) \) consists in one point possibly multiple.

Let \( m \) be the minimum of \( \deg_Z(P_1) \) and \( \deg_Z(P_2) \).
The following theorems give us an effective way to test if a curve is in a pseudo-generic position or not.

**Theorem 4** Let \( (\text{Sr}_i(X, Y, Z))_{i \in \{0, \ldots, m\}} \) be the subresultant sequence and \( (\text{sr}_i(X, Y))_{i \in \{0, \ldots, m\}} \) be the principal subresultant coefficient sequence. Let \( (\Delta_i(X, Y))_{i \in \{1, \ldots, m\}} \) be the sequence of \( \mathbb{Q}[X, Y] \) defined by the following relations

- \( \Delta_0(X, Y) = 1 ; \Theta_0(X, Y) = h(X, Y) \);
- \( \Delta_i(X, Y) = \Theta_i(X, Y) = \frac{\text{gcd}(\Theta_{i-1}(X, Y), \text{sr}_i(X, Y))}{\text{sr}_{i-1}(X, Y)} \).

For \( i \in \{1, \ldots, m\} \), let \( \mathcal{C}(\Delta_i) := \{(x, y) \in \mathbb{R}^2 \mid \Delta_i(x, y) = 0 \} \) and \( \mathcal{C}(h) := \{(x, y) \in \mathbb{R}^2 \mid h(x, y) = 0 \} \) then

1. \( h(X, Y) = \prod_{i=1}^{m} \Delta_i(X, Y) \),
2. \( \mathcal{C}(h) = \bigcup_{i=1}^{m} \mathcal{C}(\Delta_i) \).
3. \( C_R \) is in pseudo-generic position with respect to the \((x, y)-plane\) if and only if \( \forall i \in \{1, \ldots, m\} \), \( \forall (x, y) \in \mathbb{C}^2 \) such that \( \text{sr}_i(x, y) \neq 0 \) and \( \Delta_i(x, y) = 0 \), we have \( \text{Sr}_i(x, y, Z) = \text{sr}_i(x, y) \left( Z + \frac{\text{sr}_{i-1}(x, y)}{\text{sr}_i(x, y)} \right)^i \).

**Proof.** 1. By definition, \( \forall i \in \{1, \ldots, m\} \),
\[ \Delta_i(X, Y) = \frac{\Theta_i(X, Y)}{\Theta_{i-1}(X, Y)} \]
So by a trivial induction
\[ \prod_{i=1}^{m} \Delta_i(X, Y) = \Theta_0(X, Y) / \Theta_m(X, Y) \].
\( \deg_Z(P_1) = \deg(P_1) \) and \( \deg_Z(P_2) = \deg(P_2) \) imply \( \text{sr}_m(X, Y) \in \mathbb{Q}^* \) (see Remark 3).
So \( \Theta_m(X, Y) = \text{gcd}(\Theta_{m-1}(X, Y), \text{sr}_m(X, Y)) = 1 \), then
\[ h(X, Y) = \Theta_0(X, Y) / \Theta_m(X, Y) \].

2. Knowing that \( h(X, Y) = \prod_{i=1}^{m} \Delta_i(X, Y) \), so it is clear that \( \mathcal{C}(h) = \bigcup_{i=1}^{m} \mathcal{C}(\Delta_i) \).

3. Assume that \( C_R \) is in pseudo-generic position with respect to the \((x, y)-plane\). Let \( i \in \{1, \ldots, m\} \) and \( (\alpha, \beta) \in \mathbb{C}^2 \) such that \( \text{sr}_i(\alpha, \beta) \neq 0 \) and \( \Delta_i(\alpha, \beta) = 0 \). Then \( \Delta_i(X, Y) = \frac{\Theta_i(X, Y)}{\Theta_{i-1}(X, Y)} \Rightarrow \Theta_{i-1}(\alpha, \beta) = 0 \). Knowing that \( \Theta_{i-1}(X, Y) = \text{gcd}(\Theta_{i-1}(X, Y), \text{sr}_{i-1}(X, Y)) \), so it exists \( d_1, d_2 \in \mathbb{Q}[X, Y] \) such that \( \text{sr}_{i-2}(X, Y) = d_1(X, Y) \Theta_i(X, Y) \Theta_{i-1}(X, Y) \). In this way,
\[ \Theta_{i-2}(\alpha, \beta) = 0 \Rightarrow \Theta_{i-2}(\alpha, \beta) = 0 \) and \( \text{sr}_{i-1}(\alpha, \beta) = 0 \). By repeating the same argument, \( \Theta_{i-2}(\alpha, \beta) = 0 \) and \( \text{sr}_{i-1}(\alpha, \beta) = 0 \).

The following proposition is a corollary of the third result of the previous theorem. If \( C_R \) is in pseudo-generic position with respect to the \((x, y)-plane\), it gives a rational parametrization for the regular points of \( C_R \).

**Proposition 4** Assume that \( C_R \) is in pseudo-generic position with respect to the \((x, y)-plane\) and let \( (\alpha, \beta, \gamma) \in \mathbb{C}^3 \) such that \( \text{sr}_i(\alpha, \beta) \neq 0 \) and \( \Delta_i(\alpha, \beta) = 0 \). Then,
\[ \gamma := -\frac{\text{sr}_{i-1}(\alpha, \beta)}{\text{sr}_i(\alpha, \beta)} \].

**Remark 3** By construction, the parametrization given in Proposition 4 is valid when \( \text{sr}_i(\alpha, \beta) \neq 0 \). If \( \text{sr}_i(\alpha, \beta) = 0 \) then either \( \Delta_j(\alpha, \beta) = 0 \) for some \( j > i \) or \((\alpha, \beta)\) is a x-critical point of \( \mathcal{C}(\Delta_i) \) (see section 3.3).
Theorem 5 Let \((S_{r_i}(X, Y, Z))_{i \in \{0, \ldots, m\}}\) be the subresultants sequence associated to \(P_1(X, Y, Z)\) and \(P_2(X, Y, Z)\) and \(\Delta_i(X, Y)\) be the sequence of \(Q[X, Y]\) previously defined. The curve \(C_{r_i}\) is in pseudo-generic position with respect to the \((x, y)\)-plane if and only if

\[
\forall i \in \{1, \ldots, m-1\}, \forall j \in \{0, \ldots, i-1\}, \quad i(i-j) s_{r,i-1}(X, Y) - (j+1) s_{r,i+1}(X, Y) = 0 \mod \Delta_i(X, Y).
\]

Proof. Assume \(C_{r_i}\) be in pseudo-generic position. Let \(i \in \{1, \ldots, m-1\}, j \in \{0, \ldots, i-1\}, \quad (\alpha, \beta) \in \mathbb{R}^2\) such that

\[
\Delta_i(\alpha, \beta) = 0.
\]

If \(s_{r,i-1}(\alpha, \beta) = 0\), then \(s_{r,i-1}(\alpha, \beta) = 0\), consequently \((i+j) s_{r,i+1}(\alpha, \beta) - (i-j) s_{r,i-1}(\alpha, \beta) s_{r,j}(\alpha, \beta) = 0\).

If \(s_{r,i}(\alpha, \beta) \neq 0\), then according to Theorem 4 (3.)

\[
S_{r_i}(\alpha, \beta, Z) = \sum_{j=0}^{i} s_{r,i-j}(\alpha, \beta) Z^{-j} = s_{r,i}(\alpha, \beta) \left( Z + \frac{s_{r,i-1}(\alpha, \beta)}{s_{r,i-1}(\alpha, \beta)} \right)^i.
\]

Let \(\gamma := -\frac{s_{r,i-1}(\alpha, \beta)}{s_{r,i-1}(\alpha, \beta)}\), then

\[
S_{r_i}(\alpha, \beta, Z) = \sum_{j=0}^{i} s_{r,i-j}(\alpha, \beta) Z^{-j} = s_{r,i}(\alpha, \beta) (Z - \gamma)^i.
\]

Using the binomial Newton formula we obtain

\[
S_{r_i}(\alpha, \beta, Z) = \sum_{j=0}^{i} s_{r,i-j}(\alpha, \beta) Z^{-j} = s_{r,i}(\alpha, \beta) \sum_{j=0}^{i} \binom{i}{j} (-\gamma)^{-j} Z^j.\]

So by identification, it comes that

\[
\forall i \in \{1, \ldots, m-1\}, \forall j \in \{0, \ldots, i-1\},
\]

\[
i(i-j) s_{r,i}(\alpha, \beta) - (j+1) s_{r,i+1}(\alpha, \beta) = 0, \forall (\alpha, \beta), \quad \Delta_i(\alpha, \beta) = 0.
\]

The reciprocal uses the same arguments.

Remark 4 Theorem 5 shows that it is possible to check with certainty if a space algebraic curve is in pseudo-generic position or not. If it is not, we can put it in pseudo-generic position by a change of coordinates. In fact, there is only a finite number of bad changes of coordinates of the form \(X := X + \lambda Z, Y := Y + \mu Z, \quad Z := Z\), with \(\lambda, \mu \in \mathbb{Q}^*\) such that if \(C_{r_i}\) is not in pseudo-generic position then the transformed curve remains in a non-pseudo-generic position.

Let us introduce the definitions of generic position, critical, singular, regular points, apparent singularity and real singularity for a space algebraic curve.

Definition 5 Let \(M(X, Y, Z)\) be the \(2 \times 3\) Jacobian matrix with rows \((\partial X P_1, \partial Y P_1, \partial Z P_1)\) and \((\partial X P_2, \partial Y P_2, \partial Z P_2)\).

- A point \(p \in C_{r_i}\) is regular (or smooth) if the rank of \(M(p)\) is 2.
- A point \(p \in C_{r_i}\) which is not regular is called singular.
- A point \(p = (\alpha, \beta, \gamma) \in C_{r_i}\) is \(x\)-critical (or critical for the projection on the \(x\)-axis) if the curve \(C_{r_i}\) is tangent at this point to a plane parallel to the \((y, z)\)-plane. The corresponding \(x\) is called a \(x\)-critical value.

Definition 6 [Apparent singularity, Real singularity]

We call:

1. Apparent singularities: the singularities of the projected curve \(D = \Pi_z(C_{r_i})\) with at least two points as inverse-images (see figure 3).
2. Real singularities: the singularities of the projected curve \(D = \Pi_x(C_{r_i})\) with exactly one point as inverse-image (see figure 4).

Definition 7 [Generic position]

The curve \(C_{r_i}\) is in generic position with respect to the \((x, y)\)-plane if and only if

1. \(C_{r_i}\) is in pseudo-generic position with respect to the \((x, y)\)-plane,
2. \(D = \Pi_z(C_{r_i})\) is in generic position (as a plane algebraic curve) with respect to the \(x\)-direction,
3. any apparent singularity of \(D = \Pi_x(C_{r_i})\) is a node.

This notion of genericity also appears in a slightly more restrictive form in [ ].

The aim of the next section is to give an algorithm to certify the third point of the previous definition of generic position. We give also in this section an effective way to distinguish the real singularities from the apparent ones.

3.3 Distinguish real singularities and apparent singularities

In this section, we suppose that \(C_{r_i}\) is in pseudo-generic position and \(D = \Pi_z(C_{r_i})\) is in generic position as a plane algebraic curve.

Let \((\Gamma_j(X))_{j \in \{1, \ldots, n\}}\) be the sequence of \(\Gamma\) polynomials associated to the plane curve \(D\) and \((\beta_j(X))_{j \in \{1, \ldots, n\}}\) be the sequence of associated rational parametrization (see ).

Let \((S_{r,i}(X, Y, Z))_{i \in \{0, \ldots, m\}}\) be the subresultant sequence associated to \(P_1, P_2 \in Q[X, Y, Z]\). For any \((k, i) \in \{1, \ldots, m\} \times \{0, \ldots, k - 1\}\) let, \(R_{k,i}(X, Y)\) be the polynomial

\[
\begin{align*}
\text{apparent singularity} \quad &\text{real singularity} \\
\text{Figure 1: Apparent and real singularities.}
\end{align*}
\]
Lemma 1 Let \((a, b) \in \mathbb{R}^2\) such that \(s_{k, k}(a, b) \neq 0\), the polynomial 
\[S_{k, k}(a, b, Z) = \sum_{i=0}^{k} s_{k, i}(a, b) Z^i \in \mathbb{R}[Z]\] has one and only one root if and only if for every \(i \in \{0, \ldots, k-1\}\) \(R_{k, i}(a, b) = 0\).

For any \(j \in \{1, \ldots, n\}\) we define the sequences 
\((u_k(i))_{i \in \{1, \ldots, n\}}\) and \((v_k(i))_{i \in \{1, \ldots, n\}}\) by 
\[u_k(i) := \gcd(G_i(X), \Gamma_k(X, \beta_j(i))),\]
\[v_k(i) := \text{quo}(u_k(i), \Gamma_k(X)).\]

For \(k \in \{2, \ldots, j\}\) and \(i \in \{0, k-1\}\), we define \((w_k(i))\) by 
\[w_k(0) := v_k(X),\]
\[w_k(i+1) := \gcd(R_k(i, \beta_j(i)), \Gamma_k(X)).\]

Theorem 6 For any \(j \in \{1, \ldots, n\}\), let \((\Gamma_k(X))_{k \in \{1, \ldots, n\}}\) and \((\chi_k(X))\) be the sequences defined by the following relations:
\[\Gamma_1(X) = \text{quo}(\Gamma_1(X), m_1(X))\] and \(\Gamma_k(X) := w_k(X).\)
\[\chi_k(X) := \text{quo}(w_k(0), (\Gamma_k(X))).\]

1. For any root \(\alpha\) of \(\Gamma_k(X)\), the \(x\)-critical fiber \((\alpha, \beta(j))\) contain only the point \((\alpha, \beta(j))\) with \(\beta(j) := \frac{-m_1(X, \beta(j))}{w_k(0)},\) so \((\alpha, \beta(j))\) is a real apparent singularity.

2. For any root \(\alpha\) of \(\chi_k(X)\), \((\alpha, \beta(j))\) is an apparent singularity.

3. \(C_\mathcal{R}\) is in generic position if and only if for any \((j, k) \in \{2, \ldots, n\} \times \{2, \ldots, j\}\) \(\chi_k(X) = 1\).

Proof. 1. Let \(\alpha\) be a root of \(\Gamma_k(X) := w_k(X) = \gcd(R_k(X, \beta_j(i)), w_k(X)).\)

Then \(w_k(0) = 0\) if and only if \(\alpha, \beta(j)\) is a squarefree.

By induction, using the same argument, it comes that for \(i \in \{0, \ldots, k-1\}\), \(w_k(0) = 0\) if and only if \(\alpha, \beta(j)\) is a squarefree.

Knowing that \(v_k(0) := \text{quo}(u_k(0), \Gamma_k(X))\), \(u_k(X)\) and \(u_k(X)\) are squarefree, then \(w_k(0) = 0\) if and only if \(\alpha, \beta(j)\) is a squarefree.

By induction, using the same argument, it comes that for \(i \in \{0, \ldots, k-1\}\), \(\alpha, \beta(j)\) is a squarefree.

For any \((j, k) \in \{2, \ldots, n\} \times \{2, \ldots, j\}\), \(\chi_k(X) = 1\).

3.4 Lifting and connection phase

In this section, we suppose that \(C_\mathcal{R}\) is in generic position that means that in \(\mathcal{R}\) is in pseudo-generic position, \(\mathcal{D} = \Pi_1(\mathcal{C}_\mathcal{R})\) is in generic position as a plane algebraic curve and any apparent singularity of \(\mathcal{D} = \Pi_1(\mathcal{C}_\mathcal{R})\) is a node.

To compute the topology of \(\mathcal{C}_\mathcal{R}\), we first compute the topology of its projection on the \((x, y)\)-plane and in second we lift the computed topology.

As mentioned in section 2, to compute the topology of a plane algebraic curve in generic position, we need to compute its critical fibers and one regular fiber between two critical points. To obtain the topology of \(\mathcal{C}_\mathcal{R}\) we just need to lift the critical and regular fibers of \(\mathcal{D} = \Pi_1(\mathcal{C}_\mathcal{R})\).

Hereafter we explain how this lifting can be done without any supplementary computation for the regular fibers and the critical real fibers. For the special case of the apparent singular fibers, we present a new approach for the lifting and the connections.

3.4.1 Lifting of the regular points of \(\mathcal{D} = \Pi_1(\mathcal{C}_\mathcal{R})\)

The lifting of the regular fibers of \(\mathcal{D} = \Pi_1(\mathcal{C}_\mathcal{R})\) is done by using the rational parametrizations given in Proposition 4.

3.4.2 Lifting of the real singularities of \(\mathcal{D} = \Pi_1(\mathcal{C}_\mathcal{R})\)

The lifting of the real singularities of \(\mathcal{D} = \Pi_1(\mathcal{C}_\mathcal{R})\) is done by using the rational parametrizations given in 1. of Theorem 4.

3.4.3 Connection between real singularities and regular points
For a space curve in pseudo-generic position, the connections between real singularities and regular points are exactly those obtained on the projected curve using Grandine’s sweeping algorithm [8] (see figure 2).

3.4.4 Lifting of the apparent singularities

The lifting of the topology around an apparent singularity is a little more complex. Above an apparent singularity of \( D = \Pi_i(C_\mathbb{R}) \) we have firstly to compute the \( z \)-coordinates and secondly to decide which of the two branches pass over the other (see figure 3). We solve these problems by analyzing the situation at an apparent singularity.

According to Theorem 1 (2), \( D = \Pi_i(C_\mathbb{R}) = \bigcup_m C(\Delta_i) \), so an apparent singularity is a cross point of a branch of \( C(\Delta_i) \) and a branch of \( C(\Delta_j) \) with \( i, j \in \{1, \ldots, m\} \). So we have the following proposition.

Proposition 5 If \((\alpha, \beta)\) is an apparent singularity of \( D \) such that \( \Delta_1(\alpha, \beta) = \Delta_i(\alpha, \beta) = 0 \) and \( \gamma_1, \gamma_2 \) the corresponding \( z \)-coordinates. So by Proposition 3 and Proposition 5

\[
\text{sr}_{i,j}^a(\alpha, \beta) = \ldots = \text{sr}_{i,j}^a(\alpha, \beta) = \ldots = \text{sr}_{i,j}^a(\alpha, \beta) = 0.
\]

By Proposition 8 for any \((a, b, c) \in C_\mathbb{R}\) such that \( \Delta_i(a, b) = 0 \) and \( \text{sr}_{i,j}^a(a, b) \neq 0 \) we have \( c = \frac{-\text{sr}_{s+1}^a(a, b)}{\text{sr}_{s+1}^a(a, b)} \). So the function \((x, y) \rightarrow Z_i := -\frac{\text{sr}_{s+1}^a(x, y)}{\text{sr}_{s+1}^a(x, y)} \) gives the \( z \)-coordinate of any \((a, b, c) \in C_\mathbb{R}\) such that \( \Delta_i(a, b) = 0 \) and \( \text{sr}_{i,j}^a(a, b) \neq 0 \). \( \Delta_i(a, b) = 0 \) but \( \text{sr}_{i,j}^a(a, b) = 0 \), so the function \( Z_i \) is not defined on \((\alpha, \beta)\). The solution comes from the fact that the function \( Z_i \) is continuously extensible on \((\alpha, \beta)\). Let \( u_1 \) be the slope of the tangent line of \( C(\Delta_i) \) at \((\alpha, \beta)\) and \( t \in \mathbb{R}^* \). Let \( \gamma_i(t) := Z_i(\alpha, \beta + tu_1) = -\frac{\text{sr}_{s+1}^a(\alpha, \beta + tu_1)}{\text{sr}_{s+1}^a(\alpha, \beta + tu_1)} \).

Knowing that the algebraic curve \( C_\mathbb{R} \) hasn’t any discontinuity, it comes \( \lim_{t \to 0^+} \gamma_i(t) = \lim_{t \to 0^-} \gamma_i(t) = \gamma_1 \). By the same arguments, if we denote \( u_2 \) the slope of the tangent line of \( C(\Delta_j) \) at \((\alpha, \beta)\) and \( \gamma_j(t) := Z_j(\alpha, \beta + tu_2) = -\frac{\text{sr}_{s+1}^a(\alpha, \beta + tu_2)}{\text{sr}_{s+1}^a(\alpha, \beta + tu_2)} \), then \( \lim_{t \to 0^+} \gamma_j(t) = \lim_{t \to 0^-} \gamma_j(t) = \gamma_2 \).

The values \( u_1, u_2, \gamma_1 \) and \( \gamma_2 \) are computed using Taylor formulas and certified numerical approximations.

Now it remains to decide which of the two branches pass over the other. This problem is equivalent to the problem of deciding the connection around an apparent singularity. Let \((a_1, b_1, c_1)\) and \((a_2, b_2, c_2)\) the regular points that we have to connect to \((\alpha, \beta, \gamma_1)\) and \((\alpha, \beta, \gamma_2)\). The question is which of the points \((a_1, b_1, c_1)\) and \((a_2, b_2, c_2)\) will be connected to \((\alpha, \beta, \gamma_1)\) and the other to \((\alpha, \beta, \gamma_2)\) (see figure 4). In [8] Alcázar and Sendra give a solution using a second projection of the space curve but it costs a computation of a Sturm Habicht sequence of \( P_1 \) and \( P_2 \). Our solution does not use any supplementary computation. It comes from the fact that \( \gamma_1 \) is associated to \( u_1 \) and \( \gamma_2 \) to \( u_2 \). Knowing that \( u_1 \) is the slope of the tangent line of \( C(\Delta_i) \) at \((\alpha, \beta)\) and \( u_2 \) the slope of the tangent line of \( C(\Delta_j) \) at \((\alpha, \beta)\), \( \alpha, \beta, \gamma_1 \) will be connected to \((a_1, b_1, c_1)\) if \((a_1, b_1)\) is on the branch associated to \( u_1 \), then \((a_1, b_1)\) is on the branch associated to \( u_2 \), so \((\alpha, \beta, \gamma_2)\) will be connected to \((a_1, b_1, c_1)\) (see figure 4).

Remark 5 For a curve in generic position any apparent singularity is a node, so the slopes at an apparent singularity
are always distinct that is to say $u_1 \neq u_2$.

4. IMPLEMENTATION, EXPERIMENTS

A preliminary implementation of our method has been written using the Computer Algebra System Mathemagix. Results are visualized using the Axel\(^\text{1}\) algebraic geometric modeller which allows the manipulation of geometric objects with algebraic representation such as implicit or parametric curves or surfaces.

Since existing methods have no publicly available implementations, table only reports our experiments, performed on an Intel(R) Core machine clocked at 2GHz with 1GB RAM.

5. REFERENCES


\(^\text{1}\)http://axel.inria.fr

Figure 5: Running time of experiments.

Figure 6: Computed topology of curve 2 of table 5.
Figure 7: Computed topology of curve 7 of table 5.