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Submitted on 1 Jan 1985

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Phase transitions and size effects in the Ising dipolar magnet

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(Reçu le 19 juillet 1984, accepté le 26 septembre 1984)

Résumé. — Nous étudions un système dipolaire uniaxe dans une géométrie plane (plaque d'épaisseur $D$). Pour $D$ petit, on observe fréquemment une structure en rubans. Pour $D$ plus grand, les domaines se branchent. Le nombre de branchements augmente très lentement avec $D$. Nous calculons le diagramme de phases dans le plan $(H, T)$. Enfin, nous montrons comment la surface de la structure très ramifiée ($D \rightarrow \infty$) présente le même type de fluctuations critiques que le modèle de SK d'un verre de spin (susceptibilité constante à toute température en dessous de la transition).

Abstract. — We consider a uniaxial dipolar magnet in a simple geometry (infinite slab of thickness $D$). For small $D$, a commonly observed structure is the stripe structure. For larger $D$, branching occurs in the domain pattern. The number of ramifications increases very slowly with $D$. The phase diagram of these magnetic structures is studied in the $(H, T)$ plane. Finally, we show how the surface of the very ramified case ($D \rightarrow \infty$) has the same type of critical fluctuations as the (bulk) SK model of a spin glass (constant susceptibility at all temperatures below the transition).

1. Introduction.

Systems with competing interactions are presently receiving much interest, both from experimental and theoretical points of view. One may quote charge density waves systems [1] or the ANNNI model [2]. Spin glasses [3], where the competing interactions are disordered, exhibit even more involved properties. In this work, we consider the properties of an Ising dipolar magnet where the competition between short range exchange forces and long range (anisotropic) dipole forces may lead to complex domain structures. The geometry to be considered is that of a slab of infinite planar extent ($L_x, L_y \rightarrow \infty$), and of finite thickness $D$ ($L_z = D$). The uniaxial magnetization is taken along the $z$ axis.

If $D$ is not too large (see below), a very commonly observed pattern in zero field is the periodic stripe structure. This structure results from a competition between wall and (surface) demagnetizing energies. (In a strong enough magnetic field $H \parallel O\hat{z}$, the stripe structure gives way to a hexagonal bubble lattice, and then to a uniformly magnetized phase [4]). For thicker samples, the stripe structure is unstable against the formation of spikes of the opposite phase (e.g. a down spike in an up domain), and one observes branching of the domains as one approaches the surface of the sample. This instability comes about because the gain in the surface demagnetizing energy overcomes the loss due to the creation of new walls [5]. For very thick samples ($D \rightarrow \infty$), the domain pattern gets more and more ramified, and the system can get rid of its surface magnetization. Of course, this classification is only approximate and intermediate (or other) steps, such as undulations of the domain walls [6] appear to be important. Nevertheless, in spite of their simplicity, the structures depicted in figure 1 account for many experimental observations [7]. Similar patterns show up in other fields (metamagnets [8], type I superconductivity [9], ...).

Most theoretical and experimental work have dealt so far with the low temperature properties, such as the dependence of the domain periodicity on the thickness $D$ [9, 10]. This approach was recently extended up to the transition temperature $T_o$ for the stripe (or bubble) phase [4, 11]. In mean field, the phase transition was shown to be of the usual Landau type (macroscopic condensation of a single mode). Of course, finite $D$ implies that, close enough to $T_o$, the system behaves in a two-dimensional manner and will thus show strong fluctuations effects [4]. The purpose of the present paper is to extend the low temperature properties of the branched structure up to the transition temperature $T_o$, possibly in the
presence of a non-zero magnetic field $H(\parallel Oz)$. In section 2, we briefly recall the results pertaining to the stripe structure. (We will not consider the bubble phase any further). Section 3 will study the appearance at $T = 0$ K of the singly-branched structure, in the framework of the modified Lifshitz model [5a]. We extend this model to non-zero magnetic field and to multiply-branched structures, and give an estimate of the number $n_s$ of ramifications for $D$ large. In section 4, we consider the evolution of $n_s$ in the $(H, T)$ plane. This enables us to find a rich phase diagram. Finally, the very ramified structure ($D \to \infty$) is studied for which the surface magnetization goes to zero at all temperatures below $T_o$. Some scaling laws are derived for this case. This paper is a largely extended version of our previous note [12].

2. The stripe structure.

Consider the structure depicted in figure 1a. At $T = 0$ K, the half periodicity $a$ of the structure can be calculated by balancing the domain wall energy ($\sim 1/a$) against the surface demagnetizing energy ($\sim a$). More precisely, the total energy is [10]

$$E = L_x L_y \left\{ \frac{1}{2} \sum_{(x, y) \in \Omega} \frac{1}{\pi} (1 - e^{-n_D a y}) \right\}$$

where $\sigma_w = M^2 \delta$ is the wall energy of the material; $M^2$ is the saturation magnetization. The well-known $a-D$ relation is obtained by neglecting $\exp(-n_D a y)$ in equation (1) and minimizing with respect to $a$. This gives

$$a \simeq (D \delta)^{1/2}$$

and

$$E \simeq L_x L_y M^2 (D \delta)^{1/2}.$$  

All along this paper, we will adopt Kooy and Enz [10] values for $BaFe_{12}O_{19}$, namely:

$$\sigma_w = 2.8 \text{ erg cm}^{-2}$$

$$M = 345 \text{ G}.$$  

For $D = 3 \times 10^{-4}$ cm, one gets $a \simeq 0.6 \times 10^{-4}$ cm. This justifies $a \text{ posteriori}$ the approximation $\exp(-n_D a y) \ll 1$.

For non-zero temperatures, the walls are not sharp anymore and an adequate framework is given by the Ginzburg Landau (GL) approach. One finds, close to the transition temperature [4, 11]

$$F = D \sum_{q_y \neq n_D} G^{-1}_{q_y} m_q m_q + 0(m^4)$$

with

$$G^{-1}_{q_y} = Kq^2 + \frac{B}{Dq} (1 - e^{-qD}) + \frac{T - T_c}{T_c}.$$  

In equation (5), $K$ is the coefficient of the $(\nabla m)^2$ in the GL expansion, $B$ is a constant, and $T_c$ is the bulk mean field transition temperature. The equilibrium wave vector at the phase transition is obtained by minimizing (5) with respect to $q$. Neglecting once more the term $\exp(-qD)$, one gets

$$q_0 \sim \left( \frac{B}{DK} \right)^{1/3}.$$  

The transition temperature $T_0$ is therefore shifted down by an amount

$$\frac{T_0 - T_c}{T_c} \sim - \left( \frac{B^2 K}{D^2} \right)^{1/3}.$$  

Equation (6) takes interactions between walls into account, which is crucial around $T_0$. Indeed, without these interactions, one would have guessed that [11]

$$a(T) \sim (D \delta(T))^{1/2} \sim D^{1/2} \left( \frac{K}{\xi_G} \right)^{1/2} \sim (DK^{1/2})^{1/4} \left( \frac{T - T_c}{T_c} \right)^{1/4}.$$  

In short $a(T)$ decreases as $T$ increases but interactions between walls stabilize a non zero value $a(T_0) \sim D^{1/3}$ at $T_0$. The phase transition in this approach is of the usual Landau type, that is condensation of a single mode $q_0$ at $T_0$ plus possibly other symmetry related modes. For fluctuation effects the reader is referred to references [4, 11].

3. Branching at $T = 0$ K.

3.1 THE ZERO FIELD CASE. — Following Lifshitz [5a] and Kaczer [5b], we now consider the structure shown in figure 1b, where one has spikes of down spins in an up domain and vice versa. Boundary conditions on the spike equation $x(z)$ read (Fig. 2):

$$x(z = 0) = 0$$

$$x(z = l) = \frac{a}{2} \xi.$$  

Fig. 1. (a) Stripe structure : (b) Singly branched structure : (c) Multiply branched structure.
Fig. 2. — Spikes and boundary conditions in zero field.

where \( \xi \) and \( l \) are variational parameters to be determined. (Note that in zero field, there is only one \( x(z) \) for all spikes). We now turn to the equivalent of equation (1), which reads

\[
E = L_x L_z (E_1 + E_2 + E_3)
\]

where \( E_1 \) is the wall energy, \( E_2 \) the surface (demagnetizing) energy and \( E_3 \) the bulk (demagnetizing) energy. \( (E_3 \) is to be considered since \( \text{div}\ M = \frac{\partial M}{\partial z} \neq 0 \).

3.1.1 Wall energy \( E_1 \). — This contribution includes now both vertical and tilted domain walls. If \( \sigma_w = M^2 \delta \), \( (M = \text{saturation magnetization}), one gets

\[
E_1 = \sigma_w \left( \frac{D}{a} + 4 \int_0^l \frac{dz}{a} \sqrt{1 + \dot{x}^2} \right)
\]

where \( \dot{x} = dx/dz \).

3.1.2 Surface energy \( E_2 \). — The calculation of the energy of the periodic charge distribution of figure 1b is given in Appendix A for the case of a non-zero magnetic field. Setting \( a = a', \xi = \xi' \) in formulae (A.1)-(A.3), we obtain

\[
E_2 = \frac{16}{\pi^2} M^2 a \sum \frac{1}{n^3} (1 - \exp(-n\pi D/a)) \times \\
\times \left( \sin \frac{n\pi}{2} - 2 \sin \frac{n\pi}{2} \xi \right)^2.
\]

Lifshitz [5a] treated a slightly different case, namely one with closure domains at the surface (finite anisotropy system). In this model, the surface energy reads

\[
E_2 (\text{Lifshitz}) = \frac{k}{2} M^2 a (1 - 4 \xi + 6 \xi^2)
\]

where \( k \) is related to the (finite) anisotropy constant.

For the strictly « Ising » case, equation (12) applies. Note that Kaczer [5b] has claimed that a small \( \xi \) expansion of (12) would give something analogous to (13). Neglecting as usual the \( \exp \left( -n\pi \frac{D}{a} \right) \) term, we find for small \( \xi \)

\[
E_2 = \alpha M^2 a (1 - \beta \xi + \gamma \xi^2 \ln \xi + ...)
\]

which disagrees with Kaczer's result (\( \alpha, \beta, \gamma \) are numerical constants).

3.1.3 Bulk energy \( E_3 \). — Let us now calculate the magnetostatic energy due to the presence of the spikes (Fig. 3). The continuity of the normal component of the magnetic field \( B \) across the domain wall parametrized by \( x(z) \) implies that a non-zero component \( B_z \) must exist in regions 1 and 3 of figure 3b, and possibly in region 2 (symmetry arguments only imply that \( B_z = 0 \) in the middle of region 2). If \( \theta \) denotes the angle between the tangent \( TT' \) to the curve \( x(z) \) and \( Oz \), this continuity (Fig. 3a) reads:

\[
B_z \cos \theta = 2 B_z \sin \theta.
\]

Together with the equation

\[
\text{div}\ B = 0
\]

we could find \( B_x, B_y \) and therefore the magnetostatic energy \( E_3 \). However, following Lifshitz [5a] we now make the following assumption: the wall \( x(z) \) is such that \( \left. \frac{dx}{dz} \right| < 1. \) This means that the wall \( x(z) \) is such that \( \frac{dx}{dz} \ll 1. \) This means that the wall \( x(z) \) is such that \( \frac{dx}{dz} \ll 1. \) This means that the wall \( x(z) \) is such that \( \frac{dx}{dz} \ll 1. \) This means that the wall \( x(z) \) is such that

\[
| B_z | \simeq 4 \pi M \simeq \text{Cte}
\]

which in turn implies

\[
\frac{\partial B_z}{\partial x} = 0.
\]

Fig. 3. — (a) Geometry of the domain wall; \( TT' \) and \( NN' \) are respectively tangent and normal to the curve \( x(z) \). (b) Various regions used in the calculation of the bulk magnetostatic energy.
In this approximation $B_x(z)$ does not depend on $x$. Equation (15) reads:

$$B_x \cos \theta = (2 \sin \theta) 4 \pi M \quad \text{in region 3} \quad (19a)$$

$$B_x \cos \theta = -(2 \sin \theta) 4 \pi M \quad \text{in region 1} \quad (19b)$$

that is

$$|B_x| = 8 \pi M \tan \theta = 8 \pi M \frac{dx}{dz}. \quad (20)$$

In region 2, $B_x$ is zero in this approximation since it is zero for $x = 0$. In short, using a simplifying assumption $|\dot{x}(z)| \ll 1$, we are able to compute the magnetostatic energy of one spike by:

$$\int dx \frac{B_x^2}{8 \pi} \quad (21)$$

$$E = L_x L_y \left( \sigma_w \frac{(D + 4 I)}{a} + \frac{16 M^2}{2} - 2 \sum \frac{1}{n^3} \left(1 - \exp \left(-\frac{\pi n D}{a}\right)\right) \sin \frac{\pi n D}{2} \right) + \frac{32 \pi^2 M^2}{a} \int_0^\pi \left(\frac{dx}{dz}\right)^2 \left(\frac{a}{2} - x\right) dz. \quad (23)$$

As mentioned above, the Lifshitz-Kaczer model [5] adopts a simplified expression for the surface term. This does not change the physics at all; only numerical coefficients are slightly affected. The Euler Lagrange equation pertaining to (23) is:

$$\ddot{x}(a - 2 x) - \dot{x}^2 = 0 \quad (24)$$

the solution of which reads

$$x(z) = \frac{a}{2} \left(1 - \left\{1 - \frac{2}{7} \left(1 - (1 - \xi)^{3/2}\right)\right\}^{2/3}\right) \quad (25)$$

once boundary conditions (9) have been imposed.

One should now insert (25) into (23) and extremize $E$ with respect to $a, l, \xi$. The results are

(i) \(\frac{\partial E}{\partial l} = 0 \Rightarrow l = \frac{2}{3} \sqrt{\pi a^{3/2}} \delta^{-1/2} (1 - (1 - \xi)^{3/2})\).

(ii) Plugging (26) into (23) gives

$$E(a, \xi) = L_x L_y \left(\sigma_w \frac{D}{a} + \frac{16 M^2}{3} \times \sqrt{\pi a b \delta (1 - (1 - \xi)^{3/2})} + E_2 \right) \quad (27)$$

where $E_2$ is given by (12) in our model and (13) in the Lifshitz case. Note that without the surface term $E_2$, where the integral is to be taken over regions 1 and 3 of figure 3b. This approximation should be good as long as $\frac{a c}{l} \ll 1$, which turns out to be the case (section 3.2). For a periodicity $2a$, with 8 such regions all in all, one gets

$$E_3 = \frac{1}{2} a \frac{8}{8 \pi} \int_0^\pi (8 \pi M \dot{x})^2 \left(\frac{a}{2} - x\right) dz. \quad (22)$$

The total energy of the system, equation (10) can be rewritten (with $|\dot{x}(z)| \ll 1$)

$$\frac{\partial E}{\partial a} = 0 \text{ would yield } a \sim D^2/3 \delta^{1/3}. \text{ We will come back to this point when we consider the very ramified structure (Sect. 5).}$$

(iii) For $D > D_c^{(1)}$ with $D_c^{(1)} \sim b \delta$ ($b$ numerical constant of order 100), the branched structure of figure 1b has a lower energy than the stripe structure of figure 1a. At $D_c^{(1)}$, the transition is second order (1). With the values of section 2 ($\sigma_w = 2.8$ ergs cm$^{-2}$, $M = 345$ G), we get $D_c^{(1)} \approx 7 \times 10^{-4}$ cm. In the Lifshitz model, one finds $b \approx 31.58$.

3.2 THE BRANCHED-STRIPE TRANSITION IN A FIELD. —

We now extend the model of section 3.1 to the case of a non zero field $H(0/O_2)$. One needs now two domain wall equations $x(z)$ and $x'(z)$ to describe the (inequivalent) down and up spikes. See figure 4. The boundary conditions are

$$x(z = 0) = 0, \quad x(z = l) = \frac{a}{2} \xi$$

$$x'(z = 0) = 0, \quad x'(z = l') = \frac{a'}{2} \xi'$$

where one has introduced $a', \xi', l'$ as extra variational

(1) One probably expects hysteresis at the transition as in all nucleation phase transitions [13].
parameters. The total energy can be written as

\[ E = L_x L_y (E_1 + E_2 + E_3 + E_4) \]  

(28)

where \( E_1, E_2, E_3 \) have the same meaning as in (10) and \( E_4 \) is the Zeeman term.

3.2.1 Wall energy \( E_1 \). — Taking into account the new periodicity \((a + a')\) and the different types of domain walls, yield

\[ E_1 = \sigma_0 \left( \frac{2D}{a + a'} + \frac{4l(l + l')}{a + a'} \right). \]  

(29)

This result is obtained under the assumptions

\[ \left| \frac{dx}{dz} \right| \sim \frac{a^3}{l} \ll 1, \quad \left| \frac{dx'}{dz} \right| \sim \frac{a'^3}{l'} \ll 1. \]

3.2.2 Surface demagnetizing term \( E_2 \). — This term is calculated in Appendix A with the result

\[ E_2 = 2\pi D M^2 \left( \frac{a(1 - 2 \xi)}{a} + \left( \frac{a' \xi^2}{a'} - 1 \right) \right)^2 + \frac{8}{\pi} M^2 (a + a') \sum_{n} \frac{1}{a + a'} \left( 1 - \exp \left( 2 \pi n \frac{D}{a + a'} \right) \right) \psi_n \]  

(30a)

where

\[ \psi_n = \left( \sin \frac{n\pi a}{a + a'} - \sin \frac{n\pi a'}{a + a'} + (-1)^n \sin \frac{n\pi a'}{a + a'} \right)^2. \]  

(30b)

3.2.3 Bulk demagnetizing term \( E_3 \). — The calculation exactly parallels the one given in section 3.1. We get, with the same assumptions,

\[ E_3 = \frac{32}{a + a'} \left( \int_0^l dz \left( \frac{dx}{dz} \right)^2 \left( \frac{a}{2} - x \right) + \int_0^{l'} dz \left( \frac{dx'}{dz} \right)^2 \left( \frac{a'}{2} - x' \right) \right). \]  

(31)

3.2.4 The Zeeman energy \( E_4 \). — One has

\[ E_4 = -\frac{HM}{a + a'} \left( D(a - a') + \right. \]

\[ + 4 \int_0^l \frac{x(z)}{dz} dz + \left. 4 \int_0^{l'} x(z) dz \right). \]  

(32)

The strategy is the same as before : one should solve the Euler Lagrange equations for \( x(z) \) and \( x'(z) \), insert them back into \( E \) (Eq. (28)), and minimize \( E \), with respect to \((a, \xi, l), (a', \xi', l')\). This procedure is rather tedious and is detailed in Appendix B. We just quote here the first step, namely the expression of \( E(a, l, \xi, a', l', \xi') \):

\[
E = L_x L_y \left( E_1 + E_2 - HMD \frac{a - a'}{a + a'} + \frac{32}{a + a'} \left( \frac{l + l'}{2} \left( \frac{2C_+}{9} \right) - l' \left( \frac{2C_+}{9} \right) \right) + \Delta g_+ + \Delta g_- \right) \]

(33a)

with

\[ \Gamma = \frac{H}{8\pi M} \]  

(33b)
In equations (33a-d), \( C_{-}(a, \xi, l) \) and \( C_{+}(a', \xi', l') \) are implicit functions given in equations (B.5-B.8) of Appendix B. We have numerically solved the stationarity equations

\[
\frac{\partial E}{\partial l} = \frac{\partial E}{\partial l'} = \frac{\partial E}{\partial a} = \frac{\partial E}{\partial a'} = \frac{\partial E}{\partial \xi} = \frac{\partial E}{\partial \xi'} = 0 \quad (34)
\]

with the following parameters: \( \sigma_w = 2.8 \text{ ergs cm}^{-2} \), \( M = 345 \text{ G} \), \( D = 10 \times 10^{-4} \text{ cm} \). The results of this calculation can be summarized as follows:

(i) The branched structure does not vary much (small variations of the parameters) up to the critical field \( H_c \).

(ii) There is a first order « branched-stripe » phase transition at \( H_c \approx 120 \text{ G} \). To the accuracy of our calculation, the critical field \( H_c \) is such that

\[
C_+ \approx 0
\]

which with the help of equation (B.8) of Appendix B gives

\[
\Gamma_c = \frac{H_c}{8 \pi M} \approx \frac{a^2 \xi^2}{4 l'^2}. \quad (35)
\]

(iii) Metastable branched structures exist at least up to \( H \sim 1 \times 10^4 \text{ G} \) (remember that with our choice of parameters the transition stripe-paramagnet occurs at \( H \sim 3 \times 10^4 \text{ G} \)). One sees, as \( H \) is increased in this metastable phase, a trend toward the formation of the stripe phase (merging of spikes).

(iv) Typical order of magnitude are in the region of interest:

\[
a, a' \sim 1 \times 10^{-4} \text{ cm} \\
\xi, \xi' \sim 0.1 \\
l, l' \sim 0.7 \times 10^{-4} \text{ cm}.
\]

The ratios \( a\xi/l \) and \( a'\xi'/l' \) are therefore of order 1/7 which can be reasonably thought of as much smaller than 1. The Lifshitz approximation is therefore legitimate.

3.3 Multiple branching. — It should now be clear that, as one increases \( D \) more and more, other ramifications will occur in the domain pattern (Fig. 1c). It is in principal possible to extend the Lifshitz model to the case of multiple branching but this appears very cumbersome, to say the least. Lifshitz has shown that for \( D^{(2)} \approx 10 D^{(1)} \), a second branching occurs, but the mathematics are rather involved. To study multiple branching, we turn now to a simpler model due to Privorotskii [14]. This model rests, in zero field, on two hypothesis:

(i) The maximum width of the first generations is \( \frac{a}{3} \);

(ii) each spike is surrounded by two neighbouring vertical walls.

The situation covered by Privorotskii model is depicted in figure 5. Two comments are here in order. Assumption (i) is only meant to simplify our computations. It should not change the physics of our problem in an appreciable way. Assumption (ii) is an extreme case of Lifshitz assumption \( |x(z)| < 1 \) that we discussed in the singly branched structure. Since a vertical wall does not carry charges, one sees that the \((N + 1)\)th branching is not influenced (in the magnetostatic sense) by the \(N\)th branching. It allows one to view this model as a piling up of \((N)\) singly branched structures.

\[
\Delta \sigma_{+} = - \frac{2}{3} \left( \frac{g_{+}(\frac{2}{a'(1 - \xi')}) - g_{+}(\frac{2}{a'})}{\alpha(1 - \xi)} \right) \quad \text{and} \quad g_{+}(u) = \frac{\left( C_{+} u + \frac{9}{4} \Gamma \right)^{1/2}}{u^2} \quad (33c)
\]

\[
\Delta \sigma_{-} = - \frac{2}{3} \left( \frac{g_{-}(\frac{2}{a'(1 - \xi')}) - g_{-}(\frac{2}{a})}{\alpha(1 - \xi)} \right) \quad \text{and} \quad g_{-}(u) = \frac{\left( C_{-} u - \frac{9}{4} \Gamma \right)^{1/2}}{u^2}. \quad (33d)
\]

Fig. 5. — Privorotskii model. Three generations of branching are shown. Note the vertical walls, such as AB. The second generation starts when \( AC = \frac{a}{3} \), the third when \( A'C' = \frac{a}{3^2} \), etc...
structures with $\xi = 1/3$ and to calculate its energy as [12]

$$E_p^{(N)} = L_x L_y \left( \lambda M^2 a^{1/2} \delta^{1/2} \left( \frac{4}{5} - X + \frac{X^3}{5} \right) + \gamma a M^2 X^2 + M^2 \delta \frac{D}{a} \right) \quad (36)$$

where

$$\lambda = \frac{20}{3} \sqrt{\frac{1}{\pi}} \frac{\left( 1 - \left( \frac{2}{3} \right)^{3/2} \right)}{1 - 3^{-1/2}}$$

$$X = 3^{-3/2}$$

$\gamma \approx 1.71$. (We have neglected, as usual, the exponential term in the surface energy.)

It is clear that the energies in Lifshitz singly branched structure and Privorotskii multiply branched structures are very similar. See equations (27) and (36). A trivial difference concerns numerical factors in both expressions. Physically, Privorotskii's model yields a much lower surface magnetization. Furthermore, we will show in section 5 that equation (36) gives a series of critical thickness $D_c^{(N)}$ of the form $\delta f(N)$ (see equation (49)). The maximal number $n_s$ of branching compatible with this model is deduced from

$$\frac{a}{3n_s} \sim \delta \quad (37a)$$

that is

$$n_s \sim \ln \left( \frac{a}{\delta} \right). \quad (37b)$$

4. Evolution in the $(H, T)$ plane.

4.1 $T$ INCREASES. — The dependence of $D_c^{(N)}$ on $T$ can be obtained from the relation (see Eq. (49)):

$$D_c^{(N)} \sim \delta f(N).$$

As mentioned in the stripe case, one may replace $\delta$ by $K = \frac{\xi_{GL}}{\xi_{GL}}$ at finite temperatures ($K$ is the coefficient of the $(V_m)^2$ and $\xi_{GL}$ the correlation length of a Ginzburg Landau functional). One would therefore guess

$$D_c^{(N)} \sim \frac{K}{\xi_{GL}} f(N) \sim K^{1/2} \left( \frac{T_c - T}{T_c} \right)^{1/2} f(N). \quad (38)$$

We already know that equation (38) is valid up to the temperature where the walls start to interact, but the trend is clear: $D_c^{(N)}(T)$ decreases if $T$ increases, as indicated by (38). See also reference [5c]. The situation is shown in figure 6a. For each $N$, $D_c^{(N)}$ decreases if $T$ increases. Furthermore, one easily sees that two curves $D_c^{(N)}(T)$ and $D_c^{(N)}(T)$ can never intersect. The argument goes as follows: let us suppose that $N_2 > N_1$.

**Fig. 6.** — (a) Evolution of the various critical thickness $D_c^{(N)}$ with temperature. $T_c$ is the bulk mean field critical temperature and $T_{c}^{(N)}$ the true transition temperature. (b) $D_c^{(N)}$ as a function of $H$.

If the curves did intersect at a temperature $T_1$, one could have a situation with a slab of thickness $D$ such that for $T > T_1$: $D_c^{(N_2)} < D < D_c^{(N_1)}$.

The number of branching would be less than $N_1$ and more than $N_2$, which is absurd (we have assumed $N_2 > N_1$).

Let us now consider the evolution of the maximal number of branching $n_s$ with $T$. A qualitative argument, that we will make rigorous in section 5, is possible in the case where $D$ is large. In this case, the surface magnetization is negligible and minimization of $E_{p}^{(N)}$ with respect to $a$ gives

$$a \sim D^{2/3} \delta^{1/3}. \quad (39)$$

Inserting (39) in (37b), one gets

$$n_s \sim \frac{2}{3} \ln \left( \frac{D}{\delta} \right). \quad (40)$$

Applying the above recipe, one sees that $n_s(T)$ increases with $T$. The naive guess would give an infinite $n_s$ at $T_c$, but interactions between the walls stabilize a finite $n_s(T_0)$, where $T_0$ is the true critical temperature (see Sect. 5).

4.2 FIELD EVOLUTION. — In section 3, we have exhibited a transition (singly) branched-stripe in a field $H_c^{(1)}$. It therefore appears that $D_c^{(1)}(H)$ increases if $H$ increases. Moreover, the critical field $H_c^{(1)}$ was found to be (Eq. (35)):

$$H_c^{(1)} \sim 2 \pi M \frac{a^2 e^2}{l^2}. \quad (35)$$

Since the single branched structure is hardly distorted up to $H_c^{(1)}$, an order of magnitude for $H_c^{(1)}$ can be obtained by setting $a' \sim a$, $\xi' \sim \xi$, $l' \sim l$ in (35) and then to use (26). We find:

$$H_c^{(1)} \sim \frac{g}{a} \frac{M \delta}{a} \quad (41)$$

where $g(\xi)$ is a number depending on the (finite) value of $\xi$ at the transition (see Sect. 3.2). These are the predictions of the Lifshitz model. As for the multiply branched Privorotskii model, we have seen
that it contains a whole set of critical thicknesses $D^{(1)}, D^{(2)}, ..., D^{(N)}$, above which the system has respectively, 1, 2, ..., $N$, branchings. To calculate the evolution of $D^{(i)}$ as a function of $H$, it is natural to extend the trend exhibited by $D^{(1)}$, namely: $D^{(i)}$ increases if $H$ increase. Moreover, it is easy to show that two curves $D^{(i)}(H)$ and $D^{(j)}(H)$ do not intersect; the argument exactly parallels the one given in section 4.1. The situation is depicted in figure 6b. As shown in reference [12], it is possible to scale the variables in such a way that a critical field $H_c^{(N)}$ between the $(N)$-branched structure and $(N - 1)$ branched structure is expressed by

$$H_c^{(N)} \sim \frac{M}{a} \delta G(N)$$ (42)

that is, all fields in the problem have a common scale $M/a$. In view of the result of the (singly) branched-stripe transition of section 3, we expect this scale to be much smaller than the field needed for the stripe to paramagnet transition.

4.3 PHASE DIAGRAMS. — In view of figure 6, it is fairly easy to deduce qualitatively typical phase diagrams. In figure 7a is shown a phase diagram for a plate of thickness $D$ such that :

At $T = 0$ $D < D^{(1)}$ the stripe phase is stable

$T = T_1$, $D = D^{(1)}(T_1)$ the singly branched phase becomes stable.

For $T_1 < T < T_0$, $D$ possibly crosses other $D^{(i)}(T)$ curves yielding other «branched-branched» phase transitions (dashed lines in figure 7a).

In figure 7b, we have considered a plate of thickness $D$ such that :

at $T = 0$ $D^{(N+1)}(T = 0) > D > D^{(N)}(T = 0)$⇒

⇒ the $N$th branched structure is stable.

If one increases the field, figure 6b suggest that the system will undergo a cascade of first order transitions (the harmless staircase of Ref. [15]) to lesser and lesser ramified structures.

Let us stress that the phase diagrams of figure (7) are only qualitative since, for instance, we have completely neglected the bubble phase or its equivalent in the branched version, which may well be of experimental relevance.

4.4 PHASE TRANSITIONS. — We have previously mentioned how one could use a Ginzburg Landau (GL) approach to estimate the dependence of various parameters on the temperature. How would one construct a GL functional? When branching is present, the magnetization varies rapidly over small distances and one should take a coarse-grained magnetization to define an order parameter à la Landau. One possible way is to write

$$m_{GL}(x, z) \approx \frac{1}{\Delta x} \frac{1}{\Delta z} \times \int_x^{x+\Delta x} \int_z^{z+\Delta z} m_{micro}(x', z') dx' dz' ,$$ (43)

with $\Delta x \gg a, \Delta z \ll l$. See figure 8 for the resulting magnetization profile in the $z$ direction. As long as $D$ is finite, the maximal number $n_s$ of branching is finite, and the surface magnetization is finite but smaller than the bulk one. If one tries to analyse the Fourier transform $m_{GL}(q_x, q_z)$ as an expansion in the eigenmodes of the system, one finds that several symmetry unrelated modes become simultaneously critical at $T_0$, such as $(q_x = 2 \pi/a, q_z = 0)$ and $(q_x = 2 \pi/a, q_z = 2 \pi/D)$. This fact is in marked contradiction with Landau theory of second order phase transitions in infinite systems. The only cases where one has to deal with condensation into a single mode seem to be :

(a) the stripe case $(q_z = 0)$ ; (b) the infinitely ramified case $(q_z = \pi/D)$.

Fig. 7. — (a) Qualitative phase diagram for a plate of thickness $D$ such that $D = D^{(1)}(T_c)$. S is the stripe phase, B, the singly branched phase and dashed lines denote possible «branched-branched» phase transitions.

(b) The same for $D^{(N)}(T = 0) < D < D^{(N+1)}(T = 0)$. $B_N$ is the $N$-times branched phase. The region close to $T_0$ is not precisely known.

Fig. 8. — (a) Coarse-graining the magnetization (the spike is enlarged).
(b) The resulting profile in the $z$ direction.
The GL approach close to $T_0$ has been considered by Barker and Gehring (private communication). Let us point out that, as in the stripe case [4], one expects fluctuations and in particular defects to play a significant rôle at the transition.

5. The very ramified structure ($D \to \infty$).

5.1 Periodicity. — As previously stated, the $D = \infty$ case is interesting because one completely gets rid of the surface magnetization, and therefore of the surface demagnetizing energy. For instance the periodicity $a$ in zero field is found to scale as (Eq. (36))

$$a \sim D^{2/3} \delta^{1/3}$$

at $T = 0$ K. Close to the transition temperature $\delta(T)$ decreases (Sect 2). An argument very similar to the one given for the stripe case, yields, close to $T_0$:

$$a \sim D^{2/3} \left( \frac{K}{\epsilon_{GL}} \right)^{1/3}.$$  \hspace{1cm} (44)

Wall interactions start when $\epsilon_{GL} \sim a$ and one therefore expects at $T_0$:

$$a^{4/3} \sim D^{2/3} K^{1/3}$$

$$a \sim D^{1/2} K^{1/4}.$$ \hspace{1cm} (45)

The same result can also be obtained by considering the full dipolar propagator [11]

$$G^{-1}_a = \frac{T - T_e}{T_e} + Kq^2 + Gq_a^2 + B \frac{q_a^2}{q^2}$$

where $K, G, B$ are constants and

$$q^2 = q^2 + q_a^2 \left( q_a = \frac{2 \pi}{a}, q_z = \frac{\pi}{D} \right).$$

The shift of the critical temperature is

$$\frac{T_0 - T_e}{T_e} \approx - \frac{K^{1/2}}{D}.$$ \hspace{1cm} (47)

5.2 Surface magnetization. — In the large $D$ limit, the minimization of equation (36) with respect to $X$ and $a$ leads to

$$\left( \frac{q_a^2}{a} \right)^{1/2} = \frac{\lambda}{27} \left( \frac{1}{X} \right) \left( 1 - \frac{3X^2}{5} \right)$$

$$\frac{\lambda}{2} \left( \frac{A_5}{5} - X + \frac{X^3}{5} \right) \left( \frac{\delta}{a} \right)^{1/2} - \frac{\delta D}{a^2} + \gamma X^2 = 0$$

with $X = 3^{-N/2}$. Equations (48) imply

$$\frac{\lambda^4}{16 \gamma^3} \left( 1 - \frac{3X}{5} \right)^3 \left( 1 - \frac{4X}{5} \right) = \frac{D}{\delta}$$

(49)

which gives for each $X$ (i.e. each $N$) a critical thickness $D_e^{(N)} \sim \delta f(N)$ as anticipated in section 4.

For large $D$, one gets

$$X^{-3} \sim \frac{D}{\delta} \sim \frac{D}{D_e^{(1)}}.$$ \hspace{1cm} (50)

The maximum number of branching is therefore

$$n_s \sim \frac{2}{3} \ln \left( \frac{D}{D_e^{(1)}} \right)$$

in agreement with (37b) and (40). The surface magnetization, $m_s$, defined as the difference between the numbers of up and down spins is [12]

$$m_s \sim \frac{M}{3} \left( \frac{D}{D_e^{(1)}} \right)^{2/3}.$$ \hspace{1cm} (52)

In a similar way, we can derive the scaling of $H_e^{(N)}$ in equation (42) for $D$ large. We get

$$H_e^{(N)} \sim M \left( \frac{\delta}{D} \right)^{2/3} \sim M \left( \frac{D_e^{(1)}}{D} \right)^{2/3}.$$

(53)

(We recall that $H_e^{(N)}$, $N$ large, is the field needed to induce the first transition in the (very) branched structure). All these results pertain to $T = 0$. When $T$ increases, we already know that $n_s$ increases since $D_e$ decreases. More precisely, since

$$D_e^{(1)} \sim \frac{\delta}{\epsilon_{GL}} \sim \frac{K^{3/4}}{D^{1/2}}.$$ \hspace{1cm} (54)

Equation (51) becomes close to $T_0$

$$n_s(T_0) \sim \frac{2}{3} \ln \left( \frac{D^{3/2}}{K^{3/4}} \right) \sim \ln \left( \frac{D}{K^{1/2}} \right)$$

In a similar way, one gets ($T \lesssim T_0$)

$$m_s \sim M \left( \frac{K^{1/2}}{D} \right)$$

(56)

and

$$H_e^{(N)} \sim M \left( \frac{K^{1/2}}{D} \right)$$

(57)

In (56) and (57) $M$ is the bulk magnetization.

5.3 Criticality at all temperatures. — Taking equations (52) and (53) on the one hand, and equations (56) and (57) on the other, shows that both at $T = 0$ and $T \lesssim T_0$, one has

$$m_s \sim H_e^{(N)}$$

(58)

which gives a constant surface susceptibility. We believe this result to be true at all temperatures below $T_0$. In the very ramified structure ($D \to \infty$), the surface magnetization which is zero in zero field, grows as $H_e^{(N)}$ where $H_e^{(N)}$ is the first field to induce a transition in the branched structure. One may
therefore say that, at all temperatures below $T_o$, the surface has the same type of critical fluctuations as observed in the SK model of a spin glass [16]. A physical explanation is that, to ensure $m_s = 0$ in zero field, the system has to create new branchings all the way down to $T = 0$. All these results come from a mean field treatment which is almost correct for the infinite three dimensional dipolar magnet [17].

6. Conclusion.

In this paper we have studied branched magnetic structures in the $(H, T)$ plane. Our main results are summarized as follows: in the large $D$ limit, (i) $a$ decreases

$$a \sim D^{2/3} \delta^{1/3} \delta$$

whereas

$$a \sim D^{1/2} K^{1/4}.$$

(ii) $n_s$ increases

$$n_s \sim \frac{2}{3} \ln \frac{D}{\delta} \text{ whereas } n_s \sim \ln \frac{D}{K^{1/2}}.$$

(iii) $m_s$ decreases

$$m_s/M \sim (\delta/D)^{2/3} \text{ whereas } m_s/M \sim K^{1/2}/D.$$

(iv) $H_c^{(N)}$ decreases

$$H_c^{(N)}/M \sim (\delta/D)^{2/3} \text{ whereas } H_c^{(N)}/M \sim K^{1/2}/D.$$

Appendix A

We consider the following surface charge distribution (Fig. 9)

$$\sigma = + M \text{ for } \left\{ \begin{array}{l}
-\left(\frac{a + a'}{2}\right) < x < -\left(\frac{a + a'}{2}\right) + a' \varepsilon' \\
-\frac{a}{2} < x < -\frac{a}{2} \varepsilon \\
+\frac{a}{2} \varepsilon < x < +\frac{a}{2} \\
+\left(\frac{a + a'}{2}\right) - \frac{a' \varepsilon'}{2} < x < \left(\frac{a + a'}{2}\right)
\end{array} \right.$$

$$\sigma = - M \text{ otherwise.}$$

A noteworthy result obtained from the Privorotskii model is that the surface of the material undergoes critical fluctuations at all temperatures below $T_o$, in the large $D$ case. An interesting extension of this work would be to consider the semi-infinite case. We believe that the problem considered in this paper is also of interest for the following problems:

(i) spin glasses: the only way for an Ising system to go from a region of non zero magnetization to a region of zero magnetization is branching in-between. One therefore expects that frustration will imply, in some sense, branching. Furthermore, branched structure should display metastable states, slow rearrangements, etc...

(ii) ANNNI model: the situation is completely different in two and three dimensions, branched structures should play a significant rôle in the dimensional crossover.

(iii) metamagnets, type I superconductivity: in this case branching occurs in an asymmetric way since the stripe structure is already asymmetric (para-antiferro in metamagnets, normal-supra in superconductors).

Finally, we mention the possible interest of branching for the physics of magnetic superconductors [18].

Acknowledgments.

We thank G. Gehring and J. Lajzerowicz for discussions and the participants of the R.C.P. «Domaines et Parois» for comments.

Fig. 9. — Surface charge distribution used in Appendix A for the calculations of the surface energy. Half of the plate is shown.
If one expands \( \sigma(x) \) as

\[
\sigma = \frac{a_0}{2} + \sum_{n \neq 1} a_n \cos 2n\pi \frac{x}{a + a'}
\]

the surface energy is given by [10] : \[ E_2 = \frac{\pi}{2} a_0^2 D + \sum_{n \neq 1} a_n^2 \left( \frac{a + a'}{2n} \right) \left( 1 - \exp - \left( \frac{2\pi n}{a + a'} \right) \right). \tag{A.1} \]

With the above distribution, we get

\[
a_0 = \frac{2M}{a + a'} (a(1 - 2\xi) + a'(2\xi' - 1)) \tag{A.2}
\]

\[
a_n = \frac{4M}{n\pi} \left( \sin n\pi \frac{a}{a + a'} - \sin n\pi \frac{a\xi}{a + a'} + (-1)^n \sin n\pi \frac{a'\xi'}{a + a'} \right). \tag{A.3}
\]

Inserting (A.2) and (A.3) into (A.1), we easily obtain equations (12) and (30a).

**Appendix B**

**Euler-Lagrange Equations in a Field. Calculation of \( C_+ \) and \( C_- \).** — The Euler-Lagrange equations in a field are obtained with the help of \( E_3 \) and \( E_4 \), equations (31) and (32). One gets

\[
\ddot{x}(a - 2x) - \dot{x}^2 = \frac{H}{8\pi M} = \Gamma \tag{B.1a}
\]

\[
\ddot{x'}(a' - 2x') - \dot{x'}^2 = -\frac{H}{8\pi M} = -\Gamma. \tag{B.1b}
\]

Setting \( 2X = a - 2x \) and \( 2X' = a' - 2x' \) gives

\[
2X\ddot{X} + \dot{X}^2 = -\Gamma \tag{B.2a}
\]

\[
2X'\ddot{X'} + (\dot{X'})^2 = +\Gamma. \tag{B.2b}
\]

Or

\[
2\dot{Y}_e + \dot{Y}_e^2 = \varepsilon\Gamma \tag{B.3}
\]

with \( Y_+ = X' \), \( Y_- = X \). Setting \( \frac{1}{Y_e(z)} = v_\varepsilon(z) \) and integrating, we have

\[
dz = \frac{3}{2} \frac{dv_\varepsilon}{v_\varepsilon \left( C_+ v_\varepsilon + \frac{9}{4} \varepsilon\Gamma \right)} \tag{B.4}
\]

with \( C_\varepsilon \) an integration constant.

\( \varepsilon = -1 \)

Equation (B.4) gives

\[
z - z^-_0 = \frac{2}{3\Gamma} \sqrt{C_- v_- - \frac{9}{4} \Gamma} + \frac{4C_-}{9\Gamma^{3/2}} \arctg \frac{C_- v_-}{\frac{9}{4} \Gamma} - 1 = \Phi(v_-) \tag{B.5}
\]

\( z^-_0 \) and \( C_- \) are fixed by the boundary conditions at \( z = 0 \) and \( z = l \). In particular \( C_- \) is determined by

\[
l = \frac{2}{\varepsilon(1 - \xi)} - \Phi \left( \frac{2}{a} \right). \tag{B.6}
\]
or
\[ - (z - z_0^+) = \Omega(v_+) \]

\[ - l' = \Omega\left(\frac{2}{\alpha'(1 - \xi')}\right) - \Omega\left(\frac{2}{\alpha'}\right). \]

Inserting back these solutions into (31) and (32), gives after some algebraic manipulations, equations (33a-d).

References

(b) GENICON, J. L., TOURNIER, R., private communication.