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Self-avoiding walks on fractal spaces : exact results and Flory approximation

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Résumé. — Les marches sans retour (SAW) explorent le « squelette » d'un réseau fractal, à la différence des marches aléatoires. Nous montrons l'existence d'un exposant intrinsèque pour ces marches et nous examinons une approximation simple à la Flory, utilisant la dimension spectrale du squelette. Des résultats exacts pour divers réseaux fractals montrent que cette approximation n'est pas très satisfaisante, et que les propriétés des SAW dépendent d'autres caractéristiques intrinsèques du fractal. Quelques remarques sont présentées pour les marches sur les amas de percolation.

Abstract. — Self-avoiding walks (SAW) explore the backbone of a fractal lattice, while random walks explore the full lattice. We show the existence of an intrinsic exponent for SAW and examine a simple Flory approximation that uses the spectral dimension of the backbone. Exact results for various fractal lattices show that this approximation is not very satisfactory and that properties of SAW depend on other intrinsic aspects of the fractal. Some remarks are presented for SAW on percolation clusters.

1. Introduction.

The physics of structures possessing a scaling invariance is attracting growing interest. Recent work [1-3] has shown in particular that random walks on such fractal spaces have simple properties and provide a powerful probe, giving direct access to the spectral dimension $\tilde{d}$ which governs the density of states of low-energy excitations [4].

In the present paper, we make another step and study some properties of self-avoiding walks (SAW) on fractal lattices. (Note that here SAW has the established meaning of a non-intersecting chain in statistical equilibrium and therefore differs from the so-called « true » self-avoiding walk introduced by Amit *et al.* [5].)

First motivation : different walkers explore differently and tell complementary stories. Since only the end parts of SAW can lie on dead ends, the asymptotic

behaviour of their gyration radius is expected to be dominated by the structure of the backbone (i.e. doubly connected component) rather than by that of the full fractal space. An important side-remark is that different starting points are no longer equivalent as for regular lattices. Nevertheless global properties (such as the gyration radius of large SAW) are expected to be universal and independent of the starting point. On Euclidean lattices scaling relations exist between global (long distance) properties and local (short distance) properties (such as the number of closed loops); the formulation of such scaling relations is questionable for general fractal lattices, however this particular question will not be addressed here.

On percolation clusters the spectral dimension $\tilde{d}_B$ of the backbone is different from $\tilde{d}$, and this shows that in general random walkers and self-avoiding walkers probe different properties of a fractal space. More generally it is tempting to speculate that $\tilde{d}$ and $\tilde{d}_B$ are just the first two of a hierarchy of intrinsic dimensions, controlling more and more specific properties.

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Second motivation : is the remarkable success of the Flory approximation for SAW on Euclidean lattices accidental or not ? A partial answer to this question may come from the study of other lattices, such as fractal lattices.

In the following we first show that the dependence on the fractal (Hausdorff) dimension is trivial, and that an intrinsic exponent, independent of the embedding space, may be defined. This suggests a simple Flory-type approximation which is compared to exact results on several fractal lattices. The agreement is not so satisfactory as for Euclidean lattices and we are led to the conclusion that the spectral and fractal dimensions of the backbone are not sufficient to determine the properties of SAW. This in turn casts doubt on the existence of a general Flory approximation which would retain both simplicity and accuracy. Finally, we discuss some curious aspects of the statistics of SAW on percolation clusters.

2. Intrinsic properties and Flory approximation.

The mean-square radius of a random walk of N steps on a fractal lattice behaves asymptotically as

$$\langle R_N^2 \rangle \sim N^{2\nu_{\text{rw}}}, \quad (1)$$

with $\nu_{\text{rw}} = \tilde{d}/2\bar{d}$: $\bar{d}$ and $\tilde{d}$ are respectively the fractal and the spectral dimensions of the lattice [1, 2]. For self-avoiding walks it is similarly expected that the gyration radius behaves as

$$\langle R_N^2 \rangle \sim N^{2\nu}. \quad (2)$$

Now the exponent ν *a priori* depends on various properties of the backbone of the lattice :

$$\nu = \nu(\bar{d}_B, \tilde{d}_B, \dots), \quad (3)$$

where (...) could refer for instance to ramification indices [6] or other independent characteristic dimensions, to be defined. In the following only fractal objects which are their own backbone will be considered and the index B will be consequently dropped.

A first important result is that the combination $\bar{d}\nu$ is an intrinsic property, independent of the space in which the fractal is embedded, whereas $\bar{d}$ and ν both depend on this embedding.

To see this, consider the mass M of that part of the fractal lying within a rms distance $\langle R_N^2 \rangle^{1/2}$. By definition of $\bar{d}$, it is of order

$$M \sim \langle R_N^2 \rangle^{\bar{d}/2} \sim N^{\bar{d}\nu}. \quad (4)$$

In a distortion of the system $\langle R_N^2 \rangle$ changes (imagine a sheet of paper folded and crushed into a ball), but M and N are unaffected since they are just numbers of sites. The product $\bar{d}\nu$ must therefore be invariant, and it is useful to define an intrinsic expo-

nent $\hat{\nu}$ which does not depend on $\bar{d}$:

$$\nu = \frac{\tilde{d}}{\bar{d}} \hat{\nu}(\tilde{d}, \dots). \quad (5)$$

For a random walk, one has $\hat{\nu}_{\text{rw}} = 1/2$, for fractals as well as for Euclidean spaces, but for SAW this exponent depends on $\tilde{d}$, and possibly on other parameters, as soon as $\tilde{d} \leq 4$. An argument showing that excluded-volume effects are negligible for $\tilde{d} > 4$ has been given in a previous work [2].

The simplest possible assumption is that $\hat{\nu}$ depends only on $\tilde{d}$ since the spectral dimension is the single most important intrinsic dimension of a fractal space. Thus it is natural to replace the space dimension d by $\tilde{d}$ in formulae involving intrinsic properties of SAW, as a first guess.

The well known Flory formula provides a neat interpolation between the known values of ν for Euclidean lattices :

$$\nu_F = \frac{3}{\tilde{d} + 2}, \quad (6)$$

for $d \leq 4$. It is exact for $d = 1, 4$, presumably also for $d = 2$, and quite good for $d = 3$. This suggests a similar approximation for fractals, under the assumption that only $\tilde{d}$ plays a rôle :

$$\hat{\nu}_F = \frac{3}{\tilde{d} + 2}, \quad \tilde{d} \leq 4. \quad (7)$$

This is the simplest approximation that reduces to the Flory form for Euclidean spaces. For the original SAW exponent it gives :

$$\nu_F = \frac{1}{\bar{d}} \left(\frac{3\tilde{d}}{\tilde{d} + 2} \right). \quad (8)$$

In order to make contact with the standard derivation of the Flory formula 6, we may write the free energy F as the sum of a potential energy and a kinetic energy terms, each depending on the radius R of the chain :

$$F = a \frac{N^2}{R^{\tilde{d}}} + b \left(\frac{R}{R_0} \right)^x, \quad (9)$$

where $R_0 = N^{\nu_{\text{rw}}}$ is the radius of a chain in absence of excluded-volume effects, i.e. of a random walk, and x is yet to be determined. To be invariant under a distortion of the embedding space, F must depend on $\bar{d}$ only through $R^{\tilde{d}}$, so $x = \tilde{d}z$, where z is now an intrinsic exponent ($z = 2/d$ for Euclidean spaces). Minimization of F with respect to R yields

$$R^{\tilde{d}(z+1)} \sim N^2 R_0^{\tilde{d}z},$$

so

$$\nu = \frac{1}{\bar{d}} \left(\frac{2 + \tilde{d}z/2}{1 + z} \right). \quad (10)$$

The approximation proposed above is recovered by the choice $z = 2/\bar{d}$, or $x = 2\bar{d}/\bar{d} = 1/\nu_{\text{rw}}$. We have no real justification for that particular choice other than its simplicity.

We now derive exact results for various fractal lattices in order to compare them with the approximate formula 8.

3. Exact results on fractal lattices.

The statistics of SAW on non-Euclidean lattices have been previously studied by Dhar [7] and some of his examples turn out to be closely related to fractal lattices we have considered independently. His motivation was to show that the non-integral dimension appearing in renormalization-group expansions could not be identified in general with the fractal dimension $\bar{d}$. Here we wish to go one step further and unravel the role of the spectral dimension.

a) *Branching Koch curve*. — Branching Koch curves have been used by Gefen *et al.* [8] as examples of quasi-linear fractal lattices and provide good pedagogical exercises. For the curve whose iterative construction is depicted in figure 1a the mass increases by a factor of 5 when the linear size increases by the scaling ratio $\lambda = 3$, so the fractal dimension is $\bar{d} = \ln 5/\ln 3$.

To obtain $\bar{d}$, it is easiest to write the recurrence relation for the resistance r_n between the two ends of the lattice after n stages of the construction [8] :

$$r_{n+1} = K r_n. \quad (11)$$

Fig. 1. — First stages in the iterative construction of fractal lattices studied in the text : a) branching Koch curve; b) Sierpinski gasket; c) 3-simplex; d) Havlin-Ben Avraham (HBA) gasket.

Here $K = 8/3$, so the conductance scaling exponent is :

$$\beta_L = -\ln K / \ln \lambda = -\ln(8/3) / \ln 3$$

and the spectral dimension is [2] :

$$\tilde{d} = 2\bar{d}/(\bar{d} - \beta_L) \sim 1.2427. \quad (12)$$

Let $G_N(R)$ be the number of N -step SAW having the ends of the curve as extremities, at stage n ($R = \lambda^{n-1}$). The corresponding generating function is :

$$G(x, R) = \sum_{N \geq 1} x^N G_N(R) \quad (13)$$

and a recursive construction of the possible chains shows immediately that :

$$G(x, R) = G^3(x, R/\lambda) + G^4(x, R/\lambda). \quad (14)$$

This may be viewed as an exact real-space renormalization equation for the fugacity x : defining $G(x', R/\lambda) = G(x, R)$ there comes

$$G(x') = G^3(x) + G^4(x). \quad (15)$$

The initial condition is just $G_1(x) = x$. Other initial conditions may be of interest — e.g. if a more complicated, non-scaling basic unit is used instead of a unit segment [8]. Following well-known lines [9] the non-trivial fixed point of (15) gives the radius of convergence x^* of the series (13) for R going to infinity, hence the connectivity constant $\mu = (x^*)^{-1}$. One finds here

$$G^* = x^* = 0.755...; \quad \mu = 1.324...$$

The exponent ν may be obtained in the standard fashion by studying the correlation length $\xi(x)$ for x close to x^* :

$$G(x, R) \underset{R \rightarrow \infty}{\sim} \exp(-R/\xi(x)), \quad (16)$$

with

$$\xi(x) \sim (x^* - x)^{-\nu}. \quad (17)$$

This gives

$$(x^* - x')^\nu \sim \lambda(x^* - x)^\nu$$

and

$$\frac{1}{\nu} = \ln \left(\frac{dG'}{dG} \bigg|_{G^*} \right) / \ln \lambda. \quad (18)$$

For the branching Koch curve, the result is : $\nu \sim 0.891$.

b) *Two-dimensional Sierpinski gasket*. — The iterative construction of this by now familiar object is recalled in figure 1b : it has $\bar{d} = \ln 3/\ln 2$, $\bar{d} = 2 \ln 3/\ln 5$ [2]. For SAW the calculation proceeds along the same lines as above, but a slight subtlety is to be noticed : due to the excluded-volume effect, a chain

such as ABCA'BC' is forbidden. This implies that the generating function $G(x, AB)$ for all chains going from A to B is not sufficient to obtain $G(x, AC')$. We need to introduce two functions, $h(x)$ for the chains visiting all three vertices (A, B, C) and $g(x)$ for the chains going through only two vertices.

The recursion relations are then, with $g = g(x)$ and $g' = g(x')$:

$$\begin{aligned} g' &= (h + g)^2 + g^3 + 2hg^2 \\ h' &= hg^2 + 2gh^2. \end{aligned} \quad (19)$$

with initial conditions $g_1 = x$, $h_1 = x^2$.

The only non-trivial fixed point is :

$$h^* = 0, \quad g^* = (\sqrt{5} - 1)/2,$$

and the eigenvalues around that point are $\lambda_1 = g^{*2} < 1$, $\lambda_2 = 2g^* + 3g^{*2} > 2$. The variable h is therefore irrelevant for the calculation of the exponent ν , but it is necessary to take it into account to obtain the correct value of the connectivity constant. A numerical study of the flow in the (g, h) plane yields :

$$\begin{aligned} x^* &= 0.4371..., \quad \mu = 2.288... \\ \nu &= \ln 2 / \ln (2g^* + 3g^{*2}) \sim 0.798. \end{aligned}$$

Dhar [7] has studied a family of structures named « truncated n -simplices », which are closely related to the Sierpinski gaskets of various dimensionalities. They have identical fractal and spectral dimensions, though their construction may seem rather different at first. As shown in figure 1c for the 3-simplex each vertex of the Sierpinski lattice is replaced by a pair of vertices — which makes the structure not strictly scale-invariant. This modification suppresses the obstruction noted above and the system is described by a unique recursion relation :

$$G' = G^2 + G^3 \quad (20)$$

with initial condition $G_1 = x + x^2$ (the distance corresponding to the short segments between vertices is considered negligible, as done by Dhar). The fixed point and the eigenvalue of (20) are the same as for the Sierpinski, so the exponent ν is identical, but the connectivity constant is different :

$$x^* + x^{*2} = G^*, \quad \text{hence} \quad x^* = 0.4317..., \quad \mu = 2.316...$$

The connectivity constant is larger than in the Sierpinski, as it should since more configurations are allowed.

These two lattices may be said to belong to the same universality class : the splitting of vertices appears as an irrelevant perturbation that does not affect critical exponents while it changes non-universal properties like the connectivity constant μ .

c) *An analogous structure with a different symmetry.* — A related system has been studied by Havlin and

Ben-Avraham [10] in connection with random walks on fractal structures. The HBA gasket is drawn in figure 1d in such a way as to emphasize its connection with the 3-simplex. The important difference is that it possesses only reflection symmetry instead of the three-fold rotation symmetry of the Sierpinski gasket. It is interesting to investigate the effect of that lower symmetry.

The fractal dimension is unchanged : $\bar{d} = \ln 3 / \ln 2$. The spectral dimension is most conveniently obtained by studying the scaling properties of the resistance. A two-parameter recursion is now necessary, involving for instance the resistances r_{AB} and $r_{AC} = r_{BC}$. Using standard star-triangle transformations, one may write the (non-linear) recursion relations and the analysis shows that asymptotically the three-fold symmetry of the Sierpinski gasket is recovered. The scaling exponent $\beta_L = -\ln(5/3)/\ln 2$ is therefore the same as for the Sierpinski and so is $\bar{d} = 2 \ln 3 / \ln 5 \sim 1.365$, as can be checked by a direct study of the density of states for low-energy lattice vibrations. A similar conclusion holds for a gasket without any symmetry at the first stage (i.e. 3 different resistances r_{AB} , r_{AC} and r_{BC} in figure 1b). This result, which is in contradiction with the value $\bar{d} = \ln 9 / \ln(21/4) \sim 1.325$ proposed by Havlin and Ben Avraham [10], suggests that the spectral dimension is a property with a wide universal character.

To study SAW on lattice (1d), two generating functions are necessary, which we denote $t(x, AB)$ for the chains going from A to B (through C or not) and $l(x, AC)$ for the chains linking A and C (and not B). The recursion relations are :

$$\begin{aligned} l' &= l^2 + l^2 t \\ t' &= t^2 + l^2 t \end{aligned} \quad (21)$$

with the initial conditions $l_1 = x$, $t_1 = x^2$. A non-trivial symmetric fixed point exists with $l^* = t^* = (\sqrt{5} - 1)/2$; it corresponds to the Sierpinski fixed point. However, it is purely repulsive : the eigenvalues $\lambda_1 = (7 - \sqrt{5})/2$ and $\lambda_2 = \sqrt{5} - 1$ are both larger than unity. Starting from the initial conditions (l_1, t_1) the flow is toward a different non-trivial fixed point : $t^* = 0$, $l^* = 1$. Our tentative interpretation is that on large scales the SAW become elongated along directions AC or BC, and $\nu = 1$ as for a one-dimensional polymer.

It appears then that the lowering of symmetry is a relevant perturbation and that the HBA gasket does not belong to the same universality class as the Sierpinski gasket, inasmuch as SAW are concerned.

d) *Three-dimensional Sierpinski gasket.* — This fractal is a 3-d generalization of figure 1b, with tetrahedra replacing triangles. Its dimensions are $\bar{d} = 2$, $\bar{d} = 2 \ln 4 / \ln 6 = 1.547...$. To take into account the obstruction effect arising when a SAW goes through more than two vertices of the same tetrahedron, it is necessary to introduce 4 different generating functions.

The recurrence relations have to be obtained by computer, but their analysis shows that only two of these functions are non-zero at the relevant fixed point. We denote them $p(x)$ for the SAW going through only two vertices and $q(x)$ for the SAW such that one piece of the chain goes through two vertices and another piece through the two others. The recursion relations in the (p, q) space are :

$$\begin{aligned} p' &= p^2 + 2p^3 + 2p^4 + 6p^2q^2 + 4p^3q \\ q' &= p^4 + 4p^3q + 22q^4. \end{aligned} \quad (22)$$

They hold exactly for the truncated 4-simplex studied by Dhar [7], due to the absence of the obstruction effect, but the initial conditions are slightly different on both lattices (some configurations are excluded from p_1 and q_1 on the gasket). One finds numerically :

$$p^* = 0.42944..., \quad q^* = 0.049984...,$$

$$\mu = 3.815... \text{ (Sierpinski)}, \quad \mu = 3.868... \text{ (4-simplex)}.$$

The largest eigenvalue λ_2 at the fixed point gives : $\nu = \ln 2 / \ln \lambda_2 \sim 0.729$ for both structures.

It is interesting to note that a two-parameter renormalization is necessary here. Setting $q = 0$ in (22) gives an approximation $\nu \sim 0.715$ which might look satisfactory if compared with, say, Monte-Carlo results : but the correct theory has to take into account the possibility that a SAW goes twice through a given tetrahedron, even on large scales, and this changes the exponent. This effect is often neglected in finite size scaling studies of SAW on Euclidean lattices [11] and one may wonder how well this approximation is justified.

4. Discussion and application to percolation clusters.

4.1 UNIVERSALITY. — Since a lowering of the symmetry may change the properties of SAW without affecting $\tilde{d}$ (Section 3), we now restrict our attention to those systems where isotropy is retained on large scales. A graph of $\tilde{d}\nu (= \nu\tilde{d})$ versus $\tilde{d}$ is displayed in figure 2 for the Euclidean lattices and the structures studied above. We have also used a result derived by Dhar [7] for a modified rectangular lattice with $\tilde{d} = 2$, $\tilde{d} = 3/2$ and $\nu = 0.6650...$ The Flory approximation (Eq. 8) is seen to be rather good for some cases but far off for others. There is clearly a correlation between $\tilde{d}\nu$ and $\tilde{d}$ and another interpolation formula might provide a better approximation, but there is no hope that a reasonably smooth curve accounts for all results. We therefore conclude that the exponent ν depends on other properties of the fractal space than just the spectral dimension of its backbone.

This suggests that the success of the Flory formula for Euclidean lattices is somewhat accidental, and that for general spaces there exists no comparable formula combining simplicity and accuracy.

Fig. 2. — Self-avoiding walk exponent *versus* spectral dimension : $\blacktriangle$ Euclidean lattices ; $\bullet$ branching Koch curve ; $\times$ Sierpinski gaskets ; $+$ modified rectangular lattice. The Flory approximation (Eq. 8) is shown by the dashed line.

4.2 PERCOLATION CLUSTERS. — Incipient infinite clusters at the percolation threshold provide a physical realization of fractal spaces [1-3, 6]. It is of interest to outline the implications of our results for the properties of SAW on such clusters, since the question of SAW on disordered lattices is rather controversial [12-15].

Numerical simulations by Kremer [12] seem to indicate that the SAW exponent is not modified when only a fraction p of the lattice sites are allowed, except at the percolation threshold. A different conclusion is reached by Harris [14] who claim that ν is the same for all values of p , including p_c . However, in his calculations, the average over disorder is carried out separately for the numerator and denominator in the expression of $\langle R_N^2 \rangle$: this procedure is akin to annealing in spin systems and it is not clear that it gives the same result as the physical (quenched) averaging. Finally, Derrida [15] argues from his results on strips of finite width that ν is modified even by a weak disorder.

A first remark is that at the percolation threshold, $p = p_c$, a distinction should be made on SAW statistics restricted to the infinite cluster (as done by Kremer [12]) or averaged over all clusters (Harris [14]) : here only SAW on the infinite cluster will be considered.

A second remark is that the Flory-type formula proposed by Kremer [12] : $\nu_F = 3/(\tilde{d} + 2)$, with $\tilde{d}$ the fractal dimension of the full cluster, does not satisfy the general requirements discussed in section 2. Agreement with numerical simulations can only be fortuitous.

We give in table I the predictions of the approximation formula proposed above :

$$\nu_F = \frac{1}{\tilde{d}_B} \frac{3\tilde{d}_B}{\tilde{d}_B + 2}, \quad (8)$$

Table I. — Flory approximation v_F for self-avoiding walks on percolation clusters at threshold ($p = p_c$) compared with the corresponding exponent v_0 on Euclidean lattices ($p = 1$) in dimension d . The characteristic dimensions $\bar{d}$, $\bar{d}_B$, $\tilde{d}_B$ are defined in the text. The errors quoted are only indicative.

d	$\bar{d}$	$\bar{d}_B$	$\tilde{d}_B$	v_F	v_0
2	1.896	1.68 ^(a) ± 0.02	1.25 ^(a) ± 0.01	0.69 ± 0.01	3/4
3	2.52 ^(b) ± 0.02	2.0 ± 0.1	1.23 ± 0.03	0.57 ± 0.02	3/5
4	3.21 ± 0.07	2.3 ± 0.2	1.18 ± 0.05	0.49 ± 0.03	1/2
6	4	2	1	1/2	1/2

^(a) Reference 18.

^(b) Using $v_p = 0.88 \pm 0.01$ (reference 19).

as compared with the Flory exponent v_0 for pure Euclidean lattices. For high dimensions ($d \geq 6$) the known result $v = 1/2$ is recovered. It is to be noted that the physical origin of this value is quite different from the origin of $v_0 = 1/2$ for the pure system : in the latter, excluded volume effects become irrelevant for $d > 4$ and SAW behave just as random walks in a Euclidean lattice. Whereas, at $p = p_c$, the backbone itself has the structure of a linear polymer ($\bar{d}_B = 2$, $\tilde{d}_B = 1$), and repulsive interactions are fully effective in stretching SAW as much as possible.

The values of $\bar{d}$ and $\bar{d}_B$ quoted in table I have been extracted from various sources [16-19] and, except for $d = 2$, the values of $\tilde{d}_B$ have been obtained through the relation :

$$-\beta_L = \bar{d}_B \left(\frac{2}{\bar{d}_B} - 1 \right) = \bar{d} \left(\frac{2}{\bar{d}} - 1 \right), \quad (23)$$

using $\bar{d} = 4/3$ for simplicity [1, 10, 16]. Relation 23 just expresses that the conductance of a cluster is the same as that of its backbone (see Eq. 12).

Table I calls for several remarks :

i) $\bar{d}_B$ is not monotonic as a function of d , as confirmed by the expansion in $\varepsilon = 6 - d$: $\bar{d}_B = 2 + \varepsilon/21 + \dots$, using results of Harris [17].

ii) $\bar{d}_B$ decreases monotonically toward its mean-field value of 1, and differs from $\bar{d} \simeq 4/3$.

iii) v_F is lower than the Flory exponent v_0 of the pure system, but the difference is not large enough to be significant, in view of the approximations involved.

The suggestion that v at $p = p_c$ (for SAW on the infinite cluster) is equal to v_0 at $p = 1$ is therefore not inconsistent with our results on various fractals. A more detailed understanding of the universality classes for SAW on non-homogeneous lattices will be necessary to resolve that intriguing question.

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Note. — After this work was completed, we received a preprint by D. Ben-Avraham and S. Havlin entitled *Self avoiding walks on finitely ramified fractals*. These authors study the quantity $\bar{N}(R) = \sum_N N G_N(R) / \sum_N G_N(R)$, in our notations, and define an exponent v^* by : $\bar{N}(R) \sim R^{1/v^*}$. For the Sierpinski gasket they find $v^* = 1/\bar{d}$, which means that the chains occupy a finite fraction of the fractal space : this is in fact expected because this $\bar{N}(R)$ is dominated by the longest chains, not the most probable ones. In other words, they study $G(x, R)$ at $x = 1$, which corresponds to collapsed chains, rather than at the fixed point x^* , which is physically relevant for repulsive interactions.

After this work was accepted for publication, we received a preprint « Self-interacting self-avoiding walks on the Sierpinski gasket » by D. J. Klein and W. A. Seitz. These authors obtain the same value of the exponent v on the $2 - d$ gasket as we do, they show in addition that this value is not modified by the introduction of a self-interaction term.

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