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HAL Id: jpa-00208806
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Submitted on 1 Jan 1978

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CRITICAL EFFECTS IN RAYLEIGH-BÉNARD CONVECTION

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(Reçu le 23 décembre 1977, accepté le 16 mars 1978)

Résumé. — On montre que les propriétés du phénomène convectif de Rayleigh-Bénard au voisinage de son seuil critique d'instabilité sont étroitement liées à celles d'une transition de phase du second ordre. Dans le cadre des hypothèses de Landau-Hopf une théorie est développée dont l'ensemble des conséquences est vérifié par les résultats expérimentaux.

Abstract. — The properties of the convection Rayleigh-Bénard phenomenon near its critical threshold of instability are shown to be closely related to those of a second order phase transition. A theory is developed in the framework of the Landau-Hopf hypothesis, and all its predictions are verified by experimental results.

1. Introduction. — In the Rayleigh Bénard instability [1], it has been shown experimentally [2] that the local convection velocity increases, near the onset of convection, according to the power law:

$$V = V_0 \varepsilon^{1/2}$$

with $\varepsilon = R - R_c$, $R$ being the Rayleigh number and $R_c$ its critical value; in the case of a horizontal layer with rigid boundaries $R_c = 1707$. At $R = R_c$ the symmetry of the non convecting fluid with respect to infinitesimal translations and rotations is spontaneously broken and an ordered convective state is established in the form of a remarkable spatial periodicity of the velocity [2] and of the perturbation of the temperature [3, 4]; this corresponds in the case of a rectangular cell to the well known system of parallel convecting rolls. Such behaviour presents an analogy with a second order phase transition, where the order parameter should be the velocity. The exponent $1/2$ in the behaviour of the velocity (1) should be identified with the critical exponent, usually called $\beta$, its value being, in the present case, in good agreement with the phenomenological mean field approach of the Landau-Hopf model.

Similarly the Landau-Hopf approach tells us that the growth (or the decay) rate of the velocity amplitude becomes very slow as $\varepsilon$ becomes very small. This obviously corresponds to the well known critical slowing down in phase transitions. This critical behaviour in Rayleigh-Bénard convection has been studied theoretically in [6], [7] and [8]. Experimentally pointed out first in [2] and [9] this critical slowing down has been studied in detail by acoustic observations [10] and by heat flux measurements [11]. We present here the results obtained by direct measurements of the order parameter itself, that is to say of the velocity amplitude. We extend the Landau-Hopf model to the concept of influence length, which diverges as $\varepsilon$ goes to zero, and compare the theoretical predictions with the experimental values obtained from the local velocity measurements.

2. Theory. — 2.1 Landau-Hopf picture of the critical slowing down and of the influence length. — Both the critical behaviour of the influence length and the critical slowing down may be understood in the frame of the Landau-Hopf theory. There is also an obvious connection between this problem and the well known Eckhaus theory of stability for parallel rolls near the onset of convection [12].

The basic remark is that, in slightly supercritical conditions, the linear growth rate depends in a generic fashion on $\varepsilon$ and on the horizontal wavenumber. When the horizontal wavenumber is just equal to its value for the marginally stable state at $R_c$ (or at $\varepsilon = 0$), the expression for $\tau^{-1}$ (linear growth rate of the fluctuation) near $\varepsilon = 0$ is

$$\tau^{-1} = \tau_0^{-1} \varepsilon .$$

\textsuperscript{(1)}
But one knows from the theory of the linear stability, that in supercritical conditions, perturbations with a wavenumber lying in a whole interval of values around \( k = k_e \) are linearly unstable, the width of this interval collapsing to zero at \( \varepsilon = 0 \). Thus a natural extension of (1) to account for the existence of this interval is

\[
\varepsilon^{-1} = \tau_0^{-1} \left[ \varepsilon - \xi_0^2 (k - k_e)^2 \right]
\]

(2)

where the quantity \( \xi_0 \), which has the dimension of a length is computed below and does not depend on the Prandtl number \( P \).

To study the space-dependent phenomena due to a local perturbation in a system of parallel rolls, it is convenient to consider the perturbed roll system as having a well defined wavenumber \( k_e \), and a slowly modulated amplitude (or envelope). Of course this approximation implies that many rolls exist in the convective state, so that the horizontal extent of the container must be much larger than the wavelength of the rolls (in practice it has to be much larger than the height of the convection layer). We are close enough to the onset of instability to ensure that only one eigenmode is unstable for one value of the Rayleigh number although the determination of the wavelength becomes a difficult question [1] if \( R \) is so large that many modes are unstable for a finite box, and can be considered as a continuum. The existence of a band of unstable wavenumbers (in the limit of an infinite horizontal layer) is needed here in order to fit the horizontal boundary condition for a finite horizontal layer. Then \( V \) depends on \( x \) as

\[
V = \tilde{V}(x) \sin \left( k_e x + \phi \right).
\]

with

\[
\left| \frac{1}{\tilde{V}} \frac{d\tilde{V}}{dx} \right| \ll k_e.
\]

In this (realistic) picture, the term \((k - k_e)^2\) in (2) yields a second derivation of \( \tilde{V} \) with respect to \( x \), and the linear part of the Landau-Hopf equation becomes

\[
\frac{\partial \tilde{V}}{\partial \varepsilon t} = \tau_0^{-1} \left( \varepsilon \frac{\partial \tilde{V}}{\partial \varepsilon} + \xi_0^2 \frac{\partial^2 \tilde{V}}{\partial x^2} \right).
\]

Cubic terms must be added to account for the small non linearities existing near \( \varepsilon = 0 \), and one gets

\[
\frac{\partial \tilde{V}}{\partial \varepsilon t} = \tau_0^{-1} \left( \varepsilon \frac{\partial \tilde{V}}{\partial \varepsilon} + \xi_0^2 \frac{\partial^2 \tilde{V}}{\partial x^2} - \frac{\tilde{V}^3}{V_0^2} \right),
\]

(3)

where \( V_0 \) can be computed from the fluid equations by an \( \varepsilon \)-expansion near the onset of convection [1].

In order to look at spatial modulation of the system of parallel rolls, one has to solve (3) under stationary conditions. This gives

\[
\xi_0^2 \frac{d^2 \tilde{V}}{dx^2} = - \varepsilon \frac{\partial \tilde{V}}{\partial \varepsilon} + \frac{\tilde{V}^3}{V_0^2}.
\]

(4)

One readily recognizes the equation of motion of a particle located at time \( \varepsilon \) at the point \( \tilde{V} \) and moving in the potential

\[
U(\tilde{V}) = \frac{\varepsilon \tilde{V}^2}{2} - \frac{1}{4} \frac{\tilde{V}^4}{V_0^2}.
\]

Let us consider first the effect of a single lateral boundary; in this case, the conditions for the equation of the motion are \( \tilde{V} = 0 \) at \( x = 0 \) (no motion at the boundary) and \( \tilde{V} = \tilde{V}_0 \) at \( x = + \infty \) \( \varepsilon = \xi_0^2 V_0 \).

The solution is

\[
\tilde{V} = \tilde{V}_0 \tanh \left( \frac{x}{\xi} \right),
\]

where \( \xi = \xi_0 (2/\varepsilon)^{1/2} \) is the characteristic influence length that becomes infinite or critical at \( \varepsilon = 0 \). The \( \varepsilon^{-1/2} \) dependence of \( \xi \) is typical of this Landau-Hopf approach.

In a more realistic situation one should consider the effect of two lateral walls, that is the solutions of (4) under the conditions \( \tilde{V} = 0 \) at \( x = 0 \) and \( x = L \), \( L \) being the distance between the lateral walls which are parallel to the axis of the rolls. In the image of the motion of a particle in a potential, \( L \) is the time interval between two crossings at \( \tilde{V} = 0 \). This problem may have more than one solution (in particular \( \tilde{V} = 0 \) is always a solution). However a single solution is stable with respect to the dynamics described by (3); this is the convective solution (when it exists) without oscillations (or nodes) : \( \tilde{V} = 0 \) at \( x = 0 \) and \( x = L \) only. We shall not give the general form of this solution, which may be found for instance in the work presented in [13]. Let us only mention that, for a given \( \varepsilon \), such a solution exists if \( L \) is larger than some value ; this is actually due to the fact that the period of the oscillations in the potential \( U(\tilde{V}) \) has an upper bound \( 2 \pi \xi_0 \varepsilon^{-1/2} \).

If \( L \) is less than \( L_c = \pi \xi_0 \varepsilon^{-1/2} \), no convecting state exists. If \( L \) is slightly larger than \( L_c \), the envelope of the rolls is given by a sine function

\[
\tilde{V} = \frac{2 V_0 \sqrt{\varepsilon}}{\sqrt{3} L_c} \left( \sqrt{L^2 - L_c^2} \right) \sin \left( \frac{\pi x}{L} \right).
\]

Accordingly the well-known \( \varepsilon^{1/2} \) dependence of the amplitude of the rolls is reached in the range \( L \gg L_c \) only (or, \( L \) being given, for \( \varepsilon \gg \xi_0^2 \pi^2 L^2 \)) in our experimental conditions).

If, on the contrary, a perturbation \( \tilde{V} \) of non zero amplitude is imposed at the boundary of a slightly
subcritical region ($\varepsilon < 0$), this yields a system of rolls decaying exponentially in space (at least, if their amplitude is small enough to make the cubic term in (4) negligible), according to the law

$$\tilde{V}(x) = \tilde{V}(x = 0) e^{-\lambda' x}$$

where $\lambda' = \lambda_0(-\varepsilon)^{-1,2} (\varepsilon < 0)$ is to be compared with the penetration length in the supercritical condition: $\lambda = \lambda_0(2/\varepsilon)^{1,2} (\varepsilon > 0)$.

In critical condition ($\varepsilon = 0$), the boundary condition for the penetrating convection yields a system of parallel rolls whose amplitude decays according to the law

$$\tilde{V}(x) = V_0 \frac{\xi_0 \sqrt{2}}{x + x_0}$$

$x_0$ being determined by the boundary condition.

As we want to apply the preceding consideration to a real situation, we have to know explicitly the quantities called $V_0$, $\tau_0^{-1}$ and $\xi_0$. The computation of $V_0$, which defines the amplitude of the convection has been clearly done in detail [1, 14] and the theory is in good agreement with the experimental findings [2].

The quantities $\tau_0^{-1}$ and $\xi_0$ have already been computed in [6-8]. These calculations however have been done in the (unrealistic) case of free-free boundary conditions. (See however ref. [11] and our comment at the end of the subsection 2.2.) So we shall give a few details about the case of rigid-rigid boundaries which correspond to our experiments.

2.2 Determination of the coefficient $\tau_0$. Let us start with the linearized, time dependent equation in $V_z$, the vertical component of the velocity [15]

$$(P^{-1} \frac{\partial}{\partial t} - A) \left( \frac{\partial}{\partial t} - A \right) \Delta V_z = R(\beta_{xx}^2 + \beta_{yy}^2) V_z .$$

(5)

The relation between the temperature perturbation $\theta$ and $V_z$ is given by:

$$(P^{-1} \frac{\partial}{\partial t} - A) \Delta V_z = \theta$$

where we have used the scaling factors $d, \kappa/d$, $d^2/\kappa$, $\Delta T/R$ for length velocity, time and temperature respectively, $\kappa$ being the thermal diffusivity.

In the free-free case, solutions of the form:

$$V_z = e^{\sigma t} \cos kx \sin \pi z ,$$

where $\sigma = \tau^{-1}$ satisfy the boundary conditions:

$$V_z = 0, \quad \frac{\partial V_z}{\partial z} = 0 \quad \text{and} \quad \theta = 0 \quad \text{at} \quad z = \pm \frac{1}{2} ,$$

Therefore $\sigma$ is given by the solution of a second order equation:

$$\sigma^2 A - \sigma (1 + P) A^2 + P (A^3 + R k^2) = 0$$

with

$$A = - (\pi^2 + k^2)$$

$$\sigma_+ = \frac{(1 + P) A}{2} \pm \frac{(1 + P) A}{2} \times$$

$$\sqrt{1 - 4 P \left( 1 + P \right)^2 \left( 1 - \frac{R}{R_0(k)} \right)}$$

(6)

where

$$R_0(k) = \frac{\left( \pi^2 + k^2 \right)^{3/2}}{k^2} ,$$

$R = R_0(k)$ being the equation of the curve of marginal stability corresponding to $\sigma = 0$. Near the critical point : $R_0(k) \sim R_c$, and the square root in (6) expands in powers of $\varepsilon$ [13] giving at the lowest order

$$\sigma_+ = (\pi^2 + k^2)^{3/4} \frac{P}{1 + P} \varepsilon ;$$

we don’t worry about $\sigma_-$ which is always negative. In the rigid-rigid case, solutions of (5) are of the form:

$$V_z = e^{\sigma \tau} \cos kx \left( \sum_{i=1}^{3} A_i \chi_i \mu_i z \right)$$

where the coefficients $\mu_i$ are given by the solution of:

$$\sigma^2 B_i - \sigma (1 + P) B_i^2 + P(\beta_i^3 + R k^2) = 0$$

(7)

$$B_i = \mu_i^2 - k^2$$

According to the results obtained in the free-free case we expect $\sigma$ to depend on $\varepsilon$ so that the term in $\sigma^2$ can be neglected. Then we introduce $\delta \mu_i$ and $\delta B_i$, the deviation of $\mu_i$ and $B_i$ to $\mu_i$ and $B_i$:

$$\mu_i = \mu_i + \delta \mu_i \quad \text{and} \quad B_i = B_i + \delta B_i$$

with:

$$B_i = \mu_i^2 - k^2$$

and

$$B_i^3 + R k^2 = 0.$$ 

Inserting these expressions in (7) and writing $R$ in the form:

$$R = R_c + \Delta R ,$$

we obtain the expressions for $\delta B_i$ and consequently for $\delta \mu_i$ as functions of $\sigma$ and $\varepsilon$:

$$\delta B_i = \frac{(1 + P)}{3 P} B_i + \frac{1}{3} \varepsilon$$

$$\delta \mu_i = \frac{\delta B_i}{2 \mu_i} .$$
The coefficients $A_i$ are determined by the boundary conditions

$$V_x = V_x = \theta = 0 \text{ at } z = \pm \frac{1}{2}$$

$$\sum_{i=1}^{3} A_i \mu_i/2 = 0 \quad \text{(8a)}$$

$$\sum_{i=1}^{3} A_i \mu_i \sin \mu_i/2 = 0 \quad \text{(8b)}$$

$$\sum_{i=1}^{3} A_i (P^{-1} \sigma - B_i) B_i \sin \mu_i/2 = 0 \quad \text{(8c)}$$

The above expressions are expanded to first order in $\delta \mu$. Then when we form the determinant of the algebraic system $(8-a, b, c)$ we recognize the equation of the curve of marginal stability and the determinant reduces to:

$$C_1 \sigma \left( \frac{1 + P}{3} \right) + C_2 \sigma \left( \frac{1 - 2 P}{3} \right) + C_3 \varepsilon = 0.$$

The coefficients $C_1$, $C_2$ and $C_3$ which involve many complex numbers have been calculated on a computer, leading to the final expression:

$$\sigma = \frac{38,442.9 P}{1 + 1.9544 P^{-1} \varepsilon}.$$

Behringer and Ahlers [11] give without derivation a formula for $\sigma$ (their $\tau_{\text{Re}}$ is just $\sigma^{-1}$) for the case of rigid boundaries. Up to minor arithmetical differences their result coincides with ours if one puts

$$\tau_{\text{D}} = \frac{d^2 (2 \pi)}{\kappa}$$

in their formula (although this value of $\tau_{\text{D}}$ is not given explicitly).

2.3 Determination of the coefficient $\xi_0$ (Curvature of the neutral curve near the critical point). — As previously shown, in the free-free case the equation of the curve of marginal stability is:

$$R = \frac{(\pi^2 + k^2)^3}{k^2},$$

which can be expanded near the critical point $(R_c, k_c)$ by setting

$$R = R_c + \Delta R \quad \text{and} \quad k = k_c + \Delta k.$$}

The lowest-order term in the expansion of the right hand side is just $R_c$, the term in $\Delta k$ is null and finally to second order in $\Delta k$ we get:

$$\Delta R = \frac{4 R_c}{3 k_c^2} (\Delta k)^2$$

leading to:

$$\xi_0^2 = 8 d^2/3 \pi^2.$$
of the cell and \( z' \) is the vertical axis, \( d \) is the thickness of the layer).

As this study is related to critical phenomena, the measurements are made at values of \( \epsilon \) lower than 1 and the convection structure is bidimensional and made of parallel rolls with a wavenumber

\[
k \approx k_c = 3.117 \, d^{-1}
\]

3.2 Critical slowing down. — As already explained, the Landau-Hopf equation reads in an unsteady (and homogeneous) situation:

\[
\frac{d\bar{V}}{dt} = \frac{1}{\tau_0} \left( \bar{V} - \frac{\bar{V}^3}{\bar{V}_0^2} \right)
\]  

(9)

where \( \bar{V} \) is the velocity amplitude. If we consider the stationary solution \( d\bar{V}/dt = 0 \), we obtain the previously reported relation \( \bar{V} = \bar{V}_0 e^{\epsilon^{1/2}} \); an experimental illustration of this behaviour is shown on figure 1.

![Fig. 1. — Dependence of the maximum velocity \( V_x \) \text{max} versus \( \Delta T \) (\( \Delta T \) is the temperature difference applied to the layer).](image1)

Integrating equation (9), we obtain the time approach of the amplitude of the velocity to its new equilibrium value when the temperature difference \( \Delta T \) applied to the fluid layer is suddenly increased or decreased

\[
\bar{V}(t) = \bar{V}_f \exp \left( \frac{t}{\tau} \right) \left[ \exp \left( \frac{2t}{\tau} \right) + \left( \frac{\bar{V}_f}{\bar{V}_i} \right)^2 - 1 \right]^{-1/2}
\]  

(10)

\( \bar{V}_i \) and \( \bar{V}_f \) being respectively the initial and final amplitudes of the velocity.

In this calculation we have completely neglected the space dependence of the perturbation. Actually the dynamics of \( \bar{V} \) involves also a structure change, as \( \bar{V}_f \) and \( \bar{V}_i \) do not depend on \( x \) in the same manner. In the experimental situation, however, we were looking at the velocity in the middle of the cell near the maximum of the envelope, where this envelope is a very flat function of \( x \), so that \( \frac{\partial^2 \bar{V}}{\partial x^2} \) is very small with respect to 1, and may be neglected, yielding the ordinary differential equation (9) instead of (3).

Of course this approximation is no longer valid near the onset of convection, in the region where the envelope is close to a sine function.

![Fig. 2. — Time dependence of \( (V_x^2)_f \) after a sudden change of \( \Delta T \) at \( t = 0 \); the measurements corresponding to the first minutes have been dropped for they correspond to the temperature rearrangement of the layer. The full line represents the best fit with equation (10).](image2)

An example of the experimental time dependence is shown on figure 2. The velocity is measured at a fixed point in the layer (practically at a point where \( V \) is maximum \( \text{versus} \) \( x \) and \( z \), near \( x \approx L/2 \)) as a function of time after a sudden increase of \( \Delta T \). We can see that the experimental behaviour is well described by the relation (10), which is represented by the full line and from which we deduce the experimental value of \( \tau \). Note furthermore that the values of \( \tau \) obtained either by increasing or decreasing \( \Delta T \) (and then \( V \)) are very consistent.

From the dependences measured at different values of \( \Delta T \), we can obtain the variation of \( \tau \) with respect to \( \epsilon \), and as shown on figure 3:

\[
\tau = (20 \pm 4) \epsilon^{-1.03} \pm 0.05 \, s.
\]

![Fig. 3. — Dependence of the characteristic time \( \tau \) \text{versus} \( \epsilon \).](image3)

The expected power law is very well defined experimentally and it is interesting to compare the value of the pre-exponential factor, \( \tau_0 \), with the theoretical prediction

\[
\tau_0 = \frac{d^2}{\kappa} \left( 1 + \frac{1.954}{P} \right)
\]

where

\[
\kappa = \frac{d^2}{\rho c_p} \left( 1 + \frac{1.954}{P} \right)
\]
where \(d^2/K\) is the usual scaling factor; for the used oil, \(d^2/K = 320\) s and \(P\) goes to infinity; so the calculation gives

\[\tau_0 \approx 16\text{ s}.
\]

We see that measured and calculated values are in reasonably good agreement within the limit of our experimental accuracy.

3.3 CRITICAL INFLUENCE LENGTH. — As explained above, if we look at the experimental dependences of the velocity \(V = f(x)\), measured at a fixed value of \(z\), the envelope of the usual sinusoidal variation should follow a law as \(P \sim \tgh(x/\xi)\), where \(\xi\) is critical at \(\varepsilon = 0\) and depends on \(\varepsilon\) as \(\xi = \xi_0(2/\varepsilon)^{1/2}\) \(x\) is the distance from the boundary. Typical experimental measurements are shown on figure 4. One can see —

![Fig. 4. Spatial dependences of \(V_x\) (maximized versus \(z\)) with the distance \(x\) to the side wall for two different values of \(\varepsilon\).](image1)

... curve A — that for a value of \(\varepsilon = 0.6\) the behaviour of the velocity is essentially represented along the whole cell by a law such as \(V_x = (V_0^2)_{x0} \sin kx\), as is well known. For very low values of \(\varepsilon\), as shown in curve B, the influence domain due to the rigid boundary, which imposes the condition \(V \equiv 0\) at \(x = 0\), penetrates more and more deeply in the cell affecting many rolls.

This extra modulation effect is well fitted by a law such as \(P = P_0^1 \tgh(x/\xi)\) as shown on figure 5 and in good agreement with theoretical predictions. From many determinations of \(V_x = f(x)\) obtained at different values of \(\varepsilon\), we deduce the behaviour of \(\xi\) as a function of \(\varepsilon\). The result is shown on figure 6 where the experimental relation is

\[\xi = (0.54 \pm 0.05)\, d^{-0.49 \pm 0.03}.
\]

![Fig. 6. Dependence of the reduced influence length \(\xi/d\) versus \(\varepsilon\); the full line gives the theoretical prediction.](image2)

... the agreement with the expected dependence is excellent, not only for the exponent \(v = 0.5\) but also for the pre-exponential factor. Indeed, the calculated relation is

\[\xi = 0.54\, d^{-0.5}\]

which in the case of the studied layer \(d = 0.6\) cm gives

\[\xi = 0.324\, e^{-0.5}\, \text{cm}.
\]

4. Conclusion. — All the consequences of the Landau-Hopf picture applied to the Rayleigh-Bénard transition have been calculated and checked experimentally by direct measurement of the local velocity, i.e. of the order parameter. We have seen how the theoretical predictions are followed experimentally.

![Fig. 7. Dependence of the Landau potential density \(\Phi\) versus velocity amplitude at infinite \(x\), for different values of \(\varepsilon\). To know the dependence as function of the adimensionalized velocity, the velocity coordinate in this figure must be multiplied by 0.0531 (see conclusion), in fluids with \(P \gg 1\).](image3)
In particular, the experimental exponents we found are classical: the exponent \( p \) which gives the relation of the velocity amplitude with \( \varepsilon \) is 0.5; the divergence of the influence length \( \xi \) is found with the exponent \( v = 0.5 \), and finally the critical slowing down presents characteristic time diverging at \( R_c \) with the exponent \( y = 1 \) (\(^2\)).

\(^2\) Similar behaviour restricted to the order parameter and critical slowing down had been found on the Taylor instability, see [17, 18], and on the Soret driven instability [19].

All these measurements allow us to give the complete Landau functional under the form (see Fig. 7)

\[ \varphi = -\frac{1}{\tau_0} \int \left[ \frac{\varepsilon V^2}{2} - \frac{1}{4} \frac{V^4}{V_0} + \frac{1}{2} \xi_0^2 \left( \frac{\partial V}{\partial x} \right)^2 \right] \, dx \]

with

\[ \xi_0^2 = 0.148 \frac{d^2}{\kappa} \]

\[ \tau_0 = 0.063 \frac{d^2}{\kappa} \]

\[ V_0 = 10.5 \frac{\kappa}{d} \]

References