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Low frequency coupling and mode interference in an inhomogeneous lattice of finite length

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Summary

The propagation through a lattice made with two waveguides periodically coupled by perforations is studied at low-frequencies. A degree of inhomogeneity is introduced with parametrically opened diaphragms inserted into one waveguide of the lattice. Analytical results obtained thanks to the fourth-order transfer matrix formalism illustrate three physical phenomena. The first phenomena is the effect of the perforations, the second is the effect of the inhomogeneity of the lattice, and the third is the effect of the interference between propagating modes when the system have finite length. These results give analytical expressions for the insertion loss, the characteristic impedance or propagation constants, that can be of practical interest for understanding and designing non local acoustic treatments for automotive and turbofan engines. The cases of a strongly inhomogeneous lattice and almost homogeneous lattice and their transition to homogeneous and branched resonator cases are discussed.

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1. Introduction

This work aims to describe the acoustic propagation at low frequencies in a system of two coupled waveguides, with a particular focus on understanding the effect introduced by inhomogeneity of the lattice, that is made of two waveguides filled with different propagation media. The coupling is carried out periodically, with lateral perforations disposed regularly in the axial direction of the waveguides (see Fig.1). This initial system was chosen to represent by means of a discrete model, a variety of situations encountered in practice for noise mitigation by wall treatments for automotive exhaust and aircraft turbofan engine nacelles.

The analytical model for an elementary cell of the periodic lattice and for a lattice of finite length is detailed in [1]. Conventional tools for propagation in periodic media [2] make possible to establish the analytical expressions that characterize an elementary cell of the inhomogeneous lattice (eigenvectors, eigenmodes, and associated characteristic impedances, phase velocities ...). In the following expressions for the fourth-order transfer matrix and the dispersion equation of the lattice are recalled in section 2 and 3. The coupling between the two waveguides by means of lateral perforations is described rigorously by a perforation matrix [3].

By introducing boundary conditions at the ends of the lattice, the influence of the properties of one elementary cell on the potential noise mitigation (insertion loss) of a lattice of finite length is also illustrated.

![Figure 1. Inhomogeneous lattice, with two waveguides (with different propagation media), periodically coupled by lateral perforations](image)

Starting from well known (and extreme) lattice configurations used as references, namely coupled waveguides filled with air (totally homogeneous lattice) and
Helmholtz resonators branched on Waveguide 1 (totally inhomogeneous lattice), the discrete model is used to unveil how the degree of inhomogeneity modifies the properties of the lattice and its associated insertion loss. The degree of inhomogeneity of the lattice is introduced here by means of parametrically opened diaphragms inserted into Waveguide 2. The cases of a strongly inhomogeneous lattice and almost homogeneous lattice and their transition to extreme (references) case are discussed.

2. Fourth order transfer matrix of one cell of a periodic lattice

We consider two waveguides periodically coupled along their axes by lateral perforations as shown in Fig.1. In this section, we write the transfer matrix of an elementary cell (of length 2l) of the periodic lattice. The two waveguides are filled with different propagation media, and the resulting lattice is called an inhomogeneous lattice.

Kergomard et al. [8] showed that coupling between plane waves in guides 1 and 2 introduced by a lateral perforation situated at a given abscissa \( x_n \), can be described in an exact manner by a perforation matrix of fourth order. At a given frequency, the plane wave amplitudes of the acoustic pressures and velocity on the left of a lateral perforation

\[
V_L = \begin{pmatrix} V_{1L} \\ V_{2L} \end{pmatrix} = \begin{pmatrix} p_{1L} \\ v_{1L} \\ p_{2L} \\ v_{2L} \end{pmatrix},
\]

(1)

can be related to the same quantities on the right \( V_R \) of the perforation, using a fourth order perforation matrix \( P_F \) in order to write:

\[
V_L = P_F V_R.
\]

(2)

The matrix \( P_F \) takes the form [3] (anti-symmetrical orientation):

\[
P_F = \begin{pmatrix} (\gamma_1 + \gamma_2 M) & \gamma_2 (I - M) \\ \gamma_1 (I - M) & (\gamma_2 + \gamma_1 M) \end{pmatrix},
\]

(3)

with

\[
\gamma_{1,2} = \frac{S_{1,2}}{S_1 + S_2}, \quad M = I + \frac{2Z_0 Y_s}{1 - Z_0 Y_s} \begin{pmatrix} 1 & Y_s^{-1} \\ Y_s & 1 \end{pmatrix},
\]

(4)

where \( S_1 \) and \( S_2 \) are the cross sections of the waveguides, and \( Z_0 \) and \( Y_s \) introduce respectively the series (specific) impedance and shunt (specific) admittance of the lateral perforation. \( I \) is the second order identity matrix.

![Figure 2. Cylindrical Lattices of finite length: inhomogeneous lattice (top), branched Helmholtz resonators with closed cells in Waveguide 2 (center) and homogeneous lattice (bottom).](image)

The propagation of plane waves along the uncoupled portion of the waveguides, i.e. between abscissa \( x_{n+1} \) and \( x_n \) (length 2l, see Fig.1), is described by a classical fourth order transfer matrix \( T \):

\[
T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 & 0 & 0 \\ C_1 & D_1 & 0 & 0 \\ 0 & 0 & A_2 & B_2 \\ 0 & 0 & C_2 & D_2 \end{pmatrix}.
\]

(5)

The complete transfer matrix relating the plane wave amplitudes on the left of two successive perforations of the periodic lattice (see Fig.1) is written as the product of the transfer and the perforation matrix:

\[
V_{L,n} = P_F T V_{L,n+1}.
\]

(6)

(7)

Since the expressions for the transfer matrix \( T \) and the perforation matrix \( P_F \) given above are general, they can describe propagation at low frequency in a great variety of physical situations. For instance situations where Waveguides 1 and 2 are non reciprocal (for
instance by the presence of flow), exhibit viscothermal losses (equivalent fluid modeling a porous material), or include discontinuities like diaphragms (as for example in [4]).

3. Dispersion within the reciprocal periodic lattice

Kergomard and Pachebat [1] showed that thanks to the block-wise expressions of the perforation matrix (Eq.3) and the transfer matrix (Eq.4), the eigenvectors and eigenvalues of the transfer matrix $T^\lambda$ can be obtained analytically.

In the following, the lateral perforations radius is supposed to be small compared to the wavelength. The series impedance associated to anti-symmetrical profile of the flow velocity across the perforation can be ignored, by choosing $Z_a = 0$ [3] in the expression of the perforation matrix (Eq.3). Under this particular assumption, the characteristic polynomial det$(T^\lambda - \lambda I)$ = 0 gives the dispersion equation [1]

$$\frac{1}{V_s} = 2\lambda \left[ \frac{B_1}{\Delta_1^2} + \frac{\gamma}{\Delta_2} \right],$$

where $\Delta_{1,2} = \det(T_{1,2} - \lambda I) = \lambda^2 - \lambda(A_{1,2} + D_{1,2}) + \det T_{1,2}$ and $I$ is the identity matrix of order 2.

In the following, we also assume that the waveguides are symmetrical and reciprocal. These physical properties imply for the transfer matrices of Waveguides 1 and 2: $A_{1,2} = D_{1,2}$ and $\det T_{1,2} = 1$. Thus we can rewrite the dispersion equation of order four in $\lambda$ (Eq.6), as a second order polynomial in $\cosh \Gamma = (\lambda + 1/\lambda)/2$:

$$\frac{2}{V_p} = \frac{\overline{T_1}}{\cosh \Gamma - A_1} + \frac{\overline{T_2}}{\cosh \Gamma - A_2},$$

where $V_p = 2Y_pS_1S_2/(S_1 + S_2)$ is the acoustic admittance of the lateral perforation, and $\overline{T}_{1,2} = B_{1,2}/S_{1,2}$.

The discriminant $\Delta$ of the quadratic equation in $\cosh \Gamma$ (Eq.7) can be written by defining a Coupling coefficient $C$, as follows:

$$\Delta = (A_1 - A_2)^2 \left[ 1 + \frac{2\overline{T_1}}{\overline{T_1} + \overline{T_2}} C + C^2 \right],$$

where $C = \frac{1}{2} \frac{\overline{T_1} + \overline{T_2}}{A_1 - A_2}$.

The two solutions of the dispersion equation (Eq.7) are:

$$\cosh \Gamma = \frac{1}{2} \left( A_1 + A_2 \right) \pm \sqrt{\Delta},$$

$$\cosh \Gamma' = \frac{1}{2} \left( A_1 + A_2 \right) \pm \sqrt{\Delta}.$$
with totally closed diaphragms, the lattice is a series of branched resonators (Fig.2, center). But thanks to the approach presented above, we can also explore two intermediate situations by varying the diaphragm opening: an \textit{almost homogeneous} lattice (wide open diaphragms) and a \textit{strongly inhomogeneous} lattice (diaphragms with very small opening).

Waveguides 1 and 2 are co-axial cylinders filled with ambient air. The classical transfer matrix for a plane mode (pressure-velocity) along one uncoupled portion of length $l$ of lossless Waveguide 1 or 2 (no lateral perforation) is:

$$
t_{1,2} = \begin{bmatrix}
\cos(kl) & jZ_c^0 \sin(kl) \\
-\sin(kl)/Z_c^0 & \cos(kl)
\end{bmatrix}
$$

where $Z_c^0 = \rho_0 c_0$ and $k = \omega/c_0$ are the characteristic impedance and wavenumber of the medium (air) filling Waveguides 1 and 2, with $\rho_0$ its density and $c_0$ its sound velocity.

The transfer matrix $T_1$ and $T_2$ of Eq.4 are given by $T_1 = (t_1)^2$, and $T_2 = t_2D t_2$ where the acoustic mass impedance $Z_d$ of the diaphragms is defined as $Z_d = j\omega \rho S_2/2r_d \left(1 - \sqrt{\pi r_d^2/S_2}\right)$. The open ratio of the diaphragm for one cell is defined as $\sigma_d = n_d r_d^2/(r_2^2 - r_1^2)$ where $n_d$ is the number of diaphragms around the circumference of one elementary cell. Similarly, for the lateral perforations, the impedance $1/Y_s$ of one perforation is chosen as $1/Y_s = j\omega \rho/2r_s$.
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of the lateral perforations for one cell is defined as

\[ \sigma_s = n_s r_s^2 / (4l_s) \]

where \( n_s \) is the number of lateral perforations (all situated at \( x_n (n = 1..n_c) \)) of elementary cell number \( n \).

Figure 5. Insertion Loss of the \textit{strongly inhomogeneous} lattice (\( \sigma_d = 0.08 \)) of finite length. Black line indicates a stop band (no propagating eigenmodes). Solutions for two extreme lattice configurations are shown for reference: homogeneous lattice (\( \sigma_d = 1 \), dash-dot line), and branched Helmholtz resonators (\( \sigma_d = 0 \), thin solid line).

Figure 6. Insertion Loss of the \textit{almost homogeneous} lattice (\( \sigma_d = 0.8 \)) of finite length. Black line indicates a band with two propagating eigenmodes. Solutions for two extreme lattice configurations are shown for reference: homogeneous lattice (\( \sigma_d = 1 \), dash-dot line), and branched Helmholtz resonators (\( \sigma_d = 0 \), thin solid line).

5. Results

We first examine how two classical configurations can be interpreted thanks to the proposed approach. The first configuration is an homogeneous lattice and the second is a lattice with Helmholtz resonators branched on Waveguide 1. Then we will study the effect of inhomogeneity of the lattice on the propagation and insertion loss at low frequencies.

At the limit where propagation media are identical within Waveguides 1 and 2 (\( A_1 = A_2 \), Fig.2, bottom), \( C \) tends to infinity and the two eigenmodes (Eq.9) reduce for a homogeneous lattice to the following expression:

\[
\begin{cases}
\cosh \Gamma = A \\
\cosh \Gamma' = A + \frac{1}{2} Y_p(B_1 + B_2)
\end{cases}
\]

(11)

The corresponding homogeneous plane mode \( \Gamma \) and homogeneous flute mode \( \Gamma' \) are represented in Fig.3 (left) with dashed lines and dash-dot lines respectively. Analytical expressions for the normalized characteristic impedance of the plane mode \( Z_0(\omega) / (\rho c_0 / S_1) \) and the flute mode \( Z_0'(\omega) / (\rho c_0 / S_1) \), and their relative phase velocity \( v_0/v_0'(\omega) = 1m(\Gamma)c_0 / (2\omega) \) and \( v_0/v_0'(\omega) = 1m(\Gamma)c_0 / (2\omega) \) can also be obtained (not presented). Their variation is shown in Fig.3 (center) and Fig.3 (right), using the same dashed and dash-dot lines. Fig.3 shows, as already reported in [1], that a homogeneous lattice is a particular case with a non dispersive plane mode (\( c_0/v_0(\omega) \) is constant) which is always propagating (-1 < \( \cosh \Gamma < 1 \)). The flute mode \( c_0/v_0' \) tends asymptotically, when increasing frequency above its cut-on frequency \( F_n^\dagger = 1803Hz \) (\( F_n^\dagger \) is given by \( \cosh \Gamma = 1 \)), towards physical characteristics similar to the plane mode. This similarity for \( f > F_n^\dagger \) between homogeneous plane mode and flute mode results in longitudinal interferences, driven by the total length of the finite lattice, as detailed in [1], associated to resonant attenuation peaks of the insertion loss of the homogeneous lattice for \( f > F_n^\dagger \) (Fig.5, dash-dot line).

In the case of branched Helmholtz resonators, the diaphragms are totally closed. The impedance \( Z_h \) in Eq.10 tends to infinity and \( A_1 \ll A_2, B_1 \ll B_2 \). The unique mode propagating through the lattice is given by:

\[
\cosh \Gamma = A_1 + \frac{1}{2Z_h}B_1,
\]

(12)

where \( Z_h = 1/Y_p + B_2 / 2(A_2 - 1) \) is the input impedance of the Helmholtz resonator of one cell of the lattice (Fig.2, center). The series of branched resonators acts as a local reacting treatment on Waveguide 1, with a maximum effect (see Fig.3 left, thin solid line) at the Helmholtz resonance \( F_h = 912 \) Hz (given by \( Z_h = 0 \)). The lattice exhibits a stop band with \( |\cosh \Gamma| > 1 \) (when ignoring viscothermal losses) within the interval \( [F_1; F_n^\dagger] \), that corresponds also to a very small relative characteristic impedance \( Z_0(\omega) / (\rho c_0 / S_1) \) (see Fig.3 center, thin solid line) and relative phase velocity \( v_0/v_0'(\omega) \) (see Fig.3 right, thin solid line). The existence of this stop band (no propagating mode) induces a cumulative attenuation: the insertion loss shown in Fig.5 (thin solid line) is directly proportional to the number of elementary cells.
of the finite lattice (here \( n_e = 15 \), see Tab.1), and is very small outside the stop band.

In the case of a **strongly inhomogeneous** lattice, the open area of the diaphragms between two neighboring cells of the lattice represents only 8% of \( S_2 \) (that is \( \sigma_d = 0.08 \)). At low frequencies and for the plane mode (marker \( \diamond \)), Fig.3 shows that the propagation constant \( \cosh \Gamma \) Fig.3 (left), the characteristic impedance Fig.3 (center), and the relative phase velocity Fig.3 (right), are very similar to the branched Helmholtz resonator solution (thin solid line). Conversely, at frequencies above \( F_t^+ \) (given by \( \cosh \Gamma' = 1 \)), the flute mode (marker \( \lozenge \)) is very close to the branched Helmholtz resonator solution (thin solid line). This small amount of aperture (8%) into the resonators wall is sufficient to reduce dramatically the stop band from \([F_b^-; F_b^+]\) (branched Helmholtz resonators) to \([F_t^-; F_t^+]\) (where \( F_t^- \) is given by \( \cosh \Gamma = -1 \)). The stop band is indicated with black filled markers on Fig.3. Concerning the insertion loss (Fig.5), the **strongly inhomogeneous** lattice (thick line) differs significantly from the branched Helmholtz resonator case (thin solid line) only for frequencies below \( F_b^- \). In particular within the stop band \([F_b^-; F_b^+]\), the two insertion losses are very close, and the **strongly inhomogeneous** lattice may probably be described as a locally reacting liner, at least for frequencies within or close to the stop band. This point might be further investigated.

If one continues to open the diaphragms (increasing \( \sigma_d \)), the cut-off frequency of the plane mode \( F_b^- \) increases, as opposed to \( F_b^+ \) which does not depend on \( \sigma_d \). \( F_b^- \) tends to infinity for 100% opened diaphragms (homogeneous lattice). For an **almost homogeneous** lattice with an open area of the diaphragms that represents 80% of \( S_2 \) (that is \( \sigma_d = 0.8 \)), \( F_b^- \) is situated above the maximum frequency under study \( F_{\text{max}} = 3500 \) Hz. Below \( F_{\text{max}} \) the system has no stop band. It exhibits two propagating eigenmodes within \([F_a^-; F_{\text{max}}]\) (indicated with black filled markers on Fig.4), as for a homogeneous lattice (dashed and dash-dot lines). But contrary to a homogeneous lattice, even with diaphrags almost totally opened (\( \sigma_d = 0.8 \)), the plane mode (marker \( \circ \)) is dispersive: \( c_0/v_d(\omega) \) not is constant, see Fig.4 (right). The propagation constant and the relative phase velocity of the plane mode and the flute mode (marker \( \lozenge \)) are not close to each other. As a consequence, the resonant attenuation peaks of the insertion loss above \( F_t^+ \) observed for the homogeneous lattice (Fig.6 dash-dot line), due the longitudinal interference of two similar eigenmodes, are strongly reduced in the case of the **almost homogeneous** lattice (thick line).

### 6. Conclusion

The particular type of inhomogeneity (diaphragms) used here strongly affects the phase velocity of the plane eigenmode, and also modifies the stop band and two-modes band of the infinite lattice.

From a practical point of view, this approach illustrates how the inhomogeneity of the infinite lattice modifies the insertion loss of the lattice of finite length. It appears that globally, the insertion loss of both homogeneous lattice and branched Helmholtz resonators, is lowered when diaphragms are introduced.

As a consequence, even though diaphragms was a simple way to introduce parametrically the inhomogeneity within the lattice, it may not be a practical way to improve the performance of the lattice as an acoustic treatment. Nevertheless, lattice configurations including other types of inhomogeneity like for instance porous materials (described as equivalent viscothermal fluid) are in the scope of the method.

### References


