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FAMILIES OF AFFINE RULED SURFACES: EXISTENCE OF CYLINDERS

ADRIEN DUBOULOZ AND TAKASHI KISHIMOTO

Abstract. We show that the generic fiber of a family \( f: X \to S \) of smooth \( \mathbb{A}^1 \)-ruled affine surfaces always carries an \( \mathbb{A}^1 \)-fibration, possibly after a finite extension of the base \( S \). In the particular case where the general fibers of the family are irrational surfaces, we establish that up to shrinking \( S \), such a family actually factors through an \( \mathbb{A}^1 \)-fibration \( \rho: X \to Y \) over a certain \( S \)-scheme \( Y \to S \) induced by the MRC-fibration of a relative smooth projective model of \( X \) over \( S \). For affine threefolds \( X \) equipped with a fibration \( f: X \to B \) by irrational \( \mathbb{A}^1 \)-ruled surfaces over a smooth curve \( B \), the induced \( \mathbb{A}^1 \)-fibration \( \rho: X \to Y \) can also be obtained from a relative Minimal Model Program applied to a smooth projective model of \( X \) over \( B \).

Introduction

The general structure of smooth non complete surfaces \( X \) with negative (logarithmic) Kodaira dimension is not fully understood yet. For say smooth quasi-projective surfaces over an algebraically closed field of characteristic zero, it was established by Keel and McKernan [10] that the negativity of the Kodaira dimension is equivalent to the fact that \( X \) is generically covered by images of the affine line \( \mathbb{A}^1 \) in the sense that the set of points \( x \in X \) with the property that there exists a non constant morphism \( f: \mathbb{A}^1 \to X \) such that \( x \in f(\mathbb{A}^1) \) is dense in \( X \) with respect to the Zariski topology. This property, called \( \mathbb{A}^1 \)-uniruledness is equivalent to the existence of an open embedding \( X \to (\overline{X}, B) \) into a complete variety \( \overline{X} \) covered by proper rational curves meeting the boundary \( B = \overline{X} \setminus X \) in at most one point. In the case where \( X \) is smooth and affine, an earlier deep result of Miyanishi-Sugie [14] asserts the stronger property that \( X \) is \( \mathbb{A}^1 \)-ruled: there exists a Zariski dense open subset \( U \subset X \) of the form \( U \simeq \mathbb{A}^1 \times \mathbb{A}^1 \) for a suitable smooth curve \( \mathbb{A}^1 \). Equivalently, \( X \) admits a surjective flat morphism \( \rho: X \to C \) to an open subset \( C \) of a smooth projective model \( \overline{Z} \) of \( Z \), whose generic fiber is isomorphic to the affine line over the function field of \( C \). Such a morphism \( \rho: X \to C \) is called an \( \mathbb{A}^1 \)-fibration, and we say that \( \rho \) is of affine type or complete type when the base curve \( C \) is affine or complete, respectively.

Smooth \( \mathbb{A}^1 \)-uniruled but not \( \mathbb{A}^1 \)-ruled affine varieties are known to exist in every dimension \( \geq 3 \) [1]. Many examples of \( \mathbb{A}^1 \)-uniruled affine threefolds can be constructed in the form of flat families \( f: X \to B \) of smooth \( \mathbb{A}^1 \)-ruled affine surfaces parametrized by a smooth base curve \( B \). For instance, the complement \( X \) of a smooth cubic surface \( S \subset \mathbb{P}^3 \) is the total space of a family \( f: X \to \mathbb{A}^1 = \text{Spec}(\mathbb{C}[t]) \) of \( \mathbb{A}^1 \)-ruled surfaces induced by the restriction of a pencil \( \mathcal{P}: \mathbb{P}^3 \to \mathbb{P}^1 \) on \( \mathbb{P}^3 \) generated by \( S \) and three times a tangent hyperplane \( H \) to \( S \) whose intersection with \( S \) consists of a cuspidal cubic curve. The general fibers of \( f \) have negative Kodaira dimension, carrying \( \mathbb{A}^1 \)-fibrations of complete type only, and the failure of \( \mathbb{A}^1 \)-ruledness is intimately related to the fact that the generic fiber \( X_\eta \) of \( f \), which is a surface defined over the field \( K = \mathbb{C}(t) \), does not admit any \( \mathbb{A}^1 \)-fibration defined over \( \mathbb{C}(t) \). Nevertheless, it was noticed in [3, Theorem 6.1] that one can infer straight from the construction of \( f: X \to \mathbb{A}^1 \) the existence a finite base extension \( \text{Spec}(L) \to \text{Spec}(K) \) for which the surface \( X_\eta \times_{\text{Spec}(K)} \text{Spec}(L) \) carries an \( \mathbb{A}^1 \)-fibration \( \rho: X_\eta \times_{\text{Spec}(K)} \text{Spec}(L) \to \mathbb{P}^1_L \) defined over the field \( L \).

A natural question is then to decide whether this phenomenon holds in general for families \( f: X \to B \) of \( \mathbb{A}^1 \)-ruled affine surfaces parameterized by a smooth base curve \( B \), namely, does the existence of \( \mathbb{A}^1 \)-fibrations on the general fibers of \( f \) imply the existence of one on the generic fiber of \( f \), possibly after a finite extension of the base \( B \)? A partial positive answer is given by Gurjar-Masuda-Miyanishi in [3, Theorem 3.8] under the additional assumption that the general fibers of \( f \) carry \( \mathbb{A}^1 \)-fibrations of affine type. The main result in loc. cit. is derived from the study of log-deformations of suitable relative normal projective models

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\( \overline{f} : (\overline{X}, D) \to B \) of \( X \) over \( B \) with appropriate boundaries \( D \). It is established in particular that the structure of the boundary divisor of a well chosen smooth projective completion of a general closed fiber \( X_s \) is stable under small deformations, a property which implies in turn, possibly after a finite extension of the base \( B \), the existence of an \( \mathbb{A}^1 \)-fibration of affine type on the generic fiber of \( f \). This log-deformation theoretic approach is also central in the related recent work of Flener-Kaliman-Zaidenberg [2] on the classification of normal affine surfaces with \( \mathbb{A}^1 \)-fibrations of affine type up to a certain notion of deformation equivalence, defined for families which admit suitable relative projective models satisfying Kamawata’s axioms of logarithmic deformations of pairs [8]. The fact that the \( \mathbb{A}^1 \)-fibrations under consideration are of affine type plays again a crucial role and, in contrast with the situation considered in [3], the restrictions imposed on the families imply the existence of \( \mathbb{A}^1 \)-fibrations of affine type on their generic fibers.

Our main result (Theorem 6) consists of a generalization of the results in [3] to families \( f : X \to S \) of \( \mathbb{A}^1 \)-ruled surfaces over an arbitrary normal base \( S \), which also includes the case where a general closed fiber \( X_s \) of \( f \) admits \( \mathbb{A}^1 \)-fibrations of complete type only. In particular, we obtain the following positive answer to Conjecture 6.2 in [3]:

**Theorem.** Let \( f : X \to S \) be dominant morphism between normal complex algebraic varieties whose general fibers are smooth \( \mathbb{A}^1 \)-ruled affine surfaces. Then there exists a dense open subset \( S_* \subset S \), a finite étale morphism \( T \to S_* \), and a normal \( T \)-scheme \( h : Y \to T \) such that the induced morphism \( f_T = \text{pr}_T : X_T = X \times_{S_*} T \to T \) factors as

\[
    f_T = h \circ \rho : X_T \xrightarrow{\rho} Y \xrightarrow{h} T,
\]

where \( \rho : X_T \to Y \) is an \( \mathbb{A}^1 \)-fibration.

In contrast with the log-deformation theoretic strategy used in [3], which involves the study of certain Hilbert schemes of rational curves on well-chosen relative normal projective models \( \overline{f} : (\overline{X}, B) \to S \) of \( X \) over \( S \), our approach is more elementary, based on the notion of Kodaira dimension [7] adapted to the case of geometrically connected varieties defined over arbitrary base fields of characteristic zero. Indeed, the hypothesis means equivalently that the general fibers of \( f \) have negative Kodaira dimension. This property is in turn inherited by the generic fiber of \( f \), which is a smooth affine surface defined over the function field of \( S \), thanks to a standard Lefschetz principle argument. Then we are left with checking that a smooth affine surface \( X \) defined over an arbitrary base field \( k \) of characteristic zero and with negative Kodaira dimension admits an \( \mathbb{A}^1 \)-fibration, possibly after a suitable finite base extension \( \text{Spec}(k_0) \to \text{Spec}(k) \), a fact which follows immediately from finite type hypotheses and the aforementioned characterization of Miyanishi-Sugie [14].

The article is organized as follows. The first section contains a review of the structure of smooth affine surfaces of negative Kodaira dimension over arbitrary base fields \( k \) of characteristic zero. We show in particular that every such surface \( X \) admits an \( \mathbb{A}^1 \)-fibration after a finite extension of the base field \( k \), and we give criteria for the existence of \( \mathbb{A}^1 \)-fibrations defined over \( k \). These results are then applied in the second section to the study of deformations \( f : X \to S \) of smooth \( \mathbb{A}^1 \)-ruled affine surfaces: after giving the proof of the main result, Theorem 6, we consider in more detail the particular situation where the general fibers of \( f : X \to S \) are irrational. In this case, after shrinking \( S \) if necessary, we show that the morphism \( f \) actually factors through an \( \mathbb{A}^1 \)-fibration \( \rho : X \to Y \) over an \( S \)-scheme \( h : Y \to S \) which coincides, up to birational equivalence, with the Maximally Rationally Connected quotient of a relative smooth projective model \( \overline{f} : \overline{X} \to S \) of \( X \) over \( S \). The last section is devoted to the case of affine threefolds equipped with a fibration \( f : X \to B \) by irrational \( \mathbb{A}^1 \)-ruled surfaces over a smooth base curve \( B \): we explain in particular how to construct an \( \mathbb{A}^1 \)-fibration \( \rho : X \to Y \) factoring \( f \) by means of a relative Minimal Model Program applied to a smooth projective model \( \overline{f} : \overline{X} \to B \) of \( X \) over \( B \).

1. \( \mathbb{A}^1 \)-ruledness of affine surfaces over non closed field

1.1. Logarithmic Kodaira dimension.

1.1.1. Let \( X \) be a smooth geometrically connected algebraic variety defined over a field \( k \) of characteristic zero. By virtue of Nagata compactification [15] and Hironaka desingularization [5] theorems, there exists an open immersion \( X \hookrightarrow (\overline{X}, B) \) into a smooth complete algebraic variety \( \overline{X} \) with reduced SNC boundary divisor \( B = \overline{X} \setminus X \). The (logarithmic) Kodaira dimension \( \kappa(X) \) of \( X \) is then defined as the Iitaka dimension [6] of the pair
\((\overline{X}; \omega_{\overline{X}}(\log B))\) where \(\omega_{\overline{X}}(\log B) = (\det \Omega^1_{\overline{X}/k}) \otimes O_{\overline{X}}(B)\). So letting \(\mathcal{R}(\overline{X}, B) = \bigoplus_{m \geq 0} H^0(\overline{X}, \omega_{\overline{X}}(\log B)^{\otimes m})\), we have \(\kappa(X) = \text{tr.deg}_k \mathcal{R}(\overline{X}, B) - 1\) if \(H^0(\overline{X}, \omega_{\overline{X}}(\log B)^{\otimes m}) \neq 0\) for sufficiently large \(m\). Otherwise, if \(H^0(\overline{X}, \omega_{\overline{X}}(\log B)^{\otimes m}) = 0\) for every \(m \geq 1\), we set by convention \(\kappa(X) = -\infty\) and we say that \(\kappa(X)\) is negative. The so-defined element \(\kappa(X) \in \{0, \ldots, \dim_k X\} \cup \{-\infty\}\) is independent of the choice of a smooth complete model \((\overline{X}, B)\) [7].

Furthermore, the Kodaira dimension of \(X\) is invariant under arbitrary extensions of the base field \(k\). Indeed, given an extension \(k \subset k'\), the pair \((\overline{X}_{k'}, B_{k'})\) obtained by the base change \(\text{Spec}(k') \to \text{Spec}(k)\) is a smooth complete model of \(X_{k'} = X \times_{\text{Spec}(k)} \text{Spec}(k')\) with reduced SNC boundary \(B_{k'}\). Furthermore letting \(\pi : \overline{X}_{k'} \to \overline{X}\) be the corresponding faithfully flat morphism, we have \(\omega_{\overline{X}_{k'}}(\log B_{k'}) \simeq \pi^* \omega_{\overline{X}}(\log B)\) and so \(\mathcal{R}(\overline{X}_{k'}) \simeq \mathcal{R}(\overline{X}) \otimes_k k'\) by the flat base change theorem [4, Proposition III.9.3]. Thus \(\kappa(X) = \kappa(X_{k'})\).

**Example 1.** The affine line \(\mathbb{A}^1_k\) is the only smooth geometrically connected non complete curve \(C\) with negative Kodaira dimension. Indeed, let \(\overline{C}\) be a smooth projective model of \(C\) and let \(\overline{C}_\rho\) be the curve obtained by the base change to an algebraic closure \(\overline{k}\) of \(k\). Since \(C\) is non complete, \(B = \overline{C}_\rho \setminus C\) consists of a finite collection of closed points \(p_1, \ldots, p_s\), \(s \geq 1\), on which the Galois group \(\text{Gal}(\overline{k}/k)\) acts by \(k\)-automorphisms of \(\overline{C}_\rho\). Clearly, \(H^0(\overline{C}_\rho, \omega_{\overline{C}_\rho}(\log B)^{\otimes m}) \neq 0\) unless \(\overline{C}_\rho \simeq \mathbb{P}^1_{\overline{k}}\) and \(s = 1\). Since \(p_1\) is then necessarily \(\text{Gal}(\overline{k}/k)\)-invariant, \(\overline{C} \setminus C\) consists of unique \(k\)-rational point, showing that \(\overline{C} \simeq \mathbb{P}^1_{\overline{k}}\) and \(C \simeq \mathbb{A}^1_k\).

1.2. **Smooth affine surfaces with negative Kodaira dimension.**

Recall that by virtue of [14], a smooth affine surface \(X\) defined over an algebraically closed field of characteristic zero has negative Kodaira dimension if and only if it is \(\mathbb{A}^1\)-ruled: there exists a Zariski dense open subset \(U \subset X\) of the form \(U \simeq Z \times \mathbb{A}^1\) for a suitable smooth curve \(Z\). In fact, the projection \(\text{pr}_Z : U \simeq Z \times \mathbb{A}^1 \to Z\) always extends to an \(\mathbb{A}^1\)-fibration \(\rho : X \to C\) over an open subset \(C\) of a smooth projective model \(\overline{Z}\) of \(Z\). This characterization admits the following straightforward generalization to arbitrary base fields of characteristic zero:

**Theorem 2.** Let \(X\) be a smooth geometrically connected affine surface defined over a field \(k\) of characteristic zero. Then the following are equivalent:

a) The Kodaira dimension \(\kappa(X)\) of \(X\) is negative.

b) For some finite extension \(k_0\) of \(k\), the surface \(X_{k_0}\) contains an open subset \(U \simeq Z \times \mathbb{A}^1_{k_0}\) for some smooth curve \(Z\) defined over \(k_0\).

c) There exists a finite extension \(k_0\) of \(k\) and an \(\mathbb{A}^1\)-fibration \(\rho : X_{k_0} \to C_0\) over a smooth curve \(C_0\) defined over \(k_0\).

**Proof.** Clearly c) implies b) and b) implies a). To show that a) implies c), we observe that letting \(\overline{k}\) be an algebraic closure of \(k\), we have \(\kappa(X_{\overline{k}}) = \kappa(X) < 0\). It then follows from the aforementioned result of Miyanishi-Sugie [14] that \(X_{\overline{k}}\) admits an \(\mathbb{A}^1\)-fibration \(q : X_{\overline{k}} \to C\) over a smooth curve \(C\), with smooth projective model \(\overline{C}\). Since \(X_{\overline{k}}\) and \(\overline{C}\) are of finite type over \(\overline{k}\), there exists a finite extension \(k \subset k_0\) such that \(q : X_{\overline{k}} \to \overline{C}\) is obtained from a morphism \(\rho : X_{k_0} \to \overline{C}_0\) to a smooth projective curve \(\overline{C}_0\) defined over \(k_0\) by the base extension \(\text{Spec}(\overline{k}) \to \text{Spec}(k_0)\). By virtue of Example 1, \(\rho : X_{k_0} \to \overline{C}_0\) is an \(\mathbb{A}^1\)-fibration. \(\square\)

Examples of smooth affine surfaces \(X\) of negative Kodaira dimension without any \(\mathbb{A}^1\)-fibration defined over the base field but admitting \(\mathbb{A}^1\)-fibrations of complete type after a finite base extension were already constructed in [1]. The following example illustrates the fact that a similar phenomenon occurs for \(\mathbb{A}^1\)-fibrations of affine type, providing in particular a negative answer to Problem 3.13 in [3].

**Example 3.** Let \(B \subset \mathbb{P}^1_k = \text{Proj}(k[x, y, z])\) be a smooth conic without \(k\)-rational points defined by a quadratic form \(q = x^2 + ay^2 + bz^2\), where \(a, b \in k^*\), and let \(\overline{X} \subset \mathbb{P}^1_k = \text{Proj}(k[x, y, z, t])\) be the smooth quadric surface defined by the equation \(q(x, y, z) - t^2 = 0\). The complement \(X \subset \overline{X}\) of the hyperplane section \(\{t = 0\}\) is a \(k\)-rational smooth affine surface with \(\kappa(X) < 0\), which does not admit any \(\mathbb{A}^1\)-fibration \(\rho : X \to C\) over a smooth, affine or projective curve \(C\). Indeed, if such a fibration existed then a smooth projective model of \(C\) would be isomorphic to \(\mathbb{P}^1_k\); since the fiber of \(\rho\) over a general \(k\)-rational point of \(C\) is isomorphic to \(\mathbb{A}^1_k\), its closure in \(\overline{X}\) would intersect the boundary \(\overline{X} \setminus X \simeq B\) in a unique point, necessarily \(k\)-rational, in contradiction with the choice of \(B\).

In contrast, for a suitable finite extension \(k \subset k'\), the surface \(X_{k'}\) becomes isomorphic to the complement of the diagonal in \(\overline{X}_{k'} \simeq \mathbb{P}^1_{k'} \times \mathbb{P}^1_{k'}\) and hence, it admits at least two distinct \(\mathbb{A}^1\)-fibrations over \(\mathbb{P}^1_{k'}\), induced
by the restriction of the first and second projections from $\overline{X}_{k'}$. Furthermore, since $X_{k'}$ is isomorphic to the smooth affine quadric in $\mathbb{A}^3_{k'} = \text{Spec}(k'[u,v,w])$ with equation $uv - w^2 = 1$, it also admits two distinct $\mathbb{A}^1$-fibrations over $\mathbb{A}^1_{k'}$, induced by the restrictions of the projections $\text{pr}_u$ and $\text{pr}_v$.

1.3. Existence of $\mathbb{A}^1$-fibrations defined over the base field.

1.3.1. The previous example illustrates the general fact that if $X$ is a smooth geometrically connected affine surface with $\kappa(X) < 0$ which does not admit any $\mathbb{A}^1$-fibration, then there exists a finite extension $k'$ of $k$ such that $X_{k'}$ admits at least two $\mathbb{A}^1$-fibrations of the same type, either affine or complete, with distinct general fibers. Indeed, by virtue of Theorem 2, there exists a finite extension $k_0$ of $k$ such that $X_{k_0}$ admits an $\mathbb{A}^1$-fibration $\rho : X_{k_0} \to C$. Let $k'$ be the Galois closure of $k_0$ in an algebraic closure of $k$ and let $\rho_{k'} : X_{k'} \to C_{k'}$ be the $\mathbb{A}^1$-fibration deduced from $\rho$. If $\rho_{k'} : X_{k'} \to C_{k'}$ is globally invariant under the action of the Galois group $\text{Gal}(k'/k)$ on $X_{k'}$, in the sense that for every $\Phi \in \text{Gal}(k'/k)$ considered as a Galois automorphism of $X_{k'}$ there exists a commutative diagram

$$
\begin{array}{ccc}
X_{k'} & \xrightarrow{\Phi} & X_{k'} \\
\rho_{k'} \downarrow & & \downarrow \rho_{k'} \\
C_{k'} & \cong & C_{k'}
\end{array}
$$

for a certain $k'$-automorphism $\varphi$ of $C_{k'}$, then we would obtain a Galois action of $\text{Gal}(k'/k)$ on $C_{k'}$ for which $\rho_{k'} : X_{k'} \to C_{k'}$ becomes an equivariant morphism. Since $C_{k'}$ is quasi-projective and $\rho_{k}'$ is affine, it would follow from Grothendieck descent that there exists a curve $\tilde{C}$ defined over $k$ and a morphism $q : X \to \tilde{C}$ defined over $k$ such that $\rho_{k'} : X_{k'} \to C_{k'}$ is obtained from $q$ by the base change $\text{Spec}(k') \to \text{Spec}(k)$. Since by virtue of Example 1 the affine line does not have any nontrivial form, the generic fiber of $q$ would be isomorphic to the affine line over the field of rational functions of $\tilde{C}$ and so, $q : X \to \tilde{C}$ would be an $\mathbb{A}^1$-fibration defined over $k$, in contradiction with our hypothesis. So there exists at least an element $\Phi \in \text{Gal}(k'/k)$ considered as a $k$-automorphism of $X_{k'}$ such that the $\mathbb{A}^1$-fibrations $\rho_{k'} : X_{k'} \to C_{k'}$ and $\rho_{k'} \circ \varphi : X_{k'} \to C_{k'}$ have distinct general fibers.

Arguing backward, we obtain the following useful criterion:

**Proposition 4.** Let $X$ be a smooth geometrically connected affine surface with $\kappa(X) < 0$. If there exists a finite Galois extension $k'$ of $k$ such that $X_{k'}$ admits a unique $\mathbb{A}^1$-fibration $\rho : X_{k'} \to C_{k'}$ up to composition by automorphisms of $C_{k'}$, then $\rho : X_{k'} \to C_{k'}$ is obtained by base extension from an $\mathbb{A}^1$-fibration $\rho : X \to C$ defined over $k$.

**Corollary 5.** A smooth geometrically connected irrational affine surface $X$ has negative Kodaira dimension if and only if it admits an $\mathbb{A}^1$-fibration $\rho : X \to C$ over a smooth irrational curve $C$ defined over the base field $k$. Furthermore for every extension $k'$ of $k$, $\rho_{k'} : X_{k'} \to C_{k'}$ is the unique $\mathbb{A}^1$-fibration on $X_{k'}$ up to composition by automorphisms of $C_{k'}$.

**Proof.** Uniqueness is clear since otherwise $C_{k'}$ would be dominated by a general fiber of another $\mathbb{A}^1$-fibration on $X_{k'}$, and hence would be rational, implying in turn the rationality of $X$. By virtue of Theorem 2, there exists a finite Galois extension $k'$ of $k$ and an $\mathbb{A}^1$-fibration $\rho' : X_{k'} \to C'$ over a smooth curve $C'$. The latter is irrational as $X$ is irrational, which implies that $\rho' : X_{k'} \to C'$ is the unique $\mathbb{A}^1$-fibration on $X_{k'}$. So $\rho'$ descend to an $\mathbb{A}^1$-fibration $\rho : X \to C$ over a smooth irrational curve $C$ defined over $k$. \qed

2. Families of $\mathbb{A}^1$-ruled affine surfaces

2.1. Existence of étale $\mathbb{A}^1$-cylinders. This subsection is devoted to the proof of the following:

**Theorem 6.** Let $X$ and $S$ be normal algebraic varieties defined over a field $k$ of infinite transcendence degree over $\mathbb{Q}$, and let $f : X \to S$ be a dominant affine morphism with the property that for a general closed point $s \in S$, the fiber $X_s$ is a smooth geometrically connected affine surface with negative Kodaira dimension. Then there exists an open subset $S_0 \subset S$, a finite étale morphism $T \to S_0$ and a normal $T$-scheme $h : Y \to T$ such that $f_T = \text{pr}_T : X_T = X \times S_0 \times T \to T$ factors as

$$
f_T = h \circ \rho : X_T \xrightarrow{\rho} Y \xrightarrow{h} T
$$

where $\rho : X_T \to Y$ is an $\mathbb{A}^1$-fibration.
Proof. Shrinking $S$ if necessary, we may assume that $S$ is affine, that $f : X \to S$ is smooth and that $\kappa(X_s) < 0$ for every closed point $s \in S$. It is enough to show that the fiber $X_\eta$ of $f$ over the generic point $\eta$ of $S$ is geometrically connected, with negative Kodaira dimension. Indeed, if so, then by Theorem 2 above, there exists a finite extension $L$ of $K = \text{Frac}(\Gamma(S, \mathcal{O}_S))$ and an $\mathbb{A}^1$-fibration $\rho : X_\eta \times_{\text{Spec}(K)} \text{Spec}(L) \to C$ onto a smooth curve $C$ defined over $L$. Letting $T$ be the normalization of $S$ in $L$ and shrinking $T$ again if necessary, we obtain a finite étale morphism $T \to S$ such that the generic fiber of $\text{pr}_T : X_T \to T$ is isomorphic to the $\mathbb{A}^1$-fibered surface $\rho : X_\eta \times_{\text{Spec}(K)} \text{Spec}(L) \to C$ and then the assertion follows from Lemma 7 below.

Since $X$ and $S$ are affine and of finite type over $k$, there exists a subfield $k_0$ of $k$ of finite transcendence degree over $\mathbb{Q}$, and a smooth morphism $f_0 : X_0 \to S_0$ of $k_0$-varieties such that $f : X \to S$ is obtained from $f_0 : X_0 \to S_0$ by the base extension $\text{Spec}(k) \to \text{Spec}(k_0)$. The field $K_0 = \text{Frac}(\Gamma(S_0, \mathcal{O}_{S_0}))$ has finite transcendence degree over $\mathbb{Q}$ and hence, it admits a $k_0$-embedding $\xi : K_0 \hookrightarrow k$. Letting $(X_0)_{\eta_0}$ be the fiber of $f_0$ over the generic point $\eta_0 : \text{Spec}(K_0) \to S_0$ of $S_0$, the composition $\Gamma(S_0, \mathcal{O}_{S_0}) \to K_0 \hookrightarrow k$ induces a $k$-homomorphism $\Gamma(S_0, \mathcal{O}_{S_0}) \otimes_{k_0} k \to k$ defining a closed point $s : \text{Spec}(k) \to \text{Spec}(\Gamma(S_0, \mathcal{O}_{S_0}) \otimes_{k_0} k) = S$ of $S$ for which obtain the following commutative diagram

\[
\begin{array}{ccc}
(X_0)_{\eta_0} & \xrightarrow{f_0} & S_0 \\
\downarrow \xi^* & & \downarrow \eta_0 \\
\text{Spec}(k) & \xrightarrow{s} & \text{Spec}(k)
\end{array}
\]

The bottom square of the cube being cartesian by construction, we deduce that

\[(X_0)_{\eta_0} \times_{\text{Spec}(K_0)} \text{Spec}(k) \simeq X_0 \times_{S_0} \text{Spec}(k) \simeq X \times_S \text{Spec}(k) = X_s.
\]

Since by assumption, $X_s$ is geometrically connected with $\kappa(X_s) < 0$, we conclude that $(X_0)_{\eta_0}$ is geometrically connected and has negative Kodaira dimension. This implies in turn that $X_\eta$ is geometrically connected and that $\kappa(X_\eta) < 0$ as desired. \qed

In the proof of the above theorem, we used the following lemma:

**Lemma 7.** Let $f : X \to S$ be a dominant affine morphism between normal varieties defined over a field $k$ of characteristic zero. Then the following are equivalent:

a) The generic fiber $X_\eta$ of $f$ admits an $\mathbb{A}^1$-fibration $q : X_\eta \to C$ over a smooth curve $C$ defined over the fraction field $K$ of $S$.

b) There exists an open subset $S_0$ of $S$ and a normal $S_0$-scheme $h : Y \to S_0$ of relative dimension 1 such that the restriction of $f$ to $V = f^{-1}(S_0)$ factors as $f |_V = h \circ \rho : V \to Y \to S_0$ where $\rho : V \to Y$ is an $\mathbb{A}^1$-fibration.

**Proof.** If b) holds then letting $L$ be the fraction field of $Y$, we have a commutative diagram

\[
\begin{array}{ccccccc}
V_\xi = X_\xi & \rightarrow & V_\eta = X_\eta & \rightarrow & V \\
\downarrow \rho_\xi & & \downarrow \rho_\eta & & \downarrow \rho \\
\text{Spec}(L) & \xrightarrow{\xi} & C = Y_\eta & \rightarrow & Y \\
\downarrow h_\eta & & \downarrow h & & \downarrow h \\
\text{Spec}(K) & \xrightarrow{\eta} & S_0
\end{array}
\]

in which each square is cartesian. It follows that $h_\eta : C \to \text{Spec}(K)$ is a normal whence smooth curve defined over $K$ and that $\rho_\eta : X_\eta \to C$ is an $\mathbb{A}^1$-fibration. Conversely, suppose that $X_\eta$ admits an $\mathbb{A}^1$-fibration $q : X_\eta \to C$ and let $\overline{C}$ be a smooth projective model of $C$ over $K$. Then there exists an open subset $S_0$ of $S$ and a projective $S_0$-scheme $h : Y \to S_0$ whose generic fiber is isomorphic to $\overline{C}$. After shrinking $S_0$ if necessary, the rational map $\rho : V \dashrightarrow Y$ of $S_0$-schemes induced by $q$ becomes a morphism and
we obtain a factorization \( f \mid_V = h \circ \rho \). By construction, the generic fiber \( V \) of \( \rho : V \to Y \) is isomorphic to \( V \times_Y \text{Spec}(L) \cong (V \times_Y \text{Spec}(L) \times \text{Spec}(K)) \cong \mathbb{A}^1_L \) since \( V \times_Y \text{Spec}(L) \cong \mathbb{A}^1_L \). Hence \( \rho \) is an \( \mathbb{A}^1 \)-fibration. So \( \rho : V \to Y \) is an \( \mathbb{A}^1 \)-fibration.

**Example 8.** Let \( R = \mathbb{C}[s^{\pm 1}, t^{\pm 1}] \), \( S = \text{Spec}(R) \) and let \( D \) be the relatively ample divisor in \( \mathbb{P}^3_C = \text{Proj}_L(R[x, y, z]) \) defined by the equation \( x^2 + sy^2 + tz^2 = 0 \). The restriction \( h : X = \mathbb{P}^3_C \setminus D \to S \) of the structure morphism defines a family of smooth affine surfaces with the property that for every closed point \( s \in S \), \( X_s \) is isomorphic to the complement of \( \mathbb{P}^3_C \) of the smooth conic \( D_s \). In particular, \( X_s \) admits a continuum of pairwise distinct \( \mathbb{A}^1 \)-bundles on which the composition \( \rho : X \to S \) factors through an \( \mathbb{A}^1 \)-fibration \( \rho : X \to Y \) (see Lemma 7). The restriction of \( \rho \) to the fiber of \( f \) over a general closed point \( s \in S_0 \) is an \( \mathbb{A}^1 \)-fibration \( \rho_s : X_s \to Y_s \) over the normal, whence smooth, curve \( Y_s \). Since \( X_s \) is irrational and \( Y_s \) is irrational, and so \( \rho_s : X_s \to Y_s \) is the unique \( \mathbb{A}^1 \)-fibration on \( X_s \) up to composition by automorphisms of \( Y_s \). So in this case, we can identify \( \rho_s : X_s \to Y_s \) with the Maximally Rationally Connected fibration (MRC-fibration) \( \varphi : X_s \to Y_s \) of a smooth projective model \( X_s \) of \( X \) in the sense of [11, IV.5]: recall that \( \varphi \) is unique, characterized by the property that its general fibers are rationally connected and that for a general point \( y \in Y_s \) any rational curve in \( X_s \) which meets \( X_s \) is actually contained in \( X_s \). The \( \mathbb{A}^1 \)-fibration \( \rho : X \to Y \) can therefore be re-interpreted as being the MRC-fibration of a relative smooth projective model \( X_s \) of \( X \) over \( S \).

Reversing the argument, general existence and uniqueness results for MRC-fibrations allow actually to get rid of the smoothness hypothesis of a general fiber of \( f : X \to S \) and to extend the conclusion of Theorem 6 to arbitrary base fields of characteristic zero. Namely, we obtain the following characterization:

**Theorem 9.** Let \( X \) and \( S \) be normal varieties defined over a field \( k \) of characteristic zero and let \( f : X \to S \) be a dominant affine morphism with the property that for a general closed point \( s \in S \), the fiber \( X_s \) is irrational and \( \mathbb{A}^1 \)-ruled. Then there exists an open subset \( S^* \) and a normal \( S^* \)-scheme \( h : Y \to S^* \) such that the restriction of \( f \) to \( X_s = X \times_s S^* \) factors as

\[
f \mid_{X_s} = h \circ \rho : X_s \to Y = h_{\mid_{X_s}} \to S^*
\]

where \( \rho : X_s \to Y \) is an \( \mathbb{A}^1 \)-fibration.

**Proof.** Shrinking \( S \) if necessary, we may assume that for every closed point \( s \in S \), \( X_s \) is irrational and \( \mathbb{A}^1 \)-ruled, hence carrying a unique \( \mathbb{A}^1 \)-fibration \( \pi_s : X_s \to C_s \) over an irrational normal curve \( C_s \). Since \( f : X \to S \) is affine, there exists a normal projective \( S \)-scheme \( X \to S \) and an open embedding \( X \to X \) of schemes over \( S \). Letting \( W \to X \) be a resolution of the singularities of \( X \), we may assume up to shrinking \( S \) again if necessary that \( W \to S \) is a smooth morphism. We let \( j : X \to W \) be the birational map of \( S \)-schemes induced by the embedding \( X \to X \). By virtue of [11, Theorem 5.9], there exists an open subset \( W' \) of \( W \), an \( S \)-scheme \( h : W' \to S \) and a proper morphism \( \eta : W' \to Z \) such that for every \( s \in S \), the induced rational map \( \eta_s : W'_s \to Z_s \) is the MRC-fibration for \( W'_s \). On the other hand, since \( W_s \) is a smooth projective model of \( X_s \), the induced rational map \( \pi_s : X_s \to C_s \) is the MRC-fibration for \( W_s \). Consequently, for a general closed point \( z \in Z \) with \( h(z) = s \), the fiber \( W_s \) of \( \eta_s \), which is an irreducible proper rational curve contained in \( W_s \), must coincide with the closure of the image by \( j \) of a general closed fiber of \( \pi_s \). The latter being isomorphic to the affine line \( \mathbb{A}^1_C \) over the residue field \( k \) of the corresponding point of \( C_s \), we conclude that there exists an affine open subset \( U \) of \( X \) on which the composition \( \eta \circ j : U \to Z \) is a well defined morphism with general closed fibers isomorphic to affine lines over the corresponding residue fields.
So \( \mathfrak{f} \circ j : U \to Z \) is an \( \mathbb{A}^1 \)-fibration by virtue of [9]. The generic fiber of \( f : X \to S \) is thus \( \mathbb{A}^1 \)-ruled and the assertion follows from Lemma 7 above. \( \square \)

**Example 10.** Let \( h : Y \to S \) be smooth family of complex projective curves of genus \( g \geq 2 \) over a normal affine base \( S \) and let \( \mathcal{T}_{Y/S} \) be the relative tangent sheaf of \( h \). Since by Riemann-Roch \( H^0(Y_s, \mathcal{T}_{Y/S}) = 0 \) and \( \dim H^1(Y_s, \mathcal{T}_{Y/S}) = g - 1 \) for every point \( s \in S \), \( h_*\mathcal{T}_{Y/S} \) is locally free of rank \( g - 1 \) \([4, \text{Corollary III.12.9}]\) and so, \( H^1(Y, \mathcal{T}_{Y/S}) \simeq H^0(S, R^1 h_*\mathcal{T}_{Y/S}) \) by the Leray spectral sequence. Replacing \( S \) by an open subset, we may assume that \( R^1 h_*\mathcal{T}_{Y/S} \) admits a nowhere vanishing global section \( \sigma \). Via the isomorphism \( H^1(Y, \mathcal{T}_{Y/S}) \simeq \text{Ext}^1_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{T}_{Y/S}) \), we may interpret this section as the class of a non trivial extension \( 0 \to \mathcal{T}_{Y/S} \to \mathcal{E} \to \mathcal{O}_Y \to 0 \) of locally free sheaves over \( Y \). The inclusion \( \mathcal{T}_{Y/S} \to \mathcal{E} \) defines a section \( D \) of the locally trivial \( \mathbb{P}^1 \)-bundle \( \mathfrak{p} : Y = \text{Proj}(\text{Sym}_{\mathcal{O}_S} \mathcal{E}^\vee) \to Y \) and the non vanishing of \( \sigma \) guarantees that \( D \) is the support of an \( S \)-ample divisor. Indeed the \( S \)-ameneness of \( D \) is equivalent to the property that for every \( s \in S \) the induced section \( D_s \) of the \( \mathbb{P}^1 \)-bundle \( \mathfrak{p}_s : \mathfrak{X}_s \to Y_s \) over the smooth projective curve \( Y_s \) is ample. Since by construction, \( \mathfrak{p}_s |_{\mathfrak{X}_s \setminus D_s} : \mathfrak{X}_s \setminus D_s \to Y_s \) is a nontrivial torsor under the line bundle \( \text{Spec}(\text{Sym} \mathcal{T}_{Y_s}) \to Y_s \), it follows that \( D_s \) intersects positively every section \( D \) of \( \mathfrak{p}_s \) except maybe \( D_s \) itself. On the other hand, we have \( (D_s^2) = -\deg \mathcal{T}_{Y_s} = 2g(Y_s) - 2 > 0 \), and so the amplenness of \( D_s \) follows from the Nakai-Moishezon criterion and the description of the cone effective cycles on an irrational projective ruled surface given in [4, Proposition 2.20-2.21].

Letting \( X = \mathfrak{X} \setminus D \), we obtain a smooth family

\[
\mathfrak{f} = g \circ \mathfrak{p} |_{\mathfrak{X}} : X \xrightarrow{\mathfrak{p}} Y \xrightarrow{h} S
\]

where \( \mathfrak{p} |_{\mathfrak{X}} : X \to Y \) is nontrivial, locally trivial, \( \mathbb{A}^1 \)-bundle such that for every \( s \in S \), \( X_s \) is an affine surface with an \( \mathbb{A}^1 \)-fibration \( \rho_s : X_s \to Y_s \) of complete type.

In contrast with the previous example, the following proposition shows in particular that if the total space of a family of irrational \( \mathbb{A}^1 \)-ruled affine surfaces \( f : X \to S \) has finite divisor class group, then the induced \( \mathbb{A}^1 \)-fibration on a general fiber of \( f : X \to S \) is of affine type.

**Proposition 11.** Let \( X \) be a geometrically integral normal variety with finite divisor class group \( \text{Cl}(X) \) and let \( f : X \to S \) be a dominant morphism to a normal variety \( S \) with the property that for a general closed point \( s \in S \), the fiber \( X_s \) is irrational and \( \mathbb{A}^1 \)-ruled, say with unique \( \mathbb{A}^1 \)-fibration \( \pi_s : X_s \to C_s \). Then there exists an effective \( \mathbb{G}_a \cdot S \) action on \( X \) such that for a general closed point \( s \in S \), the \( \mathbb{A}^1 \)-fibration \( \pi_s : X_s \to C_s \) factors through the algebraic quotient \( \rho_s : X_s \to X_s//\mathbb{G}_a \cdot s = \text{Spec}(\Gamma(X_s, \mathcal{O}_{X_s})^{\mathbb{G}_a \cdot s}) \).

**Proof.** Let \( f |_{X_s} = h \circ \rho : X_s \xrightarrow{\rho} Y \xrightarrow{h} S_s \) be as in Theorem 9. Since \( \rho \) is an \( \mathbb{A}^1 \)-fibration, there exists an affine open subset \( U \subseteq Y \) such that \( \rho^{-1}(U) \simeq U \times \mathbb{A}^1 \) as schemes over \( U \). Since \( \rho^{-1}(U) \) is affine, its complement in \( X \) is of pure codimension 1, and the finiteness of \( \text{Cl}(X) \) implies that it is actually the support of an effective principal divisor \( \text{div}_X(a) \) for some \( a \in \Gamma(X, \mathcal{O}_X) \). Letting \( \partial \theta \) be the locally nilpotent derivation of \( \Gamma(\rho^{-1}(U), \mathcal{O}_X) \simeq \Gamma(X, \mathcal{O}_X)_{\rho} \) corresponding to the \( \mathbb{G}_a \cdot U \)-action by translations on the second factor, the finite generation of \( \Gamma(X, \mathcal{O}_X) \) guarantees that for a suitably chosen \( n \geq 0 \), \( a^n \partial \theta \) lifts to a locally nilpotent derivation \( \partial \) of \( \Gamma(X, \mathcal{O}_X) \). By construction, the restriction of \( f \) to the dense open subset \( \rho^{-1}(U) \) of \( X \) is invariant under the corresponding \( \mathbb{G}_a \)-action, and so \( f : X \to S \) is \( \mathbb{G}_a \)-invariant. For a general closed point \( s \in S \), the induced \( \mathbb{G}_a \)-action on \( X_s \) nontrivial, and its algebraic quotient \( \rho_s : X_s \to X_s//\mathbb{G}_a = \text{Spec}(\Gamma(X_s, \mathcal{O}_{X_s})^{\mathbb{G}_a \cdot s}) \) is a surjective \( \mathbb{A}^1 \)-fibration over a normal affine curve \( X_s//\mathbb{G}_a \). Since \( C_s \) is irrational, the general fibers of \( \rho_s \) and \( \pi_s \) must coincide. It follows that \( \pi_s \) is \( \mathbb{G}_a \)-invariant, whence factors through \( \rho_s \). \( \square \)

### 3. Affine threefolds fibered in irrational \( \mathbb{A}^1 \)-ruled surfaces

In this section we consider in more detail the case of normal complex affine threefolds \( X \) admitting a fibration \( f : X \to B \) by irrational \( \mathbb{A}^1 \)-ruled surfaces, over a smooth curve \( B \). We explain how to derive the variety \( h : Y \to B \) for which \( f \) factors through an \( \mathbb{A}^1 \)-fibration \( \rho : X \to Y \) from a relative minimal model program applied to a suitable projective model of \( X \) over \( B \). In the case where the divisor class group of \( X \) is finite, we provide a complete classification of such fibrations in terms of additive group actions on \( X \).
3.1. $\mathbb{A}^1$-cylinders via relative Minimal Model Program.

Let $X$ be a normal complex affine threefold and let $f : X \to B$ be a flat morphism onto a smooth curve $B$ with the property that a general closed fiber $X_b$ of $f$ is an irreducible irrational $\mathbb{A}^1$-ruled surface. We let $\pi : W \to B$ be a smooth projective model of $X$ over $B$ obtained from an arbitrary normal relative projective completion $X \to \overline{X}$ of $X$ over $B$ by resolving the singularities. We let $j : X \to W$ be the birational map induced by the open immersion $X \to \overline{X}$.

By applying a minimal model program for $W$ over $B$, we obtain a sequence of birational $B$-maps

$$W = W_0 \to W_1 \to W_2 \to \cdots \to W_\ell-1 \to W_\ell = W'',$

between $B$-schemes $\pi_i : W_i \to B$, where $\varphi_i : W_i \to W_{i+1}$ is either a divisorial contraction or a flip, and the rightmost variety $W'$ is the output of a minimal model program over $B$. The hypotheses imply that $W'$ has the structure of a Mori conic bundle $\pi : W' \to Y$ over a $B$-scheme $h : Y \to B$ corresponding to the contraction of an extremal ray of $\text{NE}(W'/B)$. Indeed, a general fiber of $\pi$ being a birationaly ruled projective surface, the output $W'$ is not a minimal model of $W$ over $B$. So $W'$ is either a Mori conic bundle over a $B$-scheme $Y$ of dimension 2 or a del Pezzo fibration over $B$, the second case being excluded by the fact that the general fibers of $\pi$ are irrational.

Proposition 12. The induced map $\rho = \pi|_X : X \to Y$ is a rational $\mathbb{A}^1$-fibration.

Proof. Since a general closed fiber $X_b$ is a normal affine surface with an $\mathbb{A}^1$-fibration $\pi_b : X_b \to C_0$ over a certain irrational smooth curve $C_0$, it follows that there exists a unique maximal affine open subset $U_b$ of $C_0$ such that $\pi_b^{-1}(U_b) \simeq U_b \times \mathbb{A}^1$ and such that the rational map $j_b : \pi_b^{-1}(U_b) \to W_b$ induced by $j$ is regular, inducing an isomorphism between $\pi_b^{-1}(U_b)$ and its image. Each step $\varphi_i : W_i \to W_{i+1}$ consists of either a flip whose flipping and flipped curves are contained in fibers of $\pi_i : W_i \to B$ and $\pi_{i+1} : W_{i+1} \to B$ respectively, or a divisorial contraction whose exceptional divisor is contained in a fiber of $\pi_i : W_i \to B$, or a divisorial contraction whose exceptional divisor intersects a general fiber of $\pi_i : W_i \to B$. Clearly, a general closed fiber of $\pi_i : W_i \to B$ is not affected by the first two types of birational maps. On the other hand, if $\varphi_i : W_i \to W_{i+1}$ is the contraction of a divisor $E_i \subset W_i$ which dominates $B$, then a general fiber of $\varphi_i|_{E_i}$ is a smooth proper rational curve. The intersection of $E_i$ with a general closed fiber $W_{i,b}$ of $\pi_i$ thus consists of proper rational curves, and its intersection with the image of the maximal affine cylinder like open subset $\pi_b^{-1}(U_b)$ of $X_b$ is either empty or composed of affine rational curves. Since $U_b$ is an irrational curve, it follows that each irreducible component of $E_i \cap (\pi_b^{-1}(U_b))$ is contained in a fiber of $\pi_b$. This implies that there exists an open subset $U_{b,0}$ of $U_b$ with the property that for every $i = 1, \ldots, \ell$, the restriction of $\varphi_i \circ \cdots \circ \varphi_1$ to $\pi_b^{-1}(U_{b,0}) \subset X_b$ is an isomorphism onto its image in $W_{i,b}$. A general fiber of $\overline{\pi} : W' \to Y$ over a closed point $y \in Y$ being a smooth proper rational curve, its intersection with $\pi_h^{-1}(y)(U_{h,0})$ viewed as an open subset of $W_{h(y)}'$ is thus either empty or equal to a fiber of $\pi_{h(y)}$. So by virtue of [9], there exists an open subset $V$ of $X$ on which $\overline{\pi}$ restricts to an $\mathbb{A}^1$-fibration $\overline{\pi}|_V : V \to Y$.

Corollary 13. Let $X$ be a normal complex affine threefold $X$ equipped with a morphism $f : X \to B$ onto a smooth curve $B$ whose general closed fibers are irrational $\mathbb{A}^1$-ruled surfaces. Then $X$ is birationally equivalent to the product of $\mathbb{P}^1$ with a family $h_0 : C_0 \to B_0$ of smooth projective curves of genus $g \geq 1$ over an open subset $B_0 \subset C_0$.

Proof. By the previous Proposition, $X$ has the structure of a rational $\mathbb{A}^1$-fibration $\rho : X \to Y$ over a 2-dimensional normal proper $B$-scheme $h : Y \to B$. In particular, $X$ is birational to $Y \times \mathbb{P}^1$. On the other hand, for a general closed point $b \in B$, the curve $Y_b$ is birational to the base $C_b$ of the unique $\mathbb{A}^1$-fibration $\pi_b : X_b \to C_0$ on the irrational affine surface $X_b$. Letting $\sigma : Y \to Y$ be a desingularization of $Y$, there exists an open subset $B_0$ of $B$ over which the composition $h \circ \sigma : Y \to Y$ restricts to a smooth family $h_0 : C_0 \to B_0$ of projective curves of a certain genus $g \geq 1$. By construction, $X$ is birational to $C_0 \times \mathbb{P}^1$.

Remark 14. Example 10 above shows conversely that for every smooth family $h : C \to B$ of projective curves of genus $g \geq 2$, there exists a smooth $\mathbb{A}^1$-ruled affine threefold $X$ birationally equivalent to $C \times \mathbb{P}^1$. Actually, in the setting of the previous Corollary 13, if we assume further that a general fiber of $f : X \to B$ carries an $\mathbb{A}^1$-fibration $\pi_0 : X_b \to C_0$ over a smooth curve $C_0$ whose smooth projective model has genus $g \geq 2$, then there exists a uniquely determined family $h : C \to B$ of proper stable curves of genus $g$ such that $X$ is birationally equivalent to $C \times \mathbb{P}^1$: indeed, the moduli stack $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ being proper.
3.2. Factorial threefolds.

**Proposition 15.** Let $X$ be a normal affine threefold with finite divisor class group $\text{Cl}(X)$ and let $f : X \to B$ be a morphism onto a smooth curve $B$ whose general closed fibers are irrational $\mathbb{A}^1$-ruled surfaces. Then there exists a factorization $f = h \circ \rho : X \to Y \to B$ where $\rho : X \to Y$ is the algebraic quotient morphism of an effective $\mathbb{G}_{a,b}$-action on $X$. In particular, a general fiber of $f$ admits an $\mathbb{A}^1$-fibration of affine type.

**Proof.** By virtue of Proposition 11, there exist an effective $\mathbb{G}_{a,b}$-action on $X$ such that for a general closed point $b \in B$, the $\mathbb{A}^1$-fibration $\pi_b : X_b \to C_b$ on $X_b$ factors through the algebraic quotient $\rho_b : X_b \to X_b//\mathbb{G}_{a,b} = \text{Spec}(\Gamma(X_b, \mathcal{O}_{X_b})^{\mathbb{G}_{a,b}})$. Since $X$ is a threefold, the ring of invariants $\Gamma(X, \mathcal{O}_X)^{\mathbb{G}_{a,b}}$ is finitely generated [16]. The quotient morphism $\rho : X \to Y = \text{Spec}(\Gamma(X, \mathcal{O}_X)^{\mathbb{G}_{a,b}})$ is an $\mathbb{A}^1$-fibration, and since $Y$ is a categorical quotient in the category of algebraic varieties, the invariant morphism $f : X \to B$ factors through $\rho$. □

**Corollary 16.** Let $f : \mathbb{A}^3 \to B$ be a morphism onto a smooth curve $B$ with irrational $\mathbb{A}^1$-ruled general fibers. Then $B$ is isomorphic to either $\mathbb{P}^1$ or $\mathbb{A}^1$ and there exists a factorization $f = h \circ \rho : \mathbb{A}^3 \to \mathbb{A}^2$, where $\rho : \mathbb{A}^3 \to \mathbb{A}^2$ is the quotient morphism of an effective $\mathbb{G}_{a,b}$-action on $\mathbb{A}^3$.

**Proof.** Since $B$ is dominated by a general line in $\mathbb{A}^3$, it is necessarily isomorphic to $\mathbb{P}^1$ or $\mathbb{A}^1$. The second assertion follows from Proposition 15 and the fact that the algebraic quotient of every nontrivial $\mathbb{G}_{a}$-action on $\mathbb{A}^3$ is isomorphic to $\mathbb{A}^2$ [13]. □

**Example 17.** In Corollary 16 above, the base curve $B$ need not be affine. For instance, the morphism

$$f : \mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z]) \longrightarrow \mathbb{P}^1, (x, y, z) \mapsto [(xz - y^2)x^2 + 1 : (xz - y^2)^3]$$

defines a family whose general member is isomorphic to the product $C_\lambda \times \mathbb{A}^1$, where $C_\lambda \subset \mathbb{A}^2 = \text{Spec}(\mathbb{C}[x, y, z])$ is the affine cubic curve with equation $(xz - y^2)^3 + \lambda((xz - y^2)x^2 + 1) = 0$. The subring $\mathbb{C}[x, y, z]$ coincides with the ring of invariants of the $\mathbb{G}_a$-action associated with the locally nilpotent $\mathbb{C}[x]$-derivation $x\partial_x + 2y\partial_z$ and $f$ is the composition of the quotient morphism $\rho : \mathbb{A}^3 \to \mathbb{A}^2 = \mathbb{A}^3//\mathbb{G}_a = \text{Spec}(\mathbb{C}[u, v]), (x, y, z) \mapsto (xz - y^2, x)$ and the morphism $h : \mathbb{A}^2 = \text{Spec}(\mathbb{C}[u, v]) \to \mathbb{P}^1, (u, v) \mapsto [uv^2 + 1 : v]^3$.

Corollary 16 above implies in particular that a general fiber of a regular function $f : \mathbb{A}^3 \to \mathbb{A}^1$ cannot be an irrational surface equipped with an $\mathbb{A}^1$-fibration of complete type only. In contrast, regular functions $f : \mathbb{A}^3 \to \mathbb{A}^1$ whose general fibers are rational and equipped with $\mathbb{A}^1$-fibrations of complete type only do exist, as illustrated by the following example.

**Example 18.** Let $f = x^3 - y^3 + z(z + 1) \in \mathbb{C}[x, y, z]$ and let $f : \mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z]) \to \mathbb{A}^1 = \text{Spec}(\mathbb{C}[\lambda])$ be the corresponding cubic morphism. The closure $\overline{S}_\lambda$ in $\mathbb{P}^3 = \text{Proj}(\mathbb{C}[x, y, z, t])$ of a general fiber $S_\lambda = f^{-1}(\lambda)$ of $f$ is a smooth cubic surface which intersects the hyperplane $H_\infty = \{t = 0\}$ along the union $B_3$ of three lines meeting at the Eckardt point $p = [0 : 0 : 1 : 0]$. Thus $S_\lambda$ is rational and a direct computation reveals that $\kappa(S_\lambda) = -\infty$. So by virtue [14], $S_\lambda$ admits an $\mathbb{A}^1$-fibration $\pi_\lambda : S_\lambda \to C_\lambda$ over a smooth rational curve $C_\lambda$. If $C_\lambda$ was affine, then there would exist a non trivial $\mathbb{G}_a$-action on $S_\lambda$ having the general fibers of $\pi_\lambda$ as general orbits. But it is straightforward to check that every automorphism of $S_\lambda$ considered as a birational self-map of $\overline{S}_\lambda$ is in fact a biregular automorphism of $\overline{S}_\lambda$ preserving the boundary $B_\lambda$. So the automorphism group of $S_\lambda$ injects into the group $\text{Aut}(\overline{S}_\lambda, B_\lambda)$ of automorphisms of the pair $(\overline{S}_\lambda, B_\lambda)$. The latter being a finite group, we conclude that no such $\mathbb{G}_a$-action exists, and hence that $S_\lambda$ only admits $\mathbb{A}^1$-fibrations of complete type. An $\mathbb{A}^1$-fibration $\pi_\lambda : S_\lambda \to \mathbb{P}^1$ can be obtained as follows: letting $B_\lambda = L_1 \cup L_2 \cup L_3, L_1$ is a member of a 6-tuple of pairwise disjoint lines whose simultaneous contraction realizes $\overline{S}_\lambda$ as a blow-up $\sigma : \overline{S}_\lambda \to \mathbb{P}^2$ of $\mathbb{P}^3$ in such a way that $\sigma(L_2)$ and $\sigma(L_3)$ are respectively a smooth conic and its tangent line at the point $p = \sigma(L_1)$. The birational transform $\pi_\lambda : \overline{S}_\lambda \dashrightarrow \mathbb{P}^1$ on $\overline{S}_\lambda$ of the pencil generated by $\sigma(L_2)$ and $2\sigma(L_1)$ restricts to an $\mathbb{A}^1$-fibration $\pi_\lambda : S_\lambda \to \mathbb{P}^1$ with two degenerate fibers: an irreducible one, of multiplicity two, consisting of the intersection with $S_\lambda$ of the unique exceptional divisor of $\sigma$ whose center is supported on $\sigma(L_3) \setminus \{p\}$, and a smooth one consisting of the intersection with $S_\lambda$ of the four exceptional divisors of $\sigma$ with centers supported on $\sigma(L_2) \setminus \{p\}$. 
FAMILIES OF AFFINE RULED SURFACES: EXISTENCE OF CYLINDERS

References


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