Estimation of deformations between distributions by minimal Wasserstein distance.

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Abstract

We consider the issue of estimating a measure observed in a deformation framework. For this we consider a parametric deformation model on an empirical sample and provide a new matching criterion for cloud points based on a generalization of the registration criterion used in [15]. We study the asymptotic behaviour of the estimators of the deformations and provide some examples to some particular deformation models.

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Contents

1 Introduction

Giving a sense to the notion of *mean behaviour* may be counted among the very early activities of statisticians. When confronted to large data sample, the usual notion of Euclidean mean is too rough since the information conveyed by the data possesses an inner geometry far from the Euclidean one. Indeed, deformations on the data such as translations, scale location models for instance or more general warping procedures prevent the use of the usual methods in data analysis. This problem arises naturally for a wide range of statistical research fields such as functional data analysis for instance in [15], [12], [19], [5] and references therein, image analysis in [22] or [3], shape analysis in [17] or [16] with many applications ranging from biology in [8] to pattern recognition [20] just to name a few.

To handle this issue without any assumption on the deformations, Sakoe and Chiba in [20] present a synchronization algorithm known as the Dynamic Time Warping (D.T.W.), aligning two curves by a time axis renormalization. When dealing with functional data observed in a regression scheme, this idea was generalized in [27].

For a better understanding of the deformations, another major direction has been investigated. It consists in modeling the deformations by a parametric warping operator, such as for instance, scale location parameters, rotations in [7], actions of parameters of Lie groups or in a more general way deformations parametrized by their coefficients on a given basis [6] or in an RKHS set [1]. Adding structure on the deformations enables to define the *mean behaviour* as the data warped by the *mean deformation*, i.e. the deformation parametrized by the mean of the parameters. Semi-parametric technics as in [15] or [26] enable to provide sharp estimation of these parameters.

The same kind of issues arises when considering the estimation of distribution functions observed with deformations. This situation occurs often in biology, for example when considering gene expression data obtained from microarray technologies. A microarray may contain thousands of spots, each one containing a few million copies of identical D.N.A. molecules that uniquely correspond to a gene. From each spot, a measure is obtained but before performing any statistical analysis on such data, it is necessary to process rough data in order to remove any systematic bias inhering to the microarray technology. A natural way to handle this phenomena is to try to remove these variations in order to align the measured densities, which proves difficult since the densities are unknown. In bioinformatics and computational biology, a method to reduce this kind of variability is known as normalization.

However, when dealing with the registration of warped distributions, the literature is scarce. We mention here the method provided for biological computational issues known as quantile normalization in [8] and the related work [14]. In [18] and [10] a criterion based on Wasserstein's distance is used to match two distributions for some particular deformation framework. In this work, we consider the extension of such parametric methods to the problem of estimating a distribution of random variables, observed in a warping framework through a precise estimation of the particular deformation parameters.

Actually, assume that we observe $i = 1, \dots, n$ samples of $j = 1, \dots, J$ independent random variables X_{ij} with distribution μ_j. Each sample is drawn from a *mean* distribution μ with some variations in the sense that there exists an unobserved warping function φ such that, for all j, we have $\mu_j = \mu \circ \varphi_j^{-1}$. To deal with this issue, we assume a parametric model for the warping function. We consider that the deformations follow a known shape which depends on parameters, specific for each sample. Hence there are parameters $\theta^\star = (\theta_1^\star, \dots, \theta_J^\star)$ such that $\varphi_j = \varphi_{\theta_j^\star}$, for all $j = 1, \dots, J$. Each $\theta_j^\star$ represents the warping effect that undergoes the j^{th} sample, which must be removed to recover the unknown distribution by inverting the warping operator. Hence, we will estimate, in a semi-parametric framework, the parameters $\theta_j^\star$.

For this, inspired by the method provided in [15], we warp the observations and construct an estimator of $\theta^\star$ by minimizing the energy needed to *align* all the distributions μ_j to the distribution μ_{j-1} . That is to say, we will minimize the cost of transport of the mass charged by μ_j on the mass charged by μ_{j-1}. Hence, to quantify the alignment between the two probabilities, it seems natural to us to consider the Wasserstein distance, see for instance in [25] or [2] for the connexions between this distance and mass transport. So we will study an estimator $\widehat{\theta}^n$ of $\theta^\star$ obtained by minimizing a criterion inspired by the D.T.W. and based on the Wasserstein distance between two probabilities.

We will obtain a result of consistency under general assumptions, in particular we will not assume the compactness of the support of μ. This estimator of $\theta^\star$ will enable us to obtain an consistent estimator of the structural distribution μ. Under stronger assumptions, we will also obtain a result of convergence in law for $\widehat{\theta}^n$.

The paper is organized as follows : the description of our model and the definitions of the estimators are given in Section 2. Section 3 is devoted to the consistency results obtained for the estimators of $\theta^\star$ and μ. In Section 4, a new framework is introduced to study the asymptotic comportment of the deformation estimates with a result about their convergence in distribution. Section 5 presents some examples of deformations which fall in the scope of our study. Finally some simulations are provided in Section 6. The proofs are postponed to a technical Appendix.

2 Statistical model for distribution deformations

In this section, we will define a model for deformations of random variables and recall some useful definitions.

First, recall the following notations. If P_n is a sequence of probabilities which converges weakly to a probability P when $n \to \infty$, we note $P_n \rightharpoonup P$. In all the paper, we denote by $\| \; \|$ the euclidean norm on $\mathbb{R}^k$ for all $k \in \mathbb{N}$, $k \geqslant 2$. Finally, for a given sample $Y = (Y_1, \dots, Y_n)$, we denote by $Y_{(1)} \leqslant \dots \leqslant Y_{(i)} \leqslant \dots \leqslant Y_{(n)}$ its order statistics.

For $i = 1, \ldots, n$ and $j = 1, \ldots, J$, set ε_{ij} unobserved real i.i.d. random variables with unknown distribution μ defined on an Borel set $I_a \subset \mathbb{R}$. We will consider deformations of these real-valued observations. Hence, we consider a family of deformation functions, indexed by parameters $\lambda \in \Lambda$, for Λ a compact and convex subset of $\mathbb{R}^d$, which warps a point x onto another point $\varphi_\lambda(x)$. The shape of the deformation is modelled by the known function φ while the amount of deformation is characterized by the parameter λ. Namely, set

$$\begin{array}{rccc} \varphi: & \Lambda \times I_a & \to & I_b \\ & (\lambda, x) & \mapsto & \varphi_\lambda(x) \end{array}$$

for I_a, I_b subsets of $\mathbb{R}$ possibly unbounded.

We assume that we observe

$$X_{ij} = \varphi_{\theta_j^\star}(\varepsilon_{ij}) \quad 1 \leqslant i \leqslant n \quad 1 \leqslant j \leqslant J, \tag{1}$$

where $\theta_j^\star$ is the unknown deformation parameter in $\Lambda \subset \mathbb{R}^d$, associated to the j-th sample.

Our aim is to estimate the parameter $\theta^\star \in \Theta = \Pi_{j=1}^J \Lambda$. For this, we will study a criterion based on a registration procedure for the distributions μ_j of each i.i.d. sample $(X_{1j}, \ldots, X_{nj})$, for all $j = 1, \ldots, J$. To compute the distance between the distributions, we will need the following probabilistic tools.

If F is a distribution function , we define the quantile function associated by

$$F^{-1}(t) = \inf\{x \in \mathbb{R}, F(x) \geqslant t\}.$$

Recall that if F_n is the empirical distribution associated to a sample $(Y_1, \ldots, Y_n)$, then we have

$$F_n^{-1}(t) = Y_{(i)} \text{ for } \frac{i-1}{n} < t \leqslant \frac{i}{n}.$$

A natural distance to measure the deformation cost to align two distributions is given by the Wasserstein distance. For $p \in \mathbb{N}^\star$, consider the following set

$$\mathcal{W}_2(\mathbb{R}^p) = \{P \text{ probability on } \mathbb{R}^p \text{ which admits a finite second order moment}\}.$$

Given two probabilities P and Q in $\mathcal{W}_2(\mathbb{R}^p)$ we denote by $\mathcal{P}(P, Q)$ the set of all probability measures π over the product set $\mathbb{R}^p \times \mathbb{R}^p$ with first (resp. second) marginal P (resp. Q).

The transportation cost with quadratic cost function, or quadratic transportation cost, between these two measures P, Q is defined as

$$\mathcal{T}_2(P, Q) = \inf_{\pi \in \mathcal{P}(P,Q)} \int \|x - y\|^2 d\pi.$$

The quadratic transportation cost allows to endow the set $\mathcal{W}_2(\mathbb{R}^p)$ with a metric by setting

$$W_2(P, Q) = \mathcal{T}_2(P, Q)^{1/2}.$$

Note that we will use W_2 metrics in this work. This choice is led by the issue of optimal matching between cloud points, see for instance in [4]. Yet other choices

$$W_r^r(P, Q) = \inf_{\pi \in \mathcal{P}(P,Q)} \int d(x, y)^r d\pi$$

are possible for different r and other distances d on $\mathbb{R}^p$. In particular, the earth-mover distance which corresponds to $r = 1$ could be used with more complicated calculations. However the study of this criterion falls beyond the scope of this paper.

Hereafter, we will consider distributions on $\mathbb{R}$. In this case the Wasserstein distance can be computed directly using the inverse distribution functions, as

$$W_2^2(P,Q) = \int_0^1 \left(F^{-1}(t) - G^{-1}(t)\right)^2 dt, \tag{2}$$

where F (resp. G) is the distribution function associated to P (resp. Q). The registration procedure we consider is an extension to point cloud estimation of the methodology pioneered in [15] and deeply studied in [26]. Wasserstein distance is actually a powerful tool to study similarities between point distributions, see in [9] or [11].

Recall that our aim is to align the law μ_j of the observations X_j. Hence a natural idea is to apply the inverse deformation operator to these observations. More precisely for all candidate θ_j, and to each observation X_{ij}, we can apply the inverse deformation of parameter θ_j. Hence we can compute for all different j's, the following random variables

$$Z_{ij}(\theta) = \varphi_{\theta_j}^{-1}(X_{ij}) \tag{3}$$

where we denote $\theta = (\theta_1, \ldots, \theta_J) \in \Theta = \Pi_{j=1}^J \Lambda$. Now, denote by $\mu_j(\theta)$ the common law of the elements of the i.i.d. sample $Z_j(\theta) = (Z_{1j}(\theta), \ldots, Z_{nj}(\theta))$. By varying the parameter θ, we will try to *align* successively the distribution $\mu_j(\theta)$ onto the previous distribution $\mu_{j-1}(\theta)$.

Let

$$\mu_j^n(\theta) = \frac{1}{n}\sum_{i=1}^n \delta_{Z_{ij}(\theta)}$$

the empirical law of the sample $(Z_{ij}(\theta))_{1\leqslant i\leqslant n}$.
We note F_j the distribution function associated with the law μ_j and F the distribution function associated with the law μ, F_j^n the empirical distribution function of the random sample $(X_{1j}, \ldots, X_{nj})$.

We introduce the following criterion

$$M: \quad \theta \mapsto M(\theta) = \frac{1}{J-1}\sum_{j=2}^J W_2^2(\mu_j(\theta), \mu_{j-1}(\theta)). \tag{4}$$

Remark that for $\theta = \theta^\star$, we get for all j $\mu_j(\theta^\star) = \mu_{j-1}(\theta^\star) = \mu$. Hence the distributions are the same for the true parameter $\theta^\star$, and the criterion M reaches its minimum at this point.

The estimation of this criterion is given by its corresponding empirical version, which is

$$M_n(\theta) = \frac{1}{J-1}\sum_{j=2}^J W_2^2\left(\mu_j^n(\theta), \mu_{j-1}^n(\theta)\right). \tag{5}$$

It can be computed using (2) and the order statistics associated to the samples $Z_j(\theta)$ as

$$M_n(\theta) = \frac{1}{J-1}\sum_{j=2}^{J}\frac{1}{n}\sum_{i=1}^{n}\left[Z_{(i)j}(\theta) - Z_{(i)j-1}(\theta)\right]^2.$$

The estimator of $\theta^\star$ is finally defined as

$$\widehat{\theta}^n \in \arg\min_{\theta\in\Theta} M_n(\theta). \tag{6}$$

Our aim is thus twofold.

- First, study the asymptotic comportment of this M-estimator.
- Then, using this estimate, recover the template measure μ with the following estimator

$$\widehat{\mu}^n = \frac{1}{J}\sum_{j=1}^{J}\left(\frac{1}{n}\sum_{i=1}^{n}\delta_{\varphi^{-1}_{\widehat{\theta}^n_j}(X_{ij})}\right) := \frac{1}{J}\sum_{j=1}^{J}\widehat{\mu}^n_j. \tag{7}$$

We point out that we restrict ourselves to distributions on $\mathbb{R}$ and not $\mathbb{R}^p$. As a matter of fact, the statistical analysis of the estimates and their asymptotic behaviour in distribution require a particular study of the asymptotic expansion of M_n that can not be achieved using the general expression of Wasserstein metrics. Indeed, we will need to express W_2 with quantile functions, estimated by the corresponding order statistics, which can only be done in the one dimensional case.

3 Consistent estimation of the deformation parameters and the distribution template

The main objective of this section is to study the consistency of the estimator defined in (6) as

$$\widehat{\theta}^n \in \arg\min_{\theta\in\Theta} M_n(\theta).$$

Consider the following assumptions

A0 : μ is a probability law on a measurable subset $I_a \subset \mathbb{R}$.

We consider deformation functions that verify

A1 : There exists a measurable subset $I_b \subset \mathbb{R}$ such that for all $\lambda \in \Lambda$, $\varphi_\lambda : \begin{array}{ccc} I_a & \to & I_b \\ x & \mapsto & \varphi_\lambda(x) \end{array}$ is invertible.

and

A2 : For all $\lambda \in \Lambda$, $\varphi_\lambda : \begin{array}{ccc} I_a & \to & I_b \\ x & \mapsto & \varphi_\lambda(x) \end{array}$ is increasing.

Next, to ensure that the criterion M is finite, the distribution of $Z_j(\theta)$, $\mu_j(\theta)$, must belong to $\mathcal{W}_2(\mathbb{R})$ for all $\theta \in \Theta$, so we need

A3 : For all $\lambda \in \Lambda$ and all $1 \leqslant j \leqslant J$, $\varphi_\lambda^{-1}(\cdot)$ is in $L^2(\mu_j)$ that is to say $\varphi_\lambda^{-1} \circ \varphi_{\theta_j^\star}(\cdot) \in L^2(\mu)$.

The two following assumptions are more technical.

A4 : For all $x \in I_a$, $\varphi_\lambda : \begin{array}{ccc} \Lambda & \to & I_b \\ \lambda & \mapsto & \varphi_\lambda^{-1}(x) \end{array}$ is continuously differentiable. We denote its partial differential with respect to the variable λ on λ_0 by $\partial\varphi_{\lambda_0}^{-1}(x)$.

A5 : The family $\left(\partial\varphi_\lambda^{-1}(\cdot)\right)_{\lambda\in\Lambda}$ has an envelop in $L^2(\mu_j)$ for all j, that is

$$\sup_{\lambda\in\Lambda} \left\|\partial\varphi_\lambda^{-1}(x)\right\| \leqslant H(x)$$

with $H \in L^2(\mu_j)$ for all j.

It remains to have the following inequality for all j

$$\sup_{\lambda\in\Lambda} \left\|\partial\varphi_\lambda^{-1} \circ \varphi_{\theta_j^\star}(x)\right\| \leqslant G(x)$$

with $G \in L^2(\mu)$.

These two last assumptions are actually regularity assumption on the deformations. They are required in order to get bounds for the empirical processes.

The last assumption is related to the identifiability of the model. More precisely it ensures that M admits an unique minimum on Θ at the parameter of interest, $\theta^\star$.

A6 For all $\theta \neq \theta^\star$ in Θ, there exists $1 \leqslant k < j \leqslant J$, and a set A such that $\mu(A) > 0$ and $\varphi_{\theta_j}^{-1} \circ \varphi_{\theta_j^\star} \neq \varphi_{\theta_k}^{-1} \circ \varphi_{\theta_k^\star}$ on A.

Finally, recall that Λ is a compact and convex subset of $\mathbb{R}^d$.

3.1 Estimation of $\theta^\star$

Assume we observe X_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, J$, defined in (1). The following theorem proves the consistency of the estimator of the deformation parameters.

Theorem 3.1. *Under assumptions* **A0** *to* **A6**, $\widehat{\theta}^n \in argmin_{\theta\in\Theta} M_n(\theta)$ *converges in probability to* $\theta^\star$ *when* n *tends to infinity .*

The estimate of $\theta^\star$ is defined as an M-estimator. Hence its study follows the classical guidelines stated for instance in [24]. More precisely, its consistency can be obtained by establishing the uniform convergence of the criterion, that is

$$\sup_{\theta\in\Theta} |M_n(\theta) - M(\theta)| \xrightarrow{n\to\infty} 0 \text{ in probability}$$

under the following condition of identifiability

$$\text{for all } \varepsilon > 0, \inf_{\Theta\cap B(\theta^\star,\varepsilon)^c} M(\theta) > 0.$$

So according to Theorem 5.7 p.45 in [24], these two results enable to obtain Theorem 3.1.

The uniform convergence is obtained through the followings steps

- We first prove the pointwise convergence of M_n to M in probability. It involves classical properties of the Wasserstein distance about the convergence of empirical measures.

- Next we obtain the following property of "uniform continuity"

$$\text{for all } \varepsilon > 0, \limsup_{n\to\infty} \mathbb{P}\left(\sup_{\|\theta^1-\theta^2\|\leqslant\nu} \left|M_n\left(\theta^1\right) - M_n\left(\theta^2\right)\right| > \varepsilon \right) \xrightarrow{\nu\to 0} 0.$$

  This part is the most important, and especially requires assumption **A5**.

We conclude by using arguments of compactness and continuity. The latter, in addition to assumption **A6**, are used to obtain the condition of identifiability. The details of the proof are given in the Appendix.

3.2 Reconstruction of the measure μ

Theorem 3.1 enables to get a sharp approximation of the true parameters of deformations with the estimator $\widehat{\theta}^n$. This entails that the observations can be aligned by computing the inverse transformation applied to the observations. Actually when n is sufficiently large, $\varphi^{-1}_{\widehat{\theta}^n_j}\left(X_{ij}\right) = \varphi^{-1}_{\widehat{\theta}^n_j} \circ \varphi_{\theta^\star_j}\left(\varepsilon_{ij}\right)$ is very close to ε_{ij}. So a natural estimator of the measure μ is given by

$$\widehat{\mu}^n = \frac{1}{J}\sum_{j=1}^{J}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\varphi^{-1}_{\widehat{\theta}^n_j}(X_{ij})}\right) = \frac{1}{J}\sum_{j=1}^{J}\widehat{\mu}^n_j$$

The following theorem proves its consistency.

Theorem 3.2. *Under assumptions* **A0** *to* **A6***,* $\widehat{\mu}^n$ *converges in the Wasserstein distance sense to the measure* μ *in probability :*

$$W_2\left(\widehat{\mu}^n, \mu\right) \xrightarrow{n\to\infty} 0 \textit{ in probability.}$$

The convergence result can be made more precise with the following proposition.

Proposition 3.3. *Under assumptions* **A0** *to* **A6***, then for all* j*,*

$$W_2\left(\widehat{\mu}^n_j, \mu\right) \xrightarrow{n\to\infty} 0 \textit{ in probability.}$$

This result comes almost directly from the consistency of the deformation parameters stated in Theorem 3.1.

4 Asymptotic analysis of the deformation parameters

4.1 Assumptions

In order to handle a larger variety of deformation models, we weaken assumption **A6**. Following [26] or [5], we restrict the set of parameters $\Theta = \Pi_{j=1}^{J}\Lambda$ to the set $\tilde{\Theta} = \{\theta \in \Theta, \theta_1 = \theta^\star\}$, which amounts to saying that we take the first sample as a reference and align all the others onto this one. So, to verify assumption **A6** it is now sufficient to show that there exists j with $2 \leqslant j \leqslant J$ such that $\varphi_{\theta_j}^{-1} \circ \varphi_{\theta_j^\star} \neq Id$ for $\theta \neq \theta^\star$ on a set of positive μ-measure.

So we study the estimator

$$\widehat{\theta}^n \in \arg\min_{\theta \in \tilde{\Theta}} M_n(\theta)$$

that is

$$\left(\widehat{\theta}_2^n, \ldots, \widehat{\theta}_J^n\right) \in \arg \min_{(\theta_2,\ldots,\theta_J)\in\Theta_{J-1}} M_n((\theta_1^\star, \theta_2, \ldots, \theta_J))$$

where $\Theta_{J-1} = \Pi_{j=2}^J \Lambda$: the parameter space has in fact dimension $J-1$. So for sake of simplicity, in this section we will write $\theta = (\theta_2, \ldots, \theta_J) \in \Theta_{J-1}$ and $M_n(\theta)$ instead of $M_n((\theta_1^\star, \theta_2, \ldots, \theta_J))$

Consider the following notations. If G is a function differentiable on Θ_{J-1} and for $\theta = (\theta_2, \ldots, \theta_J) \in \Theta_{J-1}$ we denote by $\partial_j G(\theta)$ its partial derivative with respect to the j-th variable ($2 \leqslant j \leqslant J$) at the point θ, and $DG(\theta)$ its differential at point θ.

Now we add to assumptions **A0** to **A6** the following regularity conditions on the deformation functions.

AL1 φ^{-1} is C^2 with respect to its two variables (λ, x) on $\Lambda \times I_b$. We denote by $\partial\varphi_\lambda^{-1}(x)$ its partial derivative with respect to the first variable at the point (λ, x) and by $d\varphi_\lambda^{-1}(x)$ its partial derivative with respect to the second variable at the point (λ, x).

Consider the following restrictions on the distributions μ_j, which are the distributions of observations X_{1j}.

AL2 For all j, μ_j is a law with a compact support $[\alpha; \beta] \subset I_b$.

AL3 For all j, the distribution function F_j of the law μ_j is continuously differentiable with strictly positive derivative f_j on its support.

Actually these assumptions are required to prove the convergence in distribution of the empirical quantile functions.

Note that using the relation $F_j = F \circ \varphi_{\theta_j^\star}$ which is due to **A2**, we obtain that **AL2** and **AL3** implies that F is continuously differentiable with strictly positive derivative denoted by f.

In the following, we consider that $d = 1$, that is $\theta_j^\star \in \Lambda \subset \mathbb{R}^d = \mathbb{R}$. However the following result stands for $d \geqslant 2$ with slight modifications.

4.2 Asymptotic distribution of the deformation estimates

In this section we study the convergence in law of the estimator $\widehat{\theta}^n$, which relies on a Taylor expansion for the empirical criterion. So using **AL1** we first establish that the criterion M_n is C^2 on Θ_{J-1}.

Hence, Taylor expansion of the partial derivative of M_n can be written as

$$\partial_j M_n\left(\widehat{\theta}^n\right) = \partial_j M_n\left(\theta^\star\right) + D\partial_j M_n\left(\theta^{n,j}\right)\left(\widehat{\theta}^n - \theta^\star\right)$$

for $\theta^{n,j}$ between $\widehat{\theta}^n$ and $\theta^\star$ and $2 \leqslant j \leqslant J$. But M_n admits a minimum on $\widehat{\theta}^n$, so

$$-\partial_j M_n(\theta^\star) = D\partial_j M_n\left(\theta^{n,j}\right)\left(\widehat{\theta}^n - \theta^\star\right).$$

Hence

$$-DM_n(\theta^\star) = \begin{pmatrix} D\partial_2 M_n\left(\theta^{n,2}\right) \\ \cdot \\ \cdot \\ D\partial_J M_n\left(\theta^{n,J}\right) \end{pmatrix}\left(\widehat{\theta}^n - \theta^\star\right).$$

We set $D\partial_j M_n(\theta^{n,j}) = \Phi_j(F_1^n, \ldots, F_J^n, \theta^{n,j})$.
The main part of the proof consists in showing the two asymptotic results.

- On the one hand, we will show that $\sqrt{n}(-DM_n(\theta^\star)) \rightharpoonup Z \in \mathbb{R}^{J-1}$ for some random vector Z by using a functional delta method. This part is inspired by the work of [13].
- On the other hand, we will establish that $\Phi_j(F_1^n, \ldots, F_J^n, \theta^{n,j})$ converges in probability to a deterministic quantity $\Phi_j(F_1, \ldots, F_J, \theta^\star)$ when n goes to infinity.

Let

$$\Phi = \begin{pmatrix} \Phi_2(F_1, \ldots, F_J, \theta^\star) \\ \cdot \\ \cdot \\ \Phi_J(F_1, \ldots, F_J, \theta^\star) \end{pmatrix}.$$

Then if Φ is invertible, we finally get that

$$\sqrt{n}\left(\widehat{\theta}^n - \theta^\star\right) \rightharpoonup \Phi^{-1}Z.$$

More precisely, Φ is the tridiagonal symmetric matrix in $\mathbb{R}^{J-1\times J-1}$, with the lines $\begin{pmatrix} \Phi_2(F_1, \ldots, F_J, \theta^\star) \\ \cdot \\ \cdot \\ \Phi_J(F_1, \ldots, F_J, \theta^\star) \end{pmatrix}$ where

$$\begin{aligned}\Phi_2(F_1, \ldots, F_J, \theta^\star) = \Bigg(&\frac{4}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta_2^\star}\left(F_2^{-1}(t)\right)^2 dt, \\ &\frac{-2}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta_2^\star}\left(F_2^{-1}(t)\right)\partial\varphi^{-1}_{\theta_3^\star}\left(F_3^{-1}(t)\right)dt, 0, \ldots, 0\Bigg)\end{aligned} \tag{8}$$

$$\begin{aligned}\Phi_j(F_1, \ldots, F_J, \theta^\star) = \Bigg(&0, \ldots, 0, \frac{-2}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta_j^\star}\left(F_j^{-1}(t)\right)\partial\varphi^{-1}_{\theta_{j-1}^\star}\left(F_{j-1}^{-1}(t)\right)dt, \\ &\frac{4}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta_j^\star}\left(F_j^{-1}(t)\right)^2 dt, \\ &\frac{-2}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta_j^\star}\left(F_j^{-1}(t)\right)\partial\varphi^{-1}_{\theta_{j+1}^\star}\left(F_{j+1}^{-1}(t)\right)dt, 0 \ldots, 0\Bigg)\end{aligned} \tag{9}$$

for $3 \leqslant j \leqslant J-1$, and

$$\Phi_J\left(F_1,\ldots,F_J,\theta^\star\right) = \Bigg(0,\ldots,0, \frac{-2}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta^\star_J}\left(F_J^{-1}(t)\right)\partial\varphi^{-1}_{\theta^\star_{J-1}}\left(F_{J-1}^{-1}(t)\right)dt, \qquad (10)$$
$$\frac{2}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta^\star_J}\left(F_J^{-1}(t)\right)^2 dt\Bigg).$$

Then we have that

Theorem 4.1. *Under assumptions* **A0** *to* **A6** *and* **AL1** *to* **AL3** *and if* Φ *is invertible, then*

$$\sqrt{n}\left(\widehat{\theta}^n - \theta^\star\right) \rightharpoonup \Phi^{-1}\begin{pmatrix} Z_2 \\ \cdot \\ Z_J \end{pmatrix}$$

with, for $2 \leqslant j \leqslant J-1$

$$Z_j = \frac{2}{J-1}\int_0^1 \frac{\partial\varphi^{-1}_{\theta^\star_j}\left(F_j^{-1}(t)\right)}{f(F^{-1}(t))}\left[2\mathbb{G}_j(t) - \mathbb{G}_{j-1}(t) - \mathbb{G}_{j+1}(t)\right]dt$$

and

$$Z_J = \frac{2}{J-1}\int_0^1 \frac{\partial\varphi^{-1}_{\theta^\star_J}\left(F_J^{-1}(t)\right)}{f(F^{-1}(t))}\left[\mathbb{G}_J(t) - \mathbb{G}_{J-1}(t)\right]dt,$$

where $\mathbb{G}_j$ *are independent standard Brownian bridges, for* $1 \leqslant j \leqslant J$.

If $d \geqslant 2$, we obtain the same kind of theorem but Φ becomes tridiagonal by blocks and we have to take into account in the proof that $\partial\varphi^{-1}_{\theta^\star_j}(x)$ is a vector in $\mathbb{R}^d$.

The following proposition presents two cases of invertibility for the matrix Φ if $d=1$.

Proposition 4.2. *Assume that for all* j, $\partial\varphi^{-1}_{\theta^\star_j}\left(F_j^{-1}(t)\right)$ *is not almost everywhere equal to 0. Then the matrix* Φ *defined above is invertible if one of the two following conditions is verified.*

1. *The quantity* $\int_0^1 \partial\varphi^{-1}_{\theta^\star_j}\left(F_j^{-1}(t)\right)^2 dt$ *is independent of* j *and we have* $\partial\varphi^{-1}_{\theta^\star_j}\left(F_j^{-1}(t)\right) \neq \gamma\partial\varphi^{-1}_{\theta^\star_{j+1}}\left(F_{j+1}^{-1}(t)\right)$ *for all* $2 \leqslant j \leqslant J-1$ *and for all* $\gamma \in \mathbb{R}$. *In this case,* Φ *is a diagonal dominant matrix .*

2. *For all* $2 \leqslant j \leqslant J-1$ *there exists* γ_j *such that* $\partial\varphi^{-1}_{\theta^\star_j}\left(F_j^{-1}(t)\right) = \gamma_j\partial\varphi^{-1}_{\theta^\star_{j+1}}\left(F_{j+1}^{-1}(t)\right)$.

Note that $\partial\varphi^{-1}_{\theta^\star_j}\left(F_j^{-1}(t)\right) = \partial\varphi^{-1}_{\theta^\star_j}\left(\varphi_{\theta^\star_j}\left(F^{-1}(t)\right)\right)$, so the second assumption may hold for a large variety of cases.

5 Applications

Now we provide some examples of admissible deformations, which undergo previous set of assumptions.

5.1 Example 1 : Location/scale model

$$\varphi_\lambda(x) = \lambda_2 x + \lambda_1$$

This choice of deformation corresponds to observations

$$X_{ij} = \mu_j^\star + \sigma_j^\star \varepsilon_{ij} \quad 1 \leqslant i \leqslant n \quad 1 \leqslant j \leqslant J$$

where ε_{ij} are random independent variables with mean 0 drawn from an unknown distribution μ. It corresponds to an ANOVA model with different variances.
Here $\lambda = (\lambda_1, \lambda_2) \in \Lambda \subset \mathbb{R}^2$. The deformation function φ_λ is invertible on $\mathbb{R}$ if $\lambda_2 \neq 0$. φ_λ is non decreasing if $\lambda_2 > 0$, then we must choose Λ as a compact convex subset of $\mathbb{R} \times (0; +\infty)$.

We have $\varphi_\lambda^{-1}(x) = \frac{x-\lambda_1}{\lambda_2} = \varphi_{(\frac{-\lambda_1}{\lambda_2}, \frac{1}{\lambda_2})}(x)$, and $\varphi_\lambda^{-1}(\varphi_\beta(x)) = \frac{x\beta_2+\beta_1-\lambda_1}{\lambda_2}$ which is in $L^2(\mu)$ if $\mu \in \mathcal{W}_2(\mathbb{R})$.

Moreover $\partial\varphi_\lambda^{-1}(x) = \left(\frac{-1}{\lambda_2}, \frac{\lambda_1-x}{\lambda_2^2}\right)$ and $\left\|\partial\varphi_\lambda^{-1}(x)\right\| = \sqrt{\left(\frac{-1}{\lambda_2}\right)^2 + \left(\frac{\lambda_1-x}{\lambda_2^2}\right)^2}$. Hence $\sup_{\lambda\in\Lambda} \left\|\partial\varphi_\lambda^{-1}(\cdot)\right\| \in L^2(\mu_j)$ if $\mu \in \mathcal{W}_2(\mathbb{R})$.

In conclusion, assumptions **A0** to **A5** are verified as soon as Λ is a compact convex of $\mathbb{R} \times (0; +\infty)$ and $\mu \in \mathcal{W}_2(\mathbb{R})$.

After some computations it can be shown that the matrix Φ defined in Section 4 is also invertible.

In the particular case of a translation model, $\lambda_2 = 0$, i.e $\varphi_\lambda(x) = x+\lambda$, the assumptions are easily tractable.

φ_λ is invertible on $\mathbb{R}$ for all $\lambda \in \mathbb{R}$ with $\varphi_\lambda^{-1}(x) = x - \lambda = \varphi_{-\lambda}(x)$. Assumptions **A1**, **A2** and **A4** are easily checked.

Moreover we have $\partial\varphi_\lambda^{-1}(x) = -1$, $\varphi_\lambda^{-1}(\varphi_\beta(x)) = x + \beta - \lambda$. Hence **A0**, **A3** and **A5** are verified if μ is in $\mathcal{W}_2(\mathbb{R})$, and $\Lambda \subset \mathbb{R}$ compact and convex. Finally the second condition of invertibility for the matrix Φ in Proposition 4.2 is trivially satisfied.

The case of the scale model, that is $\varphi_\lambda(x) = \lambda x$, can be considered with same reasoning. In this case, assumptions **A0** to **A5** are verified if $\mu \in \mathcal{W}_2(\mathbb{R})$, and Λ is a compact interval included in $(0; +\infty)$.

5.2 Example 2: Logarithmic transform.

$$\varphi_\lambda(x) = \lambda \log(x)$$

φ_λ is invertible from $(0; +\infty)$ to $\mathbb{R}$ for all $\lambda \neq 0$, and φ_λ is non decreasing if λ is positive: here Λ must be contained in $(0; +\infty)$ and ε take its values in $(0; +\infty)$. We have $\varphi_\lambda^{-1}(x) = \exp\left(\frac{x}{\lambda}\right)$, and $\varphi_\lambda^{-1}(\varphi_\beta(x)) = \exp\left(\frac{\beta\log(x)}{\lambda}\right) = x^{\frac{\beta}{\lambda}}$. Hence $\varphi_\lambda^{-1} \in L^2(\mu_j)$ if $\mathbb{E}\left[\varepsilon^{\frac{2\theta_j^\star}{\lambda}}\right] < \infty$ for all $\lambda \in \Lambda$.

Moreover $\partial\varphi_\lambda^{-1}(x) = \frac{-x}{\lambda^2}\exp\left(\frac{x}{\lambda}\right)$, so $\partial\varphi_\lambda^{-1}(\varphi_\beta(x)) = \frac{-\beta}{\lambda^2} x^{\frac{\beta}{\lambda}}\log(x)$, and $\sup_{\lambda\in\Lambda}\left|\partial\varphi_\lambda^{-1}(\cdot)\right| \in L^2(\mu_j)$ if $\mathbb{E}\left[\varepsilon^{\frac{2\theta_j^\star}{\lambda_{min}}}\log^2(\varepsilon)\right] < \infty$ and $\mathbb{E}\left[\varepsilon^{\frac{2\theta_j^\star}{\lambda_{Max}}}\log^2(\varepsilon)\right] < \infty$ where $\lambda_{Max} = \max\{\lambda\in\Lambda\}$ and $\lambda_{min} = \min\{\lambda\in\Lambda\}$. In this case the conditions are more restrictive on the law μ. Remark that the exponential distribution verifies these conditions.

Finally, as previously this example verifies the second condition of invertibility for the matrix Φ stated in Proposition 4.2.

5.3 Example 3 : Composition

$$\varphi_\lambda(x) = f\circ\tilde{\varphi}_\lambda(x)$$

Consider a function $\tilde{\varphi}_\lambda(x)$ which verifies all the assumptions **A1** to **A6**. Then, if f is an increasing function invertible from I_b to I_c, the deformation function $\varphi_\lambda(x) = f\circ\tilde{\varphi}_\lambda(x)$ verifies also these assumptions replacing I_b by I_c. Indeed, assumptions **A1**, **A2** and **A4** about invertibility and differentiability are immediately verified, and we have

$$\varphi_\lambda^{-1}\circ\varphi_\beta = \tilde{\varphi}_\lambda^{-1}\circ f^{-1}\circ f\circ\tilde{\varphi}_\beta = \tilde{\varphi}_\lambda^{-1}\circ\tilde{\varphi}_\beta$$

and

$$\partial\varphi_\lambda^{-1} = \partial\left(\tilde{\varphi}_\lambda^{-1}\circ f^{-1}\right) = \partial\tilde{\varphi}_\lambda^{-1}\circ f^{-1}.$$

So

$$\partial\varphi_\lambda^{-1}\circ\varphi_\beta = \partial\tilde{\varphi}_\lambda^{-1}\circ f^{-1}\circ f\circ\tilde{\varphi}_\beta = \partial\tilde{\varphi}_\lambda^{-1}\circ\tilde{\varphi}_\beta.$$

Hence assumptions of integrability (**A3**, **A5**) and identifiability (**A6**) are also verified for the function $\varphi_\lambda(x)$. Moreover, the matrix Φ which is involved in the convergence in distribution is exactly the same in the two cases.

The composition action allows a large number of new admissible deformations. For instance, the logit model $\varphi_\lambda(x) = \frac{1}{1+\exp(-\lambda x)}$ can be obtained by the composition of the scale model with the function $f(x) = \frac{1}{1+\exp(-x)}$.

The study of the example 2 gives also the conditions under which the deformation $\varphi_\lambda(x) = x^\lambda$ can be handled by our method.

6 Simulations

In this section we present some simulations obtained for several distinct deformation functions and template measures.

More precisely, we simulate $\left(\varphi_{\theta_j^\star}(\varepsilon_{ij})\right)_{1\leqslant i\leqslant n, 1\leqslant j\leqslant J}$ for different choices of functions φ, parameters $\theta_j^\star$ and law μ of the ε_{ij}. For each combination of theses parameters we fix $J = 6$, we simulate samples of size $n = 1000$ for each deformation j, and we compute the estimator $\widehat{\theta}^n$ by fixing $\widehat{\theta}_1^n = \theta_1^\star$. We compute the error $\theta_j^\star - \widehat{\theta}_j^n$ for $2\leqslant j\leqslant 6$, and we repeat 100 times this operation. In the following we present the repartition of the errors. Each box-plot corresponds to the distribution of the estimation error of one $\theta_j^\star$.

For each deformation function, we consider the three following structural laws : the Uniform law on the interval $[-1;1]$, the Binomial law of size 20 and parameter $1/3$ and the standard Gaussian distribution.

Here we consider the shift deformations $\varphi_{\theta_j^\star}(x) = x + \theta_j^\star$, and we aim at recover the parameter $\theta^\star = (10, 0.2, 3, -9, 2, -5)$.

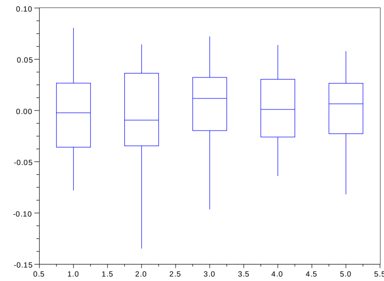

Figure 1: Uniform law and shift deformation

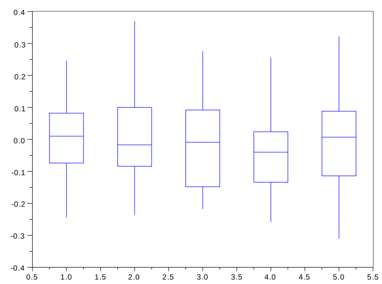

Figure 2: Binomial law and shift deformation

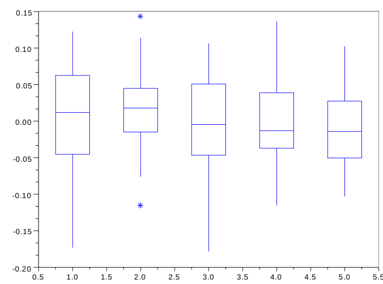

Figure 3: Gaussian law and shift deformation

In the following we first consider the logit deformation $\varphi_{\theta_j^\star}(x) = \frac{1}{1+\exp\left(\theta_j^\star x\right)}$, and after the scale deformation $\varphi_{\theta_j^\star}(x) = \theta_j^\star x$. The parameter of deformation is in each case : $\theta^\star = (1, 5, 3.2, 4.5, 1.5, 2)$.

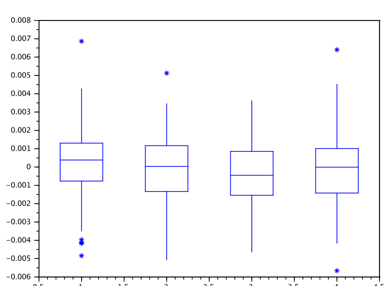

Figure 4: Uniform law and logit deformation

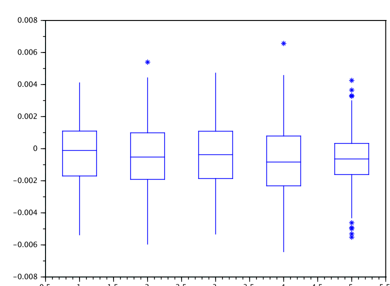

Figure 5: Binomial law and logit deformation

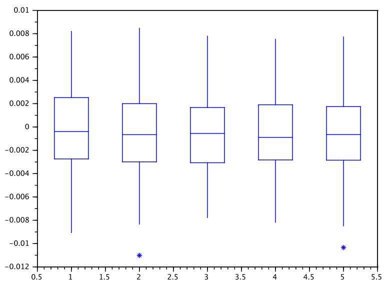

Figure 6: Gaussian law and logit deformation

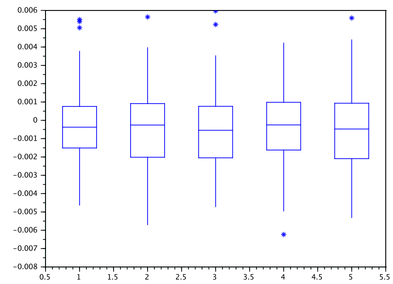

Figure 7: Binomial law and scale deformation

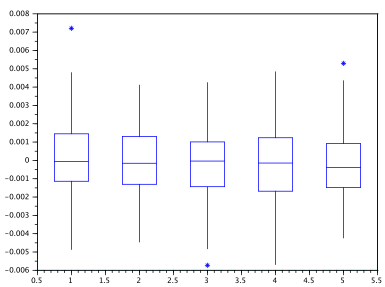

Figure 8: Uniform law and scale deformation

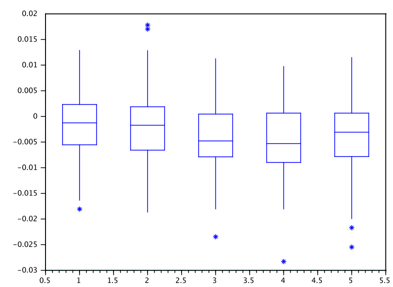

Figure 9: Gaussian law and scale deformation

As expected, the estimation procedure provides a good reconstruction of the parameters of the deformations.

A Appendix section

A.1 Proof of Theorem 3.1

We start by proving the uniqueness of the minimum of the criterion $M(\theta)$.

STEP 0 : Identifiability

We have already remarked that $M(\theta^\star) = 0 = \min_{\theta\in\Theta} M(\theta)$.

Set $\theta \in \Theta$. We have $M(\theta) = 0$ if and only if for all $2 \leqslant j \leqslant J$, $W_2^2\left(\mu_j(\theta), \mu_{j-1}(\theta)\right) = 0$, that is

$$\varphi_{\theta_j}^{-1} \circ \varphi_{\theta_j^\star} = \varphi_{\theta_{j-1}}^{-1} \circ \varphi_{\theta_{j-1}^\star} \quad \mu \text{ a.s.}$$

Hence, if we assume **A6**, that is for all $\theta \neq \theta^\star$ in Θ, there exists $1 \leqslant k < j \leqslant J$, and a set A such that $\mu(A) > 0$ and $\varphi_{\theta_j}^{-1} \circ \varphi_{\theta_j^\star} \neq \varphi_{\theta_k}^{-1} \circ \varphi_{\theta_k^\star}$ on A, then M admits an unique minimum on Θ in $\theta^\star$.

Indeed, if **A6** holds, for $\theta \neq \theta^\star$ we necessarily have an index j and a set A such as $\mu(A) > 0$ and $\varphi_{\theta_j}^{-1} \circ \varphi_{\theta_j^\star} \neq \varphi_{\theta_{j-1}}^{-1} \circ \varphi_{\theta_{j-1}^\star}$ on A . Hence in this case, $M(\theta) \neq 0$.

Now we aim to show that the empirical criterion M_n converges uniformly to M in probability. The proof follows three steps, beginning with the study of the pointwise convergence.

STEP 1

For all θ in Θ,

$$|M_n(\theta) - M(\theta)| \xrightarrow{n\to\infty} 0 \text{ in probability }.$$

Proof

Using the triangular inequality we can write

$$\begin{aligned} W_2\left(\mu_j^n(\theta), \mu_{j-1}^n(\theta)\right) \leqslant W_2\left(\mu_j^n(\theta), \mu_j(\theta)\right) &+ W_2\left(\mu_j(\theta), \mu_{j-1}(\theta)\right) \\ &+ W_2\left(\mu_{j-1}(\theta), \mu_{j-1}^n(\theta)\right) \end{aligned}$$

and

$$\begin{aligned} W_2\left(\mu_j(\theta), \mu_{j-1}(\theta)\right) \leqslant W_2\left(\mu_j(\theta), \mu_j^n(\theta)\right) &+ W_2\left(\mu_j^n(\theta), \mu_{j-1}^n(\theta)\right) \\ &+ W_2\left(\mu_{j-1}^n(\theta), \mu_{j-1}(\theta)\right). \end{aligned}$$

Hence

$$\begin{aligned} W_2\left(\mu_j(\theta), \mu_{j-1}(\theta)\right) - W_2\left(\mu_j^n(\theta), \mu_j(\theta)\right) - W_2\left(\mu_{j-1}(\theta), \mu_{j-1}^n(\theta)\right) \\ \leqslant W_2\left(\mu_j^n(\theta), \mu_{j-1}^n(\theta)\right) \\ \leqslant W_2\left(\mu_j^n(\theta), \mu_j(\theta)\right) + W_2\left(\mu_j(\theta), \mu_{j-1}(\theta)\right) + W_2\left(\mu_{j-1}(\theta), \mu_{j-1}^n(\theta)\right). \end{aligned}$$

Here we use the following result about the convergence in the Wasserstein sense of the empirical measures which is stated in [21] p. 63.

If P_n is the empirical law of an i.i.d. sample $Y_1, \ldots, Y_n$ with law $P \in \mathcal{W}_2(\mathbb{R})$, then

$$W_2(P_n, P) \xrightarrow{n\to\infty} 0 \text{ a.s.}$$

So we deduce that for all j and θ fixed, $W_2\left(\mu_j^n(\theta), \mu_{j-1}^n(\theta)\right)$ converges a.s. to $W_2\left(\mu_j(\theta), \mu_{j-1}(\theta)\right)$ when n goes to infinity.

Hence we conclude that for all θ

$$|M_n(\theta) - M(\theta)| \xrightarrow{n\to\infty} 0 \text{ a.s.}$$

and consequently the convergence in probability holds, implied by the a.s. convergence.

STEP 2

For all $\varepsilon > 0$

$$\limsup_{n\to\infty} \mathbb{P}\left(\sup_{\|\theta^1-\theta^2\|\leqslant\nu} \left|M_n\left(\theta^1\right) - M_n\left(\theta^2\right)\right| > \varepsilon\right) \xrightarrow{\nu\to 0} 0$$

Proof

Recall that

$$M_n(\theta) = \frac{1}{J-1}\sum_{j=2}^{J}\frac{1}{n}\sum_{i=1}^{n}\left[Z_{(i)j}(\theta) - Z_{(i)j-1}(\theta)\right]^2.$$

For all $2 \leqslant j \leqslant J$, define

$$M_n^j(\theta) = \frac{1}{n}\sum_{i=1}^{n}\left[Z_{(i)j}(\theta) - Z_{(i)j-1}(\theta)\right]^2.$$

For θ^1 and θ^2 in Θ, we have

$$\left|M_n\left(\theta^1\right) - M_n\left(\theta^2\right)\right| \leqslant \frac{1}{J-1}\sum_{j=2}^{J}\left|M_n^j\left(\theta^1\right) - M_n^j\left(\theta^2\right)\right|$$

and

$$\left|M_n^j\left(\theta^1\right) - M_n^j\left(\theta^2\right)\right| \leqslant \frac{1}{n}\sum_{i=1}^{n}\left|\left[Z_{(i)j}\left(\theta^1\right) - Z_{(i)j-1}\left(\theta^1\right)\right]^2 - \left[Z_{(i)j}\left(\theta^2\right) - Z_{(i)j-1}\left(\theta^2\right)\right]^2\right|.$$

It can be bounded using the equality $a^2 - b^2 = (a-b)(a+b)$ by

$$\left|M_n^j\left(\theta^1\right) - M_n^j\left(\theta^2\right)\right| \leqslant \frac{1}{n}\sum_{i=1}^{n}B_{ij}\left(\theta^1,\theta^2\right)\left[A_{ij}\left(\theta^1,\theta^2\right) + A_{ij-1}\left(\theta^1,\theta^2\right)\right],$$

where we have set

$$A_{ij}\left(\theta^1,\theta^2\right) = \left|Z_{(i)j}\left(\theta^1\right) - Z_{(i)j}\left(\theta^2\right)\right|$$

and

$$B_{ij}\left(\theta^1,\theta^2\right) = \left|Z_{(i)j}\left(\theta^1\right) - Z_{(i)j-1}\left(\theta^1\right) + Z_{(i)j}\left(\theta^2\right) - Z_{(i)j-1}\left(\theta^2\right)\right|.$$

By Cauchy-Schwarz's inequality

$$\left|M_n^j\left(\theta^1\right) - M_n^j\left(\theta^2\right)\right| \leqslant \sqrt{\frac{1}{n}\sum_{i=1}^{n}B_{ij}\left(\theta^1,\theta^2\right)^2}\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left[A_{ij}\left(\theta^1,\theta^2\right) + A_{ij-1}\left(\theta^1,\theta^2\right)\right]^2},$$

and using the triangular inequality we obtain

$$\left|M_n^j\left(\theta^1\right) - M_n^j\left(\theta^2\right)\right| \leqslant \sqrt{\frac{1}{n}\sum_{i=1}^{n}B_{ij}\left(\theta^1,\theta^2\right)^2}\left[\sqrt{\frac{1}{n}\sum_{i=1}^{n}A_{ij}\left(\theta^1,\theta^2\right)^2} + \sqrt{\sum_{i=1}^{n}A_{ij-1}\left(\theta^1,\theta^2\right)^2}\right].$$

We first consider

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}B_{ij}\left(\theta^1,\theta^2\right)^2}.$$

With the same arguments as before and using **A2**

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n} B_{ij}\left(\theta^1,\theta^2\right)^2} \leqslant \sqrt{\frac{1}{n}\sum_{i=1}^{n} \varphi_{\theta_j^1}^{-1}\left(X_{(i)j}\right)^2} + \sqrt{\frac{1}{n}\sum_{i=1}^{n} \varphi_{\theta_{j-1}^1}^{-1}\left(X_{(i)j-1}\right)^2}$$
$$+ \sqrt{\frac{1}{n}\sum_{i=1}^{n} \varphi_{\theta_j^2}^{-1}\left(X_{(i)j}\right)^2} + \sqrt{\frac{1}{n}\sum_{i=1}^{n} \varphi_{\theta_{j-1}^2}^{-1}\left(X_{(i)j-1}\right)^2}.$$

Hence

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n} B_{ij}\left(\theta^1,\theta^2\right)^2} \leqslant \sqrt{\frac{1}{n}\sum_{i=1}^{n} \varphi_{\theta_j^1}^{-1}\left(X_{ij}\right)^2} + \sqrt{\frac{1}{n}\sum_{i=1}^{n} \varphi_{\theta_{j-1}^1}^{-1}\left(X_{ij-1}\right)^2}$$
$$+ \sqrt{\frac{1}{n}\sum_{i=1}^{n} \varphi_{\theta_j^2}^{-1}\left(X_{ij}\right)^2} + \sqrt{\frac{1}{n}\sum_{i=1}^{n} \varphi_{\theta_{j-1}^2}^{-1}\left(X_{ij-1}\right)^2}.$$

So

$$\sup_{\|\theta^1-\theta^2\|\leqslant\nu} \sqrt{\frac{1}{n}\sum_{i=1}^{n} B_{ij}\left(\theta^1,\theta^2\right)^2} \leqslant 2\sup_{\lambda\in\Lambda} \sqrt{\frac{1}{n}\sum_{i=1}^{n} \varphi_{\lambda}^{-1}\left(X_{ij}\right)^2}$$
$$+2\sup_{\lambda\in\Lambda} \sqrt{\frac{1}{n}\sum_{i=1}^{n} \varphi_{\lambda}^{-1}\left(X_{ij-1}\right)^2}.$$

Now we will show that for all j

$$\sup_{\|\theta^1-\theta^2\|\leqslant\nu} \sqrt{\frac{1}{n}\sum_{i=1}^{n} B_{ij}\left(\theta^1,\theta^2\right)^2} = O_{\mathbb{P}}^{j}(1).$$

For this we use the following lemma taken from Example 3.7.3 p.38 in [23] and recalled for sake of completeness.

Lemma A.1. *Consider* $(X_i)_{i\geqslant 1}$ *i.i.d. variables defined on a space* $\mathcal{X}$ *with law* P, Γ *a compact set in* $\mathbb{R}^d$ *and* $h:\Gamma\times\mathcal{X}\mapsto\mathbb{R}$. *Assume that* $h(\cdot,x)$ *is continuous on* Γ *for* P*-almost* $x\in\mathcal{X}$ *and* $\sup_{\gamma\in\Gamma}|h(\gamma,\cdot)|\in L^1(P)$.

Then

$$\sup_{\gamma\in\Gamma}\left|\frac{1}{n}\sum_{i=1}^{n} h\left(\gamma,X_i\right) - \mathbb{E}\left[h\left(\gamma,X_1\right)\right]\right| \xrightarrow{n\to\infty} 0 \ a.s.$$

In particular $\sup_{\gamma\in\Gamma}\left|\frac{1}{n}\sum_{i=1}^{n} h\left(\gamma,X_i\right)\right| = O_{\mathbb{P}}(1)$.

Using assumption **A4**, for λ^1, $\lambda^2\in\Lambda$ we can write

$$\varphi_{\lambda^1}^{-1}(x) - \varphi_{\lambda^2}^{-1}(x) = \partial\varphi_{\lambda^{1,2}}^{-1}(x)\left(\lambda^1-\lambda^2\right)$$

for $\lambda^{1,2}$ on the segment between λ^1 and λ^2. Then

$$\left|\varphi_{\lambda^1}^{-1}(x) - \varphi_{\lambda^2}^{-1}(x)\right| \leqslant \sup_{\lambda\in\Lambda}\left\|\partial\varphi_{\lambda}^{-1}(x)\right\| \left\|\lambda^1 - \lambda^2\right\|.$$

So for all $\lambda \in \Lambda$, using **A5**

$$\left|\varphi_{\lambda}^{-1}(x)\right| \leqslant H(x)\Delta + \left|\varphi_{\lambda^0}^{-1}(x)\right|$$

where $\lambda^0 \in \Lambda$ and Δ is the diameter of Λ. Hence **A3** implies that for all j

$$\sup_{\lambda\in\Lambda}\left|\varphi_{\lambda}^{-1}(\cdot)\right|^2 \in L^1(\mu_j) \tag{11}$$

and so we can apply Lemma A.1 to $\varphi_{\lambda}^{-1}(x)^2$ and we obtain

$$\sup_{\theta^1,\theta^2\in\Theta^2}\sqrt{\frac{1}{n}\sum_{i=1}^{n} B_{ij}\left(\theta^1,\theta^2\right)^2} = O_{\mathbb{P}}^{j}(1).$$

Now we focus on $\sqrt{\frac{1}{n}\sum_{i=1}^{n} A_{ij}\left(\theta^1,\theta^2\right)^2}$.

Using again assumption **A2**, we can write

$$\begin{aligned}A_{ij}\left(\theta^1,\theta^2\right) &= \left|Z_{(i)j}\left(\theta^1\right) - Z_{(i)j}\left(\theta^2\right)\right| \\ &= \left|\varphi_{\theta_j^1}^{-1}\left(X_{(i)j}\right) - \varphi_{\theta_j^2}^{-1}\left(X_{(i)j}\right)\right|.\end{aligned}$$

Hence

$$\sum_{i=1}^{n} A_{ij}\left(\theta^1,\theta^2\right)^2 = \sum_{i=1}^{n}\left|\varphi_{\theta_j^1}^{-1}\left(X_{ij}\right) - \varphi_{\theta_j^2}^{-1}\left(X_{ij}\right)\right|^2.$$

Now using again a Taylor-Lagrange expansion

$$\begin{aligned}\left|\varphi_{\theta_j^1}^{-1}\left(X_{ij}\right) - \varphi_{\theta_j^2}^{-1}\left(X_{ij}\right)\right| &= \left|\partial\varphi_{\tilde{\theta}_{ij}^{1,2}}^{-1}\left(X_{ij}\right)\left(\theta_j^1 - \theta_j^2\right)\right| \\ &\leqslant \sup_{\lambda\in\Lambda}\left\|\partial\varphi_{\lambda}^{-1}\left(X_{ij}\right)\right\| \left\|\theta_j^1 - \theta_j^2\right\|\end{aligned}$$

so

$$\sup_{\|\theta^1-\theta^2\|\leqslant\nu}\frac{1}{n}\sum_{i=1}^{n} A_{ij}\left(\theta^1,\theta^2\right)^2 \leqslant \frac{1}{n}\sum_{i=1}^{n}\sup_{\lambda\in\Lambda}\left\|\partial\varphi_{\lambda}^{-1}\left(X_{ij}\right)\right\|^2 \nu^2.$$

But under assumption **A4** we can apply the Law of Large Numbers to get that $\frac{1}{n}\sum_{i=1}^{n}\sup_{\lambda\in\Lambda}\left\|\partial\varphi_{\lambda}^{-1}\left(X_{ij}\right)\right\|^2$ converges in probability, and so

$$\frac{1}{n}\sum_{i=1}^{n}\sup_{\lambda\in\Lambda}\left\|\partial\varphi_{\lambda}^{-1}\left(X_{ij}\right)\right\|^2 = O_{\mathbb{P}}^{j}(1).$$

In conclusion

$$\sup_{\|\theta^1-\theta^2\|\leqslant\nu}\left|M_n\left(\theta^1\right) - M_n\left(\theta^2\right)\right| \leqslant V_n\nu^2$$

where $V_n = O_{\mathbb{P}}(1)$ is independent of ν and we obtain

$$\limsup_{n\to\infty} \mathbb{P}\left(\sup_{\|\theta^1-\theta^2\|\leqslant\nu} \left|M_n\left(\theta^1\right) - M_n\left(\theta^2\right)\right| > \varepsilon\right) \xrightarrow{\nu\to 0} 0.$$

STEP 3

The function $\theta \mapsto M(\theta)$ is continuous on Θ.

Proof

Let $(\theta^n)_{n\in\mathbb{N}}$ be a sequence of Θ such that $\theta^n \xrightarrow{n\to\infty} \theta^0$. We will show that $M(\theta^n) \xrightarrow{n\to\infty} M(\theta^0)$ by proving the convergence $W_2^2(\mu_j(\theta^n), \mu_j(\theta^0)) \xrightarrow{n\to\infty} 0$. For this we will use the following equivalence.

If $(P_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{W}_2(\mathbb{R})$ and $P \in \mathcal{W}_2(\mathbb{R})$, then

$$W_2(P_n, P) \xrightarrow{n\to\infty} 0$$

if and only if

$$P_n \rightharpoonup P \text{ and } \mathbb{E}\left[X_n^2\right] \to \mathbb{E}\left[X^2\right]$$

where X_n follows the law P_n and X the law P.

This characterization of the convergence in the Wasserstein's sense is proved for instance in [21].

We first show that $\mathbb{E}\left[Z_{1j}^2(\theta^n)\right] \xrightarrow{n\to\infty} \mathbb{E}\left[Z_{1j}^2(\theta^0)\right]$. Thanks to (11) , we have for all $\theta \in \Theta$,

$$|Z_{1j}(\theta)| = \left|\varphi_{\theta_j}^{-1}(X_{1j})\right| \leqslant \tilde{H}(X_{1j})$$

with $\tilde{H}(X_{1j}) \in L^2$.

Moreover using the regularity of φ^{-1} with respect to the deformation parameter we have the a.s. convergence

$$Z_{1j}^2(\theta^n) = \varphi_{\theta_j^n}^{-1}(X_{1j})^2 \xrightarrow{n\to\infty} \varphi_{\theta_j^0}^{-1}(X_{1j})^2 = Z_{1j}^2\left(\theta^0\right).$$

Hence we obtain $\mathbb{E}\left[Z_{1j}^2(\theta^n)\right] \xrightarrow{n\to\infty} \mathbb{E}\left[Z_{1j}^2(\theta^0)\right]$.

In addition, we proved the a.s. convergence of $Z_{1j}^2(\theta^n)$ to $Z_{1j}^2(\theta^0)$, which implies the weak convergence $\mu_j(\theta^n) \rightharpoonup \mu_j(\theta^0)$.

From this we deduce that $W_2^2(\mu_j(\theta^n), \mu(\theta^0)) \xrightarrow{n\to\infty} 0$ and consequently $M(\theta^n) \xrightarrow{n\to\infty} M(\theta^0)$ if $\theta^n \xrightarrow{n\to\infty} \theta^0$: M is continuous on Θ.

CONSEQUENCE

If Θ is compact, then

$$\sup_{\theta\in\Theta} |M_n(\theta) - M(\theta)| \xrightarrow{n\to\infty} 0 \text{ in probability.}$$

Proof

Set ε and δ two real positive numbers. Thanks to the steps 2 and 3, we can choose ν_0 such that

$$\limsup_{n\to\infty} \mathbb{P}\left(\sup_{\|\theta^1-\theta^2\|\leqslant\nu_0} \left|M_n\left(\theta^1\right) - M_n\left(\theta^2\right)\right| > \varepsilon\right) \leqslant \delta$$

and

$$\sup_{\|\theta^1-\theta^2\|\leqslant\nu_0} \left|M\left(\theta^1\right) - M\left(\theta^2\right)\right| \leqslant \varepsilon.$$

With the compactness of Θ, we can find a sequence $\left(\theta^k\right)_{1\leqslant k\leqslant m}$ in Θ such that $\Theta \subset \cup_{k=1}^m B\left(\theta^k, \nu_0\right)$. Now for $\theta \in \Theta \cap B\left(\theta^p, \nu_0\right)$

$$|M_n(\theta) - M(\theta)| \leqslant |M_n(\theta) - M_n(\theta^p)| + |M_n(\theta^p) - M(\theta^p)| + |M(\theta^p) - M(\theta)|$$

$$\sup_{\theta\in\Theta} |M_n(\theta) - M(\theta)| \leqslant \sup_{\|\theta^1-\theta^2\|\leqslant\nu_0} \left|M_n\left(\theta^1\right) - M_n\left(\theta^2\right)\right| + \max_{1\leqslant k\leqslant m} \left|M_n\left(\theta^k\right) - M\left(\theta^k\right)\right| + \sup_{\|\theta^1-\theta^2\|\leqslant\nu_0} \left|M\left(\theta^1\right) - M\left(\theta^2\right)\right|$$

Hence

$$\left(\sup_{\theta\in\Theta} |M_n(\theta) - M(\theta)| > 3\varepsilon\right) \subset \left(\sup_{\|\theta^1-\theta^2\|\leqslant\nu_0} \left|M_n\left(\theta^1\right) - M_n\left(\theta^2\right)\right| > \varepsilon\right) \cup \left(\max_{1\leqslant k\leqslant m} \left|M_n\left(\theta^k\right) - M\left(\theta^k\right)\right| > \varepsilon\right).$$

So

$$\mathbb{P}\left(\sup_{\theta\in\Theta} |M_n(\theta) - M(\theta)| > 3\varepsilon\right) \leqslant \mathbb{P}\left(\sup_{\|\theta^1-\theta^2|\leqslant\nu_0} \left|M_n\left(\theta^1\right) - M_n\left(\theta^2\right)\right| > \varepsilon\right) + \sum_{k=1}^m \mathbb{P}\left(\left|M_n\left(\theta^k\right) - M\left(\theta^k\right)\right| > \varepsilon\right)$$

And with the step 1, we deduce that for all δ and $\varepsilon > 0$

$$\limsup_{n\to\infty} \mathbb{P}\left(\sup_{\theta\in\Theta} |M_n(\theta) - M(\theta)| > 3\varepsilon\right) \leqslant \delta.$$

Hence,

$$\sup_{\theta\in\Theta} |M_n(\theta) - M(\theta)| \xrightarrow{n\to\infty} 0 \text{ in probability.}$$

Finally we complete the proof as follows.

Using the result of identifiability together with the continuity of M and the compactness of Θ, we deduce that for all $\varepsilon > 0$

$$\inf_{\Theta\cap B(\theta^\star,\varepsilon)^c} M > 0.$$

Following the M-estimation theorem of [24] (th 5.7 p.45), this result combining with the uniform convergence in probability of M_n to M leads to the consistency of the estimator.

A.2 Proof of Proposition 7

We denote by μ_j^n the empirical law of the sample $(\varepsilon_{1j}, \dots \varepsilon_{nj})$. Then we can write

$$W_2\left(\widehat{\mu}_j^n, \mu\right) \leqslant W_2\left(\widehat{\mu}_j^n, \mu_j^n\right) + W_2\left(\mu_j^n, \mu\right)$$

First, the convergence of the empirical measures in the Wasserstein sense used in the step 1 implies the a.s. convergence of $W_2\left(\mu_j^n, \mu\right)$ to 0 when n tends to infinity. Indeed, assumption **A3** implies that $\mu \in \mathcal{W}_2(\mathbb{R})$.

Second, φ_λ is non decreasing for all λ, so we have

$$\begin{aligned} W_2^2\left(\widehat{\mu}_j^n, \mu_j^n\right) &= \frac{1}{n}\sum_{i=1}^n \left(\varphi_{\widehat{\theta}_j^n}^{-1}\left(\varphi_{\theta_j^\star}\left(\varepsilon_{(i)j}\right)\right) - \varepsilon_{(i)j}\right)^2 \\ &= \frac{1}{n}\sum_{i=1}^n \left(\varphi_{\widehat{\theta}_j^n}^{-1}\left(\varphi_{\theta_j^\star}\left(\varepsilon_{ij}\right)\right) - \varepsilon_{ij}\right)^2, \end{aligned}$$

and with a Taylor expansion of $\theta \mapsto \varphi_\theta^{-1}(X_{ij})$ between $\widehat{\theta}_j^n$ and $\theta_j^\star$, we obtain

$$\varphi_{\widehat{\theta}_j^n}^{-1}\left(\varphi_{\theta_j^\star}\left(\varepsilon_{ij}\right)\right) = \varepsilon_{ij} + \partial\varphi_{\widetilde{\theta}_i^{n,j}}^{-1}(X_{ij})\left(\widehat{\theta}_j^n - \theta_j^\star\right)$$

for $\widetilde{\theta}_i^{n,j}$ in the segment between $\widehat{\theta}_j^n$ and $\theta_j^\star$. So

$$W_2^2\left(\widehat{\mu}_j^n, \mu_j^n\right) \leqslant \left[\frac{1}{n}\sum_{i=1}^n \sup_{\lambda\in\Lambda}\left\|\partial\varphi_\lambda^{-1}(X_{ij})\right\|^2\right]\left\|\widehat{\theta}_j^n - \theta_j^\star\right\|^2.$$

But we showed in the step 2 that $\left[\frac{1}{n}\sum_{i=1}^n \sup_{\lambda\in\Lambda}\left\|\partial\varphi_\lambda^{-1}(X_{ij})\right\|^2\right] = O_{\mathbb{P}}(1)$, and the consistency of the estimator $\widehat{\theta}^n$ implies that $\left\|\widehat{\theta}_j^n - \theta_j^\star\right\| \xrightarrow{n\to\infty} 0$ in probability. Hence we deduce that $W_2^2\left(\widehat{\mu}_j^n, \mu_j^n\right) \xrightarrow{n\to\infty} 0$ in probability.

In conclusion,

$$W_2\left(\widehat{\mu}_j^n, \mu\right) \xrightarrow{n\to\infty} 0 \text{ in probability.}$$

To obtain Theorem 3.2, recall that $\widehat{\mu}^n = \frac{1}{J}\sum_{j=1}^J \widehat{\mu}_j^n$. The convergence $W_2\left(\widehat{\mu}^n, \mu\right) \xrightarrow{n\to\infty}$ 0 in probability is simply obtained by using again the characterization of the convergence in the Wasserstein's sense stated in the step 3.

A.3 Proof of Theorem 4.1

Here we introduce new notations.

We note $\mathbb{D}]\alpha;\beta]$ the set of distribution functions of measures that concentrate on $]\alpha;\beta]$ and $\mathbb{S}$ the Skorohod space, that is the space of cadlag functions on $\bar{\mathbb{R}}$ endowed with the supremum norm $\| \quad \|_\infty$. Recall that the cadlag functions are defined as the right continuous functions which admit a limit from the left.

$\ell_\infty(0;1)$ is the set of functions bounded on $(0;1)$, and for $I = I_b$ or $I = [\alpha;\beta]$, $\ell_\infty((0;1);I)$ is the set of functions bounded on $(0;1)$ with values in I. $\ell_{\infty,m}(0;1)$ denotes the set of bounded and measurable functions on $(0;1)$. Recall that $[\alpha;\beta] \subset I_b$.

On the spaces $\ell_\infty^J(0;1)$ and $\ell_{\infty,m}^J(0;1)$ we consider the norm $\|h\|_{\infty,J} = \max\left(\|h_1\|_\infty, \ldots, \|h_J\|_\infty\right)$ for $h = (h_1, \ldots, h_J)$. Finally we denote by Q_j^n the empirical quantile function $(F_j^n)^{-1}$.
We start by the computation of the first and second differentials of M_n.

Differentiability of $\mathbf{M_n}$

Recall that we consider $\theta = (\theta_2, \ldots, \theta_J) \in \Theta_{J-1}$, so in the following we set $\theta_1 = \theta_1^\star$. We have

$$M_n(\theta) = \frac{1}{J-1} \sum_{j=2}^{J} \frac{1}{n} \sum_{i=1}^{n} \left[\varphi_{\theta_j}^{-1}\left(X_{(i)j}\right) - \varphi_{\theta_{j-1}}^{-1}\left(X_{(i)j-1}\right)\right]^2.$$

Hence M_n is C^2 on Θ_{J-1} under **AL1**, and for all $2 \leqslant j \leqslant J-1$

$$\partial_j M_n(\theta) = \frac{2}{J-1} \frac{1}{n} \sum_{i=1}^{n} \partial\varphi_{\theta_j}^{-1}\left(X_{(i)j}\right) \left[2\varphi_{\theta_j}^{-1}\left(X_{(i)j}\right) - \varphi_{\theta_{j-1}}^{-1}\left(X_{(i)j-1}\right) - \varphi_{\theta_{j+1}}^{-1}\left(X_{(i)j+1}\right)\right]$$

and

$$\partial_J M_n(\theta) = \frac{1}{J-1} \frac{2}{n} \sum_{i=1}^{n} \partial\varphi_{\theta_J}^{-1}\left(X_{(i)J}\right) \left[\varphi_{\theta_J}^{-1}\left(X_{(i)J}\right) - \varphi_{\theta_{J-1}}^{-1}\left(X_{(i)J-1}\right)\right].$$

We can also write, for all $2 \leqslant j \leqslant J-1$

$$\partial_j M_n(\theta) = \frac{2}{J-1} \int_0^1 \partial\varphi_{\theta_j}^{-1}\left(Q_j^n(t)\right) \left[2\varphi_{\theta_j}^{-1}\left(Q_j^n(t)\right) - \varphi_{\theta_{j-1}}^{-1}\left(Q_{j-1}^n(t)\right) - \varphi_{\theta_{j+1}}^{-1}\left(Q_{j+1}^n(t)\right)\right] dt \qquad (12)$$

$$\partial_J M_n(\theta) = \frac{2}{J-1} \int_0^1 \partial\varphi_{\theta_J}^{-1}\left(Q_J^n(t)\right) \left[\varphi_{\theta_J}^{-1}\left(Q_J^n(t)\right) - \varphi_{\theta_{J-1}}^{-1}\left(Q_{J-1}^n(t)\right)\right] dt. \qquad (13)$$

So M_n is C^1 on Θ_{J-1} with differential $DM_n(\theta) = (\partial_2 M_n(\theta), \ldots, \partial_J M_n(\theta))$.
Moreover for $2 \leqslant j \leqslant J-1$

$$\partial_{j,j}^2 M_n(\theta) = \frac{1}{J-1} \frac{1}{n} \sum_{i=1}^{n} 2\partial\varphi_{\theta_j}^{-1}\left(X_{(i)j}\right) \left[2\partial\varphi_{\theta_j}^{-1}\left(X_{(i)j}\right)\right]$$

$$+ \frac{1}{J-1} \frac{1}{n} \sum_{i=1}^{n} 2\partial^2\varphi_{\theta_j}^{-1}\left(X_{(i)j}\right) \left[2\varphi_{\theta_j}^{-1}\left(X_{(i)j}\right) - \varphi_{\theta_{j-1}}^{-1}\left(X_{(i)j-1}\right) - \varphi_{\theta_{j+1}}^{-1}\left(X_{(i)j+1}\right)\right],$$

for $3 \leqslant j \leqslant J$

$$\partial_{j-1,j}^2 M_n(\theta) = \frac{1}{J-1} \frac{1}{n} \sum_{i=1}^{n} 2\partial\varphi_{\theta_j}^{-1}\left(X_{(i)j}\right) \left[-\partial\varphi_{\theta_{j-1}}^{-1}\left(X_{(i)j-1}\right)\right]$$

and for $2 \leqslant j \leqslant J-1$

$$\partial_{j+1,j}^2 M_n(\theta) = \frac{1}{J-1} \frac{1}{n} \sum_{i=1}^{n} 2\partial\varphi_{\theta_j}^{-1}\left(X_{(i)j}\right) \left[-\partial\varphi_{\theta_{j+1}}^{-1}\left(X_{(i)j+1}\right)\right]$$

finally if $k \neq j, j-1, j+1$

$$\partial^2_{k,j} M_n(\theta) = 0.$$

In addition

$$\partial^2_{J,J} M_n(\theta) = \frac{1}{J-1}\frac{1}{n}\sum_{i=1}^{n} 2\partial\varphi^{-1}_{\theta_J}\left(X_{(i)J}\right)\left[\partial\varphi^{-1}_{\theta_J}\left(X_{(i)J}\right)\right]$$

$$+\frac{1}{J-1}\frac{1}{n}\sum_{i=1}^{n} 2\partial^2\varphi^{-1}_{\theta_J}\left(X_{(i)J}\right)\left[\varphi^{-1}_{\theta_J}\left(X_{(i)J}\right) - \varphi^{-1}_{\theta_{J-1}}\left(X_{(i)J-1}\right)\right].$$

So for all $3 \leqslant j \leqslant J-1$
$D\partial_j M_n(\theta) = \left(0, \ldots, 0, \partial^2_{j-1,j} M_n(\theta), \partial^2_{j,j} M_n(\theta), \partial^2_{j+1,j} M_n(\theta), 0, \ldots, 0\right)$ with

$$\partial^2_{j,j} M_n(\theta) = \frac{4}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta_j}\left(Q^n_j(t)\right)^2 dt + \tag{14}$$

$$\frac{2}{J-1}\int_0^1 \partial^2\varphi^{-1}_{\theta_j}\left(Q^n_j(t)\right)\left[2\varphi^{-1}_{\theta_j}\left(Q^n_j(t)\right) - \varphi^{-1}_{\theta_{j-1}}\left(Q^n_{j-1}(t)\right) - \varphi^{-1}_{\theta_{j+1}}\left(Q^n_{j+1}(t)\right)\right] dt$$

and

$$\partial^2_{j-1,j} M_n(\theta) = \partial^2_{j,j-1} M_n(\theta) = \frac{-2}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta_j}\left(Q^n_j(t)\right)\partial\varphi^{-1}_{\theta_{j-1}}\left(Q^n_{j-1}(t)\right) dt. \tag{15}$$

Moreover

$$D\partial_2 M_n(\theta) = \tag{16}$$

$$\Bigg(\frac{2}{J-1}\int_0^1 2\partial\varphi^{-1}_{\theta_2}\left(Q^n_2(t)\right)^2 + \partial^2\varphi^{-1}_{\theta_2}\left(Q^n_2(t)\right)\left[2\varphi^{-1}_{\theta_2}\left(Q^n_2(t)\right) - \varphi^{-1}_{\theta_1^\star}\left(Q^n_1(t)\right) - \varphi^{-1}_{\theta_3}\left(Q^n_3(t)\right)\right] dt,$$

$$\frac{-2}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta_2}\left(Q^n_2(t)\right)\partial\varphi^{-1}_{\theta_3}\left(Q^n_3(t)\right) dt, 0, \ldots, 0\Bigg)$$

$$D\partial_J M_n(\theta) = \Bigg(0, \ldots, 0, \frac{-2}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta_J}\left(Q^n_J(t)\right)\partial\varphi^{-1}_{\theta_{J-1}}\left(Q^n_{J-1}(t)\right) dt, \tag{17}$$

$$\frac{2}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta_J}\left(Q^n_J(t)\right)^2 + \partial^2\varphi^{-1}_{\theta_J}\left(Q^n_J(t)\right)\left[\varphi^{-1}_{\theta_J}\left(Q^n_J(t)\right) - \varphi^{-1}_{\theta_{J-1}}\left(Q^n_{J-1}(t)\right)\right] dt\Bigg).$$

Recall that $\partial_j M_n(\theta) \in \mathbb{R}$. Hence, as explain in Section 4 the regularity of M_n allows a Taylor expansion

$$\partial_j M_n\left(\widehat{\theta}^n\right) = \partial_j M_n(\theta^\star) + D\partial_j M_n\left(\theta^{n,j}\right)\left(\widehat{\theta}^n - \theta^\star\right)$$

for $\theta^{n,j}$ between $\widehat{\theta}^n$ and $\theta^\star$. Using that M_n admits a minimum on $\widehat{\theta}^n$ we have

$$-\partial_j M_n(\theta^\star) = D\partial_j M_n\left(\theta^{n,j}\right)\left(\widehat{\theta}^n - \theta^\star\right),$$

hence

$$-DM_n(\theta^\star) = \begin{pmatrix} D\partial_2 M_n(\theta^{n,2}) \\ \cdot \\ \cdot \\ D\partial_J M_n(\theta^{n,J}) \end{pmatrix} \left(\widehat{\theta}^n - \theta^\star\right).$$

We set $DM_n(\theta^\star) = \Psi(F_1^n, \ldots, F_J^n)$, $D\partial_j M_n(\theta^{n,j}) = \Phi_j(F_1^n, \ldots, F_J^n, \theta^{n,j})$.

The aim of the following is to show that Ψ is Hadamard differentiable in order to apply a Delta method to get $\sqrt{n}(-DM_n(\theta^\star)) \rightharpoonup Z \in \mathbb{R}^J$ for some random vector Z.

Convergence in law of $\mathbf{DM_n}(\theta^\star)$

We have $\Psi = \chi \circ \Psi_0$ where

$$\Psi_0(F_1, \ldots F_J) = \left(F_1^{-1}, \ldots, F_J^{-1}\right)$$

is defined on $\mathbb{D}]\alpha;\beta]^J$ with values in $\ell^J_{\infty,m}((0;1),[\alpha;\beta])$.

χ is defined from $\ell^J_{\infty,m}((0;1),[\alpha;\beta])$ to $\mathbb{R}^{J-1}$ with $\chi(g) = (\chi_2(g), \ldots, \chi_J(g))$, where $g = (g_1, \ldots, g_J)$ and for $2 \leqslant j \leqslant J-1$

$$\chi_j(g_1, \ldots, g_J) = \\ \frac{2}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta_j^\star}(g_j(t)) \left[2\varphi^{-1}_{\theta_j^\star}(g_j(t)) - \varphi^{-1}_{\theta_{j-1}^\star}(g_{j-1}(t)) - \varphi^{-1}_{\theta_{j+1}^\star}(g_{j+1}(t))\right] dt$$

and

$$\chi_J(g_1, \ldots, g_J) = \frac{2}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta_J^\star}(g_J(t)) \left[\varphi^{-1}_{\theta_J^\star}(g_J(t)) - \varphi^{-1}_{\theta_{J-1}^\star}(g_{J-1}(t))\right] dt.$$

Now consider the following lemma.

Lemma A.2. *Let $G : I_b^J \to \mathbb{R}$ a continuous function. Then, if $[\alpha;\beta] \subset I_b$,*

$$\tilde{G} : \left(\ell^J_{\infty,m}((0;1);[\alpha;\beta]), \|\quad\|_{\infty,J}\right) \to \mathbb{R}$$

$$(g_1, \ldots, g_J) \mapsto \int_0^1 G(g_1(u), \ldots, g_J(u))\, du$$

is continuous. If G is continuously differentiable, then $\tilde{G}$ is Hadamard differentiable tangentially to $\ell^J_{\infty,m}((0;1))$ with

$$D\tilde{G}(g_1, \ldots, g_J)[h_1, \ldots, h_J] = \int_0^1 DG(g_1(u), \ldots, g_J(u))[h_1(u), \ldots, h_J(u)]\, du.$$

Using **AL1**, we apply this lemma to

$$G_j(x_1, \ldots, x_J) = \partial\varphi^{-1}_{\theta_j^\star}(x_j)\left[2\varphi^{-1}_{\theta_j^\star}(x_j) - \varphi^{-1}_{\theta_{j-1}^\star}(x_{j-1}) - \varphi^{-1}_{\theta_{j+1}^\star}(x_{j+1})\right]$$

for $2 \leqslant j \leqslant J-1$, and

$$G_J(x_1, \dots, x_J) = \partial\varphi^{-1}_{\theta^\star_J}(x_J)\left[\varphi^{-1}_{\theta^\star_J}(x_J) - \varphi^{-1}_{\theta^\star_{J-1}}(x_{J-1})\right]$$

so we deduce that χ is Hadamard differentiable tangentially to $\ell^J_{\infty,m}(0;1)$. Moreover, for $[k_1, \dots, k_J] \in \ell^J_{\infty,m}(0;1)$

$$\begin{aligned}
&D\chi_J(g_1, \dots, g_J)[k_1, \dots, k_J] = \\
&\quad \frac{2}{J-1}\int_0^1 d\partial\varphi^{-1}_{\theta^\star_J}(g_J(t))[k_J(t)]\left[\varphi^{-1}_{\theta^\star_J}(g_J(t)) - \varphi^{-1}_{\theta^\star_{J-1}}(g_{J-1}(t))\right] \\
&\quad +\partial\varphi^{-1}_{\theta^\star_J}(g_J(t))\left[d\varphi^{-1}_{\theta^\star_J}(g_J(t))[k_J(t)] - d\varphi^{-1}_{\theta^\star_{J-1}}(g_{J-1}(t))[k_{J-1}(t)]\right]dt
\end{aligned}$$

and for $2 \leqslant j \leqslant J-1$

$$\begin{aligned}
&D\chi_j(g_1, \dots, g_J)[k_1, \dots, k_J] = \\
&\frac{2}{J-1}\int_0^1 d\partial\varphi^{-1}_{\theta^\star_j}(g_j(t))[k_j(t)]\left[2\varphi^{-1}_{\theta^\star_j}(g_j(t)) - \varphi^{-1}_{\theta^\star_{j-1}}(g_{j-1}(t)) - \varphi^{-1}_{\theta^\star_{j+1}}(g_{j+1}(t))\right] \\
&\quad +\partial\varphi^{-1}_{\theta^\star_j}(g_1(t))\left[2d\varphi^{-1}_{\theta^\star_j}(g_j(t))[k_j(t)]\right. \\
&\quad\quad \left. -d\varphi^{-1}_{\theta^\star_{j-1}}(g_{j-1}(t))[k_{j-1}(t)] - d\varphi^{-1}_{\theta^\star_{j+1}}(g_{j+1}(t))[k_{j+1}(t)]\right]dt.
\end{aligned}$$

Under **AL2** and **AL3** we can apply Theorem A.3 in Section A.6 which ensures that Ψ_0 is Hadamard differentiable at $(F_1, \dots, F_J)$ tangentially to $C^J[\alpha;\beta]$, with

$$D\Psi_0(F_1, \dots, F_J)[h_1, \dots, h_J] = -\left(\frac{h_1 \circ F_1^{-1}}{f_1 \circ F_1^{-1}}, \dots, \frac{h_J \circ F_J^{-1}}{f_J \circ F_J^{-1}}\right)$$

for $(h_1, \dots, h_J) \in C^J[\alpha;\beta]$. Hence, with the regularity of the functions F_j, we obtain that $D\Psi_0(F_1, \dots, F_J)\left(C^J[\alpha;\beta]\right) \subset \ell^J_{\infty,m}(0;1)$. Thus we can apply the chain rule to the composed function $\Psi = \chi \circ \Psi_0$ to get that Ψ is Hadamard differentiable at $(F_1, \dots, F_J)$ tangentially to $C^J[\alpha;\beta]$ with

$$D\Psi(F_1, \dots, F_J)[h] = D\chi(\Psi_0(F_1, \dots, F_J))[D\Psi_0(F_1, \dots, F_J)[h]]$$

for $h = (h_1, \dots, h_J) \in C^J[\alpha;\beta]$.

Under **A2**, we have $F_j = F \circ \varphi^{-1}_{\theta^\star_j}$ for all j. Hence, $F_j^{-1} = \left(F \circ \varphi^{-1}_{\theta^\star_j}\right)^{-1} = \varphi_{\theta^\star_j} \circ F^{-1}$ and we obtain $\varphi^{-1}_{\theta^\star_j}\left(F_j^{-1}(t)\right) = F^{-1}(t)$ for all j. This leads to

$$\begin{aligned}
&D\Psi_J(F_1, \dots, F_J)[h_1, \dots, h_J] = \\
&\frac{2}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta^\star_J}\left(F_J^{-1}(t)\right)\left[d\varphi^{-1}_{\theta^\star_J}\left(F_J^{-1}(t)\right)\left[\frac{-h_J\left(F_J^{-1}(t)\right)}{f_J\left(F_J^{-1}(t)\right)}\right]\right. \\
&\quad\quad \left. -d\varphi^{-1}_{\theta^\star_{J-1}}\left(F_{J-1}^{-1}(t)\right)\left[\frac{-h_{J-1}\left(F_{J-1}^{-1}(t)\right)}{f_{J-1}\left(F_{J-1}^{-1}(t)\right)}\right]\right]dt,
\end{aligned}$$

$$D\Psi_j(F_1,\dots,F_J)[h_1,\dots,h_J] =$$
$$\frac{2}{J-1}\int_0^1 \partial\varphi^{-1}_{\theta_j^\star}\left(F_j^{-1}(t)\right)\left[2d\varphi^{-1}_{\theta_j^\star}\left(F_j^{-1}(t)\right)\left[\frac{-h_j\left(F_j^{-1}(t)\right)}{f_j\left(F_j^{-1}(t)\right)}\right]\right.$$
$$\left.-d\varphi^{-1}_{\theta_{j-1}^\star}\left(F_{j-1}^{-1}(t)\right)\left[\frac{-h_{j-1}\left(F_{j-1}^{-1}(t)\right)}{f_{j-1}\left(F_{j-1}^{-1}(t)\right)}\right]-d\varphi^{-1}_{\theta_{j+1}^\star}\left(F_{j+1}^{-1}(t)\right)\left[\frac{-h_{j+1}\left(F_{j+1}^{-1}(t)\right)}{f_{j+1}\left(F_{j+1}^{-1}(t)\right)}\right]\right]dt.$$

Moreover, if we differentiate the equality $F_j(x) = F\circ\varphi^{-1}_{\theta_j^\star}(x)$ we obtain that $f_j(x) = d\varphi^{-1}_{\theta_j^\star}(x)\, f\circ\varphi^{-1}_{\theta_j^\star}(x)$.

Hence $f_j(F_j^{-1}(t)) = d\varphi^{-1}_{\theta_j^\star}(F_j^{-1}(t))f\circ\varphi^{-1}_{\theta_j^\star}\left(F_j^{-1}(t)\right) = d\varphi^{-1}_{\theta_j^\star}(F_j^{-1}(t))f\circ F^{-1}(t)$, and we can simplify

$$D\Psi_J(F_1,\dots,F_J)[h_1,\dots,h_J] =$$
$$\frac{2}{J-1}\int_0^1 \frac{\partial\varphi^{-1}_{\theta_J^\star}\left(F_J^{-1}(t)\right)}{f(F^{-1}(t))}\left[h_{J-1}\left(F_{J-1}^{-1}(t)\right)-h_J\left(F_J^{-1}(t)\right)\right]dt$$

$$D\Psi_j(F_1,\dots,F_J)[h_1,\dots,h_J] =$$
$$\frac{2}{J-1}\int_0^1 \frac{\partial\varphi^{-1}_{\theta_j^\star}\left(F_j^{-1}(t)\right)}{f(F^{-1}(t))}\left[h_{j-1}\left(F_{j-1}^{-1}(t)\right)+h_{j+1}\left(F_{j+1}^{-1}(t)\right)-2h_j\left(F_j^{-1}(t)\right)\right]dt$$

With the independence of the different samples and the convergence in law of the empirical distribution functions which is stated in Theorem A.5 in Section A.6, we know that

$$\sqrt{n}\begin{pmatrix} F_1^n - F_1 \\ \cdot \\ \cdot \\ F_J^n - F_J\end{pmatrix} \rightharpoonup \begin{pmatrix} \mathbb{G}_1\circ F_1 \\ \cdot \\ \cdot \\ \mathbb{G}_J\circ F_J\end{pmatrix}$$

in the product space $(\mathbb{S}^J, \|\quad\|_{\infty,J})$ where $(\mathbb{G}_j)_{j=1}^J$ are independent standard Brownian bridges .

Hence we can apply Theorem A.4, the functional Delta method which is stated in section A.6 with the following correspondences : A is the Skohorod space, $A_\phi = \mathbb{D}]\alpha;\beta]^J$, $A_0 = C^J[\alpha;\beta]$ (we have $(\mathbb{G}_1\circ F_1\cdot,\cdot,\mathbb{G}_J\circ F_J)\in A_0$). Hence, computing $\Psi(F_1,\dots,F_J)=0$ we obtain

$$\sqrt{n}(-DM_n(\theta^\star)) \rightharpoonup -D\Psi(F_1,\dots,F_J)[\mathbb{G}_1\circ F_1,\dots,\mathbb{G}_J\circ F_J]$$

in $\mathbb{R}^{J-1}$.

Next we will show that $\Phi_j(F_1^n,\dots,F_J^n,\theta^{n,j}) \to \Phi_j(F_1,\dots,F_J,\theta^\star)$ in probability.

Convergence of $\mathbf{D^2M_n}$.

We can write $\Phi_j(F_1^n,\ldots,F_J^n,\theta^{n,j}) = \phi_j(\Psi_0(F_1^n,\ldots,F_J^n),\theta^{n,j})$ where for all $2 \leqslant j \leqslant J-1$

$\phi_j(g_1,\ldots,g_J,\theta) = \left(0,\ldots,0,\phi_j^{j-1},\phi_j^j,\phi_j^{j+1},0\ldots,0\right)$ with

$$\phi_j^j(g_1,\ldots,g_J,\theta) = \frac{4}{J-1}\int_0^1 \partial\varphi_{\theta_j}^{-1}(g_j(t))^2\,dt$$
$$+\frac{2}{J-1}\int_0^1 \partial^2\varphi_{\theta_j}^{-1}(g_j(t))\left[2\varphi_{\theta_j}^{-1}(g_j(t)) - \varphi_{\theta_{j-1}}^{-1}(g_{j-1}(t)) - \varphi_{\theta_{j+1}}^{-1}(g_{j+1}(t))\right]dt,$$

for $2 \leqslant j \leqslant J-1$

$$\begin{aligned}\phi_j^{j+1}(g_1,\ldots,g_J,\theta) =& \phi_{j+1}^j(g_1,\ldots,g_J,\theta)\\ =& \frac{-2}{J-1}\int_0^1 \partial\varphi_{\theta_j}^{-1}(g_j(t))\,\partial\varphi_{\theta_{j+1}}^{-1}(g_{j+1}(t))\,dt,\end{aligned}$$

and

$$\phi_J(g_1,\ldots,g_J,\theta)$$
$$= (0,\ldots,0\ ,\frac{-2}{J-1}\int_0^1 \partial\varphi_{\theta_J}^{-1}(g_J(t))\,\partial\varphi_{\theta_{J-1}}^{-1}(g_{J-1}(t))\,dt,$$
$$\left.\frac{2}{J-1}\int_0^1 \partial\varphi_{\theta_J}^{-1}(g_J(t))^2 + \partial^2\varphi_{\theta_J}^{-1}(g_J(t))\left[\varphi_{\theta_J}^{-1}(g_J(t)) - \varphi_{\theta_{J-1}}^{-1}(g_{J-1}(t))\right]dt\right).$$

Using **AL1** and a slight modification of Lemma A.2, we get that the functions ϕ_j are continuous on $\left(\ell_{\infty,m}^J((0;1);[\alpha;\beta]) \times \mathbb{R}^{J-1}, \max\left(\|\quad\|_{\infty,J}, \|\quad\|\right)\right)$. Moreover, for all j,

$$\theta^{n,j} \xrightarrow{n\to\infty} \theta^\star \text{ in probability}$$

and

$$\Psi_0(F_1^n,\ldots,F_J^n) \xrightarrow{n\to\infty} \Psi_0(F_1,\ldots,F_J) = \left(F_1^{-1},\ldots,F_J^{-1}\right) \text{ in probability}$$

in the space $\left(\ell_{\infty,m}^J((0;1);[\alpha;\beta]), \|\quad\|_{\infty,J}\right)$. Hence for all j,

$$\Phi_j\left(F_1^n,\ldots,F_J^n,\theta^{n,j}\right) \xrightarrow{n\to\infty} \Phi_j(F_1,\ldots,F_J,\theta^\star) \text{ in probability,}$$

with

$$\Phi_j(F_1,\ldots,F_J,\theta^\star) = \left(0,\ldots,0,\Phi_j^{j-1},\Phi_j^j,\Phi_j^{j+1},0\ldots,0\right)$$

where for $2 \leqslant j \leqslant J-1$

$$\Phi_j^j(F_1,\ldots,F_J,\theta^\star) = \frac{4}{J-1}\int_0^1 \partial\varphi_{\theta_j^\star}^{-1}\left(F_j^{-1}(t)\right)^2 dt$$
$$+\frac{2}{J-1}\int_0^1 \partial^2\varphi_{\theta_j^\star}^{-1}\left(F_j^{-1}(t)\right)\left[2\varphi_{\theta_j^\star}^{-1}\left(F_j^{-1}(t)\right)\right.$$
$$\left.-\varphi_{\theta_{j-1}^\star}^{-1}\left(F_{j-1}^{-1}(t)\right) - \varphi_{\theta_{j+1}^\star}^{-1}\left(F_{j+1}^{-1}(t)\right)\right]dt$$

$$= \frac{4}{J-1} \int_0^1 \partial\varphi_{\theta_j^\star}^{-1} \left(F_j^{-1}(t)\right)^2 dt \tag{18}$$

$$\begin{aligned} \Phi_{j+1}^j \left(F_1, \dots, F_J, \theta^\star\right) &= \Phi_j^{j+1} \left(F_1, \dots, F_J, \theta^\star\right) \\ &= \frac{-2}{J-1} \int_0^1 \partial\varphi_{\theta_j^\star}^{-1} \left(F_j^{-1}(t)\right) \partial\varphi_{\theta_{j+1}^\star}^{-1} \left(F_{j+1}^{-1}(t)\right) dt, \end{aligned} \tag{19}$$

and

$$\Phi_J \left(F_1, \dots, F_J, \theta^\star\right) = \Bigg(0, \dots, 0, \frac{-2}{J-1} \int_0^1 \partial\varphi_{\theta_J^\star}^{-1} \left(F_J^{-1}(t)\right) \partial\varphi_{\theta_{J-1}^\star}^{-1} \left(F_{J-1}^{-1}(t)\right) dt,$$

$$\frac{2}{J-1} \int_0^1 \partial\varphi_{\theta_J^\star}^{-1} \left(F_J^{-1}(t)\right)^2 + \partial^2\varphi_{\theta_J^\star}^{-1} \left(F_J^{-1}(t)\right) \left[\varphi_{\theta_J^\star}^{-1} \left(F_J^{-1}(t)\right) - \varphi_{\theta_{J-1}^\star}^{-1} \left(F_{J-1}^{-1}(t)\right)\right] dt\Bigg),$$

that is

$$\begin{aligned} \Phi_J \left(F_1, \dots, F_J, \theta^\star\right) = \Bigg(0, \dots, 0, \frac{-2}{J-1} \int_0^1 \partial\varphi_{\theta_J^\star}^{-1} \left(F_J^{-1}(t)\right) \partial\varphi_{\theta_{J-1}^\star}^{-1} \left(F_{J-1}^{-1}(t)\right) dt, \\ \frac{2}{J-1} \int_0^1 \partial\varphi_{\theta_J^\star}^{-1} \left(F_J^{-1}(t)\right)^2 dt\Bigg). \end{aligned} \tag{20}$$

So

$$\begin{pmatrix} D\partial_2 M_n \left(\theta^{n,2}\right) \\ \cdot \\ \cdot \\ D\partial_J M_n \left(\theta^{n,J}\right) \end{pmatrix} \xrightarrow{n\to\infty} \begin{pmatrix} \Phi_2 \left(F_1, \dots, F_J, \theta^\star\right) \\ \cdot \\ \cdot \\ \Phi_J \left(F_1, \dots, F_J, \theta^\star\right) \end{pmatrix} = \Phi \text{ in probability.}$$

Now set $\Sigma_n = \begin{pmatrix} D\partial_2 M_n \left(\theta^{n,2}\right) \\ \cdot \\ \cdot \\ D\partial_J M_n \left(\theta^{n,J}\right) \end{pmatrix}$. As its limit Φ is invertible and the set of invertible matrices $GL_{J-1}(\mathbb{R})$ is an open set, $\mathbb{P}\left(\Sigma_n \in GL_{J-1}(\mathbb{R})\right) \xrightarrow{n\to\infty} 1$. Moreover we have

$$\begin{aligned} \sqrt{n}\left(\widehat{\theta}_n - \theta^\star\right) =& \Sigma_n^{-1} \Sigma_n \sqrt{n}\left(\widehat{\theta}_n - \theta^\star\right) \mathbf{1}_{\{\Sigma_n \in GL_{J-1}(\mathbb{R})\}} \\ &+ \sqrt{n}\left(\widehat{\theta}_n - \theta^\star\right) \mathbf{1}_{\{\Sigma_n \notin GL_{J-1}(\mathbb{R})\}}. \end{aligned}$$

Now for all ε

$$\begin{aligned} &\mathbb{P}\left(\sqrt{n}\left\|\widehat{\theta}_n - \theta^\star\right\| \mathbf{1}_{\{\Sigma_n \notin GL_{J-1}(\mathbb{R})\}} > \varepsilon\right) \\ &\qquad\qquad \leqslant \mathbb{P}\left(\mathbf{1}_{\{\Sigma_n \notin GL_{J-1}(\mathbb{R})\}} > \varepsilon\right) \xrightarrow{n\to\infty} 0. \end{aligned}$$

Hence $\sqrt{n}\left(\widehat{\theta}_n - \theta^\star\right) \mathbf{1}_{\{\Sigma_n \notin GL_{J-1}(\mathbb{R})\}}$ converges to 0 in probability when n tends to infinity.

Moreover $\Sigma_n^{-1} \mathbf{1}_{\{\Sigma_n \in GL_{J-1}(\mathbb{R})\}} \xrightarrow{n\to\infty} \Phi^{-1}$ in probability by continuity of the inverse map. As $\Sigma_n \sqrt{n}\left(\widehat{\theta}_n - \theta^\star\right) \rightharpoonup Z$, it remains to apply Slutsky's Lemma to conclude.

A.4 Proof of Proposition 4.2

Set $g_j = \partial\varphi_{\theta_j^\star}^{-1} \circ F_j^{-1}$. We denote by $\| \quad \|_2$ the norm on $L^2(0;1)$, and $\langle\cdot,\cdot\rangle_2$ the corresponding scalar product. Then we have

$$\Phi\frac{J-1}{2} =$$
$$\begin{pmatrix} 2\|g_2\|_2^2 & -\langle g_2, g_3\rangle_2 & 0 & \dots & 0 \\ -\langle g_2, g_3\rangle_2 & 2\|g_3\|_2^2 & -\langle g_3, g_4\rangle_2 & 0 & \dots \\ 0 & \cdot & \cdot & & \\ 0 & \dots & -\langle g_{J-2}, g_{J-1}\rangle_2 & 2\|g_{J-1}\|_2^2 & -\langle g_{J-1}, g_J\rangle_2 \\ 0 & \dots & 0 & -\langle g_{J-1}, g_J\rangle_2 & \|g_J\|_2^2 \end{pmatrix}$$

In the first case we obtain :

$$\Phi\frac{J-1}{2} =$$
$$\begin{pmatrix} 2\alpha & -\langle g_2, g_3\rangle_2 & 0 & \dots & 0 \\ -\langle g_2, g_3\rangle_2 & 2\alpha & -\langle g_3, g_4\rangle_2 & 0 & \dots \\ 0 & \cdot & \cdot & & \\ 0 & \dots & -\langle g_{J-2}, g_{J-1}\rangle_2 & 2\alpha & -\langle g_{J-1}, g_J\rangle_2 \\ 0 & \dots & 0 & -\langle g_{J-1}, g_J\rangle_2 & \alpha \end{pmatrix}$$

and for all $3 \leqslant j \leqslant J$, $\alpha > \left|\langle g_{j-1}, g_j\rangle_2\right|$, that is the case of strict inequality in the Cauchy-Schwartz's inequality. Hence we immediately deduce that Φ is a diagonal dominant matrix.

The second case is the case of equality in the Cauchy-Schwartz's inequality : for all $2 \leqslant j \leqslant J$, $\left|\langle g_{j-1}, g_j\rangle_2\right| = \|g_j\|_2 \|g_{j-1}\|_2$.

Here we will show that Ker $\Phi = \{0\}$. Set $u = (u_2, \dots u_J) \in$ Ker Φ: we have $J-1$ equations given by $\Phi u = 0$.

The first one gives : $u_3 = 2\frac{\|g_2\|_2}{\|g_3\|_2} u_2$.

If we assume $u_j = (j-1)\frac{\|g_2\|_2}{\|g_j\|_2} u_2$ for $2 \leqslant j \leqslant k$, for $3 \leqslant k \leqslant J-1$ the $(k-1)$th equation gives

$$-\|g_{k-1}\|_2 \|g_k\|_2 u_{k-1} + 2\|g_k\|_2^2 u_k - \|g_{k+1}\|_2 \|g_k\|_2 u_{k+1} = 0,$$

that is, by assumption

$$-(k-2)\|g_k\|_2 \|g_2\|_2 u_2 + 2(k-1)\|g_2\|_2 \|g_k\|_2 u_2 - \|g_{k+1}\|_2 \|g_k\|_2 u_{k+1} = 0$$

and so

$$u_{k+1} = k\frac{\|g_2\|_2}{\|g_{k+1}\|_2} u_2.$$

Hence , the $J-2$ first equations implies that $u_j = (j-1)\frac{\|g_2\|_2}{\|g_j\|_2} u_2$ for $2 \leqslant j \leqslant J$.

The last one imposes that

$$-\|g_{J-1}\|_2 \|g_J\|_2 u_{J-1} + \|g_J\|_2^2 u_J = 0$$

so

$$-(J-2)\|g_J\|_2 \|g_2\|_2 u_2 + (J-1)\|g_2\|_2 \|g_J\|_2 u_2 = 0$$

and necessarily $u_2 = 0$, so $u = 0$.

A.5 Proof of Lemma A.2

We first prove the continuity of $\tilde{G}$.

Choose $g = (g_1, \ldots, g_J) \in \ell^J_{\infty,m}((0;1), [\alpha;\beta])$. G is uniformly continuous on the compact $[\alpha;\beta]^J \subset I^J_b$. For all ε, set $\nu(\varepsilon)$ such that $|x-y|_\infty = max\left(|x_1 - y_1|, \ldots, |x_J - y_J|\right) \leqslant \nu(\varepsilon)$ implies $|G(x_1,\ldots,x_J) - G(y_1,\ldots,y_J)| \leqslant \varepsilon$ if $x, y \in [\alpha;\beta]^J$.

Set $h = (h_1, \ldots, h_J) \in \ell^J_{\infty,m}((0;1), [\alpha;\beta])$ such that $\|h - g\|_{\infty,J} \leqslant \nu(\varepsilon)$. Then

$$\left|\tilde{G}(h) - \tilde{G}(g)\right| \leqslant \int_0^1 |G(g(u)) - G(h(u))|\, du \leqslant \int_0^1 \varepsilon du = \varepsilon :$$

$\tilde{G}$ is continuous.

Now we consider the Hadamard differentiability.

Let $g = (g_1, \ldots, g_J) \in \ell^J_{\infty,m}((0;1), [\alpha;\beta])$, $h = (h_1, \ldots, h_J) \in \ell^J_{\infty,m}(0;1)$ and $h^t = (h^t_1, \ldots, h^t_J)$ such that $h^t \xrightarrow{t\to 0} h \in \ell^J_{\infty,m}((0;1))$ and $g + th^t \in \ell^J_{\infty,m}((0;1), [\alpha;\beta])$ for t sufficiently small. For v and w in $\mathbb{R}^J$, we denote by $[v;w]$ the segment between these two vectors, that is

$$[v;w] = \{sv + (1-s)w, s \in [0;1]\}.$$

Recall that we have set

$$D\tilde{G}(g)[h] = \int_0^1 DG(g_1(u), \ldots, g_J(u))[h_1(u), \ldots, h_J(u)]\, du.$$

First remark that $D\tilde{G}(g_1, \ldots, g_J)$ is well definite, linear and continuous on $\ell^J_{\infty,m}(0;1)$. Next, write

$$\left|\tilde{G}\left(g + th^t\right) - \tilde{G}(g) - t\int_0^1 DG(g(u))[h(u)]\, du\right|$$

$$\leqslant \int_0^1 \left|G\left((g(u)) + t\left(h^t(u)\right)\right) - G(g(u)) - tDG(g(u))\left[h^t(u)\right]\right| du$$
$$+ \int_0^1 \left|tDG(g(u))\left[h^t(u)\right] - tDG(g(u))[h(u)]\right| du$$

$$\leqslant \int_0^1 \sup_{k(u)\in[g(u);g(u)+th^t(u)]} \|DG(k(u)) - DG(g(u))\| \left\|t\left[h^t(u)\right]\right\| du$$
$$+ \int_0^1 \|DG(g(u))\| \left\|t\left[h^t(u)\right] - t[h(u)]\right\| du$$

with the Mean theorem applied to the function $F(x) = G(g(u) + tx) - tDG((g(u))\,x$ between $x = h^t(u)$ and $x = (0, \ldots, 0)$.

Hence for $t \neq 0$

$$\frac{1}{|t|}\left|\tilde{G}(g + th) - \tilde{G}(g) - t\int_0^1 DG(g_1(u), \ldots, g_J(u))[h_1(u), \ldots, h_J(u)]\, du\right|$$

$$\leqslant \int_0^1 \sup_{k(u)\in[g(u);g(u)+th^t(u)]} \|DG(k(u)) - DG(g(u))\|\, du \left\|h^t\right\|_{\infty,J}$$
$$+ \int_0^1 \|DG(g_1(u),\dots,g_J(u))\|\, du \left\|h - h^t\right\|_{\infty,J}.$$

But for all u, $th^t(u)$ tends to 0 while t tends to 0 , and by continuity of DG we deduce that

$$\sup_{k(u)\in[g(u);g(u)+th^t(u)]} \|DG(k(u)) - DG(g(u))\| \xrightarrow{t\to 0} 0$$

for all u.

Moreover $u \mapsto DG(g_1(u),\dots,g_J(u))$ is bounded thanks to the continuity of DG and the fact that $g \in \ell^J_{\infty,m}((0;1),[\alpha;\beta])$. Same arguments leads to the fact that $u \mapsto DG(k(u))$ is bounded for k between g and $g+th^t$ if t is sufficiently small. Hence we can apply the dominated convergence theorem to obtain that

$$\int_0^1 \sup_{k(u)\in[g(u);g(u)+th^t(u)]} \|DG(k(u)) - DG(g(u))\|\, du \xrightarrow{t\to 0} 0.$$

So with the convergence of h^t we conclude that

$$\frac{1}{|t|}\left|\tilde{G}\left(g+th^t\right) - \tilde{G}(g) - \int_0^1 tDG(g(u))[h(u)]\,du\right| \xrightarrow{t\to 0} 0$$

that is, $\tilde{G}$ is Hadamard differentiable tangentially to $\ell^J_{\infty,m}(0;1)$ with

$$D\tilde{G}(g_1,\dots,g_J)[h_1,\dots,h_J] = \int_0^1 DG(g_1(u),\dots,g_J(u))[h_1(u),\dots,h_J(u)]\,du.$$

A.6 Auxiliary theorems

The following theorems are taken from [24]. The first one is Lemma 21.4 p.307.

Theorem A.3. *Set*

$$\Psi_0(F_1,\dots F_J) = \left(F_1^{-1},\dots,F_J^{-1}\right)$$

defined on $\mathbb{D}]\alpha;\beta]^J$ *with values in* $\ell^J_\infty(0;1)$

Assume that for all j, F_j has a compact support $[\alpha;\beta]$ and is continuously differentiable on its support with strictly positive derivative f_j. Then Ψ_0 is Hadamard differentiable on $(F_1,\dots F_J)$ tangentially to $C[\alpha;\beta]^J$. The derivative is the map defined on $C[\alpha;\beta]^J$:

$$(h_1,\dots,h_J) \mapsto -\left(\frac{h_1\circ F_1^{-1}}{f_1\circ F_1^{-1}},\dots,\frac{h_J\circ F_J^{-1}}{f_J\circ F_J^{-1}}\right)$$

This one is the statement of the functional Delta method labelled as Theorem 20.8 p.297.

Theorem A.4. *Let A and B normed linear spaces, and $\phi : A_\phi \subset A \to B$ Hadamard differentiable at a tangentially to A_0. Let X_n random variables with values in A_ϕ such that $r_n(X_n - a) \rightharpoonup X$, where X takes its values in A_0 and $r_n \to \infty$.*

Then $r_n(\phi(X_n) - \phi(a)) \rightharpoonup D\phi(a)X$.

And finally Donsker's Theorem corresponds to Theorem 19.3 p.266.

Theorem A.5. *If $X_1, \dots, X_n$ are i.i.d. random variables with distribution function F and empirical distribution function F_n, the sequence $\sqrt{n}\,(F_n - F)$ converges in distribution in $(\mathbb{S}, \|\cdot\|_\infty)$ to $\mathbb{G} \circ F$ where $\mathbb{G}$ is a standard Brownian bridge.*

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