Forward equations for option prices in semimartingale models

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Abstract

We derive a forward partial integro-differential equation for prices of call options in a model where the dynamics of the underlying asset under the pricing measure is described by a -possibly discontinuous- semimartingale. This result generalizes Dupire's forward equation to a large class of non-Markovian models with jumps and allows to retrieve various forward equations previously obtained for option prices in a unified framework.

Contents

Since the seminal work of Black, Scholes and Merton [6, 27] partial differential equations (PDE) have been used as a way of characterizing and efficiently computing option prices. In the Black-Scholes-Merton model and various extensions of this model which retain the Markov property of the risk factors, option prices can be characterized in terms of solutions to a backward PDE, whose variables are time (to maturity) and the value of the underlying asset. The use of backward PDEs for option pricing has been extended to cover options with path-dependent and early exercise features, as well as to multifactor models (see e.g. [1]). When the underlying asset exhibit jumps, option prices can be computed by solving an analogous partial integro-differential equation (PIDE) [2, 13].

A second important step was taken by Dupire [15, 16, 18] (see also Derman & Kani [14]) who showed that when the underlying asset is assumed to follow a diffusion process

$$dS_t = \sigma(t, S_t) dW_t$$

prices of call options (at a given date t_0) solve a *forward* PDE in the strike and maturity variables:

$$\frac{\partial C_{t_0}}{\partial T} = -r(T) K \frac{\partial C_{t_0}}{\partial K} + \frac{K^2 \sigma(T,K)^2}{2} \frac{\partial^2 C_{t_0}}{\partial K^2}$$

on $[t_0, \infty[\times]0, \infty[$ with the initial condition: $\forall K > 0 \quad C_{t_0}(t_0, K) = (S_{t_0} - K)_+$. This forward equation allows to price call options with various strikes and maturities on the same underlying asset, by solving a *single* partial differential equation. Dupire's forward equation also provides useful insights into the *inverse problem* of calibrating diffusion models to observed call and put option prices [5].

Given the theoretical and computational usefulness of the forward equation, there have been various attempts to extend Dupire's forward equation to other types of options and processes, most notably to Markov processes with jumps [2, 9, 11, 24, 8]. Most of these constructions use the Markov property of the underlying process in a crucial way.

As already noted by Dupire [17], the forward PDE holds in a more general context than the backward PDE: even if the (risk-neutral) dynamics of the underlying asset is not necessarily Markovian, but described by a continuous Brownian martingale

$$dS_t = S_t \sigma_t dW_t$$

then call options still verify a forward PDE where the diffusion coefficient is given by the local (or effective) volatility function $\sigma(t, S)$ given by

$$\sigma(t, S) = \sqrt{E[\sigma_t^2 | S_t = S]}$$

This method, also known as "Markovian projection", is linked to the construction of a Markov process which mimicks the marginal distributions of a martingale [4, 21, 26]. Such "mimicking processes" provide a method to extend the Dupire equation to non-Markovian settings.

We show in this work that the forward equation for call prices holds in a more general setting, where the dynamics of the underlying asset is described by a -possibly discontinuous- semimartingale. We present two different derivations of this result. The first derivation (Section 1) is based on a generalization [4] of Gyöngy's [21] "mimicking theorem" to semimartingales. The second derivation (Section 1.3), based on the Tanaka-Meyer formula, has the merit of requiring weaker assumptions on the local characteristics of the process. Neither of the two methods requires ellipticity or non-degeneracy of the diffusion coefficient; the results are thus applicable to pure jump models and point process models used in equity and credit risk modeling.

Our result extends the forward equation from the original diffusion setting of Dupire [16] to various examples of non-Markovian and/or discontinuous processes and implies previous derivations of forward equations [2, **?**, 9, 8, 11, 16, 17, 24, 25] as special cases. Section 2 gives examples of forward PIDEs obtained in various settings: time-changed Lévy processes, local Lévy models and point processes used in portfolio default risk modeling. In the case where the underlying risk factor follows, an Ito process or a Markovian jump-diffusion driven by a Lévy process, we retrieve previously known forms of the forward equation. In this case, our approach gives a rigorous derivation of these results under precise assumptions in a unified framework. In some cases, such as index options (Sec. 2.5) or CDO expected tranche notionals (Sec. 2.6), our method leads to a new, more general form of the forward equation valid for a larger class of models than previously studied [3, 11, 30].

The forward equation for call options is a PIDE in one (spatial) dimension, regardless of the dimension of the factors driving the underlying asset. It may thus be used as a method for reducing the dimension of the problem. The case of index options (Section 2.5) in a multivariate jump-diffusion model illustrates how the forward equation projects a high dimensional pricing problem into a one-dimensional state equation.

1 Forward PIDEs for call options

1.1 General formulation of the forward equation

Consider a (strictly positive) price process S whose dynamics under the pricing measure $\mathbb{P}$ is given by a stochastic volatility model with jumps:

$$S_T = S_0 + \int_0^T r(t) S_{t^-}\, dt + \int_0^T S_{t^-}\, \delta_t dW_t + \int_0^T \int_{-\infty}^{+\infty} S_{t^-}(e^y - 1)\tilde{M}(dt\ dy) \quad (1)$$

where $r(t) > 0$ represents a (deterministic) bounded discount rate, δ_t the (random) volatility process and M is an integer-valued random measure with compensator $\mu(dt\, dy; \omega) = m(t, dy, \omega)\, dt$, representing jumps in the log-price, and $\tilde{M} = M - \mu$ is the compensated random measure associated to M. (see [12, 29] for definitions). Both the volatility δ_t and $m(t, dy)$, which represents the intensity of jumps of size y at time t are allowed to be stochastic. In particular,

we do *not* assume the jumps to be driven by a Lévy process or a process with independent increments.

We assume the following integrability condition:

Assumption 1.

$$\forall T > 0, \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \delta_t^2\, dt + \int_0^T dt \int_{\mathbb{R}} (e^y - 1)^2 m(t, dy)\right)\right] < \infty \tag{H}$$

The value $C_{t_0}(T, K)$ at time t_0 of a call option with expiry $T > t_0$ and strike $K > 0$ is given by

$$C_{t_0}(T, K) = e^{-\int_{t_0}^T r(t)\, dt} E^{\mathbb{P}}[\max(S_T - K, 0)|\mathcal{F}_{t_0}] \tag{2}$$

As argued below (see Section 1.3) under Assumption (H), the expectation in (2) is finite.

Our main result is the following:

Theorem 1 (Forward PIDE for call options)**.** *Let ψ_t be the exponential double tail of the compensator $m(t, dy)$*

$$\psi_t(z) = \begin{cases} \int_{-\infty}^z dx\ e^x \int_{-\infty}^x m(t, du) & z < 0 \\ \int_z^{+\infty} dx\ e^x \int_x^{\infty} m(t, du) & z > 0 \end{cases} \tag{3}$$

and define

$$\begin{cases} \sigma(t, z) & = \sqrt{\mathbb{E}\left[\delta_t^2 | S_{t^-} = z\right]}; \\ \chi_{t,y}(z) & = \mathbb{E}\left[\psi_t\left(z\right) | S_{t-} = y\right] \end{cases} \tag{4}$$

Under assumption (H)*, the call option price $(T, K) \mapsto C_{t_0}(T, K)$, as a function of maturity and strike, is a solution (in the sense of distributions) of the partial integro-differential equation:*

$$\frac{\partial C_{t_0}}{\partial T} = -r(T)K\frac{\partial C_{t_0}}{\partial K} + \frac{K^2\sigma(T, K)^2}{2}\frac{\partial^2 C_{t_0}}{\partial K^2} + \int_0^{+\infty} y \frac{\partial^2 C_{t_0}}{\partial K^2}(T, dy)\, \chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right) \tag{5}$$

on $[t_0, \infty[\times]0, \infty[$ with the initial condition: $\forall K > 0 \quad C_{t_0}(t_0, K) = (S_{t_0} - K)_+$.

Remark 1. *The discounted asset price*

$$\hat{S}_T = e^{-\int_0^T r(t) dt}\, S_T,$$

is the stochastic exponential of the martingale U defined by

$$U_T = \int_0^T \delta_t\, dW_t + \int_0^T \int (e^y - 1)\tilde{M}(dt\, dy).$$

Under assumption (H)*, we have*

$$\forall T > 0, \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\langle U^c, U^c\rangle_T + \langle U^d, U^d\rangle_T\right)\right] < \infty$$

and [28, Theorem 9] implies that $(\hat{S}_T)$ is a $\mathbb{P}$-martingale.

The form of the integral term in (5) may seem different from the integral term appearing in backward PIDEs [13, 23]. The following lemma gives an interpretation of $\chi_{T,y}(z)$ in terms of call payoffs, which casts this term into a more familiar form:

Lemma 1. *Let $n(t, dz, y, \omega)\, dt$ be a random measure on $[0,T]\times\mathbb{R}\times\mathbb{R}^+$ verifying*

$$\forall t \in [0,T], \quad \int_{-\infty}^{\infty} (e^z \wedge |z|^2) n(t, dz, y, \omega) < \infty \qquad \text{a.s.}$$

$\chi_{t,y}(z)$ its exponential double tail defined as

$$\chi_{t,y}(z) = \begin{cases} \int_{-\infty}^{z} dx\ e^x \int_{-\infty}^{x} n(t, du, y) & z < 0 \\ \int_{z}^{+\infty} dx\ e^x \int_{x}^{\infty} n(t, du, y) & z > 0 \end{cases} \tag{6}$$

Then

$$\int_{\mathbb{R}} [(ye^z - K)^+ - e^z(y-K)^+ - K(e^z - 1)1_{y>K\}}] n(t, dz, y) = y\, \chi_{t,y}\left(\ln\left(\frac{K}{y}\right)\right)$$

Proof.

$$\begin{aligned} &\int_{\mathbb{R}} [(ye^z - K)^+ - e^z(y-K)^+ - K(e^z-1)1_{y>K\}}] n(t, dz, y) \\ &= \int_{\mathbb{R}} [(ye^z - K)1_{z>\ln(\frac{K}{y})\}} - e^z(y-K)1_{y>K\}} - K(e^z-1)1_{y>K\}}] n(t, dz, y) \\ &= \int_{\mathbb{R}} [(ye^z - K)1_{z>\ln(\frac{K}{y})\}} + (K - ye^z)1_{y>K\}}] n(t, dz, y) \end{aligned}$$

- If $K \geq y$, then

$$\begin{aligned} &\int_{\mathbb{R}} 1_{K\geq y\}} [(ye^z - K)1_{z>\ln(\frac{K}{y})\}} + (K - ye^z)1_{y>K\}}] n(t, dz, y) \\ = &\int_{\ln(\frac{K}{y})}^{+\infty} y(e^z - e^{\ln(\frac{K}{y})})\, n(t, dz, y) \end{aligned} \tag{7}$$

- If $K < y$, then

$$\begin{aligned} &\int_{\mathbb{R}} 1_{K<y\}} [(ye^z - K)1_{z>\ln(\frac{K}{y})\}} + (K - ye^z)1_{y>K\}}] n(t, dz, y) \\ = &\int_{\ln(\frac{K}{y})}^{+\infty} [(ye^z - K) + (K - ye^z)] n(t, dz, y) + \int_{-\infty}^{\ln(\frac{K}{y})} [K - ye^z] n(t, dz, y) \\ = &\int_{-\infty}^{\ln(\frac{K}{y})} y(e^{\ln(\frac{K}{y})} - e^z) n(t, dz, y) \end{aligned} \tag{8}$$

Using integration by parts, $\chi_{t,y}$ can be equivalently expressed as

$$\chi_{t,y}(z) = \begin{cases} \int_{-\infty}^{z} (e^z - e^u)\, n(t, du, y) & z < 0 \\ \int_{z}^{\infty} (e^u - e^z)\, n(t, du, y) & z > 0 \end{cases}$$

Hence:

$$\int_{\mathbb{R}}[(ye^z-K)^+-e^z(y-K)^+-K(e^z-1)1_{y>K}]n(t,dz,y)=y\,\chi_{t,y}\left(\ln\left(\frac{K}{y}\right)\right)$$

□

We will now present two different derivations of Theorem 1:

- The first derivation (Section 1.2) is based on a generalization [4] of Gyöngy's [21] "mimicking theorem" to discontinuous semimartingales: aside from the forward equation, this method also yields a mimicking process which has the same marginals as the price process.
- The second derivation (Section 1.3), based on the Tanaka-Meyer formula, has the merit of requiring weaker assumptions on the local characteristics of the process.

1.2 Derivation using a mimicking process

We present in this section a probabilistic proof of Theorem 1 based on a generalization [4] of Gyöngy's [21] "mimicking theorem" to semimartingales. We first construct, using results from [4], a simpler process which has the same marginal distributions as $(S_t)_{t\geq 0}$. But to proceed, we need to make stronger assumptions on the characteristics of (S_t):
$\forall T>0, \exists c_T>0, \forall t\in[0,T]$,

$$\left\{\begin{array}{ll} |\delta_t|\leq c_T \qquad a.s & (A_{1a})\\ \int(1\wedge y^2)\,m(t,dy,\omega)\leq c_T<\infty \quad a.s. & (A_{2a})\\ \int_0^T\int_{y>1}e^{2y}m(t,dy,\omega)\leq c_T<\infty \quad a.s. & (A_{3a})\end{array}\right.$$

Under these assumptions, we have the following result:

Proposition 1 (Mimicking a stochastic volatility model with jumps)**.**
Assume (A_{1a}), (A_{2a}) *and* (A_{3a}). *Define for* $t\geq 0$, $z\in\mathbb{R}$, $A\in\mathcal{B}(\mathbb{R}\backslash\{0\})$,

$$\sigma(t,z)=\sqrt{\mathbb{E}\left[\delta_t^2|S_{t^-}=z\right]};\qquad n(t,A,z)=\mathbb{E}\left[m(t,A)|S_{t^-}=z\right] \tag{9}$$

If σ *and* n *are continuous in* (t,z) *on* $[0,T]\times]0,\infty[$, *there exists a weak solution* $((Y_t)_{t\in[0,T]},\mathbb{Q}_{S_0})$ *to the stochastic differential equation*

$$Y_t=S_0+\int_0^T r(t)Y_{t-}dt+\int_0^T Y_{t-}\sigma(t,Y_{t-})dB_t+\int_0^T\int_{-\infty}^{+\infty}Y_{t-}(e^y-1)\tilde{N}(dt\,dy) \tag{10}$$

where (B_t) *is a Brownian motion,* N *an integer-valued random measure on* $[0,T]\times\mathbb{R}$ *with compensator* $n(t,dy,Y_{t-})\,dt$, $\tilde{N}$ *the compensated random measure associated with* N.

Y is the unique solution of the martingale problem for the infinitesimal generator L given, for $f \in C_0^\infty([0,T]\times\mathbb{R})$, by

$$\begin{aligned} Lf(t,x) = xr(t)\frac{\partial f}{\partial x}(t,x) + \frac{x^2\sigma(t,x)^2}{2}\frac{\partial^2 f}{\partial x^2}(t,x) \\ + \int_{\mathbb{R}^d}[f(t,xe^y) - f(t,x) - 1_{|y|\leq 1}x(e^y-1).\frac{\partial f}{\partial x}f(t,x)]n(t,dy,x) \end{aligned} \tag{11}$$

and the marginal distributions of Y mimick those of S:

$$\forall t \in [0,T], Y_t \stackrel{\mathrm{d}}{=} S_t.$$

(9) means that for any function $f : \mathbb{R} \mapsto \mathbb{R}^+$ such that $\mathbb{E}\left[\int f(y)\, m(t,dy)\right] < \infty$,

$$\mathbb{E}\left[\int f(y)\, m(t,dy)|S_{t-} = z\right] = \int f(y)\, n(t,dy,z).$$

Proof of Proposition 1. Applying the Itô formula to $X_T \equiv \ln(S_T)$, we obtain

$$\begin{aligned} X_T &= \ln(S_0) + \int_0^T \frac{1}{S_{t^-}}\, dS_t + \frac{1}{2}\int_0^T \frac{-1}{S_{t^-}^2}\,(S_{t-}\delta_t)^2\, dt & (12)\\ &+ \sum_{t\leq T}\left[\ln(S_{t^-}+\Delta S_t) - \ln(S_{t^-}) - \frac{1}{S_{t^-}}\Delta S_t\right] & (13)\\ &= \ln(S_0) + \int_0^T \left(r(t) - \frac{1}{2}\delta_t^2\right)\, dt + \int_0^T \delta_t\, dW_t & (14)\\ &+ \int_0^T\int_{-\infty}^{+\infty}(e^y-1)\tilde{M}(dt\, dy) - \int_0^T\int_{-\infty}^{+\infty}(e^y-1-y)\; M(dt\, dy) & (15) \end{aligned}$$

Note that $|e^y - 1 - y\, 1_{|y|\leq 1}| \leq \frac{y^2}{4}\, 1_{|y|<1} + C\, 1_{y<-1} + e^{2y}1_{y>1}$. Using assumptions (A_{2a}) and (A_{3a}), for all T there exists $K_T > 0$ such that

$$\int_0^T\int_{-\infty}^{\infty}\left(e^y - 1 - y\, 1_{|y|\leq 1}\right)\, m(t,dy)\, dt < K_T$$

allowing us to write

$$\begin{aligned}
&\int_0^T \int_{-\infty}^{+\infty} (e^y - 1)\tilde{M}(dt\,dy) - \int_0^T \int_{-\infty}^{+\infty} (e^y - 1 - y)\; M(dt\,dy) \\
= &\int_0^T \int_{-\infty}^{+\infty} (e^y - 1)\tilde{M}(dt\,dy) - \int_0^T \int_{-\infty}^{+\infty} \left(e^y - 1 - y\,1_{|y|\leq 1}\right)\; M(dt\,dy) \\
+ &\int_0^T \int_{|y|>1} y\, M(dt\,dy) \\
= &\int_0^T \int_{-\infty}^{+\infty} (e^y - 1)\tilde{M}(dt\,dy) - \int_0^T \int_{-\infty}^{+\infty} \left(e^y - 1 - y\,1_{|y|\leq 1}\right)\; \tilde{M}(dt\,dy) \\
- &\int_0^T \int_{-\infty}^{+\infty} \left(e^y - 1 - y\,1_{|y|\leq 1}\right)\; m(t,dy)\,dt + \int_0^T \int_{|y|>1} y\, M(dt\,dy) \\
= &\int_0^T \int_{-\infty}^{+\infty} y\,1_{|y|\leq 1}\,\tilde{M}(dt\,dy) - \int_0^T \int_{-\infty}^{+\infty} \left(e^y - 1 - y\,1_{|y|\leq 1}\right)\; m(t,dy)\,dt \\
+ &\int_0^T \int_{|y|>1} y\, M(dt\,dy)
\end{aligned}$$

We obtain the following decomposition of X_T

$$X_T = X_0 + \int_0^T \beta_t\,dt + \int_0^T \delta_t\,dW_t + \int_0^T \int_{|y|\leq 1} y\,\tilde{M}(dt\,dy) + \int_0^T \int_{|y|>1} y\,M(dt\,dy), \tag{16}$$

with

$$\beta_t = r(t) - \frac{1}{2}\delta_t^2 - \int_{-\infty}^{\infty} \left(e^y - 1 - y\,1_{|y|\leq 1}\right)\; m(t,dy)\,dt$$

Define:

$$\beta^Z(t,z) = \mathbb{E}\left[\beta_t | X_{t^-} = \ln(z)\right];$$

With the notations of [4, Theorem 1]: δ_t and β_t satisfy (H_1) and n satisfies (H_2). The continuity of $\sigma(t,z)$ and $n(t,y,z)$ in (t,z) on $[0,T]\times[0,\infty[$ guarantees the continuity of $(t,z) \to \mathbb{E}\left[\beta_t | X_{t^-} = \ln(z)\right]$. Applying [4, Theorem 1], there exists (Z_t) weak solution of

$$Z_t = X_0 + \int_0^t \beta^Z(u,V_u)\,du + \int_0^t \sigma(u,V_u)\,dB_u + \int_0^t \int_{|y|\leq 1} y\,\tilde{N}(du\,dy) + \int_0^t \int_{|y|>1} y\,N(du\,dy)$$

where N is an integer-valued random measure on $[0,T]\times\mathbb{R}$ with compensator $n(t,dy,Y_{t-})\,dt$, $\tilde{N}$ the associated compensated random measure, such that (Z_t) mimicks the marginal laws of (X_t).
Therefore, $Y_t \equiv \exp(Z_t)$ has the same marginal distributions as S_t and Y_t satisfies the following SDE:

$$Y_t = S_0 + \int_0^T r(t)Y_{t-}dt + \int_0^T Y_{t-}\sigma(t,Y_{t-})dB_t + \int_0^T \int_{-\infty}^{+\infty} Y_{t-}(e^y - 1)\tilde{N}(dtdy)$$

This ends the proof. □

Remark 2. *Conditions* (A_{1a}), (A_{2a}) *and* (A_{3a}) *are indeed stronger than* (H)*: since* $y \to (e^y-1)^2$ *is bounded on* $]-\infty,-1]$, $y \to \frac{(e^y-1)^2}{y^2}$ *is bounded on* $[-1,1]$ *and* $y \to (e^y-1)^2$ *is bounded by* $y \to e^{2y}$ *on* $]1,+\infty[$*, there exist positive constants* C_1, C_2, C_3 *such that*

$$\begin{aligned} & \frac{1}{2}\langle U^c, U^c\rangle_T + \langle U^d, U^d\rangle_T \\ = & \int_0^T \delta_t^2\, dt + \int_0^T \int (e^y-1)^2 m(dt\, dy) \\ \leq & \int_0^T \delta_t^2\, dt + C_1 \int_0^T \int_{y<-1} 1\, m(dt\, dy) + C_2 \int_{|y|\leq 1} y^2\, m(dt\, dy) \\ + & C_3 \int_0^T \int_{y>1} e^{2y}\, m(dt\, dy). \end{aligned}$$

So by [28, Theorem 9], $\hat{S} = \mathcal{E}(U)$ *is a martingale under* $\mathbb{P}$ *(see remark 1). The mimicking process preserves the martingale property: since* $\hat{Y}_T \equiv e^{-\int_0^T r(t)dt} Y_T$ *also satisfies assumption* (H)*,* $\exp(Y)$ *is a martingale under* $\mathbb{Q}_{S_0}$*.*

Proof of Theorem 1. We first note that, by replacing $\mathbb{P}$ by the conditional measure $\mathbb{P}_{|\mathcal{F}_0}$ given $\mathcal{F}_0$, we may replace the conditional expectation in (2) by an expectation with respect to the marginal distribution $p_T^S(dy)$ of S_T under $\mathbb{P}_{|\mathcal{F}_{t_0}}$. Thus, without loss of generality, we put $t_0 = 0$ in the sequel, consider the case where $\mathcal{F}_0$ only contains null sets and we denote $C_0(T,K) \equiv C(T,K)$ for simplicity. Consider $X_t = \ln(S_t)$ as in (16). Define the infitesimal generator H_T as follows: for f in $C_0^\infty(\mathbb{R})$,

$$\begin{aligned} H_T f(y) = & \left(r(T) - \frac{1}{2}\sigma(T,y)^2 - \int_{-\infty}^{\infty} \left(e^z - 1 - z\, 1_{|z|\leq 1}\right) n(T,dz,y)\, dt \right) f'(y) \\ & + \frac{\sigma(T,y)^2}{2} f''(y) + \int_{\mathbb{R}} [f(y+z) - f(y) - z f'(y)] n(T,dz,y) \end{aligned}$$

The distribution $p_T^X(dy)$ of X_T given $\mathcal{F}_0$ solves a forward Kolmogorov equation [4, Theorem 3]:

$$\frac{\partial p_T^X}{\partial T} = H_T^\star . p_T^X,$$

where $H_T^\star$ is the L^2-adjoint of H_T. For all f in $C_0^\infty(\mathbb{R})$, $\mathbb{E}[f(X_T)] = \int_{\mathbb{R}} f(y) p_T^X(dy)$ but also $\mathbb{E}[f(X_T)] = \mathbb{E}[f(\ln(S_T))] = \int_{\mathbb{R}} f(\ln(y)) p_T^S(dy)$ hence $\frac{\partial p_T^X}{\partial T}.f(y) = p_T^X(dy) H_T.f(y) = p_T^S(dy) H_T.f(\ln(y))$, leading to , after change of variable, the Kolmogorov forward equation for $((S_t), \mathbb{P}_{|\mathcal{F}_0})$:

$$\frac{\partial p_T^S}{\partial T} = L_T^\star . p_T^S,$$

where $L_T^\star$ is the L^2-adjoint of L_T, defined for all f in $C_0^\infty(\mathbb{R})$ by

$$\begin{aligned} L_T f(y) = & \; r(T) y f'(y) + \tfrac{y^2 \sigma(T,y)^2}{2} f''(y) \\ & + \textstyle\int_{\mathbb{R}} [f(ye^z) - f(y) - y(e^z-1) f'(y)] n(T,dz,y) \end{aligned}$$

We decompose L_T into a differential and an integral component: $L_T = A_T + B_T$ where

$$\begin{aligned} A_T f(y) &= r(T) y f'(y) + \frac{y^2 \sigma(T,y)^2}{2} f''(y) \\ B_T f(y) &= \int_{\mathbb{R}} [f(ye^z) - f(y) - y(e^z - 1) f'(y)] n(T, dz, y) \end{aligned}$$

Using the fact that $y\frac{\partial}{\partial y}(y-K)^+ = K1_{y>K\}} + (y-K)_+$ and $\frac{\partial^2}{\partial K^2}(y-K)^+ = \epsilon_K(y)$ where ϵ_K is a unit mass at K, we obtain

$$A_T(y-K)^+ = r(T)[K1_{y>K\}} + (y-K)^+] + \frac{y^2\sigma(T,y)^2}{2}\epsilon_K(y)$$

and

$$\begin{aligned} B_T(y-K)^+ &= \textstyle\int_{\mathbb{R}}[(ye^z - K)^+ - (y-K)^+ - (e^z-1)\left(K1_{y>K\}} + (y-K)^+\right)] n(T,z,y)\, dz \\ &= \textstyle\int_{\mathbb{R}}[(ye^z - K)^+ - e^z(y-K)^+ - K(e^z-1)1_{y>K\}}] n(T,z,y)\, dz \end{aligned}$$

Applying Lemma 1 to the random measure $n(T, dz, y)$ yields

$$B_T(y-K)^+ = \chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right)$$

(2) can be expressed as

$$C(T,K) = e^{-\int_0^T r(t)\,dt} \int_{\mathbb{R}^+} (y-K)^+ \; p_T^S(dy) \tag{17}$$

We observe that for all t in $[0,T]$:

$$\begin{aligned} & (y-K)^+ . \frac{\partial p_t^S}{\partial t}(dy) = p_t^S(dy)\, L_t . (y-K)^+ \\ = \;& p_t^S(dy) r(t)[K1_{y>K\}} + (y-K)^+] + p_t^S(dy)\frac{y^2\sigma(t,y)^2}{2}\epsilon_K(y) \\ + \;& p_t^S(dy) 1_{K\geq y\}} \int_{\ln(\frac{K}{y})}^{+\infty} y(e^z - e^{\ln(\frac{K}{y})})\, n(t,dz,y) \\ + \;& 1_{K<y\}} p_T^S(dy) \int_{-\infty}^{\ln(\frac{K}{y})} y(e^{\ln(\frac{K}{y})} - e^z) n(t,dz,y) \\ \leq \;& c_T[K1_{y>K\}} + (y-K)^+] + \frac{y^2 c_T^2}{2}\epsilon_K(y) + 1_{0<y\leq K\}}\, y(1 - e^{\ln(\frac{K}{y})})c_T + 1_{y>K\}}\, y e^{\ln(\frac{K}{y})} c_T \\ \leq \;& K_T \end{aligned} \tag{18}$$

Therefore using a dominated convergence argument, one may differentiate (17)

with respect to T:

$$\begin{aligned}\frac{\partial C}{\partial T} &= -r(T)C(T,K) + e^{-\int_0^T r(t)\,dt}\int_{\mathbb{R}^+}(y-K)^+\,\frac{\partial p_T^S}{\partial T}(dy)\\ &= -r(T)C(T,K) + e^{-\int_0^T r(t)\,dt}\int_{\mathbb{R}^+}(y-K)^+\,L_T^\star.\,p_T^S(dy)\\ &= -r(T)C(T,K) + e^{-\int_0^T r(t)\,dt}\int_{\mathbb{R}^+}p_T^S(dy)\,L_T.\,(y-K)^+\end{aligned}$$

By differentiating with respect to K, we get:

$$\frac{\partial C}{\partial K}(T,K) = -e^{-\int_0^T r(t)\,dt}\int_K^\infty p_T^S(dy) \qquad \frac{\partial^2 C}{\partial K^2}(T,dy) = e^{-\int_0^T r(t)\,dt}p_T^S(dy) \tag{19}$$

Note that, unless p_T^S has a density with respect to the Lebesgue measure, $\partial^2 C/\partial K^2$ is in general a positive measure, not a function.

$$e^{-\int_0^T r(t)\,dt}\int_{\mathbb{R}^+}p_T^S(dy)\,A_T(y-K)^+ = -r(T)K\frac{\partial C}{\partial K}+r(T)C(T,K)+\frac{K^2\sigma(T,K)^2}{2}\frac{\partial^2 C}{\partial K^2}$$

On the other hand

$$\begin{aligned}e^{-\int_0^T r(t)\,dt}\int_{\mathbb{R}^+}p_T^S(dy)\,B_T(y-K)^+ &= e^{-\int_0^T r(t)\,dt}\int_0^{+\infty}p_T^S(dy)\,\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right)\\ &= \int_0^{+\infty}y\frac{\partial^2 C}{\partial K^2}(T,dy)\,\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right)\end{aligned}$$

We conclude therefore that

$$\begin{aligned}\frac{\partial C}{\partial T} &= e^{-\int_0^T r(t)\,dt}\int_{\mathbb{R}^+}p_T^S(dy)\,(A_T+B_T)(y-K)^+\\ &= -r(T)K\frac{\partial C}{\partial K}+\frac{K^2\sigma(T,K)^2}{2}\frac{\partial^2 C}{\partial K^2}+\int_0^{+\infty}y\frac{\partial^2 C}{\partial K^2}(T,dy)\,\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right)\end{aligned}$$

□

1.3 Derivation using the Tanaka-Meyer formula

In this section we present a proof of Theorem 1 using the Tanaka-Meyer formula for semimartingales [22, Theorem 9.43] under assumption (H). Unlike the previous method, this proof does not require continuity assumptions on the coefficients of the forward PIDE.

Proof. Let $p_T^S(dy)$ be the conditional law of S_T given $\mathcal{F}_{t_0}$. As in Section 1.2, we put $t_0 = 0$ in the sequel and consider the case where $\mathcal{F}_0$ only contains null sets. We recall:

$$\begin{aligned}\frac{\partial C}{\partial K} &= -e^{-\int_0^T r(t)\,dt}\int_K^\infty p_T^S(dy) = -\mathbb{E}\left[1_{S_{t^-}>K}\right]\\ \frac{\partial^2 C}{\partial K^2}(T,dy) &= e^{-\int_0^T r(t)\,dt}p_T^S(dy)\end{aligned} \tag{20}$$

Let $L_t^K = L_t^K(S)$ be the semimartingale local time of S at K under $\mathbb{P}$ (see [22, Chapter 9] or [29, Ch. IV] for definitions). For $h > 0$, applying the Tanaka-Meyer formula to $(S_t - K)^+$ between T and $T+h$, we have

$$\begin{aligned}(S_{T+h} - K)^+ = (S_T - K)^+ + \int_T^{T+h} 1_{S_{t-}>K\}} dS_t + \frac{1}{2}(L_{T+h}^K - L_T^K) \\ + \sum_{T<t\leq T+h} (S_t - K)^+ - (S_{t-} - K)^+ - 1_{S_{t-}>K\}} \Delta S_t.\end{aligned} \tag{21}$$

As noted in Remark 1, the integrability condition (H) implies that $\hat{S}_t = e^{-\int_0^t r(s)\,ds} S_t = \mathcal{E}(U)_t$ is a martingale under $\mathbb{P}$. So (1) can be expressed as $dS_t = r(t)S_{t-}dt + d\hat{S}_t$ where the discounted price :

$$\int_T^{T+h} 1_{S_{t-}>K\}} dS_t = \int_T^{T+h} 1_{S_{t-}>K\}} d\hat{S}_t + \int_T^{T+h} r(t) S_{t-} 1_{S_{t-}>K\}} dt$$

where the first term is a martingale. Taking expectations, we get:

$$\begin{aligned}e^{\int_0^{T+h} r(t)\,dt} C(T+h,K) - e^{\int_0^T r(t)\,dt} C(T,K) \\ = \mathbb{E}\left[\int_T^{T+h} r(t) S_t\, 1_{S_{t-}>K\}} dt + \tfrac{1}{2}(L_{T+h}^K - L_T^K)\right] \\ + \mathbb{E}\left[\textstyle\sum_{T<t\leq T+h} (S_t - K)^+ - (S_{t-} - K)^+ - 1_{S_{t-}>K\}} \Delta S_t\right].\end{aligned}$$

Noting that $S_{t-} 1_{S_{t-}>K\}} = (S_{t-} - K)^+ + K 1_{S_{t-}>K\}}$, we obtain

$$\mathbb{E}\left[\int_T^{T+h} r(t) S_{t-} 1_{S_{t-}>K\}} dt\right] = \int_T^{T+h} r(t) e^{\int_0^t r(s)\,ds} \left[C(t,K) - K\frac{\partial C}{\partial K}(t,K)\right] dt$$

using Fubini's theorem and (20). As for the jump term,

$$\begin{aligned}\mathbb{E}\left[\textstyle\sum_{T<t\leq T+h} (S_t - K)^+ - (S_{t-} - K)^+ - 1_{S_{t-}>K\}} \Delta S_t\right] \\ = \mathbb{E}\left[\textstyle\int_T^{T+h} dt \int m(t,dx)\,(S_{t-}e^x - K)^+ - (S_{t-} - K)^+ - 1_{S_{t-}>K\}} S_{t-}(e^x - 1)\right] \\ = \mathbb{E}\Big[\textstyle\int_T^{T+h} dt \int m(t,dx)\big((S_{t-}e^x - K)^+ - (S_{t-} - K)^+ \\ -(S_{t-} - K)^+(e^x - 1) - K 1_{S_{t-}>K\}}(e^x - 1)\big)\Big] \\ = \textstyle\int_T^{T+h} dt \mathbb{E}\Big[\int m(t,dx)\big((S_{t-}e^x - K)^+ \\ -e^x(S_{t-} - K)^+ - K 1_{S_{t-}>K\}}(e^x - 1)\big)\Big]\end{aligned}$$

Applying Lemma 1, to the random measure m and its exponential double tail ψ:

$$\int m(t,dx)\big((S_{t-}e^x - K)^+ - e^x(S_{t-} - K)^+ - K 1_{S_{t-}>K\}}(e^x - 1)\big) = S_{t-}\,\psi_{t,S_{t-}}\left(\ln\left(\frac{K}{S_{t-}}\right)\right)$$

holds, leading to:

$$\begin{aligned}
& \mathbb{E}\left[\sum_{T<t\leq T+h}(S_t-K)^+-(S_{t-}-K)^+-1_{S_{t-}>K\}}\Delta S_t\right] \\
= & \int_T^{T+h} dt\,\mathbb{E}\left[S_{t-}\,\psi_{t,S_{t-}}\left(\ln\left(\frac{K}{S_{t-}}\right)\right)\right] \\
= & \int_T^{T+h} dt\,\mathbb{E}\left[S_{t-}\mathbb{E}\left[\psi_{t,S_{t-}}\left(\ln\left(\frac{K}{S_{t-}}\right)\right)|S_{t-}\right]\right] \\
= & \int_T^{T+h} dt\,\mathbb{E}\left[S_{t-}\chi_{t,S_{t-}}\left(\ln\left(\frac{K}{S_{t-}}\right)\right)\right] \qquad (22)
\end{aligned}$$

Let $\varphi \in C_0^\infty(]0,\infty[)$ be a test function. The occupation time formula (see [22, Theorem 9.46]) yields:

$$\int_0^{+\infty} dK\,\varphi(K)(L^K_{T+h}-L^K_T)=\int_T^{T+h}\varphi(S_{t-})d[S]^c_t=\int_T^{T+h}dt\,\varphi(S_{t-})S^2_{t-}\delta^2_t$$

$(\hat{S}_T)$ is a martingale, hence : $\mathbb{E}[S_T]<\infty$. Since $(S_{T+h}-K)^+<S_{T+h}$, $(S_T-K)^+<S_T$, $|\sum_{T<t\leq T+h}(S_t-K)^+-(S_{t-}-K)^+-1_{S_{t-}>K\}}\Delta S_t|<3S_t$ and $\mathbb{E}\left[\int_T^{T+h}1_{S_{t-}>K\}}dS_t\right]<\infty$ then (21) leads to : $\mathbb{E}\left[L^K_{T+h}-L^K_T\right]<\infty$; furthermore, since φ is bounded and has compact support, one may take expectations on both sides and apply Fubini's theorem to obtain:

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty dK\varphi(K)(L^K_{T+h}-L^K_T)\right] &= \mathbb{E}\left[\int_T^{T+h}\varphi(S_t)S^2_t\delta^2_t dt\right] \\
&= \int_T^{T+h}dt\,\mathbb{E}\left[\varphi(S_t)S^2_t\delta^2_t\right] \\
&= \int_T^{T+h}dt\,\mathbb{E}\left[\mathbb{E}\left[\varphi(S_{t-})S^2_{t-}\delta^2_t\right]|S_{t-}\right] \\
&= \mathbb{E}\left[\int_T^{T+h}dt\,\varphi(S_{t-})S^2_{t-}\sigma(t,S_{t-})^2\right] \\
&= \int_0^\infty\int_T^{T+h}\varphi(K)K^2\sigma(t,K)^2p^S_t(dK)\,dt \\
&= \int_T^{T+h}dt\,e^{\int_0^t r(s)\,ds}\int_0^\infty\varphi(K)K^2\sigma(t,K)^2\frac{\partial^2 C}{\partial K^2}(t,dK)
\end{aligned}$$

where the last line is obtained by using (20). Gathering together all the terms,

we obtain:

$$\begin{aligned}
&\int_0^\infty dK\,\varphi(K)\left[e^{\int_0^{T+h} r(t)\,dt}C(T+h,K) - e^{\int_0^T r(t)\,dt}C(T,K)\right] \\
= &\int_T^{T+h} dt\,r(t)\,e^{\int_0^t r(s)\,ds}\int_0^\infty dK\,\varphi(K)[C(t,K) - K\frac{\partial C}{\partial K}(t,K)] \\
+ &\int_T^{T+h} dt\,e^{\int_0^t r(s)\,ds}\int_0^\infty \frac{\varphi(K)}{2}K^2\sigma(t,K)^2\frac{\partial^2 C}{\partial K^2}(t,dK) \\
+ &\int_0^\infty dK\,\varphi(K)\int_T^{T+h} dt\mathbb{E}\left[S_{t-}\chi_{t,S_{t-}}\left(\ln\left(\frac{K}{S_{t-}}\right)\right)\right] \qquad (23)
\end{aligned}$$

Dividing by h and taking the limit $h \to 0$ yields:

$$\begin{aligned}
&\int_0^\infty dK\varphi(K)e^{\int_0^T r(t)\,dt}\left[\frac{\partial C}{\partial T}(T,K) + r(T)C(T,K)\right] \\
= &\int_0^\infty dK\varphi(K)e^{\int_0^T r(t)\,dt}\left[r(T)C(T,K) - r(T)K\frac{\partial C}{\partial K}(T,K)\right] \\
+ &\int_0^\infty \frac{\varphi(K)}{2}K^2\sigma(t,K)^2e^{\int_0^T r(t)\,dt}\frac{\partial^2 C}{\partial K^2}(T,dK) \\
+ &\int_0^\infty dK\,\varphi(K)\mathbb{E}\left[S_{t-}\chi_{T,S_{T-}}(\ln(\frac{K}{S_{T-}}))\right] \\
= &\int_0^\infty dK\varphi(K)e^{\int_0^T r(t)\,dt}\left[r(T)C(T,K) - r(T)K\frac{\partial C}{\partial K}(T,K)\right] \\
+ &\int_0^\infty \frac{\varphi(K)}{2}K^2\sigma(t,K)^2e^{\int_0^T r(t)\,dt}\frac{\partial^2 C}{\partial K^2}(T,dK) \\
+ &\int_0^\infty dK\,\varphi(K)e^{\int_0^T r(t)\,dt}\int_0^{+\infty} y\frac{\partial^2 C}{\partial K^2}(T,dy)\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right) \qquad (24)
\end{aligned}$$

Since this equality holds for any $\varphi \in C_0^\infty(]0,\infty[,\mathbb{R})$, $C(.,.)$ is a solution of

$$\begin{aligned}
\frac{\partial C}{\partial T}(T,K) &= \frac{K^2\sigma(t,K)^2}{2}\frac{\partial^2 C}{\partial K^2}(T,K) - r(T)K\frac{\partial C}{\partial K}(T,K) \\
&+ \int_0^{+\infty} y\frac{\partial^2 C}{\partial K^2}(dy)\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right)
\end{aligned}$$

in the sense of distributions on $[0,T]\times]0,\infty[$. □

2 Examples

We now give various examples of pricing models for which Theorem 1 allows to retrieve or generalize previously known forms of forward pricing equations.

2.1 Ito processes

When S is an Ito process i.e. when the jump part is absent, the forward equation (5) reduces to the Dupire equation [16]. This result can be derived from the mimicking theorem of Gyöngy [21] in the case where the volatility process δ_t verifies a non-degeneracy (i.e. uniform ellipticity) condition. In this case our result gives an alternative set of assumptions under which the Dupire equation holds, which do not require this non-degeneracy condition:

Proposition 2 (Dupire PDE)**.** *Consider the price process S whose dynamics under the pricing measure $\mathbb{P}$ is given by:*

$$S_T = S_0 + \int_0^T r(t) S_t dt + \int_0^T S_t \delta_t dW_t \tag{25}$$

Define

$$\sigma(t,z) = \sqrt{\mathbb{E}\left[\delta_t^2 | S_t = z\right]}$$

If

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \delta_t^2 \, dt\right)\right] < \infty \; a.s. \quad (H_1)$$

the call option price (2) *is a solution (in the sense of distributions) of the partial differential equation:*

$$\frac{\partial C_{t_0}}{\partial T} = -r(T) K \frac{\partial C_{t_0}}{\partial K} + \frac{K^2 \sigma(T,K)^2}{2} \frac{\partial^2 C_{t_0}}{\partial K^2} \tag{26}$$

on $[t_0, \infty[\times]0, \infty[$ with the initial condition: $\forall K > 0 \quad C_{t_0}(t_0, K) = (S_{t_0} - K)_+$.

Proof. It is sufficient to take $\mu \equiv 0$ in (1) then equivalently in (5). We leave the end of the proof to the reader. □

2.2 Markovian jump-diffusion models

Another important particular case in the literature is the case of a Markov jump-diffusion driven by a Poisson random measure. Andersen and Andreasen [2] derived a forward PIDE in the situation where the jumps are driven by a compound Poisson process with time-homogeneous Gaussian jumps. We will now show here that Theorem 1 implies the PIDE derived in [2], given here in a more general context allowing for a time- and state-dependent Lévy measure, as well as infinite number of jumps per unit time ("infinite jump activity").

Proposition 3 (Forward PIDE for jump diffusion model)**.** *Consider the price process S whose dynamics under the pricing measure $\mathbb{P}$ is given by:*

$$S_t = S_0 + \int_0^T r(t) S_{t-} dt + \int_0^T S_{t-} \sigma(t, S_{t-}) dB_t + \int_0^T \int_{-\infty}^{+\infty} S_{t-}(e^y - 1) \tilde{N}(dtdy) \tag{27}$$

where B_t is a Brownian motion and N a Poisson random measure on $[0,T]\times\mathbb{R}$ with compensator $\nu(dz)\,dt$, $\tilde{N}$ the associated compensated random measure. Assume:

$$\begin{cases} \sigma(.,.) \quad \text{is bounded} & (H_1') \\ \int_{y>1} e^{2y}\nu(dy) < \infty & (H_2') \end{cases}$$

Then the call option price

$$C_{t_0}(T,K) = e^{-\int_{t_0}^T r(t)\,dt} E^{\mathbb{P}}[\max(S_T - K, 0)|\mathcal{F}_{t_0}]$$

is a solution (in the sense of distributions) of the partial integro-differential equation:

$$\begin{aligned} \frac{\partial C_{t_0}}{\partial T} &= -r(T)K\frac{\partial C_{t_0}}{\partial K} + \frac{K^2\sigma(T,K)^2}{2}\frac{\partial^2 C_{t_0}}{\partial K^2} \\ &\quad + \int_{\mathbb{R}} \nu(dz)\, e^z \left[C_{t_0}(T, Ke^{-z}) - C_{t_0}(T,K) - K(e^{-z}-1)\frac{\partial C_{t_0}}{\partial K} \right] \end{aligned} \tag{28}$$

on $[t_0,\infty[\times]0,\infty[$ with the initial condition: $\forall K>0 \quad C_{t_0}(t_0,K) = (S_{t_0}-K)_+$.

Proof. We first note that, by replacing $\mathbb{P}$ by the conditional measure $\mathbb{P}_{|\mathcal{F}_0}$ given $\mathcal{F}_0$, we may replace the conditional expectation in (2) by an expectation with respect to the marginal distribution $p_T^S(dy)$ of S_T under $\mathbb{P}_{|\mathcal{F}_{t_0}}$. Thus, without loss of generality, we put $t_0 = 0$ in the sequel, consider the case where $\mathcal{F}_0$ only contains null sets and we denote $C_0(T,K) \equiv C(T,K)$ for simplicity.
By (2) differentiating in the sense of distributions with respect to K, we get:

$$\frac{\partial C}{\partial K}(T,K) = -e^{-\int_0^T r(t)\,dt}\int_K^\infty p_T^S(dy) \tag{29}$$

$$\frac{\partial^2 C}{\partial K^2}(T,dy) = e^{-\int_0^T r(t)\,dt} p_T^S(dy) \tag{30}$$

In this particular case, $m(t,dz)\,dt \equiv \nu(dz)\,dt$ and ψ_t is simply defined by:

$$\psi_t(z) \equiv \psi(z) = \begin{cases} \int_{-\infty}^z dx\; e^x \int_{-\infty}^x \nu(du) & z<0 \\ \int_z^{+\infty} dx\; e^x \int_x^\infty \nu(du) & z>0 \end{cases}$$

Then (4) yields $\chi_{t,y}(z) \equiv \chi(z) = \psi(z)$. Let now focus on the term

$$\int_0^{+\infty} y\frac{\partial^2 C}{\partial K^2}(T,dy)\,\chi\left(\ln\left(\frac{K}{y}\right)\right)$$

in (5). Applying Lemma 1:

$$\begin{aligned}
&\int_0^{+\infty} y\frac{\partial^2 C}{\partial K^2}(T,dy)\,\chi\left(\ln\left(\frac{K}{y}\right)\right)\\
=&\int_0^{\infty} e^{-\int_0^T r(t)\,dt} p_T^S(dy)\int_{\mathbb{R}}[(ye^z-K)^+ - e^z(y-K)^+ - K(e^z-1)1_{y>K\}}]\nu(dz)\\
=&\int_{\mathbb{R}} e^z\int_0^{\infty} e^{-\int_0^T r(t)\,dt}\,p_T^S(dy)[(y-Ke^{-z})^+ - (y-K)^+ - K(1-e^{-z})1_{y>K\}}]\,\nu(dz)\\
=&\int_0^{+\infty} y\frac{\partial^2 C}{\partial K^2}(T,dy)\,\chi\left(\ln\left(\frac{K}{y}\right)\right)\\
=&\int_{\mathbb{R}} e^z\left[C(T,Ke^{-z}) - C(T,K) - K(e^{-z}-1)\frac{\partial C}{\partial K}\right]\,\nu(dz) \qquad (31)
\end{aligned}$$

This ends the proof. □

2.3 Pure jump processes

We now consider price processes with no Brownian component. Assumption (H) then reduces to

$$\forall T>0,\quad \mathbb{E}\left[\exp\left(\int_0^T dt\int(e^y-1)^2 m(t\,dy)\right)\right]<\infty \qquad (H_2)$$

and the forward equation for call option becomes

$$\frac{\partial C}{\partial T}+r(T)K\frac{\partial C}{\partial K} = \int_0^{+\infty} y\frac{\partial^2 C}{\partial K^2}(T,dy)\,\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right) \qquad (32)$$

It is convenient to use the change of variable: $v=\ln y, k=\ln K$. Define, $c(k,T)=C(e^k,T)$. Then one can write this PIDE as:

$$\frac{\partial c}{\partial T}+r(T)\frac{\partial c}{\partial k} = \int_{-\infty}^{+\infty} e^{2(v-k)}\left(\frac{\partial^2 c}{\partial k^2}-\frac{\partial c}{\partial k}\right)(T,dv)\,\chi_{T,v}(k-v) \qquad (33)$$

where $\chi_{T,v}$ is defined by:

$$\chi_{T,v}(z)=\mathbb{E}\left[\psi_T(z)|S_{T-}=e^v\right]$$

with:

$$\psi_T(z)=\begin{cases}\int_{-\infty}^z dx\ e^x\int_{-\infty}^x m(T,du) & z<0\\ \int_z^{+\infty} dx\ e^x\int_x^{\infty} m(T,du) & z>0\end{cases}$$

In the case considered in [8], where the Lévy density m_Y has a deterministic separable form:

$$m_Y(t,dz,y)\,dt=\alpha(y,t)\,k(dz)\,dz\,dt \qquad (34)$$

Equation (33) allows us to recover[1] equation (14) in [8]: (33) becomes

$$\frac{\partial c}{\partial T} + r(T)\frac{\partial c}{\partial k} = \int_{-\infty}^{+\infty} \kappa(k-v) e^{2(v-k)} \alpha(e^{2v}, T) \left(\frac{\partial^2 c}{\partial k^2} - \frac{\partial c}{\partial k} \right) d(v)$$

where κ is defined as the exponential double tail of $k(u)\,du$, i.e:

$$\kappa(z) = \begin{cases} \int_{-\infty}^{z} dx\ e^x \int_{-\infty}^{x} k(u)\,du & z < 0 \\ \int_{z}^{+\infty} dx\ e^x \int_{x}^{\infty} k(u)\,du & z > 0 \end{cases}$$

The right hand side can be written as a convolution of distributions:

$$\frac{\partial c}{\partial T} + r(T)\frac{\partial c}{\partial k} = [a_T(.)\ \left(\frac{\partial^2 c}{\partial k^2} - \frac{\partial c}{\partial k} \right)] * g \qquad \text{where} \tag{35}$$

$$g(u) = e^{-2u}\kappa(u) \qquad a_T(u) = \alpha(e^{2u}, T) \tag{36}$$

Therefore, it implies that from the knowledge of $c(.,.)$ and a choice for $\kappa(.)$ we can recover a_T hence $\alpha(.,.)$. As noted by Carr et al. [8], this equation is analogous to the Dupire formula for diffusions: it enables to "invert" the structure of the jumps–represented by α– from the cross-section of option prices. Note that, like the Dupire formula, this inversion involves a double deconvolution/differentiation of c which illustrates the ill-posedness of the inverse problem.

2.4 Time changed Lévy processes

Time changed Lévy processes were proposed in [7] in the context of option pricing. Consider the price process S whose dynamics under the pricing measure $\mathbb{P}$ is given by:

$$\left(S_T \equiv e^{\int_0^T r(t)\,dt} X_T \right) \qquad X_t = \exp\left(L_{T_t}\right) \qquad T_t = \int_0^t \theta_s ds \tag{37}$$

where L_t is a Lévy process with characteristic triplet (b, σ^2, ν), N its jump measure and (θ_t) is a locally bounded positive semimartingale. We assume L and θ are $\mathcal{F}_t$-adapted.
$X_t \equiv (e^{-\int_0^T r(t)\,dt} S_T)$ is a martingale under the pricing measure $\mathbb{P}$ if $\exp(L_t)$ is a martingale which requires the following condition on the characteristic triplet of (L_t):

$$b + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} (e^z - 1 - z\,1_{|z|\leq 1})\nu(dy) = 0 \tag{38}$$

Define the value $C_{t_0}(T,K)$ at time t_0 of the call option with expiry $T > t_0$ and strike $K > 0$ of the stock price (S_t):

$$C_{t_0}(T,K) = e^{-\int_0^T r(t)\,dt} E^{\mathbb{P}}[\max(S_T - K, 0)|\mathcal{F}_{t_0}] \tag{39}$$

[1] Note however that the equation given in [8] does not seem to be correct: it involves the double tail of $k(z)\,dz$ instead of the exponential double tail.

Proposition 4. *Define*

$$\alpha(t,x) = E[\theta_t | X_{t-} = x]$$

and χ the exponential double tail of $\nu(du)$

$$\chi(z) = \begin{cases} \int_{-\infty}^{z} dx\ e^x \int_{-\infty}^{x} \nu(du) & z < 0 \\ \int_{z}^{+\infty} dx\ e^x \int_{x}^{\infty} \nu(du) & z > 0 \end{cases} \tag{40}$$

Then under assumption (H_2'), the call option price $C_{t_0} : (T,K) \mapsto C_{t_0}(T,K)$ at date t_0, as a function of maturity and strike, is a solution (in the sense of distributions) of the partial integro-differential equation:

$$\begin{aligned} \frac{\partial C}{\partial T} &= -r\alpha(T,K)K\frac{\partial C}{\partial K} + \frac{K^2\alpha(T,K)\sigma^2}{2}\frac{\partial^2 C}{\partial K^2} \\ &\quad + \int_0^{+\infty} y \frac{\partial^2 C}{\partial K^2}(T,dy)\,\alpha(T,y)\,\chi\left(\ln\left(\frac{K}{y}\right)\right) \end{aligned} \tag{41}$$

on $[t,\infty[\times]0,\infty[$ with the initial condition: $\forall K > 0 \quad C_{t_0}(t_0,K) = (S_{t_0} - K)_+$.

Proof. Using [4, Theorem 4], (L_{T_t}) writes

$$\begin{aligned} L_{T_t} &= L_0 + \int_0^t \sigma\sqrt{\theta_s} dB_s + \int_0^t b\theta_s ds \\ &\quad + \int_0^t \theta_s \int_{|z|\le 1} z\tilde{N}(ds\,dz) + \int_0^t \theta_s \int_{|z|>1} zN(ds\,dz) \end{aligned}$$

where N is an integer-valued random measure with compensator $\nu(dz)\,dt$, $\tilde{N}$ its compensated random measure.
Applying the Itô formula yields

$$\begin{aligned} X_t &= X_0 + \int_0^t X_{s-}\left[b\theta_s + \frac{1}{2}\sigma^2\theta_s\right] ds + \int_0^t X_{s-}\sigma\sqrt{\theta_s}\,dB_s \\ &+ \int_0^t X_{s-}\theta_s \int_{|z|\le 1} z\tilde{N}(ds\,dz) + \int_0^t X_{s-}\theta_s \int_{|z|>1} zN(ds\,dz) \\ &+ \int_0^t \int_{\mathbb{R}} X_{s-}\theta_s(e^z - 1 - z)N(ds\,dz) \end{aligned}$$

Under assumption (H_2'), $\int (e^z - 1 - z\,1_{|z|\le 1})\nu(dz) < \infty$, hence:

$$\begin{aligned} X_t &= X_0 + \int_0^t X_{s-}\left[b\theta_s + \frac{1}{2}\sigma^2\theta_s + \int_{\mathbb{R}}(e^z - 1 - z\,1_{|z|\le 1})\theta_s\nu(dz)\right] ds + \int_0^t X_{s-}\sigma\sqrt{\theta}\,dB_s \\ &+ \int_0^t \int_{\mathbb{R}} X_{s-}\theta_s(e^z - 1)\tilde{N}(ds\,dz) \\ &= X_0 + \int_0^t X_{s-}\sigma\sqrt{\theta_s}\,dB_s + \int_0^t \int_{\mathbb{R}} X_{s-}\theta_s(e^z - 1)\tilde{N}(ds\,dz) \end{aligned}$$

and (S_t) may be expressed as:

$$S_t = S_0 + \int_0^t S_{s-} r(s)\, ds + \int_0^t S_{s-} \sigma \sqrt{\theta_s}\, dB_s + \int_0^t \int_{\mathbb{R}} S_{s-} \theta_s (e^z - 1) \tilde{N}(ds\, dz)$$

(S_t) is now in the suitable form (1) to apply Theorem 1, which yields the result. □

2.5 Index options in a multivariate jump-diffusion model

Consider a multivariate model with d assets:

$$S_T^i = S_0^i + \int_0^T r(t) S_{t-}^i dt + \int_0^T S_{t-} \delta_t^i dW_t^i + \int_0^T \int_{\mathbb{R}^d} S_{t-}^i (e^{y_i} - 1) \tilde{N}(dt\, dy)$$

where δ^i is an adapted process taking values in $\mathbb{R}$ representing the volatility of asset i, W is a d-dimensional Wiener process, N is a Poisson random measure on $[0,T] \times \mathbb{R}^d$ with compensator $\nu(dy)\, dt$, $\tilde{N}$ denotes its compensated random measure.
The Wiener processes W^i are correlated: for all $1 \leq (i,j) \leq d$, $\langle W^i, W^j \rangle_t = \rho_{i,j} t$, with $\rho_{ij} > 0$ and $\rho_{ii} = 1$.
An index is defined as a weighted sum of the asset prices:

$$I_t = \sum_{i=1}^d w_i S_t^i$$

The value $C_{t_0}(T,K)$ at time t_0 of an index call option with expiry $T > t_0$ and strike $K > 0$ is given by

$$C_{t_0}(T,K) = e^{-\int_{t_0}^T r(t)\, dt} E^{\mathbb{P}}[\max(I_T - K, 0) | \mathcal{F}_{t_0}] \tag{42}$$

The following result generalizes the forward PDE studied by Avellaneda et al. [3] for the diffusion case to a setting with jumps:

Theorem 2. *Forward PIDE for index call options. Assume*

$$\begin{cases} \forall T > 0 \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \|\delta_t\|^2\, dt\right)\right] < \infty & (A_{1b}) \\ \int_{\mathbb{R}^d} (1 \wedge \|y\|)\, \nu(dy) < \infty \quad a.s. & (A_{2b}) \\ \int_{\|y\|>1} e^{2\|y\|} \nu(dy) < \infty \quad a.s. & (A_{3b}) \end{cases} \tag{43}$$

Let $k(.,t,dy)$ be the random measure:

$$k(t,dy) = \int_{\mathbb{R}^{d-1}-\{0\}} \ln\left(\frac{\sum_{1\leq i\leq d-1} w_i S_{t-}^i e^{y_i} + w_d S_{t-}^d e^y}{I_{t-}}\right) \nu(dy_1, \cdots, dy_{d-1}, dy)$$

and $\eta_t(z)$ its exponential double tail:

$$\eta_t(z) = \begin{cases} \int_{-\infty}^z dx\ e^x \int_{-\infty}^x k(t,du) & z < 0 \\ \int_z^{+\infty} dx\ e^x \int_x^{\infty} k(t,du) & z > 0 \end{cases} \tag{44}$$

Define:

$$\sigma(t,z) = \frac{1}{z}\sqrt{\mathbb{E}\left[\left(\sum_{i,j=1}^{d} w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right) | I_{t^-} = z\right]}; \tag{45}$$

$$\chi_{t,y}(z) = \mathbb{E}\left[\eta_t(z) \,|I_{t-} = y\right] \tag{46}$$

The index call price $(T,K) \mapsto C_{t_0}(T,K)$, as a function of maturity and strike, is a solution (in the sense of distributions) of the partial integro-differential equation:

$$\frac{\partial C}{\partial T} = -r(T)K\frac{\partial C}{\partial K} + \frac{\sigma(T,K)^2}{2}\frac{\partial^2 C}{\partial K^2} + \int_0^{+\infty} y \frac{\partial^2 C}{\partial K^2}(T,dy)\,\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right) \tag{47}$$

on $[t_0,\infty[\times]0,\infty[$ with the initial condition: $\forall K>0 \quad C_{t_0}(t_0,K) = (I_{t_0}-K)_+$.

Proof. $(B_t)_{t\geq 0}$ defined by

$$dB_t = \frac{\sum_{i=1}^d S_{t-}^i \delta_t^i dW_t^i}{\left(\sum_{i,j=1}^d \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right)^{1/2}}$$

is a continuous local martingale with quadratic variation t: by Lévy's theorem, B is a Brownian motion. Hence I may be decomposed as

$$\begin{aligned} I_T = &\sum_{i=1}^d w_i S_0^i + \int_0^T r(t) I_{t-}\, dt + \int_0^T \left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right)^{\frac{1}{2}} dB_t \\ &+ \int_0^T \int_{\mathbb{R}^d} \sum_{i=1}^d w_i S_{t-}^i (e^{y_i}-1)\tilde{N}(dt\,dy) \end{aligned} \tag{48}$$

The essential part of the proof consists in rewriting (I_t) in the suitable form (1) to apply Theorem 1.

Applying the Itô formula to $\ln(I_T)$ yields:

$$
\begin{aligned}
\ln(I_T) &= \ln(I_0) + \int_0^T \frac{1}{I_{t^-}}\, dI_t + \frac{1}{2}\int_0^T \frac{-1}{I_{t^-}^2}\left(\sum_{i,j=1}^{d} w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right) dt \\
&+ \sum_{t\le T}\left[\ln(I_{t^-} + \Delta I_t) - \ln(I_{t^-}) - \frac{1}{I_{t^-}}\Delta I_t\right] \\
&= \ln(I_0) + \int_0^T \left(r(t) - \frac{1}{2I_{t-}^2}\left(\sum_{i,j=1}^{d} w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right)\right) dt \\
&+ \int_0^T \frac{1}{I_{t-}}\left(\sum_{i,j=1}^{d} w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right)^{\frac{1}{2}} dB_t \\
&+ \int_0^T \int_{\mathbb{R}^d} \left(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} - 1\right) \tilde{N}(dt\, dy) \\
&- \int_0^T \int_{\mathbb{R}^d} \left(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} - 1 - \ln\left(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}}\right)\right) N(dt\, dy) \\
&= \ln(I_0) + \int_0^T \Big(r(t) - \frac{1}{2I_{t-}^2}\left(\sum_{i,j=1}^{d} w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right) \\
&\qquad - \int \left(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} - 1 - \ln\left(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}}\right)\right)\nu(dy)\Big)\, dt \\
&+ \int_0^T \frac{1}{I_{t-}}\left(\sum_{i,j=1}^{d} w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right)^{\frac{1}{2}} dB_t \\
&+ \int_0^T \int_{\mathbb{R}^d} \ln\left(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}}\right) \tilde{N}(dt\, dy) \qquad (49)
\end{aligned}
$$

The last equality is obtained since
$\int \left(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} - 1 - \ln\left(\frac{\sum_{1\le i\le d} w_i S_{t-}^i e^{y_i}}{I_{t-}}\right)\right)\nu(dy) < \infty$: using the convexity property of the logarithm (one recalls that $\sum_{1\le i\le d} \frac{w_i S_{t-}^i}{I_{t-}} = 1$), and the

Hölder inequality:

$$\left|\ln\left(\frac{\sum_{1\leq i\leq d} w_i S^i_{t-} e^{y_i}}{I_{t-}}\right)\right| \leq \left|\sum_{1\leq i\leq d} \frac{w_i S^i_{t-}}{I_{t-}} y_i\right| \tag{50}$$

$$\leq \sum_{1\leq i\leq d} |y_i| \leq \left(\sum_{1\leq i\leq d} y_i^2\right)^{1/2} = \|y\|, \tag{51}$$

$$\tag{52}$$

hence the functions $y \to \ln\left(\frac{\sum_{1\leq i\leq d} w_i S^i_{t-} e^{y_i}}{I_{t-}}\right)$ and $y \to \frac{\sum_{1\leq i\leq d} w_i S^i_{t-} e^{y_i}}{I_{t-}}$ are integrable with respect to $\nu(dy)$ by assumptions (A_{2b}) and (A_{3b}). We furthermore observe that

$$\begin{array}{c} \int_{\mathbb{R}^d} \left(1 \wedge \left|\ln\left(\frac{\sum_{1\leq i\leq d} w_i S^i_{t-} e^{y_i}}{I_{t-}}\right)\right|\right) \nu(dy) < \infty \quad a.s. \\ \int_0^T \int_{\|y\|>1} e^{2\left|\ln\left(\frac{\sum_{1\leq i\leq d} w_i S^i_{t-} e^{y_i}}{I_{t-}}\right)\right|} \nu(dy)\, dt < \infty \quad a.s. \end{array} \tag{53}$$

Similarly, since (A_{2b}) and (A_{3b}), for all $1 \leq i \leq d$, $\int \left(e^{y_i} - 1 - 1_{|y_i|\leq 1} y_i\right) \nu(dy) < \infty$ and $\ln\left(S^i_T\right)$ rewrites:

$$\begin{array}{rl} \ln\left(S^i_T\right) & = \ln\left(S^i_0\right) + \int_0^T \left(r(t) - \frac{1}{2}(\delta^i_t)^2 - \int_{-\infty}^{\infty} \left(e^{y_i} - 1 - 1_{|y_i|\leq 1} y_i\right) \nu(dy)\, dt\right) dt \\ & \quad + \int_0^T \delta^i_t\, dW^i_t + \int_0^T \int_{\mathbb{R}^d} y_i\, \tilde{N}(dt\, dy) \end{array}$$

Define the d-dimensional martingale $W_t = (W^1_t, \cdots, W^{d-1}_t, B_t)$. For all $1 \leq i, j \leq d-1$, $\langle W^i, W^j\rangle_t = \rho_{i,j} t$;

$$\langle W^i, B\rangle_t = \frac{\sum_{j=1}^d w_j \rho_{ij} S^j_{t-} \delta^j_t}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta^i_t \delta^j_t\, S^i_{t-} S^j_{t-}\right)^{1/2}} t$$

Define

$$\Theta_t = \begin{pmatrix} \rho_{11} & \cdots & \rho_{1,d-1} & \frac{\sum_{j=1}^d w_j \rho_{1j} S^j_{t-} \delta^j_t}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta^i_t \delta^j_t\, S^i_{t-} S^j_{t-}\right)^{1/2}} \\ \vdots & \cdots & \vdots & \vdots \\ \rho_{d-1,1} & \cdots & \rho_{d-1,d-1} & \frac{\sum_{j=1}^d w_j \rho_{d-1,j} S^j_{t-} \delta^j_t}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta^i_t \delta^j_t\, S^i_{t-} S^j_{t-}\right)^{1/2}} \\ \frac{\sum_{j=1}^d w_j \rho_{1,j} S^j_{t-} \delta^j_t}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta^i_t \delta^j_t\, S^i_{t-} S^j_{t-}\right)^{1/2}} & \cdots & \frac{\sum_{j=1}^d w_j \rho_{d-1,j} S^j_{t-} \delta^j_t}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta^i_t \delta^j_t\, S^i_{t-} S^j_{t-}\right)^{1/2}} & 1 \end{pmatrix}$$

There exists a standard Brownian motion (Z_t) such that $W_t = AZ_t$ where A is a $d \times d$ matrix verifying

$$\Theta = {}^t A\, A.$$

Define $X_T \equiv \big(\ln(S_T^1), \cdots, \ln(S_T^{d-1}), \ln(I_T)\big)$;

$$\delta = (\delta_t^1, \cdots, \delta_t^{d-1}, \frac{1}{I_{t-}} \left(\sum_{i,j=1}^{d} w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j \right)^{\frac{1}{2}});$$

$$\beta_t = \begin{pmatrix} r(t) - \frac{1}{2}(\delta_t^1)^2 - \int_{-\infty}^{\infty} (e^{y_1} - 1 - y_1)\, \nu(dy) \\ \vdots \\ r(t) - \frac{1}{2}(\delta_t^{d-1})^2 - \int_{-\infty}^{\infty} (e^{y_{d-1}} - 1 - y_{d-1})\, \nu(dy) \\ r(t) - \frac{1}{2I_{t-}^2} \sum_{i,j=1}^{d} w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j - \int_{-\infty}^{\infty} \left(\frac{\sum_{1 \le i \le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} - 1 - \ln\left(\frac{\sum_{1 \le i \le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \right) \nu(dy) \end{pmatrix}$$

and

$$\psi_t(y) = \begin{pmatrix} y_1 \\ \vdots \\ y_{d-1} \\ \ln\left(\frac{\sum_{1 \le i \le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \end{pmatrix}$$

then (X_T) may be expressed as:

$$X_T = X_0 + \int_0^T \beta_t\, dt + \int_0^T \delta_t A\, dZ_t + \int_0^T \int_{\mathbb{R}^d} \psi_t(y)\, \tilde{N}(dt\, dy) \tag{54}$$

Let us now compute the inverse ϕ of ψ:

$$\begin{aligned} \phi: \quad [0,T] \times \Omega \times \mathbb{R}^d &\mapsto \quad \mathbb{R}^d & (55) \\ (t, \omega, (y_1, \cdots, y_d)) &\quad (\, \phi_t(\omega, y)_1, \cdots, \phi_t(\omega, y)_d\,) & (56) \end{aligned}$$

For all t in $[0,T]$,for all $y \in \mathbb{R}^d$

$$\phi_t(\omega, \psi_t(\omega, y)) = \psi_t(\omega, \phi_t(\omega, y)) = y.$$

Obviously, for $1 \le i \le (d-1)$, $(\phi_t(y))_i = y_i$. For $i = d$:

$$\begin{aligned} \psi_t(\phi_t(y))_d &= \ln\left(\frac{\sum_{1 \le i \le d} w_i S_{t-}^i e^{(\phi_t(y))_i}}{I_{t-}} \right) & (57) \\ &= \ln\left(\frac{\sum_{1 \le i \le d-1} w_i S_{t-}^i e^{y_i} + w_d S_{t-}^d e^{(\phi_t(y))_d}}{I_{t-}} \right) = y_d & (58) \end{aligned}$$

Hence:

$$\phi_t(y) = \left(y_1, \cdots, y_{d-1}, \ln\left(\frac{e^{y_d} I_{t-} - \sum_{1 \le i \le d-1} w_i S_{t-}^i e^{y_i}}{w_d S_{t-}^d} \right) \right)$$

Observe that $\psi_t(.,0) = 0$, ϕ is predictable, and $\phi_t(\omega,.)$ is differentiable with Jacobian matrix $\nabla_y \phi_t(y)$:

$$(\nabla_y \phi_t(y))_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ \frac{e^{y_1} w_1 S^1_{t-}}{e^{y_1} I_{t-} - \sum_{1\le i\le d-1} w_i S^i_{t-} e^{y_i}} & \cdots & \frac{e^{y_{d-1}} w_{d-1} S^{d-1}_{t-}}{e^{y_d} I_{t-} - \sum_{1\le i\le d-1} w_i S^i_{t-} e^{y_i}} & \frac{e^{y_d} I_{t-}}{e^{y_d} I_{t-} - \sum_{1\le i\le d-1} w_i S^i_{t-} e^{y_i}} \end{pmatrix}$$

(ψ, ν) satisfies [4, Assumption (H3)]: since (A_{2b}), for all T and for all t in $[0,T]$,

$$\begin{aligned} & \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} (1 \wedge \|\psi_t(.,y)\|^2)\, \nu(dy)\, dt\right] \\ = & \int_0^T \int_{\mathbb{R}^d} 1 \wedge \left(y_1^2 + \cdots + y_{d-1}^2 + \ln\left(\frac{\sum_{1\le i\le d} w_i S^i_{t-} e^{y_i}}{I_{t-}}\right)^2\right) \nu(dy)\, dt \\ \le & \int_0^T \int_{\mathbb{R}^d} 1 \wedge (2\|y\|^2)\, \nu(dy)\, dt < \infty \end{aligned}$$

Define ν_ϕ, the image of ν by ϕ:

$$\forall A \in \mathcal{B}(\mathbb{R}^d - \{0\}) \quad \nu_\phi(\omega, t, A) = \nu(\phi_t(\omega, A)) \tag{59}$$

Applying [4, Lemma 2], X_T may be expressed as:

$$X_T = X_0 + \int_0^T \beta_t\, dt + \int_0^T \delta_t\, dZ_t + \int_0^T \int y\, \tilde{M}(dt\, dy)$$

where M is an integer-valued random measure (resp. $\tilde{M}$ its compensated random measure) with compensator

$$\mu(\omega; dt\ dy) = m(t, dy; \omega)\, dt$$

defined via its density with respect to ν_ϕ:

$$\frac{d\mu}{d\nu_\phi}(\omega, t, y) = \left| 1 - \sum_{1\le i\le d-1} \frac{w_i S^i_{t-}(\omega)}{I_{t-}(\omega)} e^{y_i - y_d} \right|^{-1}. \tag{60}$$

Considering now the d-th component of X_T, one obtains the semimartingale decomposition of $\ln(I_t)$:

$$\begin{aligned} \ln(I_T) = & \ln(I_0) + \int_0^T \Big(r(t) - \frac{1}{2I_{t-}^2} \left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta^i_t \delta^j_t\, S^i_{t-} S^j_{t-} \right) \\ & - \int \left(\frac{\sum_{1\le i\le d} w_i S^i_{t-} e^{y_i}}{I_{t-}} - 1 - \ln\left(\frac{\sum_{1\le i\le d} w_i S^i_{t-} e^{y_i}}{I_{t-}} \right) \right) \nu(dy) \Big)\, dt \\ & + \int_0^T \frac{1}{I_{t-}} \left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta^i_t \delta^j_t\, S^i_{t-} S^j_{t-} \right)^{\frac{1}{2}} dB_t + \int_0^T \int y\, \tilde{K}(dt\, dy) \end{aligned}$$

where K is an integer-valued random measure on $[0,T]\times\mathbb{R}$ with compensator $k(t,dy)\,dt$ defined by: $\forall B\in\mathcal{B}(\mathbb{R}-\{0\})$,

$$k(t,B)=\mu(t,\mathbb{R}^{d-1}\times B) \tag{61}$$

and $\tilde{K}$ its compensated random measure. Therefore

$$I_T=I_0+\int_0^T r(t)I_{t-}\,dt+\int_0^T\left(\sum_{i,j=1}^d w_iw_j\rho_{ij}\,\delta_t^i\delta_t^j\,S_{t-}^iS_{t-}^j\right)^{\frac{1}{2}}dB_t+\int_0^T\int_{\mathbb{R}^d}(e^y-1)\,I_{t-}\tilde{K}(dt\,dy)$$

The volatility of I_t satisfies for all t in $[0,T]$,

$$\frac{\sum_{i,j=1}^d w_iw_j\rho_{ij}\,\delta_t^i\delta_t^j\,S_{t-}^iS_{t-}^j}{I_{t-}^2}\leq\sum_{i,j=1}^d\rho_{ij}\,\delta_t^i\delta_t^j$$

and

$$\left|\ln\left(\frac{\sum_{1\leq i\leq d}w_iS_{t-}^ie^{y_i}}{I_{t-}}\right)\right|\leq\|y\|.$$

Hence:

$$\begin{aligned}
&\frac{1}{2}\int_0^T\frac{\sum_{i,j=1}^d w_iw_j\rho_{ij}\,\delta_t^i\delta_t^j\,S_{t-}^iS_{t-}^j}{I_{t-}^2}\,dt+\int_0^T(e^y-1)^2k(t,\,dy)\,dt\\
=\;&\frac{1}{2}\int_0^T\frac{\sum_{i,j=1}^d w_iw_j\rho_{ij}\,\delta_t^i\delta_t^j\,S_{t-}^iS_{t-}^j}{I_{t-}^2}\,dt\\
+\;&\int_0^T\int_{\mathbb{R}^d}\left(\frac{\sum_{1\leq i\leq d-1}w_iS_{t-}^ie^{y_i}+w_dS_{t-}^de^y}{I_{t-}}-1\right)^2\nu(dy_1,\cdots,dy_{d-1},dy)\,dt\\
\leq\;&\frac{1}{2}\sum_{i,j=1}^d\rho_{ij}\,\delta_t^i\delta_t^j+\int_0^T\int_{\mathbb{R}^d}(e^{\|y\|}-1)^2\nu(dy_1,\cdots,dy_{d-1},dy)\,dt
\end{aligned}$$

Using assumptions (A_{1b}), (A_{2b}) and (A_{3b}), the last inequality implies that (I_t) satisfies (H). Hence (I_t) is now in a suitable form to apply Theorem 1, which yields the result. We leave the end of the proof to the reader. □

2.6 Forward equations for CDO pricing

Portfolio credit derivatives such as CDOs or index default swaps are derivatives whose payoff depends on the total loss L_t due to defaults in a reference portfolio of obligors. Reduced-form top-down models of portfolio default risk [19, 20, 30, 10, 31] represent the default losses of a portfolio as a *marked point process* $(L_t)_{t\geq 0}$ where the jump times represents credit events in the portfolio and the jump sizes ΔL_t represent the portfolio loss upon a default event. Marked point processes with random intensities are increasingly used as ingredients in such models [19, 20, 25, 30, 31].

In all such models the loss process (represented as a fraction of the portfolio notional) may be represented as

$$L_t = \int_0^t \int_0^1 x\, M(ds\, dx)$$

where $M(dt\, dx)$ is an integer-valued random measure whose compensator $\mu(dt\, dx; \omega) = m(t, dx; \omega)\, dt$ has finite mass

$$\lambda_t(\omega) = \int_0^1 m(t, dx; \omega)$$

N_t represents the number of defaults and λ represents the default intensity i.e. the (random) jump intensity of the point process $N_t = M([0,t] \times [0,1])$. Denote by $T_1 \leq T_2 \leq ..$ the jump times of N. The *marked point process* L may also be represented as

$$L_t = \sum_{k=1}^{N_t} Z_k$$

where the "mark" Z_k taking values in $[0,1]$ is distributed according to

$$F_t(dx; \omega) = \frac{m_X(t, dx; \omega)}{\lambda_t(\omega)}$$

Note that the percentage loss L_t belongs to $[0,1]$, so $\Delta L_t \in [0, 1 - L_{t-}]$. For the equity tranche $[0, K]$, we define the expected tranche notional at maturity T as

$$C_{t_0}(T, K) = \mathbb{E}[(K - L_T)_+ | \mathcal{F}_{t_0}] \tag{62}$$

As noted in [10], the prices of portfolio credit derivatives such as CDO tranches only depend on the loss process through the expected tranche notionals. Therefore, if one is able to compute $C_{t_0}(T, K)$ then one is able to compute the values of all CDO tranches at date t_0. In the case of a loss process with constant loss increment, Cont and Savescu [11] derived a forward equation for the expected tranche notional. The following result generalizes the forward equation derived by Cont and Savescu [11] to a more general setting which allows for random, dependent loss sizes and possible dependence between the loss given default and the default intensity:

Proposition 5 (Forward equation for expected tranche notionals)**.** *Define the integer-valued random measure $M_Y(dt\, dy)$ with compensator $m_Y(t, dy, Y_{t-})\, dt$ defined by :*

$$\forall A \in \mathcal{B}([0,1]), \qquad m_Y(t, A, z) = E[m_X(t, A, .) | L_{t-} = z] \tag{63}$$

and the effective default intensity

$$\lambda^Y(t, z) = \int_0^{1-z} m_Y(t, dy, z) \tag{64}$$

The expected tranche notional $(T,K) \mapsto C_{t_0}(T,K)$, as a function of maturity and strike, is a solution of the partial integro-differential equation:

$$\frac{\partial C}{\partial T} = -\int_0^K \frac{\partial^2 C}{\partial K^2}(T,dy) \left[\int_0^{K-y} (K-y-z)\, m_Y(T,dz,y) - (K-y)\lambda^Y(T,y)\right] \tag{65}$$

on $[t_0,\infty[\times]0,1[$ *with the initial condition:*
$\forall K \in [0,1] \quad C_{t_0}(t_0,K) = (L_{t_0}-K)_+$.

Proof. By replacing $\mathbb{P}$ by the conditional measure $\mathbb{P}_{|\mathcal{F}_0}$ given $\mathcal{F}_0$, we may replace the conditional expectation in (62) by an expectation with respect to the marginal distribution $p_T(dy)$ of L_T under $\mathbb{P}_{|\mathcal{F}_{t_0}}$. Thus, without loss of generality, we put $t_0 = 0$ in the sequel and consider the case where $\mathcal{F}_0$ only contains null sets. Observe that the definition of m_Y consists in defining the compensator of the mimicking process of L_Y.
Define the operator H_T as follows: for f in $C_0^\infty([0,1])$,

$$H_T f(y) = \int_0^{1-y} [f(y+z)-f(y)] m_Y(T,dz,y)$$

Applying [4, Theorem 4] leads to the Kolmogorov forward equation for $((L_t),\mathbb{P}_{|\mathcal{F}_0})$:

$$\frac{\partial p_T}{\partial T} = H_T^\star . p_T,$$

where $H_T^\star$ is the L^2-adjoint of H_T. Hence for $y \to (K-y)^+$

$$\begin{aligned} H_T.(K-y)^+ &= \int_0^{1-y} [(K-y-z)^+ - (K-y)^+] m_Y(T,dz,y) \\ &= \int_0^{1-y} [(K-y-z)1_{K-y>z\}} - (K-y)1_{K-y>0\}}] m_Y(T,dz,y) \end{aligned}$$

If $K \geq y$,

$$\begin{aligned} H_T.(K-y)^+ &= \int_0^{K-y} [(K-y-z)-(K-y)]\, m_Y(T,dz,y) + \int_{K-y}^{1-y} [(0-(K-y)]\, m_Y(T,dz,y) \\ &= -\int_0^{K-y} z\, m_Y(T,dz,y) - (K-y)\left[\int_0^{1-y} m_Y(T,dz,y) - \int_0^{K-y} m_Y(T,dz,y)\right] \\ &= -\int_0^{K-y} (K-y-z)\, m_Y(T,dz,y) - (K-y)\lambda^Y(T,y) \end{aligned}$$

For $K < y$, $H_T.(K-y)^+ = 0$. (62) can be expressed as

$$C(T,K) = \int_0^1 (K-y)^+ \, p_T(dy) \tag{66}$$

By differentiating with respect to K, we get

$$\frac{\partial C}{\partial K} = \int_K^\infty p_T(dy) \qquad \frac{\partial^2 C}{\partial K^2}(T, dy) = p_T(dy)$$

$$\begin{aligned}
\frac{\partial C}{\partial T} &= \int_0^1 (K-y)^+ \, \frac{\partial p_T}{\partial T}(dy) \\
&= \int_0^1 (K-y)^+ \, H_T^\star . \, p_T(dy) = \int_0^1 p_T(dy) \, H_T . \, (K-y)^+ \\
&= -\int_0^K p_T(dy) \, \left[\int_0^{K-y} (K-y-z) \, m_Y(T, dz, y) - (K-y) \lambda^Y(T, y) \right]
\end{aligned}$$

Then

$$\frac{\partial C}{\partial T} = -\int_0^K \frac{\partial^2 C}{\partial K^2}(T, dy) \, \left[\int_0^{K-y} (K-y-z) \, m_Y(T, dz, y) - (K-y) \lambda^Y(T, y) \right]$$

□

In [11], loss given default (i.e. the jump size of L) is assumed constant $\delta = (1-R)/n$: the marks Z_k are then deterministic and equal to δ : $L_t = \delta N_t$ and one can compute $C(T, K)$ using the law of N_t. Setting $t_0 = 0$ and assuming as above that $\mathcal{F}_{t_0}$ only contains null sets,

$$\delta C_k(T) = C(T, K) = \mathbb{E}[(K - L_T)^+] = \mathbb{E}[(k\,\delta - L_T)^+] = \delta \mathbb{E}[(k - N_T)^+] \quad (67)$$

The compensator of L_t is $\lambda_t \, \epsilon_\delta(dz) \, dt$, where $\epsilon_\delta(dz)$ is the point mass at the point δ. The effective compensator becomes: $m_Y(t, dz, y) = E[\lambda_t | L_{t-} = y] \, \epsilon_\delta(dz) \, dt = \lambda^Y(t, y) \, \epsilon_\delta(dz)$ and the effective default intensity is $\lambda^Y(t, y) = E[\lambda_t | L_{t-} = y]$.

If we set $y = j\delta$ then : $\lambda^Y(t, j\delta) = E[\lambda_t | L_{t-} = j\delta] = E[\lambda_t | N_{t-} = j] = a_j(t)$ and $p_t(dy) = \sum_{j=0}^n q_j(t) \epsilon_{j\delta}(dy)$ with the notations in [11].
Let us focus on (65) in this case. We recall from the proof of Proposition 5 that:

$$\begin{aligned}
\frac{\partial C}{\partial T}(T, k\delta) &= \int_0^1 p_T(dy) \, H_T . \, (k\delta - y)^+ \\
&= \int_0^1 p_T(dy) \int_0^{1-y} [(k\delta - y - z)^+ - (k\delta - y)^+] \, \lambda^Y(T, y) \, \epsilon_\delta(dz) \\
&= \int_0^1 p_T(dy) \, \lambda^Y(T, y) \, [(k\delta - y - \delta)^+ - (k\delta - y)^+] \, 1_{\delta < 1-y\}} \\
&= -\delta \sum_{j=0}^n q_j(T) \, a_j(T) \, 1_{j \le k-1} \qquad (68)
\end{aligned}$$

This expression can be simplified as in [11, Proposition 2], leading to the forward equation

$$\begin{aligned}\frac{\partial C_k(T)}{\partial T} &= a_k(T)C_{k-1}(T) - a_{k-1}(T)C_k(T) - \sum_{j=1}^{k-2} C_j(T)[a_{j+1}(T) - 2a_j(T) + a_{j-1}(T)] \\ &= [a_k(T) - a_{k-1}(T)]C_{k-1}(T) - \sum_{j=1}^{k-2}(\nabla^2 a)_j C_j(T) - a_{k-1}(T)[C_k(T) - C_{k-1}(T)]\end{aligned}$$

Hence we recover [11, Proposition 2] as a special case of Proposition 5.

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