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Phenomenological Study of Dynamical Model of Traffic Flow

Masako Bando([1]), Katsuya Hasebe([1]), Ken Nakanishi([2]), Akihiro Nakayama([3]), Akihiro Shibata([4]) and Yūki Sugiyama([5])

([1])Physics Division, Aichi University, Miyoshi, Aichi 470-02, Japan
([2]) Department of Physics, Kyoto University, Kyoto 606-01, Japan
([3]) Gifu Keizai University, Ohgaki, Gifu 503, Japan
([4]) Department of Physics, Nagoya University, Nagoya 464-01, Japan
([5]) Division of Mathematical Science, City College of Mie, Tsu, Mie 514-01, Japan

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Abstract. — We show that the Car Following Model with Optimal Velocity (Optimal Velocity Model), which was proposed in our previous paper, is a successful traffic model in reproducing the characteristic features of observed traffic flow data. In our model the transition from free flow to congested flow occurs spontaneously by the collective motion of vehicles, which obey to the same dynamical equation. The observed specific discrepancy of traffic flow vs. car density graph is well understood in terms of the phase transition in our model.

1. Introduction

Traffic flow has been studied from many different points of view and there have been essentially two different types of approaches, macroscopic and microscopic ones (dynamical models). We are here mainly concerned with the latter approach, in which the global features of traffic flow are to be explained from the collective motion of individual vehicles. Among those approaches one of the most traditional treatment is the study of the Car Following Model [1–3]. These models are based on the reliable assumption that each driver controls his vehicle only according to his preceding vehicle's motion. Most models have taken the assumption that the acceleration is proportional to the relative velocity of two successive vehicles, and therefore have used essentially the first-order differential equation of motion of an individual vehicle. However, these models have difficulty in explaining the specific property of traffic flow, which has two distinct behaviors, free and congested flows.

Conventionally traffic flow is characterized by three basic quantities: velocity (v), flow (Q) and density (k). The main interests of traffic engineers are rather practical problems such as how many vehicles are allowed to run in the freeway traffic lanes with specified capacities. Measurements are made to search the relationship between the flow (or velocity) and the density. The dependence of the velocity on the density, which is usually called the k-v curve, generally shows that velocity is a decreasing function of density. The typical feature of traffic

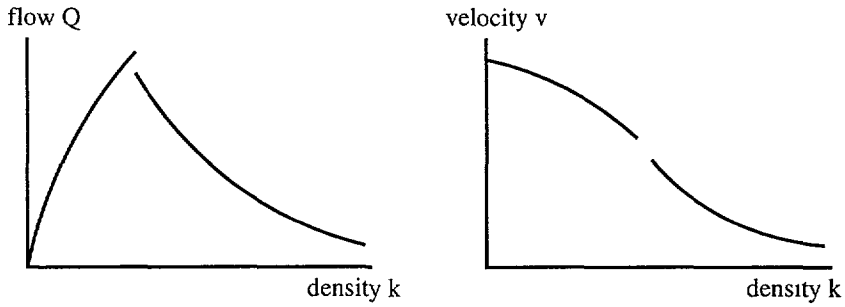

Fig. 1. — Illustration of the flow-density and velocity-density relationship of traffic flow.

flow has been expressed using the Q-k graph, which shows the relation between the flow and the density [4].

From accumulated data of measurement on freeways the following facts are commonly recognized. When the density is very low, each driver can control his vehicle almost freely, which can be called "free-flow". On the other hand, when the density is very high, each driver is strongly forced to decrease his vehicle's velocity and a "congested flow" appears. A typical k-v curve and Q-k graph are illustrated in Figure 1, neglecting the difference of quantitative details among several data. A possible discontinuity is observed between the high and low density regions and there seems to exist a critical density which separates the two regions [5–7]. We can identify these high and low density regions as the congested and the free flow.

Figure 2 is an example of the accumulated observed data taken by the Japan Highway Public Corporation (JHPC) [4]. You would find such types of graphs in most of textbooks of traffic theories. In Figure 2, we find a possible discontinuity at the occupancy $P \simeq 25\%$, which may be called a critical density as we mentioned before.

In the traditional studies of traffic flow, the behavior of two distinct region are separately explained with independent formula or models. No early works of Car Following Model could reproduce the discontinuity between free and congested flows in unified way.

In our previous paper [8, 9], we proposed a new model (hereafter we call "Optimal Velocity Model"). The essential difference from traditional Car Following Models is the introduction of an optimal velocity of a vehicle, which value is changed according to the headway distance. In our model the traffic congestion occurs spontaneously and this phenomenon can be understood as a sort of phase transition from free flow state to congested flow state. Actually, the dynamical equation of our model has two different kind of solutions. One is the homogeneous flow solution and the other is the congested flow solution, which consists of the two distinct regions; congested regions (high density) and smooth moving regions, or free regions (low density). The specific property of observed traffic flow, (Fig. 1 and Fig. 2), will be explained in this context.

The purpose of this paper is to apply our model to explain observed data and to show how well our model reproduces the Q-k graph of traffic flow, especially the behavior at the critical density. In Section 2 we explore some characteristics of observed traffic flow data. After doing a brief review of Optimal Velocity Model in Section 3, we shall apply our model to realistic traffic phenomena using the data of Chuo Motorway in Section 4. Summary and discussions will be given in Section 5.

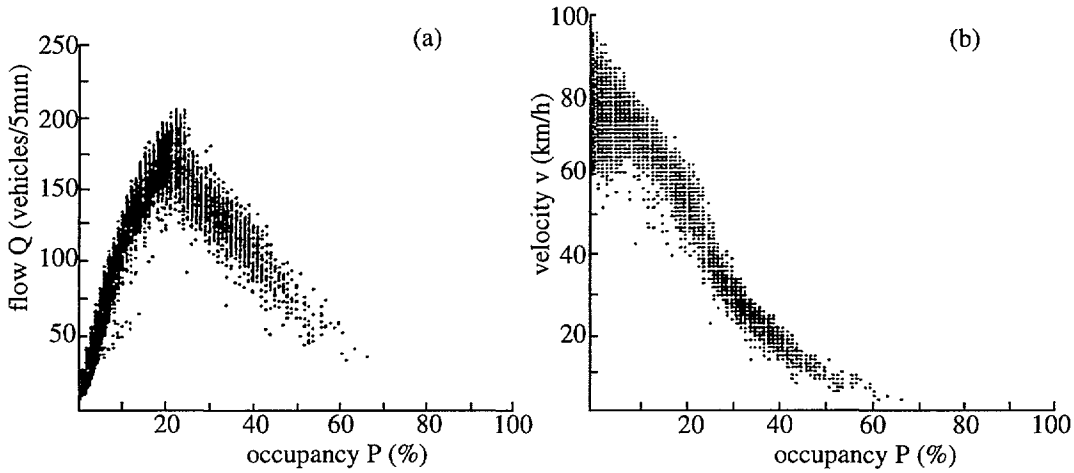

Fig. 2. — Example of flow-occupancy (a) and velocity-occupancy (b) relations obtained from the observed data taken by Japan Highway Public Corporation.

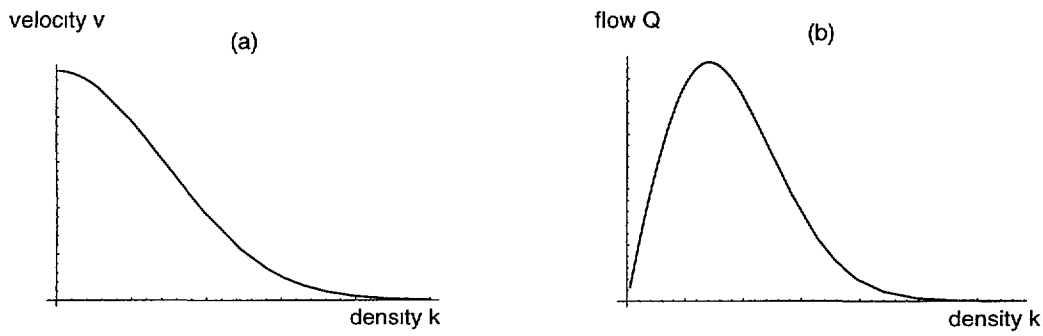

Fig. 3. — a) Drake's k-v curve; b) Q-k diagram corresponding to Drake's k-v curve.

2. Behavior of the Q-k Graph at Critical Density

In early works explaining the shape of Q-k curve, several functions have been proposed for the density-dependence of the velocity $v(k)$, by Greenshields [10], Greenberg [11], Edie [5], Underwood [12], Drew [13], Drake *et al.* [14]. Some of them are introduced semi-empirically and the others are derived from calculations of some underlying models. Among these works, the Drake curve, which is obtained from fluid dynamics, has been thought to reproduce the global features of observed data fairly well [14]. We show the illustration of Drake's curves in Figure 3. He derived the Q-k curve corresponding to a certain $v(k)$ with the assumption $Q = kv$. The curve does not reproduce a striking wedge at the critical density in the observed data (see Fig. 2). No one can obtain such wedge shape so long as he uses any continuous function for the k-v curve. Among the above works Edie's model only showed the wedge shape [5]. However, he introduced the critical point by hand.

Our model can explain the features of traffic phenomena for both congested and free flow in a unified way, i.e., to understand them from the same microscopic dynamical law. We know a lot of physical phenomena that the basic equation has a different kind of solutions, which have

quite different macroscopic aspects. This can be explained in terms of phase transition.

3. Optimal Velocity Model

Let us make a brief review of our dynamical model [8]. We investigate our model in the most simple situation where N vehicles move on a single lane circuit with a circumference L. Here we assume that road conditions are uniform along the circuit and drivers are identical.

We use the following notations; position x, velocity $\dot{x}$ and headway Δx as the basic variables for convenience. The dynamical equation for each vehicle is the following second order differential equation:

$$\ddot{x}_n = a\left\{V(\Delta x_n) - \dot{x}_n\right\}, \tag{1}$$

where x_n is the position of n-th vehicle and Δx_n denotes the headway of the n-th vehicle, $\Delta x_n = x_{n+1} - x_n$ and a is the sensitivity.

Each driver controls the acceleration of his vehicle in such a way that its velocity is maintained *at an optimal value $V(\Delta x)$ according to the headway Δx. We call this "Optimal Velocity"* ([1]). In contrast to earlier works, we do not introduce the discontinuous function $V(\Delta x)$ in order to reproduce the observed discontinuity in k-v and Q-k curves.

The Optimal Velocity function $V(\Delta x)$ should have some properties for safety driving [8]. We adopt a $\tanh(x)$-type curve as a candidate. For the convenience to fit the function with observed data, we rewrite $V(\Delta x)$ as

$$V(\Delta x) = V_0\left[\tanh m(\Delta x - b_{\mathrm{f}}) - \tanh m(b_{\mathrm{c}} - b_{\mathrm{f}})\right] \tag{2}$$

We have four parameters V_0, m, b_{f} and b_{c} in this function, which should be determined from observed data of the individual vehicle behavior. The maximum velocity for a large enough headway is given by $V_{\max} = V_0[1 - \tanh m(b_{\mathrm{c}} - b_{\mathrm{f}})]$ The headway, $\Delta x = b_{\mathrm{f}}$, corresponds to the inflection point of the Optimal Velocity function, where $V(b_{\mathrm{f}}) = V_{\max} - V_0$. The Optimal Velocity becomes zero at $\Delta x = b_{\mathrm{c}}$, which is regarded as an effective car length and is a little larger than the average length of vehicles, l_{c}, since each driver stops his vehicle before crashing:

$$b_{\mathrm{c}} = l_{\mathrm{c}} + \delta. \tag{3}$$

We have a homogeneous flow solution of equal spacing $b = L/N$ and constant velocity $V(b)$ for equation (1),

$$x_n(t) = b(n-1) + V(b)t \quad (n = 1, . \quad , N). \tag{4}$$

This solution is unstable in the case of $2V'(b) > a$, which corresponds to the area between two dashed lines as shown in Figure 5 and traffic congestion occurs spontaneously [8].

Figures 4a and 4b show typical results for $N = 100$ and $L = 200$, which are the snapshots at $t = 100$ and 1000 of the velocities of each vehicle, respectively ([2]). The homogeneous flow with a small initial disturbance develops into a congested flow with time evolution. Figure 4b indicates the spontaneous generation of congestion.

The profile of the congested flow is illustrated in the Δx - $\dot{x}$ plane (phase space). After the generation of congestion is finished, the motion of vehicles organizes the specific closed curve in the phase space. We called this "Hysteresis Loop" and the loop can be understood as some kind of a limit cycle. Here, we observe that each vehicle moves from the free region (indicated

([1]) It was called "legal velocity" in our previous papers.

([2]) Numerical solution of Figure 4 is obtained with parameter values, $V_0 = 1, m = 1, b_{\mathrm{f}} = 2$, $b_{\mathrm{c}} = 0$, and the initial condition: $x_1 = 0.1$, $x_n = b_{\mathrm{f}}(n-1)$ (for $n > 1$), $\dot{x}_n = V(b_{\mathrm{f}})$

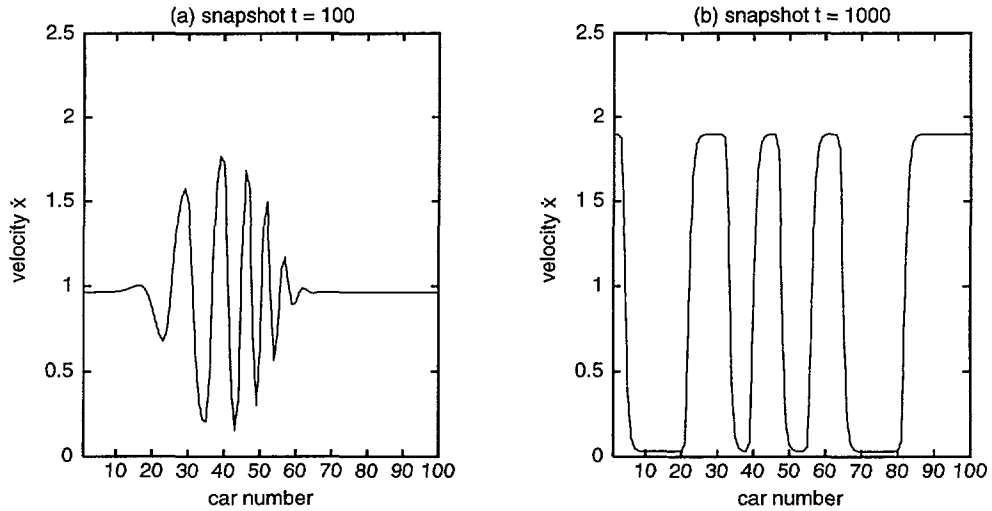

Fig. 4. — The snapshots at $t = 100$ (a) and $t = 1000$ (b). A congested flow (kink-like) solution appears as time develops.

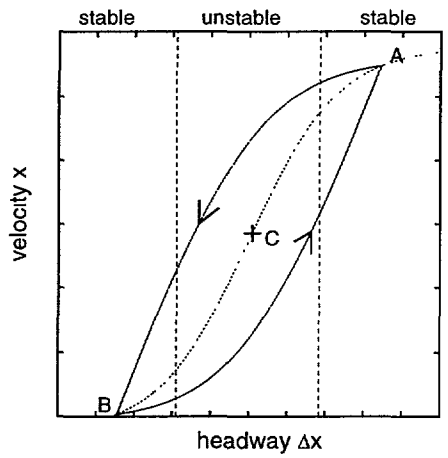

Fig. 5. — The motion of vehicles in the Δx - $\dot{x}$ plane on the "limit cycle". The dotted curve denotes $V(\Delta x)$. The stable and unstable regions of homogeneous flow solution are also shown.

by point A$\equiv$ $(\Delta x_{\mathrm{F}}, v_{\mathrm{F}})$) to the congested region (denoted by the point B$\equiv$ $(\Delta x_{\mathrm{C}}, v_{\mathrm{C}})$) on this "limit cycle", (indicated by the arrows in Fig. 5) [9] and *vice versa*. The shape of the limit cycle depends on the sensitivity a; the smaller the sensitivity is, the wider the limit cycle grows. This will be apparent in Figure 7.

4. Phenomenological Study

Now we move to the main topic in this paper. We compare the predictions of our model with freeway traffic flow data.

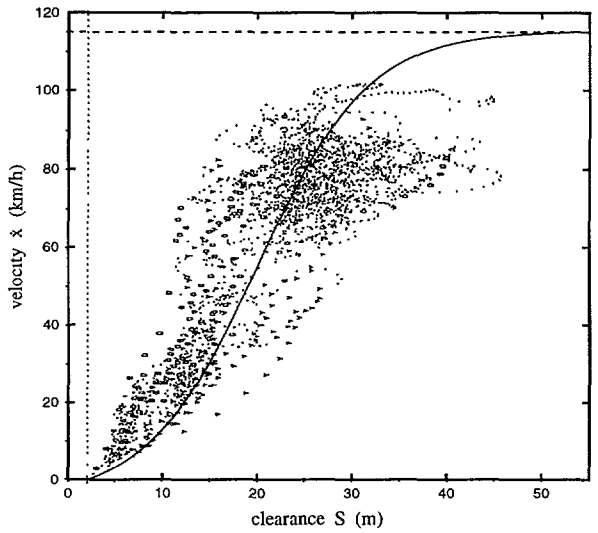

Fig. 6. — Velocity-clearance data from a car-following experiment on the Chuo Motorway, which was taken by Koshi *et al.* [6, 15]. Solid curve is the determined Optimal Velocity function V.

Our first task is to determine the values of parameters to fix the form of an Optimal Velocity function (2). This can be done with the reference of the observed data of an individual vehicle's behavior. As the most appropriate data for our purpose, we use the data of car-following experiment on the Chuo Motorway taken by "Koshi group" [6, 15]. They obtained data points on the velocity-clearance plane (Fig. 6), where the clearance S is defined as the headway subtracted by the vehicles length, l_c, (See Eq. (3));

$$S = \Delta x - l_c. \tag{5}$$

There are very few data points around 55 km/h velocity in Figure 6. In the view of our dynamical model, drivers feel difficulty to run vehicles maintaining with such velocity, which is just corresponding to the unstable velocity region of the homogeneous flow solution. So, we can infer that the inflection point is nearby such velocity. Let us fix the parameters of the Optimal Velocity function from Figure 6. The inflection point is taken as $(S, v) = (20\ \mathrm{m}, 55\ \mathrm{km/h})$. The maximal velocity is $V_{\max} = 115$ km/h (dashed line) and the minimal clearance is $S_{\min} = \delta = 2.0$ m (indicated by dotted line). From the above, we get the concrete form of the function,

$$V = 16.8[\tanh 0.0860(\Delta x - (20 + l_c)) + 0.913] \quad (\mathrm{m/s}). \tag{6}$$

The resulting function (6) is also displayed by a solid curve in Figure 6. For simulations, we should take the length of the vehicle, $l_c = 5$ m which was used in the car-following experiment on Chuo Motorway.

Next we determine the value of sensitivity a. For this purpose we make simulations for various values; $a = 1.6$, 2.0 and 2.8 using the Optimal Velocity function determined above. The results are shown in Figure 7. We choose the sensitivity $a = 2.0$ as the most appropriate value which reproduces the realistic velocity and headway of congested or free regions. Then using this value, we will make simulations of various car-densities (N/L) for the purpose of obtaining the density-flow relationship (Q-k curve).

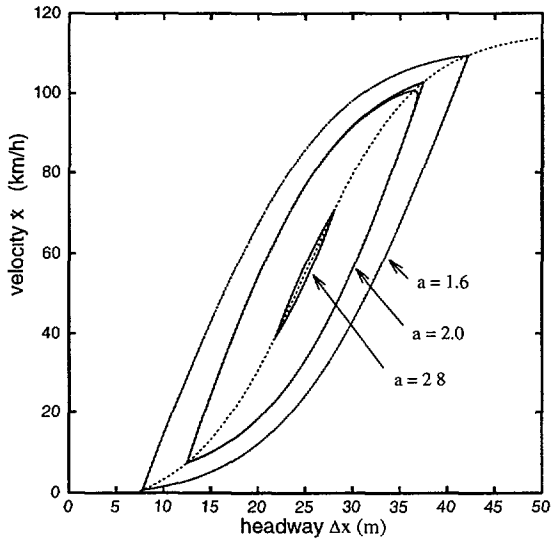

Fig. 7. — The behavior of the limit cycles obtained by varying the sensitivity input parameter a ($a = 1.6$, 2.0 and 2.8).

Before we perform the simulations, we discuss the Q-k relation of our model. First, we note the definition of quantities which should be compared with observed data. The flow Q is defined by the number of vehicles which pass through a reference point within 5 minutes. The density k is defined by the time average of inverse of headway of vehicles which pass through the reference point.

Second, we stress the Q-k relation can be predicted before we perform the simulations for various N/L. We can draw the Q-k curve of the congested flow solution (Fig. 4b or Fig. 5) as follows. We remember the limit cycle is universal, which means the values of Δx_{C} and Δx_{F} are unique for a given a and are independent on densities N/L [9]. The flow-density (Q-k) relation for the congested flow solution can be derived using these values of Δx_{C} and Δx_{F} obtained by only one simulation for some reference value of N/L.

The period T, in which a vehicle moves around the circuit, is given by

$$T = (1 + \frac{V_{\mathrm{back}} T}{L}) T_0 \quad , \tag{7}$$

where T_0 is the period in which a vehicle passes all congestion regions in the circuit ([3])

$$T_0 \simeq \frac{\Delta x_{\mathrm{F}} \cdot N_{\mathrm{F}}}{v_{\mathrm{F}} + V_{\mathrm{back}}} + \frac{\Delta x_{\mathrm{C}} \cdot N_{\mathrm{C}}}{v_{\mathrm{C}} + V_{\mathrm{back}}} \quad , \tag{8}$$

and V_{back} is the velocity of the congestion cluster moving backward

$$V_{\mathrm{back}} = \frac{\Delta x_{\mathrm{C}} \cdot v_{\mathrm{F}} - \Delta x_{\mathrm{F}} \cdot v_{\mathrm{C}}}{\Delta x_{\mathrm{F}} - \Delta x_{\mathrm{C}}} \quad . \tag{9}$$

The flow is therefore given by

$$Q = \frac{N}{T} \simeq \frac{v_{\mathrm{F}} - v_{\mathrm{C}}}{\Delta x_{\mathrm{F}} - \Delta x_{\mathrm{C}}} - V_{\mathrm{back}} \frac{N}{L} \tag{10}$$

([3]) The numbers of vehicles of congested and free regions, N_{C} and N_{F} are estimated by the equations: $N_{\mathrm{C}} + N_{\mathrm{F}} = N$ and $N_{\mathrm{C}} \Delta x_{\mathrm{C}} + N_{\mathrm{F}} \Delta x_{\mathrm{F}} = L$.

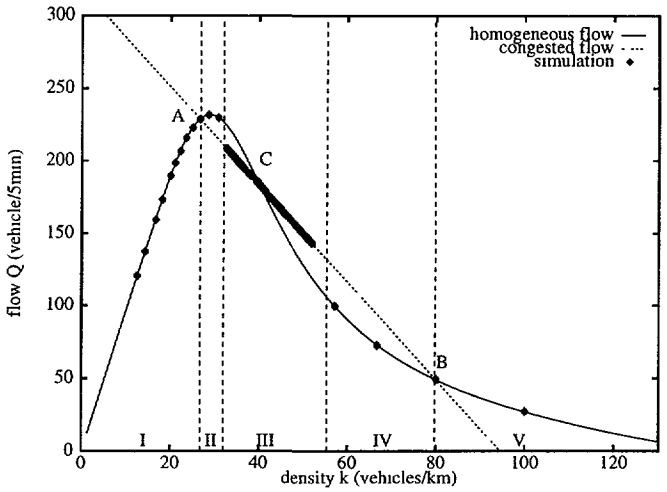

Fig. 8. — Q-k curves on the flow-density plane. The flow-density relations of the homogeneous flow solution and congested flow solution are depicted by solid and dotted curves respectively. A and B denote the free region and the congested region points respectively. C corresponds to the inflection point of Optimal Velocity function $V(\Delta x)$ (see also Fig. 5).

By substituting the value of Δx_{C} and Δx_{F} obtained from the simulation result, we find

$$Q = 318 - 3.36\ k \tag{11}$$

On the other hand, the Q-k relation of the homogeneous flow is analytically derived by using the solution (4). The flow Q is easily written as

$$Q = \frac{N}{T} = kV(1/k) = 5.04\ \{\tanh(\frac{86}{k} - 2.15) + 0.913\}\ k\ . \tag{12}$$

These two curves, equations (11) and (12), are drawn in Figure 8. The free region point (A) and the congested region point (B) correspond to the end points of the limit cycle in Figure 5. We note that the homogeneous flow solution is unstable in the region between dashed lines in Figure 8 from the analysis of linearized theory [8]. The corresponding region is shown as region III in Figure 8. The congested flow solution is expected to be realized in this region, therefore the simulation data will be plotted on the dotted line (11).

Now we finished the discussion about Q-k relation and we perform the simulations for various N/L values. We use the Optimal Velocity function (6) determined from Chuo Motorway data, and take the initial condition of simulations as the homogeneous flow with a tiny disturbance. The result of simulations is plotted also in Figure 8. Each diamond mark corresponds to one simulation, which shows a good agreement with our prediction.

We can separate the density into five regions divided by dashed lines as in Figure 8. In regions I and V, only the homogeneous solution is stable (the solid curve is realized). In region III, only the congested flow solution is stable (the dotted line is realized). On the other hand, in regions II and IV both solutions are stable. In Figure 8 the solid curve is realized because we started the simulation from the homogeneous flow. When we start from another distributions of vehicles such that congestion already exists, the congested flow solutions are stable as long as they obey to the Hysteresis loop profile. We have already confirmed this result in the previous paper [9]. In this case the dotted line is realized as well in regions II and IV.

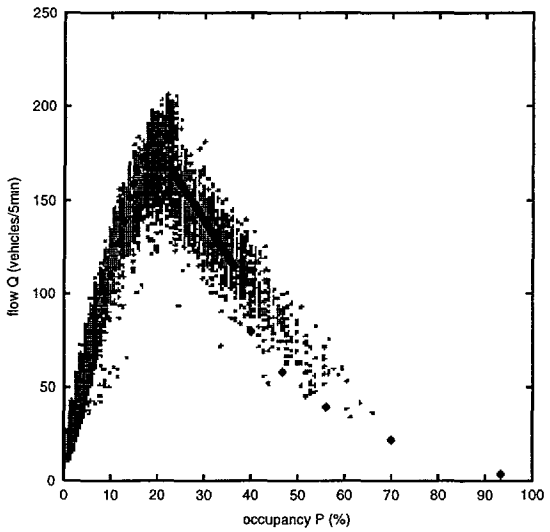

Fig. 9. — Q-k diagram on the flow-occupancy plane. The result of our simulations is shown by the diamond marks together with the observed data of Figure 2a.

Now we check our simulation data against some observed data. In principle, we should compare the simulation results with real Q-k data measured in the same freeway that we determined the Optimal Velocity function. Unfortunately we have no data of Q-k diagram given by the observation in Chuo Motorway, which should be compared with the result of the simulation using the Optimal Velocity function (6). On the other hand, we have the Q-k data of JHPC (Fig. 2), but we do not have the data like Figure 6, which can determine the Optimal Velocity function. Thus we apply the Optimal Velocity function for Chuo Motorway to the study of JHPC data (Fig. 2) with some modification. This can be done by only rescaling the overall factor V, which makes no change in the dynamics of our model. When we fit the Optimal Velocity function $V(\Delta x)$ to the JHPC data (Fig. 2b) in the same way as the case of Chuo Motorway (Fig. 6), the maximum velocity of $V(\Delta x)$ of JHPC data is about 80% smaller than the Chuo Motorway one. So we rescale the function as

$$V' = 0.8 \times V \ . \tag{13}$$

The time occupancy P, which appears in Figure 2 is related to the density; $P = l_c k$. Here we take $l_c = 7$ m as an average length of vehicles.

We perform the simulations using this function with the same procedure as the previous case. We also took the initial condition as the homogeneous flow with a tiny disturbance. We present the result of our simulations in Figure 9 by the diamond marks together with the observed data of Figure 2a. In the region where the congested flow solution is stable, the series of diamond marks are very dense and may look like a thick solid line. Our result agrees quite well with the observation and the specific discontinuity between free and congested flow is well reproduced.

5. Summary and Discussion

We have demonstrated that Optimal Velocity Model reproduces the characteristic features of the traffic flow phenomena quite well. Especially our model reproduce the specific discontinuity near the critical car-density between free and congested flows in the observed Q-k graph. This discontinuity is understood as the behaviors of two different kind of solutions of our dynamical model, which means the two different phases of our dynamical system. Thus we stress the traffic flow phenomena with congestion can be well described in the concept of phase transition.

Recently several models have been proposed to understand the phenomenon near the critical density. Navin and Hall proposed to describe the system on 3-dimensional space (k-v-Q) with a smooth function based on catastrophe theory [7]. This allows one to reproduce the discontinuity on 2-parameter plane by projecting the smooth function to this plane. The spontaneous generation of congestion, which we have demonstrated by our Optimal Velocity Model, suggests why the catastrophe theory describes the traffic flow quite well. We know some other kinds of models which can generate the traffic congestion spontaneously. Cellular Automaton models are proposed by Nagel and Schreckenberg [16], Nagatani [17] and Kikuchi, Tadaki and Yukawa [18]. In these models the lane is divided into lattice sites and each car shares a lattice site and moves from a site to site in discrete time steps according to some rule. Kikuchi *et al.* also propose another model based on coupled map lattice model [18]. These models have also been successful in generating congestion spontaneously. It would be interesting to investigate the common feature to their treatments and ours, which lead to a successful description of the traffic flow. The hydrodynamical approach based on well known Navier-Stokes equation is also proposed by Kerner and Konhäuser [19]. Under some special condition, they lead to the simple equation, which is very similar to our model. The sensitivity in our model corresponds to the inverse of relaxation time.

We should stress that our model is rather powerful in dealing with realistic traffic flow phenomena, because it is easy to include modifications to reproduce realistic traffic flow. For example, our model can include the effects of different behavior of each driver by introducing the driver-dependence of the sensitivity a_n and Optimal Velocity function $V_n(\Delta x)$.

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References

[1] Gazis D.C., Herman R. and Rothery R.W., *J. Opns. Res. Soc. Am.* **9** (1961) 545.

[2] Newell G.F., *J. Opns. Res. Soc. Am.* **9** (1961) 209.

[3] Pipes L.A., *J. Appl. Phys.* **24** (1953) 274.

[4] Traffic Engineering (in Japanese), T. Sasaki and Y. Iida, Eds. (Kokumin Kagakusha, 1992).

[5] Edie L.C., *J. Opns. Res. Soc. Am.* **9** (1961) 66.

[6] Koshi M., Iwasaki M. and Ohkura I., "Some Findings and an Overview on Vehicular Flow Characteristics", Proc. 8th Intl. Symp. on Transp. & Traffic Theory, V.F. Hurdle *et al.*, Eds. (1983) pp403-426.

[7] Navin F., Hall F., *ITE J.* (1989) 31.

[8] Bando M., Hasebe K., Nakayama A., Shibata A. and Sugiyama Y., *Phys. Rev. E* **51** (1995) 1035.

[9] Bando M., Hasebe K., Nakayama A., Shibata A. and Sugiyama Y., *Jap. J. Ind. Appl. Math.* **11** (1994) 203.

[10] Greenshields B.D., *H.R.B. Proceedings* **14** (1935) 468; Olcott E.S., *J. Opns. Res. Soc. Am.* **3** (1955) 147.

[11] Greenberg H., *J. Opns. Res. Soc. Am.* **7** (1959) 79.

[12] Underwood R.T., Quality and Theory of Traffic Flow (New Heaven: Yale Bureau of Highway Traffic, 1961) pp141-188.

[13] Drew D.R., Texas Transportation Institute Research Report 24-4 (1965).

[14] Drake J., May A.D. and Shofer J.L., Highway Research Record **154** (1967) 53.

[15] Oba T., An Experimental Study on Car-following Behavior, Thesis of Master of Engineering, Univ. of Tokyo (1988).

[16] Nagel K. and Schreckenberg M., *J. Phys. I France* **2** (1992) 2221.

[17] Nagatani T., *J. Phys. Soc. Jap.* **62** (1993) 3837.

[18] Yukawa S., Kikuchi M. and Tadaki S., *J. Phys. Soc. Jap.* **63** (1994) 3609; Yukawa S. and Kikuchi M., *J. Phys. Soc. Jap.* **64** (1995) 35.

[19] Kerner B.S. and Konhäuser P., *Phys. Rev. E* **48** (1993) R2335–R2338; Kerner B.S. and Konhäuser P., *Phys. Rev. E* **50** (1994) 54-83.