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Abstract

We study the behavior of perturbations of small nonlinear Dirac standing waves. We assume that the linear Dirac operator of reference $H = D_m + V$ has only two double eigenvalues and that degeneracies are due to a symmetry of H (theorem of Kramers). In this case, we can build a small 4-dimensional manifold of stationary solutions tangent to the first eigenspace of H.

Then we assume that a resonance condition holds and we build a center manifold of real codimension 8 around each stationary solution. Inside this center manifold any H^s perturbation of stationary solutions, with $s > 2$, stabilizes towards a standing wave. We also build center-stable and center-unstable manifolds each one of real codimension 4. Inside each of these manifolds, we obtain stabilization towards the center manifold in one direction of time, while in the other, we have instability. Eventually, outside all these manifolds, we have instability in the two directions of time.

For localized perturbations inside the center manifold, we obtain a nonlinear scattering result.

Introduction

We study the asymptotic stability of stationary solutions of a time-dependent nonlinear Dirac equation.

A localized stationary solution of a given time-dependent equation represents a bound state of a particle. Like Rañada [Ran], we call it a *particle-like solution* (PLS). Many works have been devoted to the proof of the existence of such solutions for a wide variety of equations. Although their stability is a crucial problem (in particular in numerical computation or experiment), a smaller attention has been deserved to this issue.

In this paper, we deal with the problem of stability of small PLS of the following nonlinear Dirac equation:

$$\mathrm{i}\partial_t \psi = (D_m + V)\psi + \nabla F(\psi) \tag{NLDE}$$

where ∇F is the gradient of $F : \mathbb{C}^4 \mapsto \mathbb{R}$ for the standard scalar product of $\mathbb{R}^8$. Here, D_m is the usual Dirac operator, see Thaller [Tha92], acting on $L^2(\mathbb{R}^3, \mathbb{C}^4)$

$$D_m = \alpha \cdot (-\mathrm{i}\nabla) + m\beta = -\mathrm{i}\sum_{k=1}^{3} \alpha_k \partial_k + m\beta$$

where $m \in \mathbb{R}_+^*$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, β are $\mathbb{C}^4$ hermitian matrices satisfying:

$$\begin{cases} \alpha_i\alpha_k + \alpha_k\alpha_i = 2\delta_{ik}\mathbf{1}_{\mathbb{C}^4}, & i,k \in \{1,2,3\}, \\ \alpha_i\beta + \beta\alpha_i = \mathbf{0}_{\mathbb{C}^4}, & i \in \{1,2,3\}, \\ \beta^2 = \mathbf{1}_{\mathbb{C}^4}. \end{cases}$$

Here, we choose

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} I_{\mathbb{C}^2} & 0 \\ 0 & -I_{\mathbb{C}^2} \end{pmatrix}$$

$$\text{where } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In (NLDE), V is the external potential field and $F : \mathbb{C}^4 \mapsto \mathbb{R}$ is a nonlinearity with the following gauge invariance:

$$\forall(\theta, z) \in \mathbb{R} \times \mathbb{C}^4, \quad F(e^{i\theta} z) = F(z). \tag{0.1}$$

Some additional assumptions on F and V will be made in the sequel. Stationary solutions (PLS) of (NLDE) take the form $\psi(t, x) = e^{-\mathrm{i}Et}\phi(x)$ where ϕ satisfies

$$E\phi = (D_m + V)\phi + \nabla F(\phi). \tag{PLSE}$$

We show that there exists a manifold of small solutions to (PLSE) tangent to the first eigenspace of $D_m + V$ (see Proposition 1.1 below).

In the Schrödinger case, orbital stability results (see e.g [CL82], [Wei85, Wei86] or [SS85, GSS87]) give that any solution stays near the PLS manifold. Unfortunately, orbital stability criteria applied to Schrödinger equations use the fact that Schrödinger operators are bounded from below. Hence the question of orbital stability for Dirac standing waves cannot be solved by a straightforward application of the methods used in the Schrödinger case.

Concerning the asymptotic stability, in the Schrödinger equation, the question has been solved in several cases. For small stationary solutions in the simple eigenvalue case it has been studied by Soffer and Weinstein [SW90, SW92], Pillet and Wayne [PW97] or Gustafson, Nakanishi and Tsai [GNT04]. For the two eigenvalue case under a resonance condition for an excited state, the problem has been studied by Tsai and Yau [TY02a, TY02c, TY02d, TY02b, Tsa03] or Soffer and Weinstein [SW04, SW05]. Another problem has been studied by Cuccagna [Cuc01, Cuc03, Cuc05], he considered the case of big PLS, when the linearized operator has only one eigenvalue and obtained the asymptotic stability of the manifold of ground states. Schlag [Sch04] proved that any ground state of the cubic nonlinear Schrödinger equation in dimension 3 is orbitally unstable but posseses a stable manifold of codimension 9.

We also would like to mention the works of Buslaev and Perel'mann [BP95, BP92b, BP92c, BP92a], Buslaev and Sulem [BS03, BS02], Weder [Wed00] or Krieger and Schlag [KS05] in the one dimensional Schrödinger case. Krieger and Schlag [KS05] proved a result similar to [Sch04] in the one dimensional case.

In [Bou06], we prove that there are stable directions for the PLS manifold under a non resonance assumption on the spectrum of $H := D_m + V$. This gives a stable manifold, containing the PLS manifold. But we were not able to say anything about solutions starting outside the stable manifold.

The results we present here state the existence of a stable manifold and describe the behavior of solutions starting outside of it. In fact, we prove the instability of the stable manifold. We also prove stabilization towards stationary solutions inside the stable manifold for H^s perturbation with $s > 2$. We have been able to obtain it since we impose a resonance condition (see Assumption 1.5 below), while in [Bou06], we assumed there is no resonance phenomena.

When the perturbations are localized, we can push further this study and we obtain a nonlinear scattering.

This paper is organized as follow.

In Section 1, we present our main results and the assumptions we need. Subsection 1.1, is devoted to the statement of the time decay estimates of the propagator associated with $H = D_m + V$ on the continuous subspace. One is a kind of smoothness result, in the sense of Kato (see e.g. [Kat66]), the other is a Strichartz type result. We prove these estimates with the propagation and dispersive estimates proved in [Bou06]. In subsection 1.2, we state the existence of small stationary states forming a manifold tangent to an eigenspace of H. The study of the dynamics around such states leads us to our main results, see Subsection 1.3 and 1.4. In Subsection 1.3, we split a neighborhood of a stationary state in different parts, each one giving rise to stabilization or instability. In Subsection 1.4, we state our scattering result.

To prove our theorems, we consider our nonlinear system as a small perturbation of a linear equation. More precisely in Subsection 2.2, we show that the spectral properties of the linearized operator around a stationary state, presented in Section 2, permits to obtain, like in the linear case, some properties of the dynamics around a stationary state. We obtain center, center-stable and center-unstable manifolds. In Section 3, we obtain, with our time decay estimates, a stabilization towards the PLS manifold for H^s perturbation with $s > 2$ in the center manifold. Section 4 deals with the dynamic outside the center manifold. Eventually in Section 5, we conclude our study.

Our results are the analogue, in the Dirac case, of some results of Tsai and Yau [TY02d], Soffer and Weinstein [SW90], Pillet and Wayne [PW97] and Gustafson, Nakanishi and Tsai [GNT04] about the semilinear Schrödinger equation.

1 Assumptions and statements

1.1 Time decay estimates

We generalize to small nonlinear perturbations, stability results for linear systems. These results, like in [Bou06], follow from linear decay estimates. Here we use smoothness type and Strichartz type estimates deduced from propagation and dispersive estimates of [Bou06]. Hence, we work within the same assumptions for V and $D_m + V$:

Assumption 1.1. *The potential $V : \mathbb{R}^3 \mapsto S_4(\mathbb{C})$ (self-adjoint 4×4 matrices) is a smooth function such that there exists $\rho > 5$ with*

$$\forall \alpha \in \mathbb{N}^3,\ \exists C > 0,\ \forall x \in \mathbb{R}^3,\ |\partial^\alpha V|(x) \leq \frac{C}{\langle x \rangle^{\rho+|\alpha|}}.$$

We notice that by the Kato-Rellich theorem, the operator

$$H := D_m + V$$

is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ and self-adjoint on $H^1(\mathbb{R}^3, \mathbb{C}^4)$.

We also mention that Weyl's theorem gives us that the essential spectrum of H is $(-\infty, -m] \cup [+m, +\infty)$ and the work of Berthier and Georgescu [BG87, Theorem 6, Theorem A], gives us that there is no embedded eigenvalue. Hence the thresholds $\pm m$ are the only points of the continuous spectrum which can be associated with wave of zero velocity. These waves perturb the spectral density and diminish the decay rate in the propagation and the dispersive estimates. We will work (like in [Bou06]) within the

Assumption 1.2. *The operator H presents no resonance at thresholds and no eigenvalue at thresholds.*

A resonance is a stationary solution in $H^{1/2}_{-\sigma} \setminus H^{1/2}$ for any $\sigma \in (1/2, \rho - 2)$, where H^t_σ is given by

Definition 1.1 (Weighted Sobolev space). *The weighted Sobolev space is defined by*

$$H^t_\sigma(\mathbb{R}^3, \mathbb{C}^4) = \left\{ f \in \mathcal{S}'(\mathbb{R}^3),\ \|\langle Q \rangle^\sigma \langle P \rangle^t f\|_2 < \infty \right\}$$

for $\sigma, t \in \mathbb{R}$. We endow it with the norm

$$\|f\|_{H^t_\sigma} = \|\langle Q \rangle^\sigma \langle P \rangle^t f\|_2.$$

If $t = 0$, we write L^2_σ instead of H^0_σ.

We have used the usual notations $\langle u \rangle = \sqrt{1 + u^2}$, $P = -i\nabla$, and Q is the operator of multiplication by x in $\mathbb{R}^3$.

Now let

$$\mathbf{P}_c(H) = \mathbb{1}_{(-\infty, -m| \cup [+m, +\infty)}(H)$$

be the projector associated with the continuous spectrum of H and $\mathcal{H}_c$ be its range. Using [Bou06, Theorem 1.1], we obtain a Limiting Absorption Principle which gives the H-smoothness of $\langle Q \rangle^{-1}$ in the sense of Kato:

Theorem 1.1 (Kato smoothness estimates). *If Assumptions 1.1 and 1.2 hold. Then for any $\sigma \geq 1$ and $s \in \mathbb{R}$, one has:*

$$\left\| \langle Q \rangle^{-\sigma} e^{-\mathrm{i}tH} \mathbf{P}_c(H) \psi \right\|_{L^2_t(\mathbb{R}, H^s(\mathbb{R}^3, \mathbb{C}^4))} \leq C \|\psi\|_{H^s(\mathbb{R}^3, \mathbb{C}^4)}, \tag{i}$$

$$\left\| \int_{\mathbb{R}} e^{\mathrm{i}tH} \mathbf{P}_c(H) \langle Q \rangle^{-\sigma} F(t)\, dt \right\|_{H^s(\mathbb{R}^3, \mathbb{C}^4)} \leq C \|F\|_{L^2_t(\mathbb{R}, H^s)}, \tag{ii}$$

$$\left\| \int_{s<t} \langle Q \rangle^{-\sigma} e^{-\mathrm{i}(t-s)H} \mathbf{P}_c(H) \langle Q \rangle^{-\sigma} F(s)\, ds \right\|_{L^2_t(\mathbb{R}, H^s(\mathbb{R}^3, \mathbb{C}^4))} \leq C \|F\|_{L^2_t(\mathbb{R}, H^s(\mathbb{R}^3, \mathbb{C}^4))}. \tag{iii}$$

Proof. We first prove (i). For $s = 0$, it is (see *e.g.* [ABdMG96, Proposition 7.11] or [RS78, Theorem XIII.25]) a consequence of the limiting absorption principle:

$$\sup_{\Im z \in (0,1)} \left\{ \left\| \langle Q \rangle^{-\sigma} (H - z)^{-1} P_c(H) \langle Q \rangle^{-\sigma} \right\|_2 \right\} < \infty \tag{1.1}$$

which follows from [Bou06, Theorem 1.1] or (Theorem 3.1 below) for $\sigma > 5/2$ using the fact that the Fourier transform in time of the propagator is the resolvent. Actually, the Fourier transform of

$$\langle Q \rangle^{-\sigma} e^{-\mathrm{i}t(H - \mathrm{i}\varepsilon)} \mathbf{P}_c(H) \mathbf{1}_{\mathbb{R}_+^*}(t) \langle Q \rangle^{-\sigma} f$$

in time is

$$\langle Q \rangle^{-\sigma} (H - \lambda - \mathrm{i}\varepsilon)^{-1} \mathbf{P}_c(H) \langle Q \rangle^{-\sigma} f$$

for $f \in L^2(\mathbb{R}^3, \mathbb{C}^4)$. Then we use Born expansion

$$(H - z)^{-1} = (D_m - z)^{-1} - (D_m - z)^{-1} V (D_m - z)^{-1} + (D_m - z)^{-1} V (H - z)^{-1} V (D_m - z)^{-1}$$

the limiting absorption in [IM99, Theorem 2.1(i)] (they prove the identity (1.1) for $H = D_m$ when $\sigma = 1$) and the fact that

$$\left\| (H - z)^{-1} (1 - P_c(H)) \right\|_{\mathcal{B}(L^2)} \leq \frac{1}{\inf\limits_{\lambda \in (-\infty, -m] \cup [+m, +\infty)} |z - \lambda|}$$

to obtain (1.1) for $\sigma = 1$. Hence we have concluded the proof for $s = 0$ and $\sigma \geq 1$. For $s \in 2\mathbb{Z}$ and $\sigma \geq 1$ it follows from the previous cases using boundedness of $< H >^s < D_m >^{-s}$ and $< H >^{-s} < D_m >^s$ (which follow from the boundedness of V and its derivatives) and the boundedness of $\langle Q \rangle^{\mp\sigma} [\langle Q \rangle^{\pm\sigma}, < H >^s] \langle H \rangle^{-s}$ (which follow from multicommutator estimates see [HS00, Appendix B]). The rest of the claim (i) follows by interpolation.

Estimates (i) and (ii) are equivalent by duality.

To prove estimate (iii) when $s = 0$ (the general case will follow by the same way as above), we notice that we have to prove that there exists $C > 0$ such that for all $F, G \in L^2_t(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4))$, we have

$$\left| \iint_{\mathbb{R}^2} \left\langle G(t), \langle Q \rangle^{-\sigma} e^{-\mathrm{i}(t-s)H} \mathbf{P}_c(H) \mathbf{1}_{\mathbb{R}_+^*}(t - s) \langle Q \rangle^{-\sigma} F(s) \right\rangle \, ds dt \right| \\ \leq C \|G\|_{L^2_\lambda(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4))} \|F\|_{L^2_\lambda(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4))}$$

We can suppose that F and G are smooth functions with compact support from $\mathbb{R} \times \mathbb{R}^3$ to $\mathbb{C}^4$ and we just need to prove that there exists $C > 0$ such that for all $\varepsilon > 0$, for all $F, G \in \mathcal{C}_0^\infty(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4))$, we have

$$\left| \iint_{\mathbb{R}^2} \left\langle G(t), \langle Q \rangle^{-\sigma} e^{-\mathrm{i}(t-s)(H - \mathrm{i}\varepsilon)} \mathbf{P}_c(H) \mathbf{1}_{\mathbb{R}_+^*}(t - s) \langle Q \rangle^{-\sigma} F(s) \right\rangle \, ds dt \right| \\ \leq C \|G\|_{L^2_\lambda(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4))} \|F\|_{L^2_\lambda(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4))}.$$

Then we take the limit as $\varepsilon \to 0$ and we will conclude using density arguments. Let us write $A_\varepsilon(t)$ for $\langle Q \rangle^{-\sigma} e^{-\mathrm{i}t(H - \mathrm{i}\varepsilon)} \mathbf{P}_c(H) \mathbf{1}_{\mathbb{R}_+^*}(t) \langle Q \rangle^{-\sigma}$, we have to prove

$$\left| \int_{\mathbb{R}} \langle G(t), (A_\varepsilon * F)(t) \rangle \, dt \right| \leq C \|G\|_{L^2_\lambda(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4))} \|F\|_{L^2_\lambda(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4))}.$$

Using Plancherel's identity in $L^2_t(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4))$ and $A_\varepsilon * F \in L^2_t(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4))$, we just need to prove

$$\left| \int_{\mathbb{R}} \left\langle \widehat{G}(\lambda), \widehat{A_\varepsilon * F}(\lambda) \right\rangle \, d\lambda \right| \leq C \left\| \widehat{G} \right\|_{L^2_\lambda(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4))} \left\| \widehat{F} \right\|_{L^2_\lambda(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4))}.$$

Since the Fourier transform in time of the propagator is the resolvent, F is smooth with compact support and $\varepsilon > 0$, we obtain

$$\widehat{A_\varepsilon * F}(\lambda) = \langle Q \rangle^{-\sigma} (H - \lambda - \mathrm{i}\varepsilon)^{-1} \mathbf{P}_c(H) \langle Q \rangle^{-\sigma} \widehat{F}(\lambda)$$

Hence we just have to prove

$$\left| \int_{\mathbb{R}} \left\langle \widehat{G}(\lambda), \langle Q \rangle^{-\sigma} (H - \lambda - \mathrm{i}\varepsilon)^{-1} \mathbf{P}_c(H) \langle Q \rangle^{-\sigma} \widehat{F}(\lambda) \right\rangle \, d\lambda \right| \leq C \left\| \widehat{G} \right\|_{L^2_\lambda(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4))} \left\| \widehat{F} \right\|_{L^2_\lambda(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^4))}.$$

This in turn follows from the Limiting Absorption Principle (1.1) just proved. □

To state the next result, we need the

Definition 1.2 (Besov space). *For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the Besov space $B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^4)$ is the space of all $f \in \mathcal{S}'(\mathbb{R}^3, \mathbb{C}^4)$ (dual of the Schwartz space) such that*

$$\|f\|_{B^s_{p,q}} = \left(\sum_{j \in \mathbb{N}} 2^{jsq} \|\varphi_j * f\|_p^q \right)^{\frac{1}{q}} < +\infty$$

with $\widehat{\varphi} \in \mathcal{C}_0^\infty(\mathbb{R}^n \backslash \{0\})$ such that $\sum_{j \in \mathbb{Z}} \widehat{\varphi}(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^3 \backslash \{0\}$, $\widehat{\varphi}_j(\xi) = \widehat{\varphi}(2^{-j}\xi)$ for all $j \in \mathbb{N}^*$ and for all $\xi \in \mathbb{R}^3$, and $\widehat{\varphi_0} = 1 - \sum_{j \in \mathbb{N}^*} \widehat{\varphi}_j$. It is endowed with the natural norm $f \in B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^4) \mapsto \|f\|_{B^s_{p,q}}$.

Using the Dispersive estimates of [Bou06, Theorem 1.2] and [KT98, Theorem 10.1], we obtain the

Theorem 1.2 (Strichartz estimates). *If Assumptions 1.1 and 1.2 hold. Then for any $2 \le p, q \le \infty$, $\theta \in [0,1]$, with $(1 - \frac{2}{q})(1 \pm \frac{\theta}{2}) = \frac{2}{p}$ and $(p, \theta) \neq (2, 0)$, and for any reals s, s' with $s' - s \ge \alpha(q)$ where $\alpha(q) = (1 + \frac{\theta}{2})(1 - \frac{2}{q})$, there exists a positive constant C such that*

$$\|e^{-\mathrm{i}tH} P_c(H)\psi\|_{L^p_t(\mathbb{R}, B^s_{q,2}(\mathbb{R}^3,\mathbb{C}^4))} \le C\|\psi\|_{H^{s'}(\mathbb{R}^3,\mathbb{C}^4)}, \tag{i}$$

$$\| \int e^{\mathrm{i}tH} P_c(H) F(t)\, dt \|_{H^s} \le C \|F\|_{L^{p'}_t(\mathbb{R}, B^{s'}_{q',2}(\mathbb{R}^3,\mathbb{C}^4))}, \tag{ii}$$

$$\| \int_{s<t} e^{-\mathrm{i}(t-s)H} P_c(H) F(s)\, ds \|_{L^p_t(\mathbb{R}, B^{-s}_{q,2}(\mathbb{R}^3,\mathbb{C}^4))} \le C \|F\|_{L^{\tilde{p}'}_t(\mathbb{R}, B^{\tilde{s}}_{\tilde{q}',2}(\mathbb{R}^3,\mathbb{C}^4))}, \tag{iii}$$

for any $r \in [1, \infty]$, $(\tilde{q}, \tilde{p})$ chosen like (q, p) and $s + \tilde{s} \ge \alpha(q) + \alpha(\tilde{q})$.

Proof. This is a consequence of [KT98, Theorem 10.1] applied to $U(t) = e^{-\mathrm{i}tH} P_c(H)$, using [Bou06, Theorem 1.2] or Theorem 3.2 below and

$$B^{(1+\frac{\theta}{2})(1-\frac{2}{q})+s}_{q,2} \hookrightarrow (H^s, B^{1+\theta/2+s}_{1,2})_{2/((1\pm\theta/2)p),2}$$

continuously for $p \ge 2$ ($p \neq 2$ if $\theta = 0$) and $1/q = 1 - 1/((1 \pm \theta/2)p)$. For these embeddings, we refer to the proof of [BL76, Theorem 6.4.5] as well as the properties of the real interpolation (see [BL76] or [Tri78]). More precisely for $\theta = 0$ or 1 it is obvious. In the other cases, we work like in proof of [BL76, Theorem 6.4.5(3)]:

We use [BL76, Theorem 6.4.3] ($B^s_{p,2}$ is a retract of $l^s_2(L^p)$ for $s \in \mathbb{R}$ and $p, q \in [1, \infty]$) and [BL76, Theorem 5.6.2] (about the interpolation of $l^s_2(L^p)$ spaces) with [BL76, Theorem 5.2.1] (about the interpolation of L^p spaces). Then we conclude using the injection of L^p spaces into some Lorentz spaces [BL76, Section 1.3 & Exercice 1.6.8].

In the case $\theta \neq 0$, the proof is actually simpler. We can prove it using the usual TT^* method and the Hölder inequality instead of the Hardy-Littlewood-Sobolev inequality.

□

1.2 The manifold of PLS

We study the following nonlinear Dirac equation

$$\begin{cases} \mathrm{i}\partial_t \psi = H\psi + \nabla F(\psi) \\ \psi(0, \cdot) = \psi_0. \end{cases} \tag{1.2}$$

with $\psi \in \mathcal{C}^1(I, H^1(\mathbb{R}^3, \mathbb{C}^4))$ for some open interval I which contains 0 and $H = D_m + V$. The nonlinearity $F : \mathbb{C}^4 \mapsto \mathbb{R}$ is a differentiable map for the real structure of $\mathbb{C}^4$ and hence the ∇ symbol has to be understood for the real structure of $\mathbb{C}^4$. For the usual hermitian product of $\mathbb{C}^4$, one has

$$DF(v)h = \Re\langle \nabla F(v), h \rangle.$$

If F has a gauge invariance (see Equation (0.1) or Assumption 1.4 below), this equation may have stationary solutions *i.e.* solution of the form $e^{-\mathrm{i}Et}\phi_0$ where ϕ_0 satisfies the nonlinear stationary equation:

$$E\phi_0 = H\phi_0 + \nabla F(\phi_0).$$

We will notice that the Dirac operator D_m have an interesting invariance property due to its matrix structure. This invariance can be shared by some perturbed Dirac operators and gives a consequence of a theorem of Kramers, see [BH92, Par90]. Indeed if we introduce K the antilinear operator defined by:

$$K\begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} \sigma_2\overline{\psi} \\ \sigma_2\overline{\chi} \end{pmatrix} \text{ with } \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{1.3}$$

The operator D_m commutes with K. So if V also commutes with K, we obtain that the eigenspaces of H are always of even dimension. Here we work with the

Assumption 1.3. *The potential V commutes to K. The operator $H := D_m + V$ has only two double eigenvalues $\lambda_0 < \lambda_1$, with $\{\phi_0, K\phi_0\}$ and $\{\phi_1, K\phi_1\}$ as associated orthonormalized basis.*

We also need the

Assumption 1.4. *The function $F : \mathbb{C}^4 \mapsto \mathbb{R}$ is in $\mathcal{C}^\infty(\mathbb{R}^8, \mathbb{R})$ and satisfies $F(z) = O(|z|^4)$ as $z \to 0$. Moreover, it has the following invariance properties:*

$$\forall z \in \mathbb{C}^4,\ \forall \theta \in \mathbb{R},\ F(Kz) = F(z),\ F(e^{i\theta}z) = F(z).$$

We obtain the

Proposition 1.1 (PLS manifold). *If Assumptions 1.1–1.4 hold. Then for any $\sigma \in \mathbb{R}^+$, there exist Ω a neighborhood of 0 in $\mathbb{C}^2$, a smooth map*

$$h : \Omega \mapsto \{\phi_0, K\phi_0\}^\perp \cap H^2(\mathbb{R}^3, \mathbb{C}^4) \cap L^2_\sigma(\mathbb{R}^3, \mathbb{C}^4)$$

and a smooth map $E : \Omega \mapsto \mathbb{R}$ such that $S((u_1, u_2)) = u_1\phi_0 + u_2 K\phi_0 + h((u_1, u_2))$ satisfy for all $U \in \Omega$,

$$HS(U) + \nabla F(S(U)) = E(U)S(U), \tag{1.4}$$

with the following properties

$$\begin{cases} h((u_1,u_2)) = \left(\frac{u_1}{|(u_1,u_2)|} Id_{\mathbb{C}^4} + \frac{u_2}{|(u_1,u_2)|} K\right) h\left((|(u_1,u_2)|, 0)\right), \quad \forall U = (u_1,u_2) \in \Omega, \\ h(U) = O(|U|^2), \\ E(U) = E(|U|), \\ E(U) = \lambda_0 + O(|U|^2). \end{cases}$$

Proof. This result is adapted from [PW97, Proposition 2.2] after the reduction due to the invariance of the problem with respect to K. □

Moreover, we have

Lemma 1.1 (exponential decay). *For any $\beta \in \mathbb{N}^4$, $s \in \mathbb{R}^+$ and $p, q \in [1, \infty]$. There exist $\gamma > 0$, $\varepsilon > 0$ and $C > 0$ such that for all $U \in B_{\mathbb{C}^2}(0, \varepsilon)$ one has*

$$\|e^{\gamma\langle Q\rangle}\partial_U^\beta S(U)\|_{B^s_{p,q}} \le C\|S(U)\|_2,$$

where $\partial^\beta_{(u_1,u_2)} = \frac{\partial^{|\beta|}}{\partial^{\beta_1}\Re u_1 \partial^{\beta_2}\Im u_1 \partial^{\beta_3}\Re u_2 \partial^{\beta_4}\Im u_2}$.

Proof. This is proved like in [Bou06, Lemma 4.1], where we used ideas of [His00]. □

1.3 The unstable manifold and the stabilization

Each stationary solution previously introduced has, like in [Bou06], a stable manifold. Under the following assumption, we can prove that the stable manifold is unstable, that is to say that a small perturbation of a stationary solution starting outside of this manifold leaves any neighborhood of this stationary solution. We work with the

Assumption 1.5. *The resonant condition*

$$|\lambda_1 - \lambda_0| > \min\{|\lambda_0 + m|, |\lambda_0 - m|\}$$

holds. Moreover, we have the Fermi Golden Rule

$$\Gamma(\phi) = \lim_{\substack{\varepsilon\to 0,\\ \varepsilon>0}} \left\langle d^2F(\phi)\phi_1, \Im\left((H-\lambda_0)+(\lambda_1-\lambda_0)-\mathrm{i}\varepsilon\right)^{-1} P_c(H) d^2F(\phi)\phi_1 \right\rangle > 0 \tag{1.5}$$

for any non zero eigenvector ϕ associated with λ_0.

In this assumption, the notation d^2F denotes the differential of ∇F with respect to the real structure of $\mathbb{C}^4$.

Let us introduce the linearized operator $JH(U)$ around a stationary state $S(U)$:

$$H(U) = H + d^2F(S(U)) - E(U).$$

We notice that the operator $H(U)$ is not $\mathbb{C}$-linear but only $\mathbb{R}$-linear. Hence we work with the space $L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4)$ instead of $L^2(\mathbb{R}^3, \mathbb{C}^4)$ by writing

$$\begin{pmatrix} \Re\phi \\ \Im\phi \end{pmatrix}$$

instead of ϕ. The multiplication by $-\mathrm{i}$ becomes the operator

$$J = \begin{pmatrix} 0 & -I_{\mathbb{R}^4} \\ I_{\mathbb{R}^4} & 0 \end{pmatrix}.$$

Now we mention some spectral properties of the *real operator* $JH(U)$ in $L^2(\mathbb{R}^3, \mathbb{C}^4 \times \mathbb{C}^4)$ (the complexified of $L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4)$) which are needed to state and to understand our main theorem. These properties will be proven in subsection 2.

Proposition 1.2 (Spectrum of $JH(U)$). *The operator $JH(U)$ in $L^2(\mathbb{R}^3, \mathbb{C}^4 \times \mathbb{C}^4)$ has a four dimensional geometric kernel and four double eigenvalues $E_1(U)$, $\overline{E_1}(U)$, $-E_1(U)$ and $-\overline{E_1}(U)$ with $\Re E_1(U) > 0$.*

The eigenspaces associated with $E_1(U)$ and $\overline{E_1}(U)$ are conjugated via the complex conjugation. The same holds for $-E_1(U)$ and $-\overline{E_1}(U)$.

The rest of the spectrum is the essential (or continuous) spectrum. We write $\mathcal{H}_c(U)$ for the space associated with the continuous spectrum. The space $\mathcal{H}_c(U)$ is the orthogonal of the previous eigenspaces and the geometric kernel of $JH(U)$ and is invariant by the complex conjugation.

Proof. See subsection 2 below. □

We will work on the real part of the sum the eigenspaces associated with $E_1(U)$ and $\overline{E_1}(U)$: $X_u(U) \subset L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4)$, we introduce a real basis $(\xi_i(U))_{i=1,\dots,4}$ of $X_u(U)$. We will also work in the real part of the sum of the eigenspaces associated with $-E_1(U)$ and $-\overline{E_1}(U)$: $X_s(U) \subset L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4)$, we introduce a real basis $(\xi_i(U))_{i=5,\dots,8}$ of $X_s(U)$.

We can state our main theorems which will be proved in the sections 2, 3, 4 and 5.

Theorem 1.3 (Central manifold and asymptotic stability). *If Assumptions 1.1–1.5 hold. Then, for $s > \beta+2 > 2$ and $\sigma > 3/2$, there exist $\varepsilon > 0$, a continuous map $r : B_{\mathbb{C}^2}(0,\varepsilon) \mapsto \mathbb{R}$ with $r(U) = O(|U|^2)$, $C > 0$, $\mathcal{V}$ a neighborhood of $(0,0)$ in*

$$\mathcal{S} = \{(U,z)\,;\; U \in B_{\mathbb{C}^2}(0,\varepsilon),\; z \in \mathcal{H}_c(U) \cap B_{H^s}(0, r(U))\}$$

endowed with the metric of $\mathbb{C}^2 \times H^s$ and a map $\Psi : \mathcal{V} \mapsto \mathbb{R}^8$, smooth on $\mathcal{V} \setminus (0,0)$ satisfying for any non zero $U \in B_{\mathbb{C}^2}(0,\varepsilon)$

$$\|\Psi(U,z)\| = O(\|z\|_{H^s}^2)$$

for all $z \in \mathcal{H}_c(U) \cap B_{H^s}(0, r(U))$ with $(U,z) \in \mathcal{V}$ such that the following is true.

For any initial condition of the form

$$\psi_0 = S(U_0) + z_0 + A \cdot \xi(U_0)$$

with $(U_0, z_0) \in \mathcal{V}$ *and* $A = \Psi(U_0, z_0)$, *there exists a solution* $\psi \in \cap_{k=0}^{2} \mathcal{C}^k\left(\mathbb{R}, H^{s-k}\right)$ *of* (1.2) *with initial condition* ψ_0 *and this solution is unique in* $L^\infty((-T,T), H^s(\mathbb{R}^3, \mathbb{C}^4))$ *for any* $T > 0$.

Moreover, we have for all $t \in \mathbb{R}$

$$\psi(t) = e^{-\mathrm{i}\int_0^t E(U(v))\, dv} S(U(t)) + \varepsilon(t) \tag{1.6}$$

with $\left\|\dot{U}\right\|_{L^q(\mathbb{R})} \leq C\|z_0\|_{H^s}^2$ *for all* $q \in [1, \infty]$, $\lim_{t\to\pm\infty} U(t) = U_{\pm\infty}$ *and*

$$\max\left\{ \|\varepsilon\|_{L^\infty(\mathbb{R}^\pm, H^s)}\,,\ \|\varepsilon\|_{L^2(\mathbb{R}^\pm, H^s_{-\sigma})}\,, \|\varepsilon\|_{L^2(\mathbb{R}^\pm, B^\beta_{\infty,2})} \right\} \leq C\|z_0\|_{H^s}.$$

Theorem 1.4 (Center stable end center unstable manifold). *With the same assumptions and notations as Theorem 1.3, let* $\mathcal{CM}$ *be the graph of* $(U, z) \in \mathcal{V} \mapsto S(U) + z + \Psi(U, z) \cdot \xi(U)$ *then for the set*

$$\widetilde{\mathcal{S}} = \{(U, z, p)\,;\ U \in B_{\mathbb{C}^2}(0, \varepsilon),\ z \in \mathcal{H}_c(U) \cap B_{H^s}(0, r(U)), p \in B_{\mathbb{R}^4}(0, r(U)), \}$$

endowed with the metric of $\mathbb{C}^2 \times H^s \times \mathbb{R}^4$, *there exist* $C > 0$, *neighborhoods* $\mathcal{W}_\pm$ *of* $(0, 0, 0)$ *in* $\widetilde{\mathcal{S}}$ *and maps* $\Phi_\pm : \mathcal{W}_\pm \mapsto \mathbb{R}^8$, *smooth on* $\mathcal{W}_\pm \setminus \{(0, 0, 0)\}$ *with*

$$\|\Phi_\pm(U, z, p)\| = O(\|z\|_{H^s}^2 + \|p\|^2)$$

for all $(U, z, p) \in \mathcal{W}_\pm$ *such that for any initial condition of the form*

$$\psi_0 = S(U_0) + z_0 + (p_+, p_-).\xi(U_0)$$

not in $\mathcal{CM}$, *the following is true.*

1. *If* $(U_0, z_0, p_+) \in \mathcal{W}_+$ *and* $p_- = \Phi_+(U_0, z_0, p_+)$ *(resp. If* $(U_0, z_0, p_-) \in \mathcal{W}_-$ *and* $p_+ = \Phi_-(U_0, z_0, p_-)$*) then for any small neighborhood* $\mathcal{O}$ *of* $S(U_0)$ *containing* ψ_0 *there exist* $t_\pm(\psi_0) > 0$ *and a solution* $\psi_+ \in \cap_{k=0}^{2} \mathcal{C}^k([-t_+; +\infty), H^{s-k})$ *(resp.* $\psi_- \in \cap_{k=0}^{2} \mathcal{C}^k((-\infty; t_-], H^{s-k})$*) of* (1.2) *with initial condition* ψ_0 *and this solution is unique in* $L^\infty((-T', T), H^s(\mathbb{R}^3, \mathbb{C}^4))$ *for any* $T > 0$ *(resp* $T \in (0, t_-)$*) and any* $T' \in (0, t_+)$ *(resp* $T' < 0$*).*

   Moreover, there exist $C > 0$, $\phi_\pm(t) \in \mathcal{CM}$ *and* $\rho_+(t) \in X_s(U_0)$ *(resp.* $\rho_-(t) \in X_u(U_0)$*)) for all* $t > -t_-$ *(resp for all* $t < t_+$*) such that* $\psi_\pm(t) = \phi_\pm(t) + \rho_\pm(t)$ *with*

   $$\|\rho_\pm(t)\|_{H^s} \leq C\, \|\rho_\pm(0)\|_{H^s}\, e^{\mp\gamma t} \text{ as } t \to \pm\infty \quad \text{and} \quad \psi_\pm(\mp t_\pm) \notin \mathcal{O}$$

   where γ *is in a ball around* $1/2\Gamma(U_0)$, *the radius of which is* $O(|U_0|^6)$*).*

   We also have

   $$\phi_\pm(t) = e^{-\mathrm{i}\int_0^t E(U_\pm(v))\, dv} S(U_\pm(t)) + \varepsilon_\pm(t), \quad \forall t > t - (\text{resp. } \forall t < t+)$$

   with $\left\|\dot{U}_+\right\|_{L^q((-t_-,+\infty))} \leq C(\|z_0\|_{H^s} + \|\rho_\pm(0)\|_{H^s})^2$ *(resp.* $\left\|\dot{U}_-\right\|_{L^q((-\infty,t_+))} \leq C(\|z_0\|_{H^s} + \|\rho_\pm(0)\|_{H^s})^2$*) for all* $q \in [1, \infty]$, $\lim_{t\to\pm\infty} U_\pm(t) = U_{\pm\infty}$ *and*

   $$\max\left\{ \|\varepsilon_+\|_{L^\infty((-t_-,+\infty), H^s)}\,,\ \|\varepsilon_+\|_{L^2((-t_-,+\infty), H^s_{-\sigma})}\,, \|\varepsilon_+\|_{L^2((-t_-,+\infty), B^\beta_{\infty,2})} \right\} \leq C\left(\|z_0\|_{H^s} + \|\rho_\pm(0)\|_{H^s}\right),$$

   $$\left(\text{resp. } \max\left\{ \|\varepsilon_-\|_{L^\infty((-\infty,t_+), H^s)}\,,\ \|\varepsilon_-\|_{L^2((-\infty,t_+), H^s_{-\sigma})}\,, \|\varepsilon_-\|_{L^2((-\infty,t_+), B^\beta_{\infty,2})} \right\} \leq C\left(\|z_0\|_{H^s} + \|\rho_\pm(0)\|_{H^s}\right) \right).$$

2. *If $(U_0, z_0, p_+) \in \mathcal{W}_+$ and $p_- \neq \Phi_+(U_0, z_0, p_+)$ or $(U_0, z_0, p_-) \in \mathcal{W}_-$ and $p_+ \neq \Phi_-(U_0, z_0, p_-)$, then there exist $t_+(\psi_0) > 0$, $t_-(\psi_0) < 0$ and a unique solution ψ of (1.2) with initial condition ψ_0 such that for any small neighborhood $\mathcal{O}$ of $S(U_0)$ containing ψ_0, $\phi \in \cap_{k=0}^{2} \mathcal{C}^k([t_-; t_+], H^{s-k})$ with $\psi(t_+) \notin \mathcal{O}$ and $\psi(t_-) \notin \mathcal{O}$. This solution is unique in $L^\infty((T', T), H^s(\mathbb{R}^3, \mathbb{C}^4))$ for any $T \in (0, t_+)$ and any $T' \in (t_-, 0)$.*

The first theorem shows, as in [Bou06], that perturbations in the direction of the continuous subspace, except four directions, relax towards stationary solutions. We have excluded four directions in the continuous subspace, which, due to resonance phenomena, induce orbital instability. The second theorem tells us what happens for perturbations in the directions of an excited state and in the four directions of the continuous spectrum for which we haven't the stabilization. We thus study eight directions: four of them give a manifold on which there hold exponential stabilization in positive time and orbital instability in negative time, while the four others give a manifold on which there hold exponential stabilization in negative time and orbital instability in positive time. Outside these manifolds, we have orbital instability in both negative and positive time.

Remark 1.1. *In the Theorem, we notice that when $U_0 = 0$ then $z_0 = 0$ and $p = 0$ so the theorem do not say anything for this case. In fact, the charge conservation gives the orbital stability of 0. But we cannot extend the previous results to 0 since we can build a manifold of stationary states tangent to the eigenspace associated with λ_1 similarly to Proposition 1.1.*

1.4 The nonlinear scattering

If we choose a localized z_0, we are able to expand further (1.6) as stated by the following theorems also proved in sections 2, 3, 4 and 5.

Theorem 1.5. *With the assumptions and the notations of Theorem 1.3, for the set*

$$\mathcal{S}_\sigma = \left\{(U, z) ;\ U \in B_{\mathbb{C}^2}(0, \varepsilon),\ z \in \mathcal{H}_c(U) \cap B_{H^s_\sigma}(0, r(U))\right\}$$

endowed with the metric of $\mathbb{C}^2 \times H^s_\sigma$, there exists a neigborhhood $\mathcal{V}_\sigma$ of $(0,0)$ in $\mathcal{S}_\sigma$ such that the following is true. If $A = \Psi(U_0, z_0)$ with $(U_0, z_0) \in \mathcal{V}_\sigma$, there exist $\mathcal{V}_\sigma^\pm$ open neighborhoods of $(0,0)$ in $\mathcal{S}_\sigma$ and $(U_{\pm\infty}; z_{\pm\infty}) \in \mathcal{V}_\sigma^\pm$, such that

$$|V_{\pm\infty} - U_0| \leq C\|z_0\|^2_{H^s_\sigma},\ \|z_{\pm\infty} - z_0\|_{H^s} \leq C\|z_0\|^2_{H^s_\sigma},$$

and for all $t \in \mathbb{R}$

$$\psi(t) = e^{-\mathrm{i}tE(V_{\pm\infty})} S(V_\pm(t)) + e^{JtE(V_{\pm\infty})} e^{JtH(V_{\pm\infty})} z_{\pm\infty} + \varepsilon_\pm(t)$$

with

$$\begin{aligned}
\left|\dot{V}_\pm(t) + \mathrm{i}\left(E(V_\pm(t)) - E(V_{\pm\infty})\right)\right| &\leq \frac{C}{\langle t\rangle^2}\|z_0\|^2_{H^s_\sigma},\\
|V_\pm(t) - V_{\pm\infty}| &\leq \frac{C}{\langle t\rangle}\|z_0\|_{H^s_\sigma},\\
\max\left\{\|\varepsilon_\pm(t)\|_{H^s},\ \|\varepsilon_\pm(t)\|_{H^s_{-\sigma}}, \|\varepsilon_\pm(t)\|_{B^\beta_{\infty,2}}\right\} &\leq \frac{C}{\langle t\rangle^2}\|z_0\|^2_{H^s_\sigma}\\
\text{and } \left\|e^{-JtH(V_{\pm\infty})} e^{J\int_0^t (E(V_\pm(s)) - E(V_{\pm\infty})\, ds} \varepsilon_\pm(t)\right\|_{H^s_{\frac{3}{2}}} &\leq \frac{C}{\langle t\rangle^{\frac{1}{2}}}\|z_0\|^2_{H^s_\sigma}
\end{aligned}$$

for all $t \in \mathbb{R}$.

Moreover, the maps

$$(U_0; z_0) \in \mathcal{V}_\sigma \mapsto (V_{\pm\infty}; z_{\pm\infty}) \in \mathcal{V}_\sigma^\pm$$

are bijective.

Remark 1.2. *The fact that z_0 is localized gives us the convergence of*

$$\int_0^t E(U(v))\,dv - tE(U_{\pm\infty})$$

as $t \to \pm\infty$ and allows us to obtain an asymptotic profile for the dispersive part of the perturbed solution ϕ.

What we call the nonlinear scattering result is essentially the fact that the maps

$$(U_0; z_0) \in \mathcal{V}_\sigma \mapsto (U_{\pm\infty}; z_{\pm\infty}) \in \mathcal{V}_\sigma^{\pm},$$

are well defined and bijective (actually the surjectivity is called asymptotic completness).

Using wave operators for the couple $(JH(U), JD_m)$, we can obtain an expansion of the form $\psi(t) = e^{-\mathrm{i}\int_0^t E(U(v))\,dv} S(U_{\pm\infty}) + e^{-\mathrm{i}tD_m} z_\pm + \varepsilon_\pm(t)$ but we will only have

$$\|z_{\pm\infty} - z_0\|_{H^s} \le C\|z_0\|_{H^s}$$

and

$$\max\left\{ \|\varepsilon_\pm\|_{L^\infty(\mathbb{R}^\pm, H^s)},\ \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, H^s_{-\sigma})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, B^\beta_{\infty,2})} \right\} \le C\|z_0\|_{H^s},$$

or using wave operators for the couple $(JH(U), JH)$ we can obtain an expansion of the form $\psi(t) = e^{-\mathrm{i}\int_0^t E(U(v))\,dv} S(U_{\pm\infty}) + e^{-\mathrm{i}tH} z_\pm + \varepsilon_\pm(t)$ with

$$\|z_{\pm\infty} - z_0\|_{H^s} \le C\left(|U_0| + \|z_0\|_{H^s}\right)\|z_0\|_{H^s}$$

and

$$\max\left\{ \|\varepsilon_\pm\|_{L^\infty(\mathbb{R}^\pm, H^s)},\ \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, H^s_{-\sigma})}, \|\varepsilon_\pm\|_{L^2(\mathbb{R}^\pm, B^\beta_{\infty,2})} \right\} \le C\left(|U_0| + \|z_0\|_{H^s}\right)\|z_0\|_{H^s}.$$

But in these cases, we cannot obtain a nice asymptotic completness statement.

We have (in the previous expansions)

$$\max\left\{ \sup_{t\in\mathbb{R}} \left(\left\| e^{JtH(U_{\pm\infty})} z_{\pm\infty} \right\|_{H^s} \right),\ \sup_{t\in\mathbb{R}} \left(\langle t\rangle^{3/2} \left\| e^{JtH(U_{\pm\infty})} z_{\pm\infty} \right\|_{H^s_{-\sigma}} \right), \right.$$
$$\left. \sup_{t\in\mathbb{R}} \left(\langle t\rangle^{3/2} \left\| e^{JtH(U_{\pm\infty})} z_{\pm\infty} \right\|_{B^\beta_{\infty,2}} \right) \right\} \le C\|z_{\pm\infty}\|_{H^s}.$$

This follows from Lemma 3.13 and Lemma 3.14.

Outside the center manifold, we can also have an expansion of the same type. But due to the presence of exponentially stable and unstable directions, one cannot expect a scattering result of the same type. Actually we cannot obtain the injectivity of the corresponding mappings. We have the

Theorem 1.6. *With the assumptions and the notations of Theorem 1.4, for the sets*

$$\widetilde{\mathcal{S}}_\sigma = \left\{ (U, z, p)\,;\ U \in B_{\mathbb{C}^2}(0,\varepsilon),\ z \in \mathcal{H}_c(U) \cap B_{H^s_\sigma}(0, r(U)), p \in B_{\mathbb{R}^4}(0, r(U)), \right\}$$

endowed with the metric of $\mathbb{C}^2 \times H^s_\sigma \times \mathbb{R}^4$, there exist $C > 0$, neighborhoods $\mathcal{W}^\pm_\sigma$ of $(0,0,0)$ in $\mathcal{S}_\sigma$ such that the following is true.

If $\psi_0 \notin \mathcal{CM}$, $(U_0, z_0, p_+) \in \mathcal{W}^+_\sigma$ and $p_- = \Phi_+(U_0, z_0, p_+)$ (resp. $(U_0, z_0, p_-) \in \mathcal{W}^-_\sigma$ and $p_+ = \Phi_-(U_0, z_0, p_-)$) then there exist $C > 0$, $\phi_\pm(t) \in \mathcal{CM}$ and $\rho_\pm(t) \in X_s(U_0)$ for all $t > -t_-$ (resp for all $t < t_+$) such that $\psi_\pm(t) = \phi_\pm(t) + \rho_\pm(t)$ with

$$\|\rho_\pm(t)\|_{H^s} \le C\,\|\rho_\pm(0)\|_{H^s}\,e^{\mp\gamma t} \text{ as } t \to \pm\infty \quad \text{and} \quad \psi(\mp t_\pm) \notin \mathcal{O}$$

where γ is in a ball around $1/2\Gamma(U_0)$, the radius of which is $O(|U_0|^6)$), there exist $(V_{\pm\infty}; z_{\pm\infty}) \in \mathcal{S}$ such that

$$\begin{aligned} |V_{\pm\infty} - U_0| &\le C\left(\|z_0\|_{H^s_\sigma} + \|\rho_\pm(0)\|_{H^s}\right)^2, \\ \|z_{\pm\infty} - z_0\|_{H^s} &\le C\left(\|z_0\|_{H^s_\sigma} + \|\rho_\pm(0)\|_{H^s}\right)^2, \end{aligned}$$

and for all $t > -t_-$ (resp for all $t < t+$)

$$\phi_\pm(t) = e^{-\mathrm{i}tE(V_{\pm\infty})} S(V_\pm(t)) + e^{JtE(V_{\pm\infty})} e^{JtH(V_{\pm\infty})} z_{\pm\infty} + \varepsilon_\pm(t)$$

with

$$\left|\dot{V}_{\pm}(t) + \mathrm{i}\left(E(V_{\pm}(t)) - E(V_{\pm\infty})\right)\right| \leq \frac{C}{\langle t\rangle^2}\left(\|z_0\|_{H^s_\sigma} + \|\rho_{\pm}(0)\|_{H^s}\right)^2,$$

$$|V_{\pm}(t) - V_{\pm\infty}| \leq \frac{C}{\langle t\rangle}\left(\|z_0\|_{H^s_\sigma} + \|\rho_{\pm}(0)\|_{H^s}\right)^2,$$

$$\max\left\{\|\varepsilon_{\pm}(t)\|_{H^s},\ \|\varepsilon_{\pm}(t)\|_{H^s_{-\sigma}}, \|\varepsilon_{\pm}(t)\|_{B^{\beta}_{\infty,2}}\right\} \leq \frac{C}{\langle t\rangle^2}\left(\|z_0\|_{H^s_\sigma} + \|\rho_{\pm}(0)\|_{H^s}\right)^2$$

$$\text{and } \left\|e^{-JtH(V_{\pm\infty})}e^{J\int_0^t (E(V_{\pm}(s))-E(V_{\pm\infty})\,ds}\varepsilon_{\pm}(t)\right\|_{H^s_{\frac{3}{2}}} \leq \frac{C}{\langle t\rangle^{\frac{1}{2}}}\left(\|z_0\|_{H^s_\sigma} + \|\rho_{\pm}(0)\|_{H^s}\right)^2$$

for all $t > -t_-$ (resp for all $t < t_+$).

2 Linearized operator and exponentially stable and unstable manifolds

We study the dynamics associated with (1.2) around a stationary state. We will use spectral properties of the linearized operator around a stationary state.

2.1 The spectrum of the linearized operator

Here we study the spectrum of the linearized operator associated with Equation (1.2) around a stationary state $S(U)$. Let us recall

$$H(U) = H + d^2F(S(U)) - E(U)$$

where d^2F is the differential of ∇F. The operator $H(U)$ is $\mathbb{R}-$linear but not $\mathbb{C}-$linear. Replacing $L^2(\mathbb{R}^3, \mathbb{C}^4)$ by $L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4)$ with the inner product obtained by taking the real part of the inner product of $L^2(\mathbb{R}^3, \mathbb{C}^4)$, we obtain a symmetric operator. We then complexify this real Hilbert space and obtain $L^2(\mathbb{R}^3, \mathbb{C}^4 \times \mathbb{C}^4)$ with its canonical hermitian product. This process transforms the operator $-\mathrm{i}$ into

$$J = \begin{pmatrix} 0 & Id_{\mathbb{C}^4} \\ -Id_{\mathbb{C}^4} & 0 \end{pmatrix}.$$

For $\phi \in L^2(\mathbb{R}^3, \mathbb{R}^4 \times \mathbb{R}^4) \subset L^2(\mathbb{R}^3, \mathbb{C}^4 \times \mathbb{C}^4)$, we still write ϕ instead of

$$\begin{pmatrix} \Re\phi \\ \Im\phi \end{pmatrix}.$$

The extension of $H(U)$ to $L^2(\mathbb{R}^3, \mathbb{C}^4 \times \mathbb{C}^4)$ is also written $H(U)$ and is now a real operator. The extension of K (see (1.3)) is also written K.

The linearized operator associated with Equation (1.2) around the stationary state $S(U)$ is given by $JH(U)$. We shall now study its spectrum.

Differentiating (1.4), we have that for $U = (u_1, u_2) \in \Omega$

$$\mathcal{H}_0(u_1, u_2) = \mathrm{Span}\left\{\frac{\partial}{\partial\Re u_1}S(u_1, u_2),\ \frac{\partial}{\partial\Im u_1}S(u_1, u_2), \frac{\partial}{\partial\Re u_2}S(u_1, u_2),\ \frac{\partial}{\partial\Im u_2}S(u_1, u_2)\right\}$$

is invariant under the action of $JH(U)$. Differentiating the gauge invariance property for S, we notice that $JS(U) \in \mathcal{H}_0(U)$, differentiating the gauge invariance property for F, we also obtain

$$JH(U)JS(U) = 0$$

and differentiating (1.4), we obtain for any $\beta \in \mathbb{N}^4$ with $|\beta| = 1$:

$$JH(U)\partial^{\beta}_U S(U) = (\partial^{\beta}_U E)(U)JS(U).$$

The space $\mathcal{H}_0(U)$ is contained in the geometric null space of $JH(U)$, in fact it is exactly the geometric null space as proved in the sequel.

Now we state our results on the spectrum of $JH(U)$. The first deals with the excited states part, we have the

Proposition 2.1. *If Assumptions 1.1–1.5 hold. Let*

$$\Gamma(U) = \lim_{\substack{\varepsilon\to 0,\\ \varepsilon>0}} \left\langle d^2F(S(U))\phi_1, \Im\left((H-\lambda_0)+(\lambda_1-\lambda_0)-\mathrm{i}\varepsilon\right)^{-1} P_c(H) d^2F(S(U))\phi_1 \right\rangle$$

for any sufficiently small U. Then there exists a map $E_1 : \ B_{\mathbb{C}^2}(0,\varepsilon) \mapsto \ \mathbb{R}$ with

$$\begin{cases} \Im E_1(U) = (\lambda_1 - E(U)) + O(|U|^4) \\ \Re E_1(U) = 1/2\Gamma(U) + O(|U|^6) \end{cases}$$

such that $E_1(U)$, $\overline{E_1}(U)$, $-E_1(U)$ and $-\overline{E_1}(U)$ are double eigenvalues of $JH(U)$ and we have $E_1(U) = E_1(|U|)$.

For any $s\in\mathbb{R}$, there exist smooth maps $k^{\pm} : \ B_{\mathbb{C}^2}(0,\varepsilon) \mapsto \left\{\begin{pmatrix} \phi_1 \\ -i\phi_1 \end{pmatrix}\right\}^{\perp} \cap H^s$ such that

$$k_{\pm}(U) - \left((H-E(U)) + \mathrm{i}E_1(U)\right)^{-1} P_c(H) d^2F(S(U)) \frac{|U|}{\sqrt{2}} \left(\frac{u_1}{|U|} Id_{\mathbb{C}^4} + \frac{u_2}{|U|} K\right) \begin{pmatrix} \phi_1 \\ -i\phi_1 \end{pmatrix} \tag{2.1}$$

is $O(|U|^5)$ for any $\sigma \geq 1$. For any $U = (u_1, u_2) \in B_{\mathbb{C}^2}(0,\varepsilon)$

$$k_{\pm}(U) = \frac{|U|}{\sqrt{2}} \left(\frac{u_1}{|U|} Id_{\mathbb{C}^4} + \frac{u_2}{|U|} K\right) k_{\pm}((|U|,0))$$

and defining for any $U = (u_1,u_2) \in B_{\mathbb{C}^2}(0,\varepsilon)$:

$$\Phi_{\pm}(U) = \frac{|U|}{\sqrt{2}} \left(\frac{u_1}{|U|} Id_{\mathbb{C}^4} + \frac{u_2}{|U|} K\right) \begin{pmatrix} \phi_1 \\ -i\phi_1 \end{pmatrix} + k_{\pm}(U),$$

we have

- $\{\Phi_+(U), K\Phi_+(U)\}$ *is a basis of the eigenspace associated with $E_1(U)$,*
- $\left\{\overline{\Phi_+(U)}, K\overline{\Phi_+(U)}\right\}$ *is a basis of the eigenspace associated with $\overline{E_1(U)}$,*
- $\{\Phi_-(U), K\Phi_-(U)\}$ *is a basis of the eigenspace associated with $-E_1(U)$,*
- $\left\{\overline{\Phi_-(U)}, K\overline{\Phi_-(U)}\right\}$ *is a basis of the eigenspace associated with $-\overline{E_1(U)}$.*

Moreover for any $\beta\in\mathbb{N}^4$, $s\in\mathbb{R}^+$ and $p,q\in[1,\infty]$. There exist $\gamma>0$, $\varepsilon>0$ and $C>0$ such that for all $U\in B_{\mathbb{C}^2}(0,\varepsilon)$, one has

$$\|e^{\gamma\langle Q\rangle}\partial_U^{\beta}\Phi_{\pm}(U)\|_{B^s_{p,q}} \leq C\|S(U)\|_2, \tag{2.2}$$

where $\partial^{\beta}_{u_1,u_2} = \frac{\partial^{|\beta|}}{\partial^{\beta_1}\Re u_1 \partial^{\beta_2}\Im u_1 \partial^{\beta_1}\Re u_2 \partial^{\beta_2}\Im u_2}$.

Proof. For this proof, we use ideas of the proof of [TY02d][Theorem 2.2]. The equation to solve for excited states is:

$$\left(JH(U) - z\right)\phi = 0. \tag{2.3}$$

Since the proof is similar for all cases, we restrict the study to U of the form $(|U|,0)$ and (dividing by $|U|$) to solutions the form $\phi = S_1 + \eta$ where S_1 is the normalized eigenvector of JH:

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 \\ -i\phi_1 \end{pmatrix}$$

and $\eta\in\{S_1\}^{\perp}$, the orthogonal relation is taken in fact with respect to J (but since $JS_1 = \mathrm{i}S_1$, we can take it in the usual way). For $z\in\mathbb{C}\setminus\mathrm{i}\mathbb{R}$, we obtain the equation

$$\eta = \left(J(H-E(U)) - z\right)^{-1} P_1^{\perp} W(U)\left\{S_1+\eta\right\} \tag{2.4}$$

with $P_1^\perp$ the orthogonal projector, with respect to J, into $\{S_1\}^\perp$ and $W(U) = JH(U) - J(H - E(U))$. We notice that $\{S_1\}^\perp$ is invariant under the action of $J(H - E(U))$. To solve this equation in η for a fixed u and z, we notice that if

$$\Re z > 0 \text{ and } |\Im z| \geq m,$$

the series

$$k(U,z) = (J(H-E(U))-z)^{-1} P_1^\perp \sum_{k\geq 0} \left(-W(U)\left(J(H-E(U))-z\right)^{-1} P_1^\perp\right)^k W(U)S_1,$$

is convergent in L^2 for sufficiently small $|U|$ and $|\Im z| = O(|U|^2)$ using the Limiting Absorption Principle (1.1) and the bound of the resolvent $\left\|(H-z')^{-1}\right\| \leq |\Im z'|^{-1}$ in L^2. Hence, we have a solution of (2.4).

Then we solve the equation in z. We obtain from Equation (2.3) the equation

$$\langle (JH(U)-z)\,\phi, S_1\rangle = 0 \text{ with } \phi = S_1 + k(U,z),$$

we infer

$$\begin{aligned}
z &= \langle JH(U)S_1, S_1\rangle + \langle JH(U)k(U,z), S_1\rangle \\
&= \mathrm{i}\,(\lambda_1-\lambda_0) + \langle W(U)S_1, S_1\rangle \\
&\quad + \sum_{k\geq 0}\left\langle JH(U)\left(J(H-E(U))-z\right)^{-1} P_1^\perp \left(-W(U)\left(J(H-E(U))-z\right)^{-1} P_1^\perp\right)^k W(U)S_1, S_1\right\rangle \\
&= \mathrm{i}\,(\lambda_1-\lambda_0) + \langle W(U)S_1, S_1\rangle \\
&\quad + \sum_{k\geq 0}\left\langle P_1^\perp \left(-W(U)\left(J(H-E(U))-z\right)^{-1} P_1^\perp\right)^k W(U)S_1, S_1\right\rangle \\
&\quad + \sum_{k\geq 0}\left\langle (W(U)+z)\left(J(H-E(U))-z\right)^{-1} P_1^\perp \left(-W(U)\left(J(H-E(U))-z\right)^{-1} P_1^\perp\right)^k W(U)S_1, S_1\right\rangle.
\end{aligned}$$

Since $P_1^\perp S_1 = 0$, we introduce the function

$$\begin{aligned}
f(z) &= \mathrm{i}\,(\lambda_1-\lambda_0) + \langle W(U)S_1, S_1\rangle \\
&\quad + \sum_{k\geq 0}\left\langle W(U)\left(J(H-E(U))-z\right)^{-1} P_1^\perp \left(-W(U)\left(J(H-E(U))-z\right)^{-1} P_1^\perp\right)^k W(U)S_1, S_1\right\rangle.
\end{aligned}$$

Since $JS_1 = -\mathrm{i}S_1$, we obtain that $\Re\langle W(U)S_1, S_1\rangle = 0$, so for $z \in \mathbb{C}\setminus \mathrm{i}\mathbb{R}$ we have

$$\begin{aligned}
\Re f(z) &= \Re\left\langle W(U)\left(J(H-E(U))-z\right)^{-1} P_1^\perp W(U)S_1, S_1\right\rangle + O(|U|^6) \\
&= \Im\left\langle d^2F(S(U))\left((H-E(U))+zJ\right)^{-1} P_1^\perp d^2F(S(U))S_1, S_1\right\rangle + O(|U|^6).
\end{aligned}$$

Then using (1.5) and

$$((H-E(U))+zJ)^{-1} = \frac{1}{2}\left(((H-E(U))-\mathrm{i}z)^{-1}(I_{C^2}+\mathrm{i}J) + ((H-E(U))+\mathrm{i}z)^{-1}(I_{C^2}-\mathrm{i}J)\right),$$

we obtain

$$\begin{aligned}
&\Im\left\langle d^2F(S(U))\left((H-E(U))+zJ\right)^{-1} P_1^\perp d^2F(S(U))S_1, S_1\right\rangle \\
&\qquad = \frac{1}{2}\Im\left\langle d^2F(S(U))\left((H-E(U))-\mathrm{i}z\right)^{-1} d^2P_1^\perp F(S(U))S_1, S_1\right\rangle \\
&\qquad + \frac{1}{2}\Im\left\langle d^2F(S(U))\left((H-E(U))+\mathrm{i}z\right)^{-1} P_1^\perp d^2F(S(U))S_1, S_1\right\rangle \\
&\qquad\qquad - \Im\left\langle d^2F(S(U))\left((H-E(U))^2+z^2\right)^{-1} zJP_1^\perp d^2F(S(U))S_1, S_1\right\rangle,
\end{aligned}$$

and so using regularity results of the resolvent of [GM01][Theorem 1.7], we obtain

$$\Im\left\langle d^2F(S(U))\left((H-E(U))+(\mathrm{i}(\lambda_1-\lambda_0)+0)J\right)^{-1}P_1^{\perp}d^2F(S(U))S_1,S_1\right\rangle$$
$$=\frac{1}{2}\Im\left\langle d^2F(S(U))\left((H-E(U))+(\lambda_1-\lambda_0)-\mathrm{i}0\right)^{-1}P_c(H)d^2F(S(U))S_1,S_1\right\rangle.$$

Using Assumption 1.5, the limiting absorption principle (1.1) and regularity results of [GM01][Theorem 1.7], we obtain

$$\Re f(z)=1/2\Gamma(U)+O(|U|^6)$$

for z in a ball of radius of order $|U|^2$ around $\mathrm{i}\,(\lambda_1-\lambda_0)$ and for small U. We also prove by the same way

$$\Im f(z)=(\lambda_1-\lambda_0)+O(|U|^4)$$

for z in a ball of radius of order $|U|^2$ around $\mathrm{i}\,(\lambda_1-\lambda_0)$ and for small U.

So we have proved that for sufficiently small U, f leaves a ball around $\mathrm{i}\,(\lambda_1-\lambda_0)+1/2\Gamma(U)$ invariant. With the same ideas, we prove that it is a contraction. Therefore, we have a fixed point $E_1(U)$ of each U. Then we choose $k_+(U)=|U|k(U,E_1(U))$. Using the complex conjugation, we obtain the eigenvalue $\overline{E_1(U)}$ and its associated eigenspace.

The estimate on (2.1) is proved usinf $|\Re E_1(U)|=O(|U|^2)$, the Limiting Absorption Principle (1.1) and the bound of the resolvent $\left\|(H-z')^{-1}\right\|\leq|\Im z'|^{-1}$ in L^2.

Using Weyl's sequences, we prove that the essential spectrum of $(JH(U))^*=-H(U)J$, for small U, is the essential spectrum of $-HJ=-JH$. So z with non zero real part is in the spectrum of $(JH(U))^*$ if and only if it is an isolated eigenvalue. Then to obtain $-E_1(U)$ and $-\overline{E_1(U)}$, we notice that $E_1(U)$ and $\overline{E_1(U)}$ are eigenvalues of $(JH(U))^*$. Using the symmetry : $J(JH(U))=-(JH(U))^*J$, we show that any eigenvector ϕ of $(JH(U))^*$ associated with λ, $J\phi$ is an eigenvector of $JH(U)$ associated with $-\lambda$. Hence repeating the previous proof for $(JH(U))^*$, we obtain k_-.

The exponential decay works like in Lemma 1.1 □

Remark 2.1. *If $F(z)$ is homogeneous of order p then there exist $\varepsilon,\Gamma_1,\Gamma_2>0$ such that for all $U\in B_{\mathbb{C}^2}(0,\varepsilon)$*

$$|U|^{p-2}\,\Gamma_1\leq\Gamma(U)\leq|U|^{p-2}\,\Gamma_2.$$

We just write $S((u_1,u_2))=u_1\phi_0+u_2K\phi_0+h((u_1,u_2))$, expand $\Gamma(U)$ and use Assumption 1.5 with the regularity results of the resolvente from [GM01][Theorem 1.7].

This gives

$$k_{\pm}(U)-\left((H-E(U))+\mathrm{i}E_1(U)\right)^{-1}P_c(H)d^2F(S(U))\frac{|U|}{\sqrt{2}}\left(\frac{u_1}{|U|}Id_{\mathbb{C}^4}+\frac{u_2}{|U|}K\right)\begin{pmatrix}\phi_1\\-i\phi_1\end{pmatrix}$$

is $O(|U|^{7-p})$ in $B(L^2_{-\sigma})$ for any $\sigma\in\mathbb{R}^+$.

The following proposition deals with the essential spectrum of our linearized operator.

Proposition 2.2. *If Assumptions 1.1–1.5 hold. For any sufficiently small non zero $U\in\mathbb{C}^2$, let*

$$\mathcal{H}_1(U)=\operatorname{span}\left\{\Phi_+(U),K\Phi_+(U),\overline{\Phi_+(U)},K\overline{\Phi_+(U)},\Phi_-(U),K\Phi_-(U),\overline{\Phi_-(U)},K\overline{\Phi_-(U)}\right\}.$$

The orthogonal space of $\mathcal{H}_0(U)\oplus\mathcal{H}_1(U)$ with respect to the product associated to J

$$\mathcal{H}_c(U)=\{\mathcal{H}_0(U)\oplus\mathcal{H}_1(U)\}^{\perp_J}$$

is invariant under the action of $JH(U)$.

We also have for $\mathbf{P}_c(U)$, the orthogonal projector onto $\mathcal{H}_c(U)$ with respect to J, and for $U'\in B_{\mathbb{C}^2}(U,\varepsilon)$, with sufficiently small $\varepsilon>0$, that

$$\mathbf{P}_c(U)|_{\mathcal{H}_c(U')}:\mathcal{H}_c(U')\mapsto\mathcal{H}_c(U)$$

is an isomorphism and is a bounded operator from $H^s_\sigma(\mathbb{R}^3,\mathbb{C}^8)$ or $B^s_{p,q}(\mathbb{R}^3,\mathbb{C}^8)$ to itself for any reals $s,\sigma\in\mathbb{R}$ and $p,q\in[1,\infty]$, the inverse $R(U',U)$ is continuous with respect to U and U' for these norms.

Moreover, there exists $C > 0$ such that we have

$$\begin{aligned}
\|\psi\|_X &\leq C \|P_c(U)\psi\|_X\,, && (2.5)\\
&\qquad \forall \psi \in \mathcal{H}_c(U) \text{ with } X = H^s_\sigma(\mathbb{R}^3, \mathbb{C}^8) \text{ or } B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^8), \\
&\qquad \forall s, \sigma \in \mathbb{R}, \forall p, q \in [1, \infty], \\
\int_{\mathbb{R}} \|\langle Q\rangle^{-\sigma} e^{sJH(U)} \mathbf{P}_c(U)\psi\|^2 \, ds &\leq C\|\psi\|_2^2, \ \forall \psi \in L^2, \ \forall \sigma \geq 1, && (2.6)\\
\|e^{tJH(U)} \mathbf{P}_c(U)\psi\| &\leq C\|\psi\|_2, \ \forall t \in \mathbb{R}, \ \forall \psi \in L^2, && (2.7)
\end{aligned}$$

and $\mathcal{H}_c(U)$ contains no eigenvector.

Remark 2.2. *We use the same notation for $\mathcal{H}_c(U)$ and its real part which appears in our main theorems. We just notice that $\mathcal{H}_c(U)$ appears when we discuss spectral properties in our proof. Then when we talk about dynamical properties, we deal with its real part. We remind that the real part of $\mathcal{H}_c(U)$ is left invariant by $JH(U)$.*

Proof. We prove that there is no other eigenvector, by proving that smoothness estimate (2.6) takes place over

$$\mathcal{H}_c(U) = \{\mathcal{H}_0(U) \oplus \mathcal{H}_1(U)\}^{\perp_J}.$$

First we prove that

$$\mathbf{P}_c((U))|_{\mathcal{H}_c(U')} : \mathcal{H}_c(U') \mapsto \mathcal{H}_c(U)$$

is an isomorphism. To prove it, we exhibit an inverse $R(U', U)$ which is the projector onto $\mathcal{H}_c(U')$ associated with the decomposition $\mathcal{H}_0(U) \oplus \mathcal{H}_1(U) \oplus \mathcal{H}_c(U')$ of $L^2(\mathbb{R}^3, \mathbb{C}^8)$. Indeed we have $\{\mathcal{H}_0(U) \oplus \mathcal{H}_1(U)\} \cap \mathcal{H}_c(U') = \{0\}$ when U' and U are close one to each other and $\mathrm{codim}\mathcal{H}_c(U') = \dim\mathcal{H}_0(U) \oplus \mathcal{H}_1(U)$, hence we have a decomposition of $L^2(\mathbb{R}^3, \mathbb{C}^8)$ into closed subspaces hence the associated projectors are continuous. So $R(U', U)$ should be of the form

$$R(U', U) = Id + \sum_i |J\xi_i(U)\rangle \langle \alpha_i(U', U)|$$

where $\xi_i(U)$ is a basis of the eigenspaces of $JH(U)$ and $(\alpha_i(U', U))_i$ solve the equations

$$J\xi_j(U') + \sum_i \langle J\xi_i(U), J\xi_j(U')\rangle \, \alpha_i(U', U) = 0.$$

Such α exists because the matrix $\left(\langle J\xi_i(U), J\xi_j(U')\rangle\right)_{i,j}$ is a Gramm matrix when $U = U'$ and otherwise a small perturbation of such matrices for U and U' close one to each other and hence is invertible.

The boundedness of R in $\mathcal{B}(H^s_\sigma(\mathbb{R}^3, \mathbb{C}^8))$ or $\mathcal{B}(B^s_{p,q}(\mathbb{R}^3, \mathbb{C}^8))$ follows from the exponential decay of eigenvectors and their derivatives, the continuity of R follows from the continuity of the eigenvector with respect to the parameters U or U' see Proposition 1.1, Lemma 1.1 and Proposition 2.1.

Let us now consider the orthogonal projector P_c on associated with the continuous subspace of JH. Since the eigenvector of JH are exponentially decaying, we can extend P_c to obtain an operator of $L^2_{\pm\sigma}$ into itself. The same is true for $P_c(U)$ and hence we can consider the extension of $\mathcal{H}_c(U)$ to $L^2_{\pm\sigma}$. We still call it $\mathcal{H}_c(U)$. For all $\psi \in \mathcal{H}_c(U)$:

$$\|\psi\|_{L^2_{-\sigma}} \leq \|P_c\psi\|_{L^2_{-\sigma}} + \|(1 - P_c)\psi\|_{L^2_{-\sigma}}.$$

Since $1 - P_c$ is the projector into the eigenspaces of H and ψ is orthogonal to the eigenvectors of $JH(U)$, we obtain that there exists $c > 0$ such that

$$\|(1 - P_c)\psi\|_{L^2_{-\sigma}} \leq c|U| \, \|\psi\|_{L^2_{-\sigma}}.$$

Indeed, using Proposition 2.1, we obtain a $c' > 0$ such that for sufficiently small non zero U, we have

$$\left|\left\langle \psi, J\frac{1}{\sqrt{2}}\left(\frac{u_1}{|U|} Id_{\mathbb{C}^4} + \frac{u_2}{|U|} K\right) \begin{pmatrix} \phi_1 \\ -i\phi_1 \end{pmatrix} \right\rangle\right| \leq \frac{1}{|U|} \left|\left\langle \psi, J\frac{\Phi_+(U) + \Phi_-(U)}{2} \right\rangle\right| + c' \, |U| \, \|\psi\|_{L^2_{-\sigma}}.$$

Hence since ψ is orthogonal to $\Phi_+(U)$ and $\Phi_-(U)$, we obtain that the projection of ψ in the second eigenspace of H is small, since they are invariant under the action of J. Using Proposition 1.1, we obtain the same thing for the first one.

Hence for a sufficiently small non zero U, we obtain Estimate (2.5) for $X = L^2_{-\sigma}$ with $\sigma > 0$. The rest of Estimate (2.5) follows by the same way using the exponential decay of eigenvectors (Estimate (2.2) and Lemma 1.1). We infer

$$\begin{aligned}
&\|\langle Q\rangle^{-\sigma} e^{tJH(U)}\mathbf{P}_c(U)\psi\| \\
&\quad\leq C\|\langle Q\rangle^{-\sigma}\mathbf{P}_c e^{tJH(U)}\mathbf{P}_c(U)\psi\| \\
&\quad\leq C\|\langle Q\rangle^{-\sigma}\mathbf{P}_c e^{-\mathrm{i}t(H-E(U))}\mathbf{P}_c(U)\psi\| \\
&\qquad+C\|\langle Q\rangle^{-\sigma}\int_0^t \mathbf{P}_c e^{-\mathrm{i}(t-s)(H-E(U))} D\nabla F(S(U)) e^{sJH(U)}\mathbf{P}_c(U)\psi\, ds\|
\end{aligned}$$

Using estimate (i) and (iii) of Theorem 1.1, we obtain the estimate (2.6) for sufficiently small U:

$$\int_{\mathbb{R}} \|\langle Q\rangle^{-\sigma} e^{-sJH(U)}\mathbf{P}_c(U)\psi\|^2\, ds \leq C\|\psi\|^2.$$

Hence there is no eigenvector in the range of $P_c(U)$. Using the inequalities (2.6), the conservation law for H and Duhamel's formula :

$$e^{JtH(U)} = e^{-\mathrm{i}t(H-E(U))} + \int_0^t e^{-\mathrm{i}(t-s)(H-E(U))} Jd^2\nabla F(S(U)) e^{JsH(U)}\, ds,$$

we prove the estimate (2.7). □

Since $\mathcal{H}_c(U)$ is closed and $\operatorname{codim}\mathcal{H}_c(U) = \dim\{\mathcal{H}_0(U)\oplus\mathcal{H}_1(U)\}$ and $\mathcal{H}_c(U)\cap\{\mathcal{H}_0(U)\oplus\mathcal{H}_1(U)\} = \{0\}$, we obtain $\mathcal{H}_0(U)\oplus\mathcal{H}_1(U)\oplus\mathcal{H}_c(U) = L^2(\mathbb{R}^3,\mathbb{C}^8)$ and the

Proposition 2.3. *Suppose that Assumptions 1.1–1.5 hold. Then the space $\mathcal{H}_0(U)$ is the geometric null space of $JH(U)$.*

2.2 Stable, unstable and center manifold

We can now obtain results similar to those of Bates and Jones [BJ89]. We notice that we won't prove that the Cauchy problem (1.2) is locally wellposed for initial condition outside some manifolds (built below). In fact it can be proved with the methods we present here or by generalizing to our case the results of and Vega [EV97].

We have that $JH(U)$ as an operator in $L^2(\mathbb{R}^3,\mathbb{R}^8)$ is a closed densely defined operator that generates a continuous semigroup on $L^2(\mathbb{R}^3,\mathbb{R}^8)$. The spectrum of $JH(U)$ in $L^2(\mathbb{R}^3,\mathbb{R}^8)$ is the same as $JH(U)$ in $L^2(\mathbb{R}^3,\mathbb{C}^8)$ and so it splits in three parts:

$$\begin{aligned}
\sigma_s(U) &= \{\lambda\in\sigma(JH(u)),\ \Re\lambda<0\} = \left\{-E_1(U), -\overline{E_1(U)}\right\}\\
\sigma_c(U) &= \{\lambda\in\sigma(JH(u)),\ \Re\lambda=0\} = \mathrm{i}\,\{\mathbb{R}\setminus(-E(U),E(U))\}\\
\sigma_u(U) &= \{\lambda\in\sigma(JH(u)),\ \Re\lambda>0\} = \left\{E_1(U), \overline{E_1(U)}\right\}
\end{aligned}$$

each one is associated with a spectral real subspace, respectively

$$\begin{aligned}
X_s(U) &= \operatorname{span}_{\mathbb{R}}\{\Re\Phi_-(U), \Im\Phi_-(U), K\Re\Phi_-(U), K\Im\Phi_-(U)\}\\
X_u(U) &= \operatorname{span}_{\mathbb{R}}\{\Re\Phi_+(U), \Im\Phi_+(U), K\Re\Phi_+(U), K\Im\Phi_+(U)\}\\
X_c(U) &= \Re\mathcal{H}_0(U)\oplus\Re\mathcal{H}_c(U)
\end{aligned}$$

where we used the notation $\Re\Psi = (1/2)\left(\Psi+\overline{\Psi}\right)$ and $\Im\Psi = -(\mathrm{i}/2)\left(\Psi-\overline{\Psi}\right)$ and $\Re X = \{\Re\Psi, \Psi\in X\}$, the real part of the space X. The spaces $X_s(U)$ and $X_u(U)$ are finite dimensional. Let us write $\pi^c(U)$, $\pi^s(U)$ and $\pi^u(U)$ for the projector associated with the decomposition $X_c(U)\oplus X_s(U)\oplus X_u(U)$. Since the eigenvectors belongs also to L^2_σ for any $\sigma\in\mathbb{R}$, the projector $P_c(U)$ and $\pi^c(U)$ can be defined in L^2_σ for any real σ. We can extend, by this way, the spaces $\mathcal{H}_c(U)$ and $X_c(U)$ to L^2_σ for any $\sigma\in\mathbb{R}$. We have the

Lemma 2.1. *If Assumptions 1.1–1.5 hold. Then any $\sigma \in \mathbb{R}$, there exist $r, C_1, C_2 > 0$ such that for all $t \in \mathbb{R}$, we have*

$$C_1 e^{-\gamma(U)t} \leq \left\| e^{tJH(U)} \pi^s(U) \right\|_{\mathcal{B}(L^2_\sigma)} \leq C_2 e^{-\gamma(U)t}, \tag{2.8}$$

$$C_1 e^{\gamma(U)t} \leq \left\| e^{tJH(U)} \pi^u(U) \right\|_{\mathcal{B}(L^2_\sigma)} \leq C_2 e^{\gamma(U)t}, \tag{2.9}$$

$$\left\| e^{tJH(U)} \pi^c(U) \right\|_{\mathcal{B}(L^2_\sigma)} \leq C_2 \langle t \rangle^r, \tag{2.10}$$

where $\gamma(U) = \Re E_1(U)$.

Proof. The statements for the spaces $X^s(U)$ and $X^u(U)$ follows from (2.2).

The statement about $X^c(U)$ is a little more complicate. We notice that we are not looking for an optimal r.

First, the result for $e^{-\mathrm{i}t(D_m+V)}$ in L^2_σ with $\sigma \in 2\mathbb{N}$ follows from [Tha92, Theorem 8.5] (see also Proposition 3.1 below), which is based on the charge conservation. The case $\sigma \in \mathbb{R}$ follows by duality and interpolation.

Then for $e^{tJH(U)}\pi^c(U)$, we use Duhamel's formula :

$$e^{JtH(U)}\pi^c(U) = e^{-\mathrm{i}t(H-E(U))}\pi^c(U) + \int_0^t e^{-\mathrm{i}(t-s)(H-E(U))} J d^2\nabla F(S(U)) e^{JsH(U)}\pi^c(U)\, ds,$$

then the assertion for $e^{tJH(U)}\pi^c(U)$ follows from the assertion for $e^{-\mathrm{i}t(D_m+V)}$, the charge conservation of $e^{tJH(U)}P_c(U)$ (see (2.7)), the fact that $e^{tJH(U)}S(U) = S(U)$, $e^{tJH(U)}\partial_U^\beta S(U) = \partial_U^\beta S(U) + t\partial_U^\beta E(U)S(U)$ and Lemma 1.1. □

By now we do not restrict our study to the space $L^2(\mathbb{R}^3, \mathbb{R}^8)$, we extend it to $L^2_\sigma(\mathbb{R}^3, \mathbb{R}^8)$ for any $\sigma \in \mathbb{R}$, but we still write $\mathcal{H}_c(U)$ and $X_c(U)$ for the extensions of these spaces to $L^2_\sigma(\mathbb{R}^3, \mathbb{R}^8)$ for any $\sigma \in \mathbb{R}$.

We now study the behavior of the solutions in L^2_σ of (1.2) centered around $S(U)$:

$$\partial_t \phi = JH(U)\phi + JN(U, \phi) \tag{2.11}$$

where $H(U) = H + d^2F(S(U)) - E(U)$ and $N(U, \phi) = \nabla F(S(U) + \phi) - \nabla F(S(U)) - d^2F(S(U))\phi$ and d^2F is the differential of ∇F.

In this subsection, we study a modified equation which coincides with (2.11) as long as the solution stays in a neigborhood of a small $S(U)$:

$$\partial_t \phi = JH(U)\phi + JN_\varepsilon(U, \phi) \tag{2.12}$$

where $N_\varepsilon(U, \eta) = \rho(\varepsilon^{-1}\eta)N(U, \eta)$ and ρ is a smooth function with compact support around 0.

We state the

Proposition 2.4 (Center-Stable Manifold). *If Assumptions 1.1–1.5 hold. Then for any sufficiently small non zero U, there exists around $S(U)$ a unique invariant smooth center-stable manifold $W^{cs}(U)$ for (2.12) build as a graph with value in $X_u(U)$ and tangent to $S(U) + X_c(U) \oplus X_s(U)$ at $S(U)$.*

Any solution $\phi \in L^2_\sigma$ of (2.12) initially in the neighborhood of $S(U)$ tends as $t \to -\infty$ to $W^{cs}(U)$ with

$$dist_{L^2_\sigma}(\phi(t), W^{cs}(U)) = O(e^{\gamma t}) \text{ as } t \to -\infty$$

for any $\gamma \in (0, \gamma(U))$, any $s, \sigma \in \mathbb{R}$ and for any sufficiently small neighborhood V of $S(U)$ any solution in V not in $W^{cs}(U)$ leaves V in finite positive time.

Remark 2.3. *For any $s \in \mathbb{R}^+$, due to the exponential decay of eigenvectors, even if $\phi \notin H^s_\sigma$, there exists $\psi \in W^{cs}(U)$ such that $\phi - \psi \in H^s_\sigma$ and we have*

$$dist_{H^s_\sigma}(\phi(t), W^{cs}(U)) = O(e^{\gamma t}) \text{ as } t \to -\infty$$

as shown in the following proof.

If we only consider small solutions, we obtain a locally invariant manifold for the equation (2.11), that is to say that for any initial condition in the manifold there exist a corresponding solution of (2.11) which stays in this manifold in a small interval of time around 0. We notice that in the following proofs the size of this invariant manifold, which is given by ε, is a function of U and this function is $O(\gamma(U))$. By now, we call this function r.

Proof. Our proof is an adaptation of the one of Bressan [Bre] and we refer to it for more details. We make the proof only for the case $\sigma = 0$, the proof in the general case is similar.

First we prove that there is a global solution of the equation (2.12) which do not grow much as $t \to +\infty$. We look for solution as a fixed point:

$$y(t) = \mathcal{G}_\varepsilon(y_0, y)(t)$$

for any $y_0 \in X_s(U) \oplus X_c(U)$ where for small positive ε

$$\mathcal{G}_\varepsilon(y_0, \eta)(t) = e^{tJH(U)}y_0 + \int_0^t e^{(t-s)JH(U)}\pi^c(U)JN_\varepsilon(U, \eta(s))\, ds \\ + \int_0^t e^{(t-s)JH(U)}\pi^s(U)JN_\varepsilon(U, \eta(s))\, ds - \int_t^{+\infty} e^{(t-s)JH(U)}\pi^u(U)JN_\varepsilon(U, \eta(s))\, ds,$$

with $\pi^*(U)$ the projector into $X^*(U)$ with respect to the decomposition $\oplus_{*\in\{c,s,u\}}X^*(U)$.

Let us introduce for $\gamma(U) = \Re E_1(U)$ and any Γ smaller than $\gamma(U)$, the space

$$Y_\Gamma = \left\{ y : \mathbb{R} \mapsto L^2(\mathbb{R}^3, \mathbb{C}^4),\ \exists C > 0,\ \|y(t)\|_2 \leq Ce^{\Gamma|t|},\ \forall t \in \mathbb{R} \right\}.$$

For sufficiently small $\varepsilon > 0$, the map $\mathcal{G}_\varepsilon(y_0, \cdot)$ leaves Y_Γ invariant and is continuous for the norm

$$N_\Gamma : y \mapsto \sup_{t\in\mathbb{R}} \left\{ \|y(t)\|_2 e^{-\Gamma|t|} \right\}.$$

Moreover, it is a strict contraction for sufficiently small U and $\varepsilon > 0$. Actually we choose ε as a function of Γ which is $O(\Gamma)$. In fact since $Y_\Gamma \subset Y_{\Gamma'}$ for $\Gamma < \Gamma'$, we obtain that ε as a function of U is a $O(\gamma(U))$. This proves the existence of the fixed point y.

Then we fix $h_U^{cs}(y_0) = y(0) - y_0$. The invariance of the graph of h_U^{cs} by the flow of Equation (2.12) is immediate.

Now we prove the smoothness property. We have $N_\varepsilon(U, \eta)$ is l times differentiable in η from $Y_{\Gamma'}$ to Y_Γ if $(l+1)\Gamma' \leq \Gamma$ and $\mathcal{G}_\varepsilon$ is l times differentiable from $X_c(U) \oplus X_s(U) \times Y_{\Gamma''}$ to Y_Γ if $2l\Gamma'' \leq \Gamma$ (see [Bre]). We introduce the family $(\eta_n)_{n\in\mathbb{N}}$ satisfying

$$\eta_0 = 0 \text{ and } \eta_{n+1} = \mathcal{G}_\varepsilon(y_0, \eta_n).$$

This sequence converge to y (the fixed point) in Y_Γ. Moreover, as functions of y_0, the convergence is uniform in Y_Γ (endowed with the norm N_Γ) on bounded sets of $X_s(U) \oplus X_c(U)$.

We want to prove that the sequence of their derivatives of order k with respect to η also converges in Y_Γ on bounded sets for any $\Gamma < \gamma(U)$. We prove it by induction in k. So suppose that $(\partial^j \eta_n)_{n\in\mathbb{N}}$ is converging in Y_Γ for all $j < k$ and any $\Gamma < \gamma(U)$. Then we have that (see [Bre])

$$\partial\eta_n = \partial\mathcal{G}_\varepsilon(y_0, \eta_{n-1}) = L + M\left(\partial_\eta N_\varepsilon(U, \eta_{n-1})\partial\eta_{n_1}\right)$$
$$\partial^k\eta_n = \partial^k\mathcal{G}_\varepsilon(y_0, \eta_{n-1}) = M\left(\partial_\eta N_\varepsilon(U, \eta_{n-1})\partial^k\eta_{n-1} + \Psi_k(\eta_{n-1}, \dots, \partial^{k-1}\eta_{n-1})\right),\ \forall k \geq 2$$

with $L = e^{tJH(U)}$ and

$$(M\eta)(t) = -\int_0^t e^{(t-s)JH(U)}\pi^{cs}(U)J\eta(s)\, ds + \int_t^{+\infty} e^{(t-s)JH(U)}\pi^u(U)J\eta(s)\, ds$$

and Ψ_k a smooth function of k parameter. Hence since $M \circ \partial_\eta N_\varepsilon(U, y_{n-1})$ is a strict contraction in Y_Γ for sufficiently small ε and U (once more ε is a $O(\gamma(U))$), this proves the convergence of the sequence of k-th derivatives in Y_Γ on bounded sets for any $\Gamma < \gamma(U)$. Hence the sequences of derivatives of $(\eta_n)_{n\in\mathbb{N}}$ in Y_Γ on bounded sets for any $\Gamma < \gamma(U)$. This gives the differentiability at any order of $y(0) = h(y_0)$. This also gives, since $N(U, \eta) = O(|\eta|^2)$ around zero, that $h(y_0) = O(|y_0|^2)$ around zero.

Now we want prove that $W^{cs}(U)$ is attractive in negative time. In fact $W^{cs}(U)$ is the graph of a smooth function $h : X^{cs} \mapsto X^u(U)$. Let η be such that $S(U) + \eta$ is a solution of (1.2), we have

$$\partial_t\eta = JH(U)\eta + JN_\varepsilon(U, \eta).$$

$$\eta = y + r = y + h(y) + z$$

with $y = \pi^{cs}(U)\eta$ and we have the following equation for $z \in X^u(U)$

$$\partial_t z = JH(U)z + M(U, y, z)$$

where

$$M(U, y, z) = \pi^u(U) \{JN_\varepsilon(U, \eta) - JN_\varepsilon(U, y + h(y))\} - Dh(y)\pi^{cs}(U) \{JN_\varepsilon(U, \eta) - JN_\varepsilon(U, y + h(y))\}.$$

Using Duhamel's formula, we obtain

$$z(t) = e^{tJH(U)}z(0) + \int_0^t e^{(t-s)JH(U)}M(U, y(s), z(s))\, ds.$$

We obtain since $z \in X^u(U)$

$$\|z(t)\| \leq e^{\gamma(U)t} \|z(0)\| + C \int_0^t e^{(t-s)\gamma(U)} \|M(U, y(s), z(s))\|\, ds$$

and so for $\gamma \in (0, \gamma(U))$

$$e^{-\gamma t} \|z(t)\| \leq \|z(0)\| + C|U| \sup_{s\in[0,t]} \left\{e^{-\gamma s} \|z(s)\|\right\} + o\big(\sup_{s\in[0,t]} \left\{e^{-\gamma s} \|z(s)\|\right\}\big)$$

where C do not depend of U and z. Hence if $z(0)$ and U are small, we have that there exists $c > 0$ such that $\|z(t)\| \leq ce^{\gamma t}$ for all $t \leq 0$. We notice that since $X^u(U) \subset H^s_\sigma$ for any $s \in \mathbb{R}^+$ and is finite dimensional (see Lemma 2.1), the time decay in L^2_σ gives also a time decay in H^s_σ for any $s \in \mathbb{R}^+$.

Now choose V a sufficiently small neighborhood of 0 and ϕ a solution of (2.12) initially in V but not in $W^{cs}(U)$. Suppose it stays in V in positive time. We obtain that $\phi \in Y_\Gamma$. We have

$$\begin{aligned}\phi(t) = {} & e^{tJH(U)} \left(\pi^s(U) + \pi^c(U)\right) \phi(0) \\ & + \int_0^t e^{-sJH(U)}\pi^s(U)JN_\varepsilon(U, \phi(s))\, ds + \int_0^t e^{-sJH(U)}\pi^c(U)JN_\varepsilon(U, \phi(s))\, ds \\ & + e^{tJH(U)}\pi^u(U) \left(\pi^u(U)\phi(0) + \int_0^\infty e^{-sJH(U)}\pi^u(U)JN_\varepsilon(U, \phi(s))\, ds\right) \\ & - \int_t^\infty e^{-sJH(U)}\pi^u(U)JN_\varepsilon(U, \phi(s))\, ds.\end{aligned}$$

with

$$\pi^u(U)\phi(0) + \int_0^\infty e^{-sJH(U)}\pi^u(U)JN_\varepsilon(U, \phi(s))\, ds \neq 0.$$

Hence we obtain with (2.9), that $\phi(t)$ exponentially tends to infinity in norm. This is a contradiction so ϕ leaves V in finite time. □

Then reversing the time direction that is to say replacing H by $-H$ and F by $-F$, we obtain with this theorem a locally invariant center unstable manifold with the corresponding properties:

Proposition 2.5 (Center-Unstable Manifold). *If Assumptions 1.1–1.5 hold. Then for any sufficiently small non zero U, there exists around $S(U)$ a unique smooth invariant center unstable manifold $W^{cu}(U)$ for (2.12), build as a graph with value in $X_S(U)$ and tangent to $S(U)+X_c(U)\oplus X_u(U)$ at $S(U)$.*

Any solution $\phi \in L^2_\sigma$ of (2.12) initially in the neighborhood of $S(U)$ tends as $t \to +\infty$ to $W^{cu}(U)$ with for any $s \in \mathbb{R}^+$

$$dist_{H^s_\sigma}(\phi(t), W^{cu}(U)) = O(e^{-\gamma t}) \text{ as } t \to +\infty$$

and for $\gamma \in (0, \gamma(U))$, any $s, \sigma \in \mathbb{R}$ and for any V sufficiently small neighborhood of $S(U)$ any solution in V not in $W^{cu}(U)$ leaves V in finite negative time.

We can build by the same way a center manifold which is the intersection of the previous:

Proposition 2.6 (Center Manifold). *If Assumptions 1.1–1.5 hold. Then for any sufficiently small non zero U, there exists around $S(U)$ a unique smooth invariant center manifold $W^c(U)$ for* (2.12)*, build as a graph with value in $X_s(U) \oplus X_u(U)$ and tangent to $S(U) + X_c(U)$ at $S(U)$.*

Moreover, we have $W^c(U) = W^{cs}(U) \cap W^{cu}(U)$ and $W^c(U)$ contains the part of the PLS manifold which is in a small neighborhood of $S(U)$.

Proof. We build the center manifold with the same method as in the previous cases. We can also build a center-unstable manifold inside center-stable manifold. More precisely, let $h^s_U : X_c(U) \oplus X_s(U) \mapsto X_u(U)$ be the map defining center-stable manifold and $h^u_U : X_c(U) \oplus X_u(U) \mapsto X_s(U)$ be the map defining center-unstable manifold. A solution $y = S(U) + y_c + y_s + y_u$ with $y_* \in X_*(U)$ for $* \in \{c, s, u\}$ is in the center-stable manifold if $y_u = h^s_U(y_c, y_s)$. Hence to obtain a center-unstable manifold inside center-stable manifold one has to solve, for each y_c, the equation

$$y_s = h^u_U(y_c, h^s_U(y_c, y_s)),$$

which can be solve inside a small ball for small y_c and small U by means of the fixed point theorem, since $h^*_U(y_c, z)$ is a $O(|y_c|^2 + |z|^2)$ around zero for $* \in \{s, u\}$.

By the same way, we can also build a center-stable manifold inside the center-unstable manifold.

Using the uniqueness of the center manifold, we obtain that this two manifolds are equal to the center manifold and $W^c(U) = W^{cs}(U) \cap W^{cu}(U)$.

Then any stationary states in a small neighborhood of $S(U)$ converges to $W^{cs}(U)$ and $W^{cu}(U)$ using the stabilization results of the Proposition 2.4 and Proposition 2.5. Hence, we have that it belongs to $W^{cs}(U) \cap W^{cu}(U) = W^c(U)$. □

In the two following sections, we study the dynamic respectively inside and outside the center manifold.

3 The dynamic inside the center manifold

In this section, we prove that the dynamic inside the center manifold around $S(V_0)$, for small non zero V_0, relaxes towards the PLS manifold. To this end, we use Theorem 1.1 and Theorem 1.2 about the time decay of the propagator associated with H.

3.1 Decomposition of the system

Like in [Bou06], we decompose a solution $\phi \in W^c(V_0)$ of the equation (1.2) with respect to the spectrum of $JH(U)$, with U specified in the sequel, and we study the equations for these different parts of the decomposition. We introduce

$$\mathcal{H}_0^{\perp_J}(u_1, u_2) = \left\{ \eta \in L^2(\mathbb{R}^3, \mathbb{C}^8), \left\langle J\eta, \frac{\partial}{\partial \Re u_1} S(u_1, u_2) \right\rangle = \left\langle J\eta, \frac{\partial}{\partial \Im u_1} S(u_1, u_2) \right\rangle \right.$$
$$\left. = \left\langle J\eta, \frac{\partial}{\partial \Re u_2} S(u_1, u_2) \right\rangle = \left\langle J\eta, \frac{\partial}{\partial \Im u_2} S(u_1, u_2) \right\rangle = 0 \right\}.$$

In fact, we have

$$\mathcal{H}_0^{\perp_J}(U) = \mathcal{H}_1(U) \oplus \mathcal{H}_c(U)$$

which is invariant under the action of $JH(U)$. We recall that $\mathcal{H}_1(U)$ is defined in Proposition 2.1 and $\mathcal{H}_c(U)$ in Proposition 2.2. We have the

Lemma 3.1. *If Assumptions 1.1–1.4 hold. Let $s, \sigma \in \mathbb{R}$ there exist $\varepsilon, \varepsilon' > 0$ such that for the manifold*

$$\Sigma = \left\{ (U, \eta),\ U \in B_{\mathbb{C}^2}(0, \varepsilon'),\ \eta \in \mathcal{H}_0^{\perp_J}(U) \right\}$$

endowed with the metric of $\mathbb{C}^2 \times H^s_\sigma$ and any function $\phi \in B_{H^s_\sigma}(0, \varepsilon)$, there exist a unique $(U, \eta) \in \Sigma$ with

$$\phi = S(U) + \eta.$$

Moreover, there exists a neighborhood $\mathcal{O}$ of $(0, 0) \in \Sigma$ such that the mapping $\phi \mapsto (U, \eta) \in \mathcal{O}$ is smooth.

Proof. The prove that Σ is manifold, we use Proposition 2.2 which gives that it is locally isomorph to some open subset of $\mathbb{C}^2 \times \mathcal{H}_c$ endowed with the metric of $\mathbb{C}^2 \times H^s_\sigma$. Then this is a consequence of the inverse function theorem like in [GNT04, Lemma 2.3]. ☐

For any solution ϕ of (1.2) on an interval of time I containing 0, we write for $t \in I$

$$\phi(t) = e^{-\mathrm{i}\int_0^t E(U(s))\,ds} \left(S(U(t)) + \eta(t)\right).$$

where $\eta(t) \in \mathcal{H}_0^{\perp_J}(U(t))$ and we want to solve the equation

$$\begin{array}{rcl} \mathrm{i}\partial_t \eta &=& \{H - E(U)\}\,\eta + \{\nabla F(S(U)+\eta) - \nabla F(S(U))\} - \mathrm{i}dS(U)\dot{U} \\ &=& \left\{H + d^2F(S(U)) - E(U)\right\}\eta + N(U,\eta) - \mathrm{i}dS(U)\dot{U} \end{array} \tag{3.1}$$

for $\eta(t) \in \mathcal{H}_0^{\perp_J}(U(t))$. Here d^2F is the differential of ∇F and dS the differential of S in $\mathbb{R}^2$. To close the system, we need the equation for U. This follows from the condition

$$\langle \eta(t), JdS(U(t))\rangle = 0.$$

After a time derivation (like in[Bou06]), we obtain the equation:

$$\dot{U}(t) = -A(U(t),\eta(t))\langle N(U(t),\eta(t)), dS(U(t))\rangle.$$

where

$$A(U,\eta) = \left[\langle JdS(U), dS(U)\rangle - \langle J\eta, d^2S(U)\rangle\right]^{-1}$$

the matrix $[\langle JdS(U(t)), dS(U(t))\rangle - \langle J\eta(t), d^2S(U(t))\rangle]$ is invertible for small $|U(t)|$ and $\|\eta(t)\|_2$ since we have

$$[\langle JdS(U(t)), dS(U(t))\rangle - \langle J\eta(t), d^2S(U(t))\rangle] = \begin{pmatrix} J & 0_2 \\ 0_2 & J \end{pmatrix} + O(|U(t)| + \|\eta(t)\|_2).$$

Lemma 3.2. *For any $s, s', \sigma \in \mathbb{R}$, any $p, q \in [1,\infty]$, any $V_0 \in \mathbb{C}^2 \setminus \{0\}$ sufficiently small there exist $\varepsilon, \varepsilon' > 0$, such that for the manifold*

$$\mathcal{S}(V_0,\varepsilon) = \left\{(U,z);\ U \in B_{\mathbb{C}^2}(V_0,\varepsilon), z \in \mathcal{H}_c(U) \cap B_{H^{s'}_\sigma}(0,\varepsilon')\right\},$$

endowed with the metric of $\mathbb{C}^2 \times H^{s'}_\sigma$, there exists a unique map $g : S(V_0,\varepsilon) \mapsto B^s_{p,q}(\mathbb{R}^3,\mathbb{C}^4)$ which is smooth and satisfies for all $(U,z) \in S(V_0,\varepsilon)$, $g(U,z) \in \mathcal{H}_1(U)$, $z + g(U,z) \in \mathcal{H}_0^{\perp_J}(U)$, and $S(U) + z + g(U,z) \in W^c(V_0)$. Moreover, we have $\|g(U,z)\|_{B^s_{p,q}} = O(\|z\|^2_{H^{s'}})$.

Proof. The fact that $\mathcal{S}(V_0,\varepsilon)$ is manifold here is proved like in Lemma 3.1.

Then if h^c is the function for which $W^c(V_0)$ is the graph. Any $\phi \in L^2(\mathbb{R}^3,\mathbb{R}^8)$ can be written in the form $S(V_0) + \tilde{U}\cdot DS(V_0) + \xi + \rho$ with $\rho \in \mathcal{H}_1(V_0)$ and $\xi \in \mathcal{H}_c(V_0)$. It can be also written in the form $S(U) + z + r$ with $r \in \mathcal{H}_1(U)$ and $z \in \mathcal{H}_c(U)$. These two decompositions in fact defines two bijective smooth maps in sufficiently small sets (for the first we have a linear decomposition, for the second see Lemma 3.1). We write Ψ for the first and Φ for the second. Then $f = \Psi \circ \Phi^{-1}$ has 3 components following the decomposition $\mathcal{H}_0(V_0) \oplus \mathcal{H}_1(V_0) \oplus \mathcal{H}_c(V_0)$, we write them (f_1, f_2, f_3). Then g is the solution of the implicit equation in r

$$F(U,z,r) = f_2(U,z,r) - h^c(f_1(U,z,r), f_3(U,z,r)) = 0$$

which can be solved by the implicit function theorem in $H^{s'}_\sigma$ since $\partial_r F(V_0,0,0)$ is invertible from $\mathcal{H}_1(V_0)$ to itself because $\partial_r f_2(V_0,0,0)$ ($f_2(V_0,r,0) = r$) is invertible from $\mathcal{H}_1(V_0)$ to itself and $Dh_c(0,0) = 0$.

The smoothness of g in the Besov spaces follows from the fact that $g(U,z) \in \mathcal{H}_1(U)$ and the exponential decay for excited states and their derivatives given by (2.2).

Then we notice that for any U close to V_0, the previous proof can be applied to $W^c(U)$. It shows that $W^c(U)$, $W^c(V_0)$ are in a neighborhood of $S(V_0)$ the graph of a two functions on $\mathcal{S}(V_0,\varepsilon)$ equal up to a translation in $\mathbb{C}^2$ of the first argument. Hence their graphs are equal, so locally $W^c(U) = W^c(V_0)$. The last assertion then follows from the fact that at $S(U)$, $W^c(U)$ is tangent to $S(U) + X^c(U)$ and $X^c(U)$ is orthogonal to $\mathcal{H}_1(U)$. ☐

Hence decomposing η with respect to the spectrum of $JH(U)$, we write

$$\eta(t) = g(U(t), z(t)) + z(t)$$

with $z \in \mathcal{H}_c(U) \cap L^2(\mathbb{R}^3, \mathbb{R}^8)$. We obtain the system

$$\begin{cases} \dot{U} = -A(U,\eta)\langle N(U,\eta), dS(U)\rangle \\ \partial_t z = JH(U)z + \mathbf{P}_c(U)JN(U,\eta) \\ \qquad +\mathbf{P}_c(U(v))dS(U(v))A(U(v),\eta(v))\langle N(U(v),\eta(v)), dS(U(v))\rangle + (dP_c(U))A(U,\eta)\langle N(U,\eta), dS(U)\rangle\eta \end{cases}$$

with

$$\eta(t) = z(t) + g\left(U(t), z(t)\right).$$

We notice that this equation is defined only for z small with real values and U small. We now study this system.

3.2 The stabilization towards the PLS manifold

We now show that any solution of (1.2) which belongs to the center manifold $W^c(V_0)$, for a small non zero V_0, stabilizes as $t \to \pm\infty$ towards the manifold of the stationary states inside $W^c(V_0)$. To this end, we will use Theorem 1.1 and Theorem 1.2 to prove that z tends to zero in some sense.

Let us define for any ε, $\delta > 0$

$$\mathcal{U}(\varepsilon,\delta) = \left\{ U \in \mathcal{C}^1(\mathbb{R}, B_{\mathbb{C}^2}(V_0,\varepsilon)),\ \|\dot{U}\|_{L^1(\mathbb{R})\cap L^\infty(\mathbb{R})} \leq \delta^2 \right\}$$

and for any $U \in \mathcal{U}(\varepsilon,\delta)$, let s, β be such that $s' > \beta + 2 > 2$ and $\sigma > 3/2$,

$$\mathcal{Z}(U,\delta) = \Bigg\{ z \in \mathcal{C}(\mathbb{R}, L^2(\mathbb{R}^3,\mathbb{R}^8)),\ z(t) \in \mathcal{H}_c(U(t)),$$
$$\max\left[\|z\|_{L^\infty(\mathbb{R},H^s)}, \|z\|_{L^2(\mathbb{R},H^s_{-\sigma})}, \|z\|_{L^2(\mathbb{R},B^\beta_{\infty,2})}\right] \leq \delta \Bigg\},$$

and ε, δ are small enough to ensure that for $U \in \mathcal{U}(\varepsilon,\delta)$ and $z \in \mathcal{Z}(U,\delta)$

$$S(U) + z + g(U,z) \in W^c(V_0) \cap B_{H^s}(S(V_0), r(V_0)),$$

where g is defined by Lemma 3.2 and r in Remark 2.3. It will appear later that δ is of the same order as $\|z_0\|_{H^s}$ (see Lemma 3.8 below).

3.2.1 some useful lemma

In the rest of our study, we will need some technical lemmas, which we collect here.

Lemma 3.3. *If Assumptions 1.1–1.4 hold. Let $\sigma, \sigma' \in \mathbb{R}$, $s > 1$ and $p, \widetilde{p_1}, p_1, p_2, q \in [1,\infty]$ such that*

$$\frac{1}{p} + \frac{s}{3} \geq \frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{p}.$$

and

$$\frac{1}{p} + \frac{s}{3} \geq \frac{1}{\widetilde{p_1}}.$$

Then there exist $\varepsilon > 0$ and $C > 0$ such that for all $U \in B_{\mathbb{C}}(0,\varepsilon)$ and $\eta \in B^s_{p_2,q}(\mathbb{R}^3,\mathbb{R}^8) \cap L^\infty(\mathbb{R}^3,\mathbb{R}^8)$ with $\langle Q\rangle^\sigma \eta \in B^s_{p_1,q}(\mathbb{R}^3,\mathbb{R}^8)$ and $\langle Q\rangle^{\sigma'}\eta \in B^s_{\widetilde{p_1},q}(\mathbb{R}^3,\mathbb{R}^8)$, we have

$$\|\langle Q\rangle^\sigma N(U,\eta)\|_{B^s_{p,q}} \leq C\left(s,F,|U| + \|\eta\|_{L^\infty}\right)|U|\,\|\eta\|_{L^\infty}\left\|\langle Q\rangle^{\sigma'}\eta\right\|_{B^s_{\widetilde{p_1},q}}$$
$$+ C\left(s,F,|U| + \|\eta\|_{L^\infty\cap B^s_{p_2,q}}\right)\|\eta\|^2_{L^\infty}\|\langle Q\rangle^\sigma\eta\|_{B^s_{p_1,q}}. \quad (3.2)$$

Proof. We recall the definition

$$N(U,\eta) = \nabla F(S(U)+\eta) - \nabla F(S(U)) - d^2F(S(U))\eta.$$

We have

$$N(U,\eta) = \int_0^1\int_0^1 d^3F(S(U)+\theta'\theta\eta)\cdot\eta\cdot\theta\eta\, d\theta' d\theta,$$

or

$$N(U,\eta) = \frac{1}{2}d^3F(S(U))\cdot\eta\cdot\eta + \int_0^1\int_0^1 d^4F(S(U)+\theta''\theta'\theta\eta)\cdot\theta'\theta\eta\cdot\eta\cdot\theta\eta\, d\theta'' d\theta' d\theta,$$

Then we use for $s\in\mathbb{R}_+^*$, p, p_1, p_2, $\in[1,\infty]$ such that $\frac{1}{p}+\frac{s}{3}\geq\frac{1}{p_1}+\frac{1}{p_2}\geq\frac{1}{p}$,

$$\|uv\|_{B^s_{p,q}} \leq C\|u\|_{B^s_{p_1,q}}\|v\|_{B^s_{p_2,q}},$$

and for $s>1$, we use [EV97, Proposition 2.1]

$$\|d^kF(\psi)\|_{B^s_{p_2,q}} \leq C\left(s,F,\|\psi\|_{L^\infty}\right)\|\psi\|_{B^s_{p_2,q}},$$

for $k=3$ or $k=4$ and $d^4F(z)=O(|z|)$, otherwise we decompose $d^4F(z)=A+O(|z|)$ where A is a constant operator.

Eventually using Lemma 1.1 and

$$\left\|\langle Q\rangle^\sigma|\eta|^l\right\|_{B^s_{p_1,q}} \leq C\|\eta\|^{l-1}_{L^\infty}\|\langle Q\rangle^\sigma\eta\|_{B^s_{p_1,q}},$$

for $l\in\mathbb{N}$, we conclude the proof. □

Lemma 3.4. *If Assumptions 1.1–1.4 hold. Let* $\sigma\in\mathbb{R}$, $s>1$, $p,p_1,p_2,q\in[1,\infty]$ *and* $\sigma_1,\sigma_2\in\mathbb{R}$ *such that*

$$\frac{1}{p}+\frac{s}{3}\geq\frac{1}{p_1}+\frac{1}{p_2}\geq\frac{1}{p}.$$

Then there exist $\varepsilon>0$ *and* $C>0$ *such that for all* $U\in B_{\mathbb{C}}(0,\varepsilon)$ *and* $\eta\in B^s_{p,q}(\mathbb{R}^3,\mathbb{R}^8)\cap L^\infty(\mathbb{R}^3,\mathbb{R}^8)$ *with* $\langle Q\rangle^{\sigma_1}\eta\in B^s_{p_1,q}(\mathbb{R}^3,\mathbb{R}^8)$ *and* $\langle Q\rangle^{\sigma_2}\eta\in B^s_{p_2,q}(\mathbb{R}^3,\mathbb{R}^8)$*, we have*

$$\begin{aligned}&\|< Q >^\sigma\left(\nabla F(S(U)+\eta)-\nabla F(S(U)-\nabla F(\eta))\right\|_{B^s_{p,q}}\\ &\qquad\leq C(s,F,|U|+\|\eta\|_{L^\infty})\left(|U|+\|< Q >^{\sigma_1}\eta\|_{B^s_{p_1,q}}\right)|U|\,\|< Q >^{\sigma_2}\eta\|_{B^s_{p_2,q}}.\end{aligned}$$

Proof. The proof is similar to the one of Lemma 3.3. □

Lemma 3.5. *If Assumptions 1.1–1.4 hold. Let* $\sigma\in\mathbb{R}$, $s>1$ *and* $p,q\in[1,\infty]$ *such that* $sp\geq3$*. Then there exist* $\varepsilon>0$ *and* $C>0$ *such that for all* U, $U'\in B_{\mathbb{C}^2}(0,\varepsilon)$ *and* η, $\eta'\in B^s_{p,q}(\mathbb{R}^3,\mathbb{R}^8)$*, we have*

$$\begin{aligned}\|\langle Q\rangle^\sigma\{N(U,\eta)-N(U',\eta')\}\|_{B^s_{p,q}} &\leq C\left(s,F,|U|+|U'|+\|\eta\|_{B^s_{p,q}}+\|\eta'\|_{B^s_{p,q}}\right)\times\\ &\times\Bigg\{\left(\|\langle Q\rangle^{\sigma_1}\eta\|_{B^s_{p,q}}+\|\langle Q\rangle^{\sigma_1}\eta'\|_{B^s_{p,q}}\right)^2\left(|U-U'|+\|\langle Q\rangle^{\sigma_2}(\eta-\eta')\|_{B^s_{p,q}}\right)\\ &+\left(|U|+|U'|+\left\|\langle Q\rangle^{\sigma_1'}\eta\right\|_{B^s_{p,q}}+\left\|\langle Q\rangle^{\sigma_1'}\eta'\right\|_{B^s_{p,q}}\right)\times\\ &\times\left(\|\langle Q\rangle^{\sigma_2'}\eta\|_{B^s_{p,q}}+\|\langle Q\rangle^{\sigma_2'}\eta'\|_{B^s_{p,q}}\right)\|\langle Q\rangle^{\sigma_3'}(\eta-\eta')\|_{B^s_{p,q}}\Bigg\},\end{aligned}$$

with $2\sigma_1+\sigma_2=\sigma_1'+\sigma_2'+\sigma_3'=\sigma$ *if* $< Q >^w\eta$, $< Q >^w\eta'\in B^s_{p,q}(\mathbb{R}^3,\mathbb{R}^8)$ *for* $w\in\{\sigma_1,\sigma_2,\sigma_1',\sigma_2',\sigma_3'\}$.

Proof. Using the identity

$$N(u,\eta) = \int_0^1 \int_0^1 d^3F(S(u)+\theta'\theta\eta)\cdot\eta\cdot\theta\eta\, d\theta' d\theta.$$

we can restrict the study to $d^3F(\phi) - d^3F(\phi')$. If $F = O(|z|^5)$, we have

$$\|\langle Q\rangle^\sigma \left(d^3F(\phi) - d^3F(\phi')\right)\|_{B^s_{p,q}} \le \int_0^1 \|d^4F(\phi + t(\phi-\phi'))\|_{B^s_{p,q}} \|\langle Q\rangle^\sigma(\phi-\phi')\|_{B^s_{p,q}}\, dt.$$

Then since $s > 1$ and $sp \ge 3$, we use

$$\|d^4F(\psi)\|_{B^s_{p,q}} \le C(s, F, \|\psi\|_{B^s_{p,q}}).$$

Using Lemma 1.1, we conclude the proof when $F = O(|z|^5)$.

Otherwise, if F is an homogeneous polynomial of order 4, the proof is easily adaptable since d^4F is a constant tensor.

The case $F = O(|z|^4)$ follows by summing the two previous one since as a function of $u \in \mathbb{R}^8$, $F(u) = Au^{\otimes 4} + O(|u|^5)$. ☐

Lemma 3.6. *If Assumptions 1.1–1.4 hold. Let* $\sigma \in \mathbb{R}$, $s \in \mathbb{R}$ *and* $p, q \in [1,\infty]$. *Then there exist* $\varepsilon > 0$, $M > 0$ *and* $C > 0$ *such that for all* $U, U' \in B_{\mathbb{C}^2}(0,\varepsilon)$ *and* η, $\eta' \in B_{L^2(\mathbb{R}^3,\mathbb{R}^8)}(0,M)$ *with* $\langle Q\rangle^\sigma\{\eta-\eta'\} \in B^s_{p,q}(\mathbb{R}^3,\mathbb{R}^8)$, *one has*

$$|A(U,\eta) - A(U',\eta')| \le C\left\{|U-U'| + \|\langle Q\rangle^\sigma\{\eta-\eta'\}\|_{B^s_{p,q}}\right\}. \tag{3.3}$$

Proof. We recall that

$$A(U,\eta) = [\langle JdS(U), dS(U)\rangle - \langle J\eta, d^2S(U)\rangle]^{-1}.$$

We have

$$\begin{aligned}A(U,\eta) - A(u',\eta') = &-[\langle JdS(U), dS(U)\rangle - \langle J\eta, d^2S(U)\rangle]^{-1}\times\\ &\times\left\{\langle JdS(U), dS(U)\rangle - \langle J\eta, d^2S(U)\rangle - \langle JdS(U'), dS(U')\rangle + \langle J\eta', d^2S(U')\rangle\right\}\times\\ &\times[\langle JdS(U'), dS(U')\rangle - \langle J\eta', d^2S(U')\rangle]^{-1}.\end{aligned}$$

The lemma then follows from Lemma 1.1. ☐

3.2.2 Global wellposedness for z

Let $U \in \mathcal{U}(\varepsilon,\delta)$ and $z_0 \in \mathcal{H}_c(U(0)) \cap H^s$. Let us write $U_\infty = \lim_{t\to+\infty} U(t)$, we define $\mathcal{T}_{U,z_0}(z)$ by

$$\begin{aligned}\mathcal{T}_{U,z_0}(z)(t) = &e^{-\mathrm{i}tH+\mathrm{i}\int_0^t E(U(r))\,dr} z_0 + \int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c(U(v))J\nabla F(\eta(v))\,dv\\ &+\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c(U(v))J\{\nabla F(S(U(v))+\eta(v)) - \nabla F(S(U(v)) - \nabla F(\eta(v))\}\,dv\\ &+\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c(U(v))dS(U(v))A(U(v),\eta(v))\langle N(U(v),\eta(v)), dS(U(v))\rangle\,dv\\ &-\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}(d\mathbf{P}_c(U(v))\dot U(v)\eta(v)\,dv.\end{aligned}$$

with

$$\eta(t) = z(t) + g\left(U(t), z(t)\right)$$

First, we have a local wellposedness result for z with the

Lemma 3.7. *If Assumptions 1.1–1.5 hold. Then there exist* $\varepsilon_0 > 0$ *and* $\delta_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \delta_0)$, $U \in \mathcal{U}(\delta, \varepsilon)$ *and* $z_0 \in B_{H^s(0,\delta)} \cap \mathcal{H}_c(U(0))$ *there are* $T^{\pm}(z_0, U) > 0$ *and a solution*

$$z \in \cap_{k=0}^{2} \mathcal{C}^k((-T^-(z_0, U); +T^+(z_0, U)), H^{s-k}(0, \delta))$$

of the equation

$$\begin{cases} \partial_t z = JH(U)z + \mathbf{P}_c(U)JN(u, \eta) - (d\mathbf{P}_c(U))\dot{U}\eta, \\ z(0) = z_0, \end{cases} \tag{3.4}$$

where $\eta(t) = z(t) + g\,(U(t), z(t))$.

Moreover, z *is unique in* $L^\infty((-T', T), H^s)$ *for any* $T \in (0, T^+(z_0, U))$ *and* $T' \in (0, T^-(z_0, U))$ *and we have if* $T^+(z_0, U) < +\infty$ *then*

$$\lim_{t \to T^+(z_0,U)} \|z(t)\|_{H^s} \geq \delta$$

and if $T^-(z_0, U) = +\infty$ *then*

$$\lim_{t \to -T^-(z_0,U)} \|z(t)\|_{H^s} \geq \ \delta.$$

Proof. It is a consequence of the fix point theorem applied to $\mathcal{T}_{U,z_0}$:

Using Lemmas 3.3, 3.5 and 3.6 with the Estimate (2.8)–(2.10) and the properties of g given by Lemma 3.2, we obtain that $\mathcal{T}_{U,z_0}$ leaves a small ball in H^s invariant and is a contraction inside this ball.

Hence there exists a unique solution defined on the interval $[-T, T]$. Classical arguments permit to extend the solution over a maximal interval $(-T^-(z_0, U), T^+(z_0, U))$ such that if $T^+(z_0, U) < \infty$ then necessarily the solution should leave a small ball in H^s at time $T^+(z_0, U)$. □

We have now a global wellposedness result as stated in the

Lemma 3.8. *If Assumptions 1.1–1.5 hold. There exist* $\varepsilon_0 > 0$, $\delta_0 > 0$ *and* $C > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \delta_0)$, $U \in \mathcal{U}(\varepsilon, \delta)$ *and* $z_0 \in B_{H^s}(0, \delta) \cap \mathcal{H}_c(U(0))$ *we obtain for the Cauchy problem* (3.4), $T^+(U, z_0) = +\infty$, $T^-(U, z_0) = +\infty$, $z \in \mathcal{Z}(U, \delta)$ *and*

$$\max\left[\|z\|_{L^\infty(\mathbb{R},H^s)}, \|z\|_{L^2(\mathbb{R},H^s_{-\sigma})}, \|z\|_{L^2(\mathbb{R},B^\beta_{\infty,2})}\right] \leq C\,\|z_0\|_{H^s}.$$

Proof. We have $(1 - P_c(U))z \equiv 0$ because its time derivative is zero and $(1 - P_c(U(0)))z(0) = 0$.

Let us introduce for any $0 < T < T^+(U, z_0)$, the function

$$m(T) = \sup_{t\in(-T,T)} \left\{\|z\|_{L^\infty((-T,T),H^s)}\,,\ \|z\|_{L^2((-T,T),H^s_{-\sigma})}\,,\ \|z\|_{L^2((-T,T),B^\beta_{\infty,2})}\right\}$$

First, we study the estimation of $L^2((-T, T), H^s_{-\sigma})$. We use Estimate (2.5) and the estimates of the Theorem B.1.

$$\begin{aligned}
&\|z\|_{L^2((-T,T),H^s_{-\sigma})} \\
&\leq\ C_0\,\|z_0\|_{H^s} + C\left\|\mathbf{P}_c\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c(U(v))J\nabla F(\eta(v))\,dv\right\|_{L^2((-T,T),H^s_{-\sigma})} \\
&\quad +C\,\|\nabla F(S(U)+\eta) - \nabla F(S(U) - \nabla F(\eta)\|_{L^2((-T,T),H^s_\sigma)} \\
&\quad +C\,\|dS(U)A(U,\eta)\langle N(U,\eta), dS(U)\rangle\|_{L^2((-T,T),H^s_\sigma)} \\
&\quad +C\left\|(d\mathbf{P}_c(U)\dot{U}\eta\right\|_{L^2((-T,T),H^s_\sigma)}.
\end{aligned}$$

We now study the estimation of the third term of the right hand side

$$\begin{aligned}
&\left\|\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c\mathbf{P}_c(U(v))J\nabla F(\eta(v))\,dv\right\|_{L^2_t((-T,T),H^s_{-\sigma})} \\
&\leq\ \int_{-T}^{T}\left\|e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c\mathbf{P}_c(U(v))J\nabla F(\eta(v))\right\|_{L^2_t((-T,T),H^s_{-\sigma})}dv \\
&\leq\ C(U)\,\|\nabla F(\eta)\|_{L^1((-T,T),H^s)} \\
&\leq\ C(U)\,\|\eta\|^2_{L^2((-T,T),L^\infty)}\,\|\eta\|_{L^\infty((-T,T),H^s)}\,,
\end{aligned}$$

where we used Theorem B.1 Estimate (ii). Hence for the $L^2H^s_{-\sigma}$ estimate, we obtain

$$\begin{aligned}\|z\|_{L^2((-T,T),H^s_{-\sigma})} &\le C_0\|z_0\|_{H^s} + C\|\eta\|^2_{L^2((-T,T),L^\infty)}\|\eta\|_{L^\infty((-T,T),H^s)}\\ &+C\left(\|U\|_\infty + \|\eta\|_{L^\infty((-T,T),H^s_{-\sigma})}\right)\|U\|_\infty\|\eta\|_{L^2((-T,T),H^s_{-\sigma})}\\ &+C\|\eta\|^2_{L^2((-T,T),L^\infty)} + C\left\|\dot U\right\|_{L^2}\|\eta\|_{L^\infty((-T,T),H^s)},\end{aligned}$$

using Lemma 3.2, we obtain

$$\|z\|_{L^2((-T,T),H^s_{-\sigma})} \le C_0\|z_0\|_{H^s} + Cm(T)^3 + Cm(T)^2 + C\varepsilon m(T) + C\delta^2 m(T),$$

where C depends of $\|U\|_\infty$ and $\|\eta\|_{L^\infty((-T,T),H^s)}$.

Then, we estimate the H^s norm. Using Estimate (2.5), we have

$$\begin{aligned}\|z(t)\|_{H^s} \le \|z_0\|_{H^s} &+ \int_{-T}^{T}\|\nabla F(\eta(v))\|_{H^s}\,dv\\ &+\Big\|\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c(U(v))\times\\ &\times J\{\nabla F(S(U(v))+\eta(v)) - \nabla F(S(U(v)) - \nabla F(\eta(v))\}\,dv\Big\|_{H^s}\\ &+\int_{-T}^{T}\|dS(U(v))A(U(v),\eta(v))\langle N(U(v),\eta(v)),dS(U(v))\rangle\|_{H^s}\,dv\\ &+\int_{-T}^{T}\left\|(d\mathbf{P}_c(U(v))\dot U(v)\eta(v)\right\|_{H^s}\,dv.\end{aligned}$$

to estimate the third term of the right hand side, we use the H-smoothness estimates, more precisely Theorem B.1 Estimate (ii) and then we use Lemma B.14:

$$\begin{aligned}&\left\|\int_0^t e^{-\mathrm{i}(t-v)H+\mathrm{i}\int_v^t E(U(r))\,dr}\mathbf{P}_c(U(v))J\{\nabla F(S(U(v))+\eta(v)) - \nabla F(S(U(v))) - \nabla F(\eta(v))\}\,dv\right\|_{H^s}\\ &\le\left\|\int_0^t e^{\mathrm{i}vH-\mathrm{i}\int_0^v E(U(r))\,dr}\mathbf{P}_c(U(v))J\{\nabla F(S(U(v))+\eta(v)) - \nabla F(S(U(v))) - \nabla F(\eta(v))\}\,dv\right\|_{H^s}\\ &\qquad\le C\|\{\nabla F(S(U)+\eta) - \nabla F(S(U) - \nabla F(\eta)\|_{L^2((-T,T),H^s_\sigma)}\\ &\qquad\le C\left(\|U\|_\infty + \|\eta(v)\|_{L^\infty((-T,T),H^s)}\right)\|U\|_\infty\|\eta\|_{L^2((-T,T),H^s_{-\sigma})}\end{aligned}$$

Hence for the $L^\infty H^s$ estimate, we obtain

$$\begin{aligned}\|z(t)\|_{H^s} \le \|z_0\|_{H^s} &+ C\|\eta\|_{L^\infty((-T,T),H^s)}\|\eta\|^2_{L^2((-T,T),L^\infty)}\\ C&\left(\|U\|_{L^\infty((-T,T))} + \|\eta(v)\|_{L^\infty((-T,T),H^s)}\right)\|U\|_{L^\infty((-T,T))}\|\eta\|_{L^2((-T,T),H^s_{-\sigma})}\\ &+C\|\eta\|^2_{L^2((-T,T),L^\infty)} + \left\|\dot U\right\|_{L^1((-T,T))}\|\eta\|_{L^\infty((-T,T),H^s)},\end{aligned}$$

using Lemma 3.2, we obtain

$$\|z(t)\|_{H^s} \le \|z_0\|_{H^s} + Cm(T)^3 + Cm(T)^2 + C\varepsilon m(T) + C\delta^2 m(T),$$

where C depends of $\|U\|_\infty$ and $\|\eta\|_{L^\infty((-T,T),H^s)}$.

For the $L^2B^\beta_{\infty,2}$ estimate, by Proposition 2.2 and Theorem 1.2, we have for any $\varepsilon > 0$, any $p_\varepsilon > 3/\varepsilon$ and $\theta_\varepsilon = \frac{4}{p_\varepsilon - 2}$

$$\begin{aligned}\|z\|_{L^2((-T,T),B^\beta_{\infty,2})} &\le \|z\|_{L^2((-T,T),B^{\beta+\varepsilon}_{p_\varepsilon,2})}\\ &\le C_0\|z_0\|_{H^{\beta+1+\varepsilon+\theta_\varepsilon/2}} + C\left\|d^2F(S(U))\cdot\eta\right\|_{L^2((-T,T),B^{\beta+2+\varepsilon+\theta_\varepsilon}_{p'_\varepsilon,2})}\\ &+C\|N(U,\eta)\|_{L^1((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})}\\ &+C\|dS(U)A(U,\eta)\langle N(U,\eta),dS(U)\rangle\|_{L^1((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})}\\ &+C\left\|(d\mathbf{P}_c(U))\dot U\eta\right\|_{L^1((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})}\,dv.\end{aligned}$$

With Lemma 3.3 and 3.4, we infer

$$\begin{aligned}
\|z\|_{L^2(\mathbb{R},B^{\beta}_{\infty,2})} &\le C_0\|z_0\|_{H^{\beta+1+\varepsilon+\theta_\varepsilon/2}} + C|U|_\infty \|\eta\|_{L^2((-T,T),H^{\beta+2+\varepsilon+\theta_\varepsilon}_{-\sigma})}\\
&+C|U|_\infty \|z\|_{L^2((-T,T),L^\infty)} \|z\|_{L^2((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})}\\
&+C(|U|_\infty + \|\eta\|_{L^\infty((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})}) \|\eta\|^2_{L^2((-T,T),L^\infty)} \|z\|_{L^\infty((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})}\\
&+C(|U|_\infty + \|\eta\|_{L^\infty((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})}) \|\eta\|_{L^2((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2}_{-\sigma})} \|\eta\|_{L^\infty((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})}\\
&+C\|\dot{U}\|_{L^1} \|\eta\|_{L^\infty((-T,T),H^{\beta+1+\varepsilon+\theta_\varepsilon/2})},
\end{aligned}$$

we infer since for small $\varepsilon > 0$, $s \ge \beta + 2 + \varepsilon + \theta_\varepsilon$ and using Lemma 3.2,

$$\|z\|_{L^2((-T,T),B^{\beta}_{\infty,2})} \le C_0\|z_0\|_{H^{\beta+1+\varepsilon+\theta_\varepsilon/2}} + Cm(T)^3 + Cm(T)^2 + C\varepsilon m(T) + C\delta^2 m(T).$$

Hence we obtain

$$m(T) \le C_0\|z_0\|_{H^{\beta+1+\varepsilon+\theta_\varepsilon/2}} + C\varepsilon m(T) + C\delta^2 m(T) + Cm(T)^3 + Cm(T)^2,$$

where C_0 do not depend of m and C is a nondecreasing function of $\|z\|_{L^\infty((-T,T),H^s)}$ and $\|U\|_\infty$ and hence it can be bounded by a nondecreasing function of m.

If $\|z_0\|_{H^s}$ is small then $m(0)$ is small and $m(T)$ stay small. Therefore we have that $z \in \mathcal{Z}(U,\delta)$ if $\|z_0\|_{H^s}$is small enough for any δ and ε are small enough and

$$\max\left[\|z\|_{L^\infty(\mathbb{R},H^s)}, \|z\|_{L^2(\mathbb{R},H^s_{-\sigma})}, \|z\|_{L^2(\mathbb{R},B^{\beta}_{\infty,2})}\right] \le f(\|z_0\|_{H^s})$$

where f is such that there exists $C > 0$ with

$$f(\|z_0\|_{H^s}) \le C\,\|z_0\|_{H^s}.$$

□

The solution z just found is a function of z_0 and U, writing it $z[z_0, U]$, we have the following important property given by the

Lemma 3.9. *If Assumptions 1.1–1.5 hold. Then for any $T > 0$, there exist $\varepsilon_0 > 0$, $\delta_0 > 0$, $C > 0$ and $\kappa \in (0,1)$ such that for any $\varepsilon \in (0,\varepsilon_0)$, $\delta \in (0,\delta_0)$, U, $U' \in \mathcal{U}(\varepsilon,\delta)$, $z_0 \in \mathcal{H}_c(U(0))$, $z_0' \in \mathcal{H}_c(U'(0))$, $z \in \mathcal{Z}(U,\delta)$ and $z' \in \mathcal{Z}(U',\delta)$, one has*

$$\begin{aligned}
&\|z[z_0',U'] - z[z_0,U]\|_{L^\infty((-T;T),H^s)\cap L^2((-T;T),L^\infty)\cap L^2((-T;T),H^s_{-\sigma})}\\
&\qquad\le C\,\|z_0 - z_0'\|_{H^s} + \kappa\left\{\|U - U'\|_{L^\infty((-T;T))} + \left\|\dot{U} - \dot{U}'\right\|_{L^\infty((-T;T))}\right\}.
\end{aligned}$$

Proof. We use the technics of the previous lemma. □

3.2.3 Global wellposedness for U and its stabilization

Here we want to solve the equation for U. We notice that z and α have been built in the previous section and are functions of U and $z_0 \in \mathcal{H}_c(U(0))$. Let us introduce for any $U_0 \in B_{\mathbb{C}}(0,\varepsilon)$ the function on $\mathcal{U}(\varepsilon,\delta)$:

$$f_{U_0}(U)(t) = U_0 - \int_0^t A(U(v),\eta(v))\langle N(U(v),\eta(v)), dS(U(v))\rangle\, dv,$$

where $\eta = z(t) + g\,[U(t), z(t)]$. We have the

Lemma 3.10. *If Assumptions 1.1–1.5 hold. There exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$, $\delta \in (0,\delta_0)$, the function f_{U_0} maps $\mathcal{U}(\varepsilon,\delta)$ into itself if U_0 and $z_0 \in H^s \cap \mathcal{H}_c(U_0)$ are small enough.*

Proof. By means of Lemma 3.3, we obtain

$$\|\partial_t f_{U_0}(U)\|_{L^1(\mathbb{R})\cap L^\infty(\mathbb{R})} \le C\,\|N(U(v),\eta(v))\|_{L^1(\mathbb{R},H^s_{-\sigma})\cap L^\infty(\mathbb{R},H^s)} \le \delta^2.$$

and

$$\|f_{U_0}(U)\|_{L^\infty(\mathbb{R})} \le |U_0| + C\,\|N(U(v),\eta(v))\|_{L^1(\mathbb{R},H^s)} \le |U_0| + \delta^2,$$

hence for sufficiently small U_0 and δ, we obtain the lemma. □

The function f_{U_0} has also a local Lipshitz property as stated by the

Lemma 3.11. *If Assumptions 1.1–1.5 hold. For any $T > 0$, there exist $\varepsilon_0 > 0$, $\delta_0 > 0$ and $\kappa \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \delta_0)$, $U, U' \in \mathcal{U}(\varepsilon, \delta)$, for any $z_0 \in \mathcal{H}_c(U(0)) \cap H^s$, for any $z_0' \in \mathcal{H}_c(U'(0)) \cap H^s$ small enough, for U_0, U_0' small enough, such that*

$$\left|f_{U_0}(U) - f_{U_0'}(U')\right|_{L^\infty((-T;T))} + \left|\partial_t f_{U_0}(U) - \partial_t f_{U_0'}(U')\right|_{L^1((-T;T))} \\ \leq |U_0 - U_0'| + \kappa \left(\|U - U'\|_{L^\infty((-T;T))} + \left\|\dot{U} - \dot{U}'\right\|_{L^1((-T;T))} + \|z_0 - z_0'\|_{H^s} \right).$$

Proof. This a straightforward consequence of Lemma 3.5, 3.6 and 3.9. ☐

We now obtain the

Lemma 3.12. *If Assumptions 1.1–1.5 hold. There exists $\varepsilon > 0$ and $\delta > 0$ such that for any $U_0 \in \mathbb{C}$ small and $z_0 \in \mathcal{H}_c(U_0) \cap H^s_\sigma$ small, the equation*

$$\begin{cases} \dot{U} &= -A(U,\eta)\langle N(U,\eta), dS(U)\rangle, \\ U(0) &= U_0, \end{cases} \tag{3.5}$$

where $\eta(t) = z(t) + g\left[U(t), z(t)\right]$, has a unique solution in $\mathcal{U}(\delta, \varepsilon)$. Moreover, there exists $C > 0$ such that

$$|U_{\pm\infty} - U_0| \leq C \|z_0\|^2_{H^s}.$$

Proof. This is also a the fixed point result for f_{U_0}. Let us fix $T > 0$ and consider, for any $V \in \mathcal{U}(\delta, \varepsilon)$ with sufficiently small $\delta > 0$ and $\varepsilon > 0$, the sequence:

$$\begin{cases} V_{n+1} = f_{U_0}(V_n), \ \forall n \in \mathbb{N} \\ V_0 = V; \end{cases}$$

for any $n \in \mathbb{N}$, $V_n \in \mathcal{U}(\delta, \varepsilon)$. With Lemma 3.11, the fixed point theorem give us the convergence for the norms of $L^\infty((-T, T))$ and $\dot{W}^{1,1}((-T, T))$ of $(V_n)_{n\in\mathbb{N}}$.

Then we notice that for any $T' \in \mathbb{R}$, we have

$$V_{n+1}(t) = f_{f_{U_0}(V_n)(T')}(V_n)(t - T').$$

Since for $T' \in (-T; T)$, $(f_{U_0}(V_n)(T'))$ is a Cauchy sequence, the Lemma 3.11 give us the convergence of (V_n) for the norms of $L^\infty((T' - T; T' + T))$ and $\dot{W}^{1,1}((T' - T; T' + T))$.

Iterating this process, we obtain that the sequence converges uniformly locally in time and we prove the lemma since the other statements are classical. We just notice that the last statement follow from the fact that there exists $C > 0$ such that

$$\int_{\mathbb{R}^\pm} \left|\dot{U}(v)\right| \, dv \leq \int_{\mathbb{R}^\pm} |A(U(v), \eta(v))\langle N(U(v), \eta(v)), dS(U(v))\rangle| \, dv \leq C \|z_0\|^2_{H^s}.$$

☐

3.2.4 The asymptotic profile of z

In this section, our aim is to precise the asymptotic profile of z when z_0 is localized. First we state the

Proposition 3.1. *There exists $\varepsilon > 0$, such that for all $U \in B_{\mathbb{C}^2}(0, \varepsilon)$ and $\alpha \in \mathbb{R}^+$ there exists $C > 0$ such that*

$$\left\|\langle Q\rangle^\alpha e^{JtH(U)}\psi\right\| \leq C_\alpha \sum_{\beta=0}^{\alpha} \langle t\rangle^\beta \left\|\langle Q\rangle^{\alpha-\beta}\psi\right\|$$

for any $\psi \in L^2(\mathbb{R}^3, \mathbb{C}^8)$.

Proof. From Proposition 2.2, we obtain the result for $\alpha = 0$, then we just need the result the estimate

$$\left\| Q^\alpha e^{JtH(U)} \psi \right\|^2 \leq C_\alpha^2 \sum_{0 \leq \beta \leq \alpha} |t|^{2|\beta|} \left\| Q^{\alpha-\beta} \psi \right\|^2$$

for any $\psi \in L^2(\mathbb{R}^3, \mathbb{C}^8)$, $\alpha \in \mathbb{N}^3$ and some $C > 0$ independent of ψ. The rest of the proposition will follow by interpolation.

For $U = 0$, this follows by an iterated proof from the identity

$$\frac{d}{dt} e^{\mathrm{i}tH} Q e^{-\mathrm{i}tH} = e^{\mathrm{i}tH} \alpha e^{-\mathrm{i}tH}$$

where α is the 3-vector of Dirac Pauli matrices defined in the introduction. For $U \neq 0$, we use the same proof with the exponential decay of Proposition 1.1. □

We can improve Lemma 3.13, if we use [Bou06, Theorem 1.2] and [Bou06, Theorem 1.1], which we repeat here :

Theorem 3.1 (Theorem 1.1 of [Bou06]: Propagation for perturbed Dirac dynamics). *Assume that Assumptions 1.1 and 1.2 hold and let be $\sigma > \frac{5}{2}$. Then one has*

$$\| e^{-\mathrm{i}tH} \mathbf{P}_c(H) \|_{B(L^2_\sigma, L^2_{-\sigma})} \leq C \langle t \rangle^{-\frac{3}{2}}.$$

We also have

Proposition 3.2 (Proposition 2.2 of [Bou06]: Propagation far from thresholds). *Suppose that Assumption 1.1 holds. Then for any $\chi \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C}^4)$ bounded with support in $\mathbb{R} \setminus (-m; m)$ and for any $\sigma \geq 0$, there is $C > 0$ such that*

$$\| e^{-\mathrm{i}tH} \chi(H) \|_{B(L^2_\sigma, L^2_{-\sigma})} \leq C \langle t \rangle^{-\sigma}.$$

Using Duhamel's formula like in Proposition 2.2 and interpolating with estimate (2.7), we obtain the

Corollary 3.1. *Assume that Assumptions 1.1 and 1.2 hold and let $\theta \geq 0$ and $\sigma > \frac{5}{2}\theta$. Then there exists $\varepsilon > 0$, such that for all $U \in B_{\mathbb{C}^2}(0, \varepsilon)$ one has*

$$\| e^{JtH(U)} \mathbf{P}_c(U) \|_{B(L^2_\sigma, L^2_{-\sigma})} \leq C \langle t \rangle^{-\frac{3\theta}{2}}.$$

Theorem 3.2 (Theorem 1.1 of [Bou06]: Dispersion for perturbed Dirac dynamics). *Assume that Assumptions 1.1 and 1.2 hold. Then for $p \in [1, 2]$, $\theta \in [0, 1]$, $s - s' \geq (2 + \theta)(\frac{2}{p} - 1)$ and $q \in [1, \infty]$ there exists a constant $C > 0$ such that*

$$\| e^{-\mathrm{i}tH} \mathbf{P}_c(H) \|_{B^s_{p,q}, B^{s'}_{p',q}} \leq C \left(K(t)\right)^{\frac{2}{p} - 1}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$, and

$$K(t) = \begin{cases} |t|^{-1+\theta/2} & \text{if } |t| \in (0, 1], \\ |t|^{-1-\theta/2} & \text{if } |t| \in [1, \infty). \end{cases}$$

Using, once more Duhamel's formula, the previous theorem and corollary 3.1, we obtain the

Corollary 3.2. *Assume that Assumptions 1.1 and 1.2 hold and let be $p \in [1, 2]$, $\theta \in [0, 1]$, $s - s' \geq (2 + \theta)(\frac{2}{p} - 1)$, $q \in [1, \infty]$ and $\sigma > \max\{\frac{3}{2}, (\frac{2}{p} - 1)(1 + \frac{\theta}{2})\}$. Then there exists $\varepsilon > 0$, such that for all $U \in B_{\mathbb{C}^2}(0, \varepsilon)$ one has*

$$\| e^{JtH(U)} \mathbf{P}_c(U) \|_{H^s_\sigma, B^{s'}_{p',q}} \leq C \left(K(t)\right)^{\frac{2}{p} - 1}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$, and

$$K(t) = \begin{cases} |t|^{-1+\theta/2} & \text{if } |t| \in (0, 1], \\ |t|^{-1-\theta/2} & \text{if } |t| \in [1, \infty). \end{cases}$$

Proof. We first prove it for $U = 0$. We have to study the high and low energy part in a different manner. For the low energy part, we iterate twice Duhamel's formula with respect to D_m in order to use Theorem 3.1 and Theorem 3.2 for the free case.

In the high energy part, we use also Duhamel's formula. But, we use Poposition Theorem 3.2 for the free case and Proposition 3.2.

Then for $U \neq 0$, we work like for Estimate (2.7). □

We obtain the

Lemma 3.13. *If Assumptions 1.1–1.5 hold. There exist $\varepsilon_0 > 0$, $\delta_0 > 0$ and $C > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \delta_0)$, $U_0 \in B_{\mathbb{C}^2}(0, \varepsilon)$ and $z_0 \in B_{H^s_\sigma}(0, \delta) \cap \mathcal{H}_c(U_0)$ we obtain for the Cauchy problem* (3.4) *(with U the solution of* (3.5)*) a global solution z such that*

$$\max \left[\sup_{t\in\mathbb{R}}(\|z(t)\|_{H^s}),\ \sup_{t\in\mathbb{R}}(\langle t\rangle^{3/2}\|z(t)\|_{H^s_{-\sigma}}),\ \sup_{t\in\mathbb{R}}\langle t\rangle^{3/2}\|z(t)\|_{B^\beta_{\infty,2}}),\ \sup_{t\in\mathbb{R}}(\langle t\rangle^{-3/2}\|z(t)\|_{H^s_{3/2}}) \right] \le C\, \|z_0\|_{H^s_\sigma}.$$

Proof. The proof is similar to the one of Lemma 3.8 with some adaptations involving the norm H^s_σ, we also refer to the proof of [Bou06, Lemma 5.5].

Let

$$t \mapsto \xi_\pm(t) = e^{J\int_0^t (E(U(v)) - E(U_{\pm\infty}))\, dv} z(t)$$

and

$$t \mapsto V_\pm(t) = e^{-\mathrm{i}\int_0^t (E(U(v)) - E(U_{\pm\infty}))\, dv} U(t).$$

We use exactly the same method as the one of Lemma 3.8, applied to

$$\begin{aligned}
\xi_\pm(t) = e^{JtH(V_{\pm\infty})} z_0 + \int_0^t e^{J(t-s)H(V_{\pm\infty})} \mathbf{P}_c(V_\pm(v)) J \left(d^2F(S(V_\pm(v))) - d^2F(S(V_{\pm\infty}))\right) \xi_\pm(v)\, dv \\
+ \int_0^t e^{J(t-s)H(V_{\pm\infty})} \mathbf{P}_c(V_\pm(v)) J N(V_\pm(v), \tilde{\eta}_\pm(v))\, dv \\
+ \int_0^t e^{J(t-s)H(V_{\pm\infty})} \mathbf{P}_c(V_\pm(v)) dS(V(v)) A(V_\pm(v), \tilde{\eta}_\pm(v)) \langle N(V_\pm(v), \tilde{\eta}_\pm(v)), dS(V_\pm(v))\rangle\, dv \\
- \int_0^t e^{J(t-s)H(V_{\pm\infty})} (d\mathbf{P}_c(V_\pm(v))) A(V_\pm(v), \tilde{\eta}_\pm(v)) \langle N(V_\pm(v), \tilde{\eta}_\pm(v)), dS(V_\pm(v))\rangle \tilde{\eta}_\pm(v)\, dv,
\end{aligned}$$

with $\tilde{\eta}_\pm(t) = e^{J\int_0^t (E(U(v)) - E(U_{\pm\infty}))\, dv}\, (z(t) + g(U(t), z(t)))$, but using the previous time decay estimates.

There are two differences:

One is in the estimate of the $H^s_{-\sigma}$. In fact before using the time decay estimates for $e^{-\mathrm{i}tH} P_c(H)$, we split the space associated with the continuous spectrum in two parts : one associated with energy closed to the thresholds and one associated to the rest of the spectrum. In the first part, we use the fact that $\sigma > 3/2$ to estimate the $H^s_{-\sigma}$ by the $B^\beta_{\infty,2}$ norm since we work with bounded energies. In the second part, since we work far from thresholds, we use Proposition 3.2 after estimating the $H^s_{-\sigma}$ by the $H^s_{-3/2}$.

The other difference is in the estimation of the $B^\beta_{\infty,2}$ norm. We use Corollary 3.2 for $e^{JtH(V_{\pm\infty})} z_0$ and Theorem 3.2 for the integrals. □

We have that $\lim_{\pm\infty} U = U_{\pm\infty}$ exist. If $z_0 \in H^s_\sigma$ then the associated solution U satisfies

$$\left|\dot{U}\right| \le \frac{C}{\langle t\rangle^3} \|z_0\|_{H^s_\sigma}$$

and we have

$$\int_0^t (E(U(v)) - E(U_{\pm\infty})\ dv \to E_{\pm\infty} \text{ as } t \to \pm\infty$$

for some real $E_{\pm\infty}$. We introduce

$$V_\pm(t) = e^{-\mathrm{i}\int_0^t (E(U(v)) - E(U_{\pm\infty}))\, dv} U(t),$$

they have a limit as $t \to \pm\infty$ respectively as being

$$V_{\pm\infty} = e^{-\mathrm{i}E_{\pm\infty}} U_{\pm\infty}.$$

Then we notice that we can also obtain an asymptotic profile for $e^{\mathrm{i}tH+\mathrm{i}tE(U_\infty)}z(t)$ if z_0 is localized. But we prefer to obtain a scattering result with respect to $e^{JtH(V_\infty))}$. We have the

Lemma 3.14. *If Assumptions 1.1–1.5 hold. Then there exist* $\varepsilon_0>0,\ \delta_0>0$ *such that for any* $\varepsilon\in(0,\varepsilon_0)$, $\delta\in(0,\delta_0)$, $U_0\in B_{\mathbb{C}^2}(0,\varepsilon)$ *and* $z_0\in B_{H^s_\sigma}(0,\delta)\cap\mathcal{H}_c(U_0)$ *and for the solution* z *of* (3.4) *(with* U *the solution of* (3.5)*) given in Lemma 3.7 the following limit*

$$z_{\pm\infty}=\lim_{t\to\pm\infty}e^{-JtH(V_{\pm\infty})}e^{J\int_0^t(E(U(v))-E(U_{\pm\infty}))\,dv}z(t)$$

exists in H^s*. Moreover, we have* $z_{\pm\infty}\in\mathcal{H}_c(V_{\pm\infty})\cap H^s_\sigma$ *and there exists* $C>0$ *such that*

$$\begin{aligned}\max\Big[&\|e^{-J\int_0^t(E(U(v))-E(U_{\pm\infty}))\,dv}e^{JtH(V_{\pm\infty})\,dr}z_{\pm\infty}-z(t)\|_{H^s},\\ &\|e^{-J\int_0^t(E(U(v))-E(U_{\pm\infty}))\,dv}e^{JtH(V_{\pm\infty})\,dr}z_{\pm\infty}-z(t)\|_{H^s_{-\sigma}},\\ &\|e^{-J\int_0^t(E(U(v))-E(U_{\pm\infty}))\,dv}e^{JtH(V_{\pm\infty})\,dr}z_{\pm\infty}-z(t)\|_{B^\beta_{\infty,2}}\Big]\le\frac{C}{\langle t\rangle^2}\|z_0\|^2_{H^s_\sigma}\end{aligned}$$

and

$$\|z_{\pm\infty}-e^{-JtH(V_{\pm\infty})}e^{J\int_0^t(E(U(v))-E(U_{\pm\infty}))\,dv}z(t)\|_{H^s_{3/2}}\le\frac{C}{\langle t\rangle^{\frac12}}\|z_0\|^2_{H^s_\sigma}.$$

Proof. Let

$$t\mapsto\xi_\pm(t)=e^{J\int_0^t(E(U(v))-E(U_{\pm\infty}))\,dv}z(t)$$

and

$$t\mapsto V_\pm(t)=e^{-\mathrm{i}\int_0^t(E(U(v))-E(U_{\pm\infty}))\,dv}U(t).$$

Using exactly the same method as the one of Lemma 3.8, applied to

$$\begin{aligned}e^{-JtH(V_{\pm\infty})}\xi_\pm(t)=z_0&+\int_0^t e^{-JsH(V_{\pm\infty})}\mathbf{P}_c(V_\pm(v))J\left(d^2F(S(V_\pm(v)))-d^2F(S(V_{\pm\infty}))\right)\xi_\pm(v)\,dv\\ &+\int_0^t e^{-JsH(V_{\pm\infty})}\mathbf{P}_c(V_\pm(v))JN(V_\pm(v),\tilde\eta_\pm(v))\,dv\\ &+\int_0^t e^{-JsH(V_{\pm\infty})}\mathbf{P}_c(V_\pm(v))dS(V(v))A(V_\pm(v),\tilde\eta_\pm(v))\langle N(V_\pm(v),\tilde\eta_\pm(v)),dS(V_\pm(v))\rangle\,dv\\ &-\int_0^t e^{-JsH(V_{\pm\infty})}(d\mathbf{P}_c(V_\pm(v)))A(V_\pm(v),\tilde\eta_\pm(v))\langle N(V_\pm(v),\tilde\eta_\pm(v)),dS(V_\pm(v))\rangle\tilde\eta_\pm(v)\,dv,\end{aligned}$$

with $\tilde\eta_\pm(t)=e^{J\int_0^t(E(U(v))-E(U_{\pm\infty}))\,dv}\,(z(t)+g(U(t),z(t)))$, we prove that the limits

$$\lim_{t\to\pm\infty}e^{-JtH(V_{\pm\infty})}\xi_\pm(t)=z_{\pm\infty}$$

exist. If we use the method of Lemma 3.13, we obtain the estimates on the convergence of $e^{JtH(V_{\pm\infty})}z_{\pm\infty}-\xi_\pm(t)$. Then for multiplying by $e^{-J\int_0^t(E(U(v))-E(U_{\pm\infty}))\,dv}$, we obtain the estimates and the convergence of

$$e^{-J\int_0^t(E(U(v))-E(U_{\pm\infty}))\,dv}e^{-JtH(V_{\pm\infty})}z_{\pm\infty}-z(t).$$

Then since $(1-P_c(U(t)))\,z(t)=0$, we have $(1-P_c(V_{\pm\infty}))\,z_{\pm\infty}=0$ and hence $z_{\pm\infty}$ belongs to $\mathcal{H}_c(V_{\pm\infty})$. □

4 The dynamic outside the center manifold

We can make the same study in the center stable manifold and the center unstable manifold but only in one direction of time. Let us explain it for the center stable manifold in positive time since it is similar for the center unstable manifold. Actually it is equivalent if we revert the time direction.

We just give a sketch of the proof since it is similar to the previous study. Using the idea of the proof of exponential stabilization for Proposition 2.4, we write any solution ψ in the form $\phi+\rho+f(\phi,\rho)$ with

ϕ in the center manifold, $\rho \in X_s(V_0)$ and f a function to be precised but ensuring that we are in the center stable manifold.

Indeed $W^c(V_0)$ is the graph of a smooth function $h^c : X_c(V_0) \mapsto X_s(V_0) \oplus X_u(V_0)$ and $W^{cs}(V_0)$ is the graph of a smooth function $h^u : X_c(V_0) \oplus X_s(V_0) \mapsto X_u(V_0)$. Let ν be such that $\psi = S(V_0) + \nu$ satisfy (1.2), we have

$$\partial_t \nu = JH(V_0)\nu + JN(V_0, \nu).$$

$$\begin{aligned} \nu &= y + h_c(y) + \rho + h^u(y, h^c(y) + \rho) \\ &= \phi(y) - S(V_0) + (\rho - \pi^s(V_0)h^c(y)) + (h^u(y, h^c(y) + \rho) - \pi^u(V_0)h^c(y)) \\ &= \phi(y) - S(V_0) + \rho + f(y, \rho) \end{aligned}$$

with $y = \pi^c(V_0)\nu = \pi^c(V_0)(\psi - S(V_0))$ and $\phi(y) = S(V_0) + y + h^c(y)$ is in the center manifold and $\rho \in X_s(V_0)$. We have the following equation for ρ

$$\partial_t \rho = JH(V_0)\rho + M(V_0, y, \rho) \tag{4.1}$$

where

$$\begin{aligned} M(V_0, y, \rho) = \pi^s(V_0)\{&JN(V_0, y + h^c(y) + \rho + f(y, \rho)) - JN(V_0, y + h^c(y))\} \\ &- \pi^s(V_0)Dh^c(y)\pi^c(V_0)\{JN(V_0, y + h^c(y) + \rho + f(y, \rho)) - JN(V_0, y + h^c(y))\}. \end{aligned}$$

Then we obtain for ϕ the equation

$$\partial_t \phi = JH\phi + J\nabla F(\phi) + R(\phi, \rho)$$

$$R(\phi, \rho) = J\nabla F(\phi + \rho + f(y, \rho)) - J\nabla F(\phi) - Jd^2F(S(V_0))\rho - M(V_0, \pi^c(V_0)(\phi - S(V_0)), \rho)$$

with notice that $|R(\phi, \rho)| \le C(\|\phi\|_{H^s}, \|\rho\|_{L^\infty})|\rho|$.

Working like in 3, we write $\phi = S(U) + \eta$ with $\eta = z + g(U, z)$ and we have the following equations for U and z:

$$\begin{cases} \dot{U} = -A(U, \eta)\langle N(U, \eta) - JR(U, \eta, \rho), dS(U)\rangle \\ \partial_t z = JH(U)z + \mathbf{P}_c(U)JN(U, \eta) + \mathbf{P}_c(U(v))dS(U(v))A(U(v), \eta(v))\langle N(U(v), \eta(v)) \\ \quad -JR(U, \eta, \rho), dS(U(v))\rangle + (dP_c(U))A(U, \eta)\langle N(U, \eta) - JR(U, \eta, \rho), dS(U)\rangle\eta + \mathbf{P}_c(U)R(U, \eta, \rho) \end{cases}$$

with

$$\eta(t) = z(t) + g(U(t), z(t)).$$

where g is defined by Lemma 3.2 and

$$R(U, \eta, \rho) = R(S(U) + \eta, \rho)$$

These equations are similar to those we have studied but with an extra term coming from R which is exponentially decaying in positive time. Indeed, let us introduce for any $T_0 < 0$ and $\gamma \in (0, \gamma(V_0))$ and $\delta > 0$ the set

$$\mathcal{R}_{T_0,\gamma}(\delta) = \left\{\rho \in \mathcal{C}((T_0, +\infty), X_s(V_0)),\ |\rho(t)|_{H^s} \le \delta e^{-\gamma t}, \forall t > T_0\right\},$$

we study Equation (4.1) in $\mathcal{R}_{T_0,\gamma}(\delta)$ with small initial condition ρ_0. We also define for any $\varepsilon > 0$

$$\mathcal{U}_{T_0}(\varepsilon, \delta) = \left\{U \in \mathcal{C}^1((T_0, +\infty), B_{\mathbb{C}^2}(V_0, \varepsilon)),\ \|\dot{U}\|_{L^1((T_0,+\infty)) \cap L^\infty((T_0,+\infty))} \le \delta^2\right\}$$

and for any $U \in \mathcal{U}_{T_0}(\varepsilon)$, let s, β be such that $s > \beta + 2 > 2$ and $\sigma > 3/2$,

$$\mathcal{Z}_{T_0}(U, \delta) = \Bigg\{z \in \mathcal{C}((T_0, +\infty), L^2(\mathbb{R}^3, \mathbb{R}^8)),\ z(t) \in \mathcal{H}_c(U(t)),$$

$$\max\left[\|z\|_{L^\infty((T_0,+\infty), H^s)}, \|z\|_{L^2((T_0,+\infty), H^s_{-\sigma})}, \|z\|_{L^2((T_0,+\infty), B^\beta_{\infty,2})}\right] \le \delta\Bigg\},$$

and ε, δ are small enough to ensure that for $U \in \mathcal{U}(\varepsilon, \delta)$ and $z \in \mathcal{Z}(U, \delta)$

$$S(U) + z + g(U, z) \in W^c(V_0) \cap B_{H^s}(S(V_0), r(V_0)).$$

For a sufficiently small T_0, we solve the equation for z first and then the one for ρ and eventually the one for U using the method of Section 3. This gives us the desired exponential decay for ρ as well as similar results for U and z.

We notice that instead of Lemma 3.9, we obtain the

Lemma 4.1. *If Assumptions 1.1–1.5 hold. Then for any $T > 0$, there exist $T_0 > 0$, $\varepsilon_0 > 0$, $\delta_0 > 0$, $C > 0$ and $\kappa \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \delta_0)$, U, $U' \in \mathcal{U}_{T_0}(\varepsilon, \delta)$, $\rho, \rho' \in \mathcal{R}_{T_0,\gamma}$ $z_0 \in \mathcal{H}_c(U(0))$, $z_0' \in \mathcal{H}_c(U'(0))$, $z \in \mathcal{Z}_{T_0}(U, \delta)$ and $z' \in \mathcal{Z}_{T_0}(U', \delta)$, one has*

$$\begin{aligned}\|z[z_0', U', \rho'] - z[z_0, U, \rho]\|_{L^\infty((T_0,T),H^s)\cap L^2((T_0,T),L^\infty)\cap L^2((T_0,T),H^s_{-\sigma})} \\ \leq C\|z_0 - z_0'\|_{H^s} + \kappa\Big\{ \|U - U'\|_{L^\infty((T_0,T),\mathbb{C}^2} + \left\|\dot{U} - \dot{U}'\right\|_{L^\infty((T_0,T),\mathbb{C}^2)} \\ + \left\|e^{\gamma t}(\rho - \rho')(t)\right\|_{L_t^\infty((T_0,T),X^s(V_0))}\Big\}.\end{aligned}$$

Then for ρ as a function of U, z_0 and ρ_0 (the initial condition for ρ), we obtain the

Lemma 4.2. *If Assumptions 1.1–1.5 hold. Then for any $T > 0$ there exist $T_0 > 0$, $\varepsilon_0 > 0$, $\delta_0 > 0$, $C > 0$ and $\kappa \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \delta_0)$, U, $U' \in \mathcal{U}_{T_0}(\varepsilon, \delta)$, $r_0, r_0' \in X_s(V_0)$ $z_0 \in \mathcal{H}_c(U(0))$, $z_0' \in \mathcal{H}_c(U'(0))$, $z \in \mathcal{Z}_{T_0}(U, \delta)$ and $z' \in \mathcal{Z}_{T_0}(U', \delta)$, one has*

$$\begin{aligned}\left\|e^{\gamma t}\left(\rho[z_0', U', \rho_0'] - \rho[z_0, U, \rho_0]\right)\right\|_{L_t^\infty((T_0,T),X^s(V_0))} \\ \leq C\|z_0 - z_0'\|_{H^s} + \kappa\left\{\|U - U'\|_{L^\infty((T_0,T),\mathbb{C}^2} + \left\|\dot{U} - \dot{U}'\right\|_{L^\infty((T_0,T),\mathbb{C}^2)} + \|\rho_0 - \rho_0'\|_{L^2}\right\}.\end{aligned}$$

We also notice that the proof gives the wellposedness of Equation (4.1) in $\mathcal{R}_{T_0,\gamma}(\delta)$ with small initial condition ρ_0 and that there exists $C > 0$ such that the solution ρ satisfies

$$\|\rho(t)\|_{H^s} \leq C\|\rho_\pm(0)\| e^{-\gamma t}, \forall t > T_0.$$

The asymptotic behaviour of U and z are obtained like in the previous section when z_0 is localized.

5 End of the proof of main theorems

We notice that the small *locally invariant* center manifold build in Section 2.2 for Equation (2.11) is now a small *invariant* (globally in time) center manifold. Indeed, we have just proved the stabilization towards the PLS manifold, this ensures that a solution in the center manifold will stay inside this manifold in the two direction of time.

Now let us consider $\mathcal{C}M$ as being the union of all these small globally invariant center manifolds and 0. Using the uniqueness of center manifold and Lemma 3.2, we prove that $\mathcal{C}M \setminus \{0\}$ is a manifold. Now we generalize Lemma 3.2 by the

Lemma 5.1. *For any $s, s', \sigma \in \mathbb{R}$ and $p, q \in [1, \infty]$, there exist $\varepsilon > 0$, a continuous map $r : B^2_{\mathbb{C}}(0, \varepsilon) \mapsto \mathbb{R}^+$ with $r(U) = O(\Gamma(U))$ and a continuous map $\Psi : S \mapsto \mathcal{C}M$ where*

$$\mathcal{S}_\sigma = \Big\{(U, z);\ U \in B_{\mathbb{C}^2}(0, \varepsilon), z \in \mathcal{H}_c(U) \cap B_{H^{s'}_\sigma}(0, r(U))\Big\}$$

is endowed with the metric of $\mathbb{C}^2 \times H^{s'}_\sigma$.

Moreover Ψ is bijective from $\mathcal{S}$ to an open neigborhood of $(0, 0)$ in $\mathcal{C}M$ and smooth on $\mathcal{S} \setminus \{(0, 0)\}$. For all $U \in B_{\mathbb{C}^2}(0, \varepsilon)$, there exists $C > 0$ such that for all $z \in \mathcal{H}_c(U) \cap B_{H^{s'}}(0, r(U))$, $\Psi(U, z) \in \mathcal{H}_1(U)$, $z + \Psi(U, z) \in \mathcal{H}_0(U)^\perp$, $S(U) + z + \Psi(U, z) \in \mathcal{C}M$. For sufficiently small non zero U, we have $\|\Psi(U, z)\|_{B^s_{p,q}} = O(\|z\|^2_{H^{s'}})$ for $z \in H^{s'}$ such that $(U, z) \in \mathcal{S}$.

Proof. The proof works like for Lemma 3.2. The statements for r follow from Remark 2.3. $\square$

The scattering result follows from a one to one correspondence of the initial profile with the asymptotic profile as stated in the

Proposition 5.1. *If Assumptions 1.1–1.5 hold. There exist $\varepsilon > 0$ and a continuous map $r : B^2_{\mathbb{C}}(0,\varepsilon) \mapsto \mathbb{R}^+$ with $r(U) = O(\Gamma(U))$ and $\mathcal{V}_\sigma$, $\mathcal{V}_\pm$ neighborhoods of $(0,0)$ in*

$$S_\sigma = \left\{(U,z)\,;\; U \in \mathbb{C}^2,\; z \in \mathcal{H}_c(U) \cap B_{H^s_\sigma}(0, r(U))\right\}$$

endowed with the norm of $\mathbb{C}^2 \times H^s_\sigma$ such that the maps

$$\mathcal{P}_\pm : \begin{pmatrix} U_0 \\ z_0 \end{pmatrix} \in \mathcal{V}_\sigma \mapsto \begin{pmatrix} V_{\pm\infty} \\ z_{\pm\infty} \end{pmatrix} \in \mathcal{V}_\pm$$

are bijections and are smooth on $\mathcal{V}_0 \setminus \{(0,0)\}$.

Proof. We choose for example

$$\mathcal{V}_\sigma = \left\{(U,z)\,;\; U \in B_{\mathbb{C}^2}(0,\varepsilon),\; z \in \mathcal{H}_c(U) \cap B_{H^s_\sigma}(0, r(U))\right\}$$

for some positive ε and we work on the manifold $\mathcal{V}_\sigma \setminus \{(0,0)\}$ which is locally isomorphic to an open set of $\mathbb{C}^2 \times \mathcal{H}_c(U) \cap H^s_\sigma$. We write

$$\mathcal{P}^{U_0}_\pm(U,z) = (U,z) + \mathcal{R}^{U_0}_\pm(U,z)$$

Since

$$\|(U_\infty, z_\infty) - (U_0, z_0)\|_{H^s_\sigma} = O\left(|U_0|^2 + \|z_0\|^2_{H^s_\sigma}\right),$$

we only need to prove the statement locally. Hence we prove that in a neighborhood of $(U_0, 0)$. The maps $\mathcal{P}^{U_0}_\pm(U,z) \mapsto (Id_{\mathbb{C}^2}, P_c(U_0))\mathcal{P}_\pm(U, R(U,U_0)z)$ are bijective (P_c and R are defined in Proposition 2.2).

To prove that $\mathcal{P}^{U_0}_\pm$ is bijective (*i.e.* the scattering exists). Let us prove it for $\mathcal{P}^{U_0}_+$ (it is similar fo $\mathcal{P}^{U_0}_-$). It is enough to prove that the following system has a unique solution in an open neighborhood of $(0,0)$ in $\mathcal{S}_\sigma$:

$$V_\pm(t) = V_{\pm\infty} + \int_t^\infty A(V_\pm(v), e^{JsH(V_{\pm\infty})}\tilde{\eta}_\pm(v))\langle N(U(v), e^{JsH(V_{\pm\infty})}\tilde{\eta}_\pm(v)), dS(V_\pm(v))\rangle\, dv,$$

and

$$\begin{aligned}
\tilde{\xi}_+(t) = z_\infty - \int_t^\infty e^{-JsH(V_{+\infty})}\mathbf{P}_c(V_+(v))J\left(d^2F(S(V_+(v))) - d^2F(S(V_{+\infty}))\right)e^{JsH(V_{+\infty})}\tilde{\xi}_+(v)\,dv \\
- \int_t^\infty e^{-JsH(V_{+\infty})}\mathbf{P}_c(V_+(v))JN(V_+(v), e^{JsH(V_{+\infty})}\tilde{\eta}_+(v))\,dv \\
- \int_t^\infty e^{-JsH(V_{+\infty})}\mathbf{P}_c(V_+(v))dS(V(v))A(V_+(v), e^{JsH(V_{+\infty})}\tilde{\eta}_+(v))\langle N(V_+(v), e^{JsH(V_{+\infty})}\tilde{\eta}_+(v)), dS(V_+(v))\rangle\,dv \\
+ \int_t^\infty e^{-JsH(V_{+\infty})}(d\mathbf{P}_c(V_+(v)))A(V_+(v), e^{JsH(V_{+\infty})}\tilde{\eta}_+(v))\langle N(V_+(v), e^{JsH(V_{+\infty})}\tilde{\eta}_+(v)), dS(V_+(v))\rangle e^{JsH(V_{+\infty})}\tilde{\eta}_+(v)\,dv,
\end{aligned}$$

with $\tilde{\eta}_+(t) = \tilde{\xi}_+(t) + e^{-JsH(V_{+\infty})}g\left(V_+(t), e^{JsH(V_{+\infty})}\tilde{\xi}_+(t)\right)$.

This system can be solved by a fixed point argument in the set of function such that

$$\max\Big[\sup_{t\in\mathbb{R}}(\|z_{+\infty} - \tilde{\xi}_+(t)\|_{H^s}),\ \sup_{t\in\mathbb{R}}\langle t\rangle^{3/2}\|z_{+\infty} - \tilde{\xi}_+(t)\|_{H^s_{-\sigma}},$$
$$\sup_{t\in\mathbb{R}}\langle t\rangle^{3/2}\|z_{+\infty} - \tilde{\xi}_+(t)\|_{B^\beta_{\infty,2}},\ \sup_{t\in\mathbb{R}}(\langle t\rangle^{-3/2}\|z_{+\infty} - \tilde{\xi}_+(t)\|_{H^s_{3/2}})\Big]$$

and

$$\langle t\rangle^2\,|V_+(t) - V_{+\infty}|$$

are small with the method we used in Lemma 3.14. $\square$

For the same reasons the small locally invariant center-stable manifold build in Section 2.2 is invariant in positive time. We can also consider the union of these manifolds, and we can obtain a map Φ_+ similar to the map Ψ built in Lemma 5.1. The instability in negative time is in fact a consequence of Proposition 2.5.

The corresponding conclusion holds for the center unstable manifold.

The statements on the instability outside these manifolds follow from Propositions 2.4 and 2.5.

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