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Abstract

We present a global treatment of the analytical Bethe ansatz for $\mathrm{gl}(N)$ spin chains admitting on each site an arbitrary representation. The method applies for closed and open spin chains, and also to the case of soliton non-preserving boundaries.

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1 Introduction

The aim of this short note is to review the “global” treatment for analytical Bethe Ansatz that has been introduced in [1, 2]. It is a unified presentation which applies to any $gl(\mathcal{N})$ spin chain (whatever the quantum spaces are), and to general integrable boundary condition (periodic, soliton preserving or soliton non-preserving), provided $K^+(\lambda) = \mathbb{I}$. This will be achieved by the use of only the algebraic structure described by the monodromy matrix.
To give a first insight to the technique we develop, we first review in section 2 the usual closed spin chain associated to the XXX Heisenberg model (generalized to $gl(\mathcal{N})$), adopting a view point which will make clear the generalization to a general closed spin chain (section 3). Then, we will study the case of a general open spin chain, with ”usual” (soliton preserving) integrable boundary conditions (section 4). The case of a general open spin chain with soliton non-preserving boundary conditions is presented in section 5. Finally, we conclude in section 6 on open questions.

2 Closed spin chain revisited

As already mentioned, the (generalized) Heisenberg spin chain has been well studied, both from analytical and algebraic Bethe ansatz (see references in e.g. [1]). We just review some of these well-known results to introduce the algebraic set-up which will be used in the following.
The goal is to compute the spectrum of the spin chain described by the generalized XXX $gl(\mathcal{N})$ spin chain Hamiltonian $H \propto \sum_{n=1}^{\ell} P_{n-1,n}$, which describes ℓ spins interacting with their nearest neighbours.
For $gl(2)$, we recover the usual XXX Hamiltonian:

$$H = \frac{1}{2}\sum_{n=1}^{\ell} \left(\sigma^x_{n-1}\sigma^x_n + \sigma^y_{n-1}\sigma^y_n + \sigma^z_{n-1}\sigma^z_n + 1 \right) ,$$

where σ are the Pauli matrices, and n labels the sites (with the site 0 identified with the site ℓ).
The starting point is the monodromy matrix $T_a(\lambda) = R_{a1}(\lambda)R_{a2}(\lambda)\cdots R_{a\ell}(\lambda)$, where a denotes the auxiliary space and 1,..,ℓ the quantum spaces carrying the spin variables (fundamental representation of $gl(\mathcal{N})$). $R_{a1}(\lambda)$ is the R-matrix of the Yangian of $gl(\mathcal{N})$, $R_{12}(\mu) = \mu - iP_{12}$ where P_{12} is the permutation and μ the so-called spectral parameter. In the spin chain context, it is viewed as the scattering matrix of a ‘test-particle’ with a spin of the chain (see below).
The monodromy matrix obeys the Yangian exchange relations:

$$R_{ab}(\lambda-\mu)\, T_a(\lambda)\, T_b(\mu) = T_b(\mu)\, T_a(\lambda)\, R_{ab}(\lambda-\mu)$$

These exchange relations are enough to show that the transfer matrix $t(\lambda) = tr_a T_a(\lambda)$ satisfies $[t(\lambda), t(\mu)] = 0$, that is to say it generates (upon expansion in λ) ℓ commuting (independent) quantities. Since $H = \frac{d}{d\lambda} \ln t(\lambda)|_{\lambda=0}$, this proves the integrability of the spin chain model.

2.1 Analytical Bethe Ansatz

Once one has proven the integrability of the Heisenberg spin chain, a natural question is to look for the spectrum of $t(\lambda)$. For this purpose, one introduces the reference state (so-called pseudo-vacuum) $\omega = v_1 \otimes v_1 \otimes \cdots \otimes v_1$ with $v_1^t = (1, 0, \ldots, 0)$, which is an eigenvector of the transfer matrix:

$$t(\lambda)\omega = \Lambda^0(\lambda)\omega \quad \text{with} \quad \Lambda^0(\lambda) = (\lambda + i)^\ell + (\mathcal{N} - 1)\, \lambda^\ell$$

Then, the major hypothesis of the analytical Bethe ansatz states that <u>all</u> eigenvalues of the transfer matrix can be described by the formula

$$\Lambda(\lambda) = (\lambda + i)^\ell A_0(\lambda) + \sum_{k=1}^{\mathcal{N}-1} \lambda^\ell\, A_k(\lambda),$$

where $A_k(\lambda)$ are the dressing functions, to be determined.
The determination of the dressing functions relies on the following assumption: the dressing functions are "simple" rational functions with simple poles only. Help in fixing these dressing functions is given by the requirement that all eigenvalues $\Lambda(\lambda)$ are analytical, i.e. the residues at each possible "pole" vanish, and the fusion procedure (which provides the transfer matrix for a higher dimensional auxiliary space starting from a $\mathcal{N}$-dimensional auxiliary space). These constraints and requirement are enough to deduce the form of the dressing functions:

$$A_k(\lambda) = \prod_{n=1}^{M_{k-1}} \frac{\lambda - \lambda_n^{(k-1)} + \frac{i\,(k+1)}{2}}{\lambda - \lambda_n^{(k-1)} + \frac{i\,(k-1)}{2}} \;\; \prod_{n=1}^{M_k} \frac{\lambda - \lambda_n^{(k)} + \frac{i\,(k-2)}{2}}{\lambda - \lambda_n^{(k)} + \frac{i\,k}{2}}$$

where M_j are integers and $\lambda_n^{(k)}$ are solution to the <u>Bethe ansatz equations</u> (BAE)

$$\prod_{m=1}^{M_{k-1}} e_{-1}\Big(\lambda_n^{(k)}\text{-}\lambda_m^{(k-1)}\Big) \prod_{\substack{m=1\\ m\neq n}}^{M_k} e_2\Big(\lambda_n^{(k)}\text{-}\lambda_m^{(k)}\Big) \prod_{m=1}^{M_{k+1}} e_{-1}\Big(\lambda_n^{(k)}\text{-}\lambda_m^{(k+1)}\Big) = \begin{cases} \Big(e_1\,\big(\lambda_n^{(k)}\big)\Big)^\ell \\ 1 \end{cases}$$

where the first line corresponds to $k = 1$ and the second to $k \neq 1$. We introduced $e_n(\lambda) = \dfrac{\lambda + \frac{in}{2}}{\lambda - \frac{in}{2}}$. Each solution to these BAE gives a set of dressing functions, which will themselves allow us to describe the complete spectrum of the transfer matrix.

This method can be (and has been) applied, case by case, for several spin chain models, so that one is led to address the following question: can one extend the method to all integrable spin chains once for all? This is the question we answer in the following sections.

3 "Global" treatment

The key point in a treatment valid for all spin chains is the following remark: the integrability and the spectrum of the spin chain model rely only on the algebraic structure of the Yangian ($RTT = TTR$ exchange relation). The way to tackle the problem will be in the "translation" of all the above procedure into algebraic properties. An immediate advantage of the procedure will be a treatment valid for $gl(\mathcal{N})$ spin chains whatever the quantum spaces are. The starting point is the assumption that the monodromy matrix of a spin chain is a finite dimensional irreducible representation of the Yangian. This simple assumption will be enough to deduce most of the properties needed for the analytical Bethe ansatz. Indeed, the classification of such representations tells us that one has to consider a product of evaluation representations, that we first describe.
An evaluation representation (of the Yangian) is constructed from irreducible representation $\pi(\alpha)$ of $gl(\mathcal{N})$, characterized by the weight $\alpha = (\alpha_1, \dots, \alpha_{\mathcal{N}})$. It is convenient to gather the represented $gl(\mathcal{N})$ generators into a single $\mathcal{N}\times\mathcal{N}$ matrix

$$\mathbb{G} = \sum_{i,j=1}^{\mathcal{N}} \pi(\alpha)(e_{ij})\, E_{ij}\,,$$

where E_{ij} are elementary $\mathcal{N}\times\mathcal{N}$ matrices (with 1 in position i,j), e_{ij} are the $gl(\mathcal{N})$ generators, and $\pi(\alpha)(e_{ij})$ their representation. Then, the representation of the Yangian generators is given by

$$\pi_\lambda(\alpha)\,\mathcal{L}(\lambda) = \lambda\,\mathbb{I} + \mathbb{G}$$

Here, it is interpreted as the scattering matrix of a test-particle (of momentum λ and carrying a spin in the $gl(\mathcal{N})$ fundamental representation) with a spin (on a site of the chain) in the representation $\pi(\alpha)$ of $gl(\mathcal{N})$.
The monodromy matrix takes the form

$$T_a(\lambda) = \pi_{\lambda+a_1}(\alpha^{(1)}) \otimes \pi_{\lambda+a_2}(\alpha^{(2)}) \otimes \cdots \otimes \pi_{\lambda+a_\ell}(\alpha^{(\ell)})\,\Delta^{(\ell)}\mathcal{L}(\lambda)$$

where a_j are free parameters (inhomogeneity parameters in spin chain context). It describes the response of the whole spin chain to the test-particle.
This form leads us to introduce the notion of a "global" (unrepresented) monodromy matrix

$$\mathcal{T}_a(\lambda) = \Delta^{(\ell)}\mathcal{L}(\lambda) = \mathcal{L}_{a1}(\lambda)\mathcal{L}_{a2}(\lambda)\cdots\mathcal{L}_{a\ell}(\lambda)$$

It is easy to show that this global monodromy matrix obeys the Yangian exchange relations as soon as $\mathcal{L}(\lambda)$ does. Correspondingly, we introduce a "global" transfer matrix

$$t(\lambda) = tr_a \mathcal{T}(\lambda).$$

The Yangian exchange relations show that $[t(\lambda), t(\mu)] = 0$: one gets the integrability of the model. Moreover, the exchange relations of the monodromy matrix imply that the Lie algebra $gl(\mathcal{N})$ commutes with $t(\lambda)$: the model has a $gl(\mathcal{N})$ symmetry.

3.1 Spectrum and general structure of the BAE

Now that the integrability of the model is established, we turn to the determination of the spectrum of the transfer matrix, within the framework of analytical Bethe ansatz. For such a purpose, one has to characterize more precisely the irreducible finite representations of the Yangian. They are characterized by the notion of Drinfel'd polynomials: a representation is associated by (i.e. in one-to-one correspondence with) the Drinfel'd polynomials

$$P_k(\lambda) = \prod_{n=1}^{\ell} \left(\lambda + a_n - \alpha_k^{(n)} \right) \ , \ k = 1, .., \mathcal{N} - 1$$

The weight $(\mu_1(\lambda), \ldots, \mu_{\mathcal{N}}(\lambda))$ of the representation is then determined through the relations

$$\frac{\mu_i(\lambda)}{\mu_{i+1}(\lambda)} = \frac{P_i(\lambda)}{P_i(\lambda + 1)}$$

The reference state (pseudo-vacuum) is the highest vector ω of the representation, whose eigenvalue under the transfer matrix is $\Lambda^0(\lambda) = \sum_{k=1}^{\mathcal{N}} P_k(\lambda)$.

Then one assumes the following form for all the eigenvalues:

$$\Lambda(\Lambda) = \sum_{k=1}^{\mathcal{N}} P_k(\lambda)\ A_k(\lambda).$$

The dressing functions $A_k(\lambda)$ are supposed to be "simple" rational functions with simple poles. Requiring analyticity of the eigenvalues, and using fusion (which in the present algebraic context amounts to the existence of a center in the Yangian) fixes the form of the dressing functions

$$A_k(\lambda) = \prod_{n=1}^{M_{k-1}} \frac{\lambda - \lambda_n^{(k-1)} + \frac{i\,(k+1)}{2}}{\lambda - \lambda_n^{(k-1)} - \frac{k-1}{2}} \prod_{n=1}^{M_k} \frac{\lambda - \lambda_n^{(k)} + \frac{i\,(k-2)}{2}}{\lambda - \lambda_n^{(k)} - \frac{k}{2}}$$

where the parameters $\lambda_n^{(k)}$ obey the Bethe Ansatz Equations (BAE):

$$\prod_{m=1}^{M_{k-1}} e_{-1}\Big(\lambda_n^{(k)}\text{-}\lambda_m^{(k-1)}\Big) \prod_{\substack{m=1\\ m\neq n}}^{M_k} e_2\Big(\lambda_n^{(k)}\text{-}\lambda_m^{(k)}\Big) \prod_{m=1}^{M_{k+1}} e_{-1}\Big(\lambda_n^{(k)}\text{-}\lambda_m^{(k+1)}\Big) = \frac{P_k\Big(\lambda_n^{(k)} - \frac{i\,k}{2}\Big)}{P_{k+1}\Big(\lambda_n^{(k)} - \frac{i\,k}{2}\Big)}$$

This reproduces the results found by other methods [3, 4]. Let us remark that the left-hand side of the BAE reflects the Lie algebra dependence (through the Cartan matrix of $gl(\mathcal{N})$), while their right-hand side shows up a representation dependence (through Drinfel'd polynomials).
Note that the choice of a closed spin chain model is fixed by the choice of the quantum spaces, i.e. the choice of the Drinfel'd polynomials $P_k(\lambda)$. Once these polynomials are given, the spectrum of the transfer matrix is fixed through the resolution of the BAE.
We give hereafter some examples (more can be found in [1]).

3.2 Examples

3.2.1 Closed spin chain in the fundamental representation

The usual closed spin chain corresponds to spins in the fundamental representation. The Hamiltonian is given by the well-known formula

$$H = \frac{d}{d\lambda}\,\Big(ln\ t(\lambda)\Big)\Big|_{\lambda=0}\,. \tag{1}$$

In this case, we have $\alpha^{(n)} = (1,0,\ldots,0)$, for $1 \leq n \leq \ell$. Then, the Drinfel'd polynomials read

$$P_k(\lambda) = \begin{cases} \displaystyle\prod_{j=1}^{\ell}(\lambda + a_j + i) & , \quad k = 1 \\ \displaystyle\prod_{j=1}^{\ell}(\lambda + a_j) & , \quad k \neq 1 \end{cases} \tag{2}$$

Plugging these expressions in the Bethe equations, we recover the usual Bethe equations for closed spin chains, as given in section 2.1.

3.2.2 Closed spin chain for non-fundamental representations

One can generalize the above example to the case where all the spins belong to the same (not necessarily fundamental) representation, given by

$$\alpha^{(1)} = \alpha^{(2)} = \ldots = \alpha^{(\ell)} = (\alpha_1, \alpha_2, \ldots, \alpha_{\mathcal{N}})\,. \tag{3}$$

This leads to the following Drinfel'd polynomials

$$P_k(\lambda) = (\lambda + i\alpha_k)^\ell \;\text{ so that }\; \frac{P_k\left(\lambda_n^{(k)} - \frac{i\,k}{2}\right)}{P_{k+1}\left(\lambda_n^{(k)} - \frac{ir\,k}{2}\right)} = \left[e_{\beta_k^-}\left(\lambda_n^{(k)} - i\frac{k-\beta_k^+}{2}\right)\right]^\ell \;, \quad (4)$$

with $\beta_k^\pm = \alpha_k \pm \alpha_{k+1}$. In particular, we recover the result given in [12] about the XXX higher spin chains.

4 Open spin chains and reflection algebra

The technique can be applied to the case of open spin chains. Here, the algebraic structure, instead of being the Yangian algebra $Y(\mathcal{N})$, is the reflection algebra [5] $\mathcal{B}(\mathcal{N},\mathcal{M})$, whose generators (gathered into a matrix $\mathcal{B}(\lambda)$) obey the exchange relations (reflection equation):

$$R_{ab}(\lambda_a-\lambda_b)\,\mathcal{B}_a(\lambda_a)\,R_{ba}(\lambda_a+\lambda_b)\,\mathcal{B}_b(\lambda_b) = \mathcal{B}_b(\lambda_b)\,R_{ab}(\lambda_a+\lambda_b)\,\mathcal{B}_a(\lambda_a)\,R_{ba}(\lambda_a-\lambda_b)$$

The reflection algebra is a coideal subalgebra of the Yangian, so that the "global" open spin chain monodromy matrix takes the form:

$$\begin{aligned}\mathcal{B}_a(\lambda) &= \mathcal{L}_{a1}(\lambda)\;\ldots\;\mathcal{L}_{a\ell}(\lambda)\;K_a(\lambda)\;\mathcal{L}_{a\ell}^{-1}(-\lambda)\;\ldots\;\mathcal{L}_{a1}^{-1}(-\lambda)\\ &= \Delta^{(\ell)}B(\lambda) = \mathcal{T}_a(\lambda)\,K_a(\lambda)\,\mathcal{T}_a(-\lambda)^{-1}\end{aligned}$$

where $K(\lambda)$ is a matricial solution to the reflection equation.
The classification of such matrices has been done [6], it implies that diagonalisable matrices must be of the form (ξ is a free parameter):

$$K(\lambda) = U\;(\,\lambda\,\mathbb{E}_{\mathcal{M}} + \xi\,\mathbb{I}_{\mathcal{N}}\,)\,U^{-1} \equiv U\,D_\xi(\lambda)\,U^{-1}\,,$$
$$\text{with }\; \mathbb{E}_{\mathcal{M}} = diag(\underbrace{1,\ldots,1}_{\mathcal{M}},\underbrace{-1,\ldots,-1}_{\mathcal{N}-\mathcal{M}})\,.$$

The complete classification of $\mathcal{B}(\mathcal{N},\mathcal{M})$ finite dimensional irreducible representations has been done in [7] and uses the evaluation representations presented in previous section.
As in the closed spin chain case, one defines a "global" transfer matrix

$$b(\lambda) = tr_a\left(\mathcal{B}_a(\lambda)\right)$$

which contains the Hamiltonian

$$H = -\frac{1}{2}\frac{d}{d\lambda}b(\lambda)\Big|_{\lambda=0}.$$

Again, the underlying algebraic structure is sufficient to prove

$$[b(\lambda), b(\mu)] = 0,$$

i.e. the integrability of the models, and the $gl(\mathcal{M}) \oplus gl(\mathcal{N}-\mathcal{M})$ Lie algebra symmetry.
Its spectrum depends on $K(\lambda)$ only through $D_\xi(\lambda)$.
Note that a more general treatment can be done using a $K^+(\lambda)$ matrix solution to a dual reflection equation, the transfer matrix being $b(\lambda) = tr_a K_a^+(\lambda)\mathcal{B}_a(\lambda)$. The present techniques still applies if one supposes that $K(\lambda)$ and $K^+(\lambda)$ commute (for instance when they are both diagonal), see [1]. The case presented here corresponds to $K^+(\lambda) = \mathbb{I}$ (which is indeed a solution to the dual reflection equation).

4.1 BAE for open spin chains

The pseudo-vacuum is the highest weight vector of the representation, with eigenvalue

$$\Lambda^0(\lambda) = \sum_{k=1}^{\mathcal{N}} g_k(\lambda)\ \beta_k(\lambda),$$

where $\beta_k(\lambda)$ play the role of Drinfel'd polynomials for reflection algebra and

$$g_k(\lambda) = \frac{2\lambda(2\lambda + i\mathcal{N})}{(2\lambda + ik - i)(2\lambda + ik)} D_{kk}(\lambda)$$

correspond to the chosen matrix $K(\lambda) = UD(\lambda)U^{-1}$.
The form of the eigenvalues is assumed to be

$$\Lambda(\lambda) = \sum_{k=1}^{\mathcal{N}} g_k(\lambda)\ \beta_k(\lambda)\ A_k(\lambda),$$

with dressing functions $A_k(\lambda)$ to be determined.
Analyticity and fusion lead to

$$\begin{aligned} A_k(\lambda) &= \prod_{n=1}^{M_{k-1}} \frac{\lambda + \lambda_n^{(k-1)} + \frac{i(k+1)}{2}}{\lambda + \lambda_n^{(k-1)} + \frac{i\,(k-1)}{2}} \quad \frac{\lambda - \lambda_n^{(k-1)} + \frac{i(k+1)}{2}}{\lambda - \lambda_n^{(k-1)} + \frac{i\,(k-1)}{2}} \\ &\times \prod_{n=1}^{M_k} \frac{\lambda + \lambda_n^{(k)} + \frac{ik}{2} - i}{\lambda + \lambda_n^{(k)} + \frac{i\,k}{2}} \quad \frac{\lambda - \lambda_n^{(k)} + \frac{ik}{2} - i}{\lambda - \lambda_n^{(k)} + \frac{i\,k}{2}} \end{aligned}$$

where the $\lambda_n^{(k)}$ parameters obey the BAE:

$$\prod_{m=1}^{M_{k-1}} \widetilde{e}_{-1}\left(\lambda_n^{(k)},\lambda_m^{(k-1)}\right) \prod_{\substack{m=1\\ m\neq n}}^{M_k} \widetilde{e}_{2}\left(\lambda_n^{(k)},\lambda_m^{(k)}\right) \prod_{m=1}^{M_{k+1}} \widetilde{e}_{-1}\left(\lambda_n^{(k)},\lambda_m^{(k+1)}\right)$$
$$= \frac{\beta_k\left(\lambda_n^{(k)}-\frac{i\,k}{2}\right)}{\beta_{k+1}\left(\lambda_n^{(k)}-\frac{i\,k}{2}\right)} \times \begin{cases} -e_{-\mathcal{M}-2i\xi}\left(\lambda_m^{(\mathcal{M})}\right) & \text{if } k=\mathcal{M} \\ 1 & \text{otherwise} \end{cases}$$

where $\widetilde{e}_n(\lambda,\mu)=e_n\,(\lambda-\mu)\,e_n\,(\lambda+\mu)$.
Again, the left-hand side is linked to the $gl(\mathcal{N})$ Cartan matrix, while the right-hand side is related to the chosen representations.
The choice of an open spin chain model is now determined by two types of data:
1- The choice of the quantum spaces, i.e. the choice of the Drinfel'd polynomials $\beta_k(\lambda)$
2- The choice of the boundary condition, i.e. the choice of $K(\lambda)$ (in fact the eigenvalues of $K(\lambda)$), which fixes $g_k(\lambda)$.
Then, the spectrum of the transfer matrix is given by the solutions to the BAE.

5 Soliton non-preserving open spin chains

We now turn to the case of integrable spin chains with a boundary such that a soliton reflects into an anti-soliton (and vice-versa) [8]. The underlying algebraic structure is now the twisted Yangian [9, 10], whose exchange relations are ($\rho=-\frac{\mathcal{N}}{2}$):

$$R_{ab}(\lambda_a-\lambda_b)\,\mathcal{S}_a(\lambda_a)\,R_{ab}^{t_a}(-\lambda_a-\lambda_b+i\rho)\,\mathcal{S}_b(\lambda_b)$$
$$=\mathcal{S}_b(\lambda_b)\,R_{ab}^{t_a}(-\lambda_a-\lambda_b+i\rho)\,\mathcal{S}_a(\lambda_a)\,R_{ba}(\lambda_a-\lambda_b)$$

The "global" monodromy matrix reads $\mathcal{S}_a(\lambda)=\mathcal{T}_a(\lambda)\,\widetilde{K}_a\,\mathcal{T}_a{}^{t_a}(-\lambda+i\rho)=\Delta^{(\ell)}S(\lambda)$, where $\widetilde{K}_a(\lambda)$ is a matricial solution to the twisted Yangian exchange relations. It can be proven that it must be a constant matrix $\widetilde{K}_a$.
The "global" transfer matrix is now[2] $s(\lambda)=tr_a\mathcal{S}_a(\lambda)$.
Performing the same construction starting from the highest weight vector and using the classification of the twisted Yangian finite dimensional irreducible representations [11], we get the following form for the eigenvalues

$$\Lambda^0(\lambda)=\sum_{k=1}^{\mathcal{N}} g_k(\lambda)\;\sigma_k(\lambda) \qquad ; \qquad \Lambda(\lambda)=\sum_{k=1}^{\mathcal{N}} g_k(\lambda)\;\sigma_k(\lambda)\;A_k(\lambda),$$

[2] A more general treatment can be done using a $\widetilde{K}^+(\lambda)$ matrix solution to a dual exchange relation, the transfer matrix being $s(\lambda)=tr_a\widetilde{K}_a^+(\lambda)\mathcal{S}_a(\lambda)$, see [2].

where $\sigma_k(\lambda)$ are the Drinfel'd polynomials for the twisted Yangian and $g_k(\lambda)$ are related to the matricial solution $\widetilde{K}_a(\lambda)$.
Analyticity and fusion allow us to determine the dressing functions and to get the BAE. The generic form of these laters being (see [2] for the complete set):

$$\prod_{m=1}^{M_{k-1}} \widetilde{e}_{-1}\left(\lambda_n^{(k)}, \lambda_m^{(k-1)}\right) \prod_{\substack{m=1 \\ m\neq n}}^{M_k} \widetilde{e}_2\left(\lambda_n^{(k)}, \lambda_m^{(k)}\right) \prod_{m=1}^{M_{k+1}} \widetilde{e}_{-1}\left(\lambda_n^{(k)}, \lambda_m^{(k+1)}\right)$$
$$= \frac{g_k(\lambda_p^{(k)} + \frac{\hbar k}{2})\sigma_k(\lambda_p^{(k)} + \frac{\hbar k}{2})}{g_{k+1}(\lambda_p^{(k)} + \frac{\hbar k}{2})\sigma_{k+1}(\lambda_p^{(k)} + \frac{\hbar k}{2})}$$

6 Conclusion and perspectives

We have formulated a "global" treatment for analytical Bethe Ansatz for any $gl(\mathcal{N})$ spin chain (whatever the quantum spaces are), and applicable to general integrable boundary condition (periodic, soliton preserving or soliton non-preserving), provided $K^+(\lambda) = \mathbb{I}$.
This proves the integrability of the spin chains on general ground, allows us to compute their symmetry algebra and leads to the exact form of BAE for all these spin chains.
It remains to find, within this formalism, an expression for a local Hamiltonians. Indeed, although the BAE provide the spectrum of the transfer matrix (hence of all Hamiltonians), local Hamiltonians are obtained using higher dimensional auxiliary spaces, so that their expression in term of global transfer matrix is still lacking.
An interesting point is also the question of the completeness of the spectrum, which seems to be equivalent to the irreducibility of the representation. It is at least a necessary condition.
Finally, generalization of this approach to other algebras should be done: the case of $U_q(\widehat{gl}(\mathcal{N}))$ is under investigation, while the case of super-Yangian $\mathcal{Y}(gl(\mathcal{N}|\mathcal{M}))$ seems to be an easy task, although indecomposable representations may lead to new problems[3].

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