The epimorphisms of the category Haus are exactly the image-dense morphisms

Jérôme Lapuyade-Lahorgue

The epimorphisms of the category Haus are exactly the image-dense morphisms

Jérôme Lapuyade-Lahorgue

1 Introduction

In this document, we define the category **Haus** as the category whose objects are Hausdorff topological spaces and morphisms are the continous maps. The category **Top** is the category whose objects are the topological spaces and the morphisms are the continous maps. The definition of epimorphisms and the details about category theory can be found in [1].

2 Image-dense morphisms of Haus are epimorphisms

Proposition 1. *Let A and B be two Hausdorff topological spaces and let $A \xrightarrow{f} B$ be a continuous map. If the closure $\overline{f(A)}$ of $f(A)$ is such $B = \overline{f(A)}$, then f is an epimorphism.*

Proof. Let $B \xrightarrow{g} C$ and $B \xrightarrow{h} C$ two continuous maps between two Hausdorff topological spaces such $g \neq h$. Then there exists $b \in B$ such $g(b) \neq h(b)$. As C is Hausdorff, then there exists two disjoint open subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ of C such $g(b) \in \mathcal{O}_1$ and $h(b) \in \mathcal{O}_2$. We deduce that $b \in g^{-1}(\mathcal{O}_1) \cap h^{-1}(\mathcal{O}_2)$. As $B = \overline{f(A)}$, there exists $a \in A$ such $f(a) \in g^{-1}(\mathcal{O}_1) \cap h^{-1}(\mathcal{O}_2)$. And as $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint, $g \circ f(a) \neq h \circ f(a)$. Consequently, f is an epimorphism. $\square$

3 The category Haus is a reflective and full subcategory of Top

The fullness of **Haus** in **Top** is trivial. We have to prove the reflectivity.

Proposition 2. *Let C be a topological space and let $\sim$ be the equivalence relation defined by:*

$$x \sim y \Leftrightarrow \text{ for any continous map } C \xrightarrow{g} D \text{ where } D \text{ Hausdorff, } g(x) = g(y),$$

then the projection:

$$C \xrightarrow{r} C/\sim,$$

where $C/\sim$ is provided with the quotient-topology, is a reflection from the ***Top****-object C to the* ***Haus****-object $C/\sim$.*

Proof. Let $C \xrightarrow{f} D$ a continuous map to a Hausdorff space D. If $r(x) = r(y)$, then, by definition of the equivalence relation $\sim$, $f(x) = f(y)$. Consequently, any $C \xrightarrow{f} D$ is compatible with $\sim$ and by consequence, there exists a unique function (ie. **Set**-morphism) $\overline{f}$ such the following diagram:

$$\begin{tikzcd} C \arrow[r, "r"] \arrow[d, "f"] & C/\sim \arrow[dl, "\overline{f}"] \\ D & \end{tikzcd}$$

commutes. We have to show that $\overline{f}$ is a continuous map and that $C/\sim$ is a Hausdorff space.
$\overline{f}$ is a continuous map:
We recall that $\mathcal{O}$ is an open subset of $C/\sim$ is and only if $r^{-1}(\mathcal{O})$ is an open subset of C (see [2], Theorem 4.1 of the chapitre I). Let U an open subset of D, then $r^{-1}(\overline{f}^{-1}(U)) = f^{-1}(U)$ is an open set of C. We deduce that $\overline{f}^{-1}(U)$ is an open set of $C/\sim$ and by consequence, $\overline{f}$ is continous.
$C/\sim$ is a Hausdorff space:
Let $r(x) \neq r(y)$, then there exists $C \xrightarrow{f} D$ continous map to D an Hausdorff space such $f(x) \neq f(y)$. D is Hausdorff, then there exists two disjoint open-sets $\mathcal{O}_x$ and $\mathcal{O}_y$ of D such $f(x) \in \mathcal{O}_x$ and $f(y) \in \mathcal{O}_y$. It implies that $r(x) \in \overline{f}^{-1}(\mathcal{O}_x)$ and $r(y) \in \overline{f}^{-1}(\mathcal{O}_y)$. The subsets $\overline{f}^{-1}(\mathcal{O}_x)$ and $\overline{f}^{-1}(\mathcal{O}_y)$ are clearly disjoint open subsets of $C/\sim$. □

We will denote $H(C)$ the set $C/\sim$ and it is called the "Hausdorff quotient" of C.

Corollary 1. *A topological space C is Hausdorff if and only if C and $H(C)$ are homeomorphic.*

Proof. As **Haus** is a full subcategory of **Top**, one can use the proposition I.4.20 of [1]. □

4 The epimorphisms of Haus are image-dense morphims

Proposition 3. *Let $A \xrightarrow{f} B$ an epimorphism of* ***Haus****, then $B = \overline{f(A)}$.*

Proof. Let $B \xrightarrow{q} B/\overline{f(A)}$ be the quotient morphism such $q(x) = q(y)$ if and only if ($x = y$ and $x, y \notin \overline{f(A)}$) or ($x, y \in \overline{f(A)}$) and $B/\overline{f(A)} \xrightarrow{r} H\left(B/\overline{f(A)}\right)$ be the Hausdorff quotient of $B/\overline{f(A)}$.

We have $r \circ q \circ f$ is constant and as f is an epimorphism of **Haus**, then the **Haus**-morphism $r \circ q$ is constant. So, either q is constant or r is constant. If q is constant, then $B = \overline{f(A)}$.
Suppose that $B \neq \overline{f(A)}$ and r is constant. Let $x \in B \backslash \overline{f(A)}$. As $\overline{f(A)}$ is a closed subset of B, then there exists an open subset $\mathcal{O}$ of B such $x \in \mathcal{O} \subset B \backslash \overline{f(A)}$. By definition of q, for x_1 and x_2 in $\mathcal{O}$, $q(x_1) = q(x_2)$ if and only if $x_1 = x_2$. It implies that $q^{-1}(q(\mathcal{O})) = \mathcal{O}$ and $q(\mathcal{O})$ is a Hausdorff open subset of $B/\overline{f(A)}$. We deduce that $r \circ q(\mathcal{O})$ has at least one point which is different from $r \circ q(\overline{f(A)})$. Consequently, r takes at least two different values; this fact is contradictory with r constant. □

References

[1] J. Adámek, H. Herrlich, and G. E. Strecker. *Abstract and Concrete Categories. The Joy of Cats.* 2004.

[2] C. Godbillon. *Eléments de topologie algébrique.* Hermann. Editeurs des Sciences et des Arts, 1997.