Introduction to statistical methods in signal and image processing
Florence Forbes

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Introduction to statistical methods in signal and image processing

Florence Forbes

d Florence. forbes@inria.fr

INRIA Mistis team & Lab. Jean Kuntzman
Université Grenoble Alpes
http://mistis.inrialpes.fr
A methodological framework for inverse problems

Linear Models: convolution (image restoration), projection (tomography), mixtures (source separation), Laplace and Fourier transform (NMR, MRI)

Inversion: instability, non-unicity or existence of the solution

→ Ill-posed problem

Regularization: add constraints/hypothesis on the seek solution

- Bayesian inference: \( p(z|x) \propto p(x|z)p(z) \)
- Penalized criterion minimization: \( F(z) = L(z, x) + \beta R(z) \)
Overview: Part 1- Introduction to Bayesian tools

- Introduction
- Statistical inference
  - Learning and decision
  - Maximum likelihood
- Bayesian set up
  - prior, posterior, etc.
- Bayesian inference strategies
  - Point estimators
  - Fully Bayesian treatment
- Prior distributions
  - Conjugate priors and exponential family
  - Noninformative and Jeffreys’ priors
- Tractability of posteriors
Overview: Part 2- Probabilistic graphical models

- Directed graphs: Bayesian networks
- Conditional independence and Markov properties
- Undirected graphs: Markov random fields
- Inference and learning
- Illustration: image segmentation
Introduction
Illustration: Audio-Visual Scene Analysis

- Estimate the number of audio-visual objects
- Localize and track every object
- Determine auditory activity and visibility
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Observed Data

Right camera image:  
Left camera image:

Left microphone signal:  
Right microphone signal:
Visual Features Extraction

An image pair produces a set of visual observations
\[ f = \{ f_m \}_{m=1}^M \in \mathbb{R}^3; \]
\[ f = (u, v, d): \ u, v - \text{image coordinates, } d - \text{disparity} \]
Auditory Features Extraction

ITD = interaural time difference

An ITD detection algorithm [H. Christensen, 2007] produces for a 10ms interval of audio signals one auditory observation $g_k \in \mathbb{R}$.
Audio-Visual Generative Model

$f = \{ f_m \}_{m=1}^{M}$

3D visual observation space

$g = \{ g_k \}_{k=1}^{K} \in \mathbb{R}$

1D auditory observation space

$\{ s_1, s_2, \ldots, s_N \}$

3D object space
Why statistical modelling in Audio-Visual Scene Analysis?

- Observations are strongly affected by noise: detector errors, occlusions, reverberations, ambient sounds, can be accounted for with some probability distributions.

\[
P(\mathbf{f}_m | A_m = n; \mathbf{s}_n) = \mathcal{N}(\mathbf{f}_m; \mathcal{F}(\mathbf{s}_n), \Sigma_n);
\]

\[
P(g_k | B_k = n; \mathbf{s}_n) = \mathcal{N}(g_k; \mathcal{G}(\mathbf{s}_n), \Gamma_n);
\]

- Dynamically changing environment: can be accounted for with some prior knowledge, eg. on motion cues, trajectories are continuous, smooth, etc...

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Introduction to statistical methods in signal and image processing
Statistical Model formulation

\[ \{s_1, \ldots, s_N\} \] are tying parameters

Simultaneous clustering in auditory and visual observation spaces

Model parameters: Determine \( N \) and \( s_1, \ldots, s_N \)

\[ \theta = \{s_1, \ldots, s_N, \Sigma_1, \ldots, \Sigma_N, \Gamma_1, \ldots, \Gamma_N, \pi_1, \ldots, \pi_{N+1}, \lambda_1, \ldots, \lambda_{N+1}\} \]
Real Data Results

Meeting scenario

- Estimated speaker locations and their auditory activity for a quasi-stationary scene
- Error rates for auditory activity detection: ‘missed target’ = 0.16, ‘false alarm’ = 0.14
- Localization error: within 5cm
Real Data Results

Simple tracking scenario

- Simple dynamic scene - results on the previous frame are used to initialize the model for the next frame
- Error rates for auditory activity detection: ‘missed target’ = 0.13, ‘false alarm’ = 0.43
- Localization error: within 10cm
Real Data Results

Cocktail party scenario

- Complex dynamic scene - may fail!
- Explicit dynamic model is required!
Real Data Results

Cocktail party scenario

- Complex dynamic scene - may fail!
- Explicit dynamic model is required!
Illustration: MR Brain scan segmentation

Assign each voxel to a class (label) (among K classes)

**Tissue segmentation (WM, GM, CSF)**

- Mâtière Blanche (MB)
- Mâtière Grise (MG)
- Liquide Céphalo-Rachidien (LCR)

→ Cortex 3D reconstruction

**Structure segmentation**

- Corne Frontale (LCR)
- Noyau Caudé (MG)
- Putamen (MG)
- Système Ventriculaire (LCR)
- Thalamus (MG)

→ Useful for:
  - Distinguishing Cortex GM from Nuclei GM
  - Volumetric studies
  - ...

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Introduction to statistical methods in signal and image processing
**Tissue segmentation**

- **Global** estimation of **Gaussian** intensity models for CSF, WM, GM.

- Intensity inhomogeneities are modeled by a unique bias field (e.g., Multiplicative) to be estimated.

Constrain with an atlas.
Statistical inference
From a given set of observation \( x = (x_1, \ldots, x_N) \), learn a model that best describes the data

- **Probabilistic parametric model:**
  \( x = (x_1, \ldots, x_N) \) generated from a probability distribution \( f(x|\theta) \)

  \[
  x = (x_1, \ldots, x_N) \sim f(x|\theta)
  \]

  associated **likelihood:** \( l(\theta|x) = f(x|\theta) \) viewed as a function of \( \theta \)

- **Learning:** estimating \( \theta \)
  e.g. that maximizes \( l(\theta|x) \) (**Maximum likelihood** inference)
Decision

Once a model is learned, decide about:

- The occurrence of an "event",
- Classify,
- Or find the value of a variable, etc.

Example 1: Linear model

Assume \( x = Kz + \epsilon \)

\( z = \) clean signal, \( z \sim f(z|\theta) \)

\( \epsilon = \) noise, \( \epsilon \sim f(\epsilon|\phi) \)

\( x = \) noisy observed signal

Goal: obtain an estimate for \( z (\hat{z}) \)
Example 2: Classification

e.g. 2 groups of objects (people)

\[ \theta_1 \rightarrow f(x|\theta_1) \rightarrow x \in g_1 \]
\[ \theta_2 \rightarrow f(x|\theta_2) \rightarrow x \in g_2 \]

Training data: observations in \( g_1 \) and in \( g_2 \) \( \rightarrow \hat{\theta}_1, \hat{\theta}_2 \)

Goal: given \( x^{new} \), decide to which group it belongs

(ie. compute \( p(g|x^{new}, \hat{\theta}_1, \hat{\theta}_2)) \)
Maximum likelihood estimation

- We observe $N$ realizations $x_1, \ldots, x_N$ of a variable $X$
- Decide on a parametric model for $X$: $f(x|\theta)$
- Estimate $\theta$ by maximizing $l(\theta|x)$ or $\log l(\theta|x)$

Example 1: Linear Gaussian model $z = Kx + \epsilon$ and $\epsilon \sim \mathcal{N}(\mu_\epsilon, \Sigma_\epsilon)$

$$
\log f(z|\theta) = \log \mathcal{N}(Kx + \mu_\epsilon, \Sigma_\epsilon) \propto -(z - Kx - \mu_\epsilon)^T \Sigma_\epsilon^{-1} (z - Kx - \mu_\epsilon)
$$

$$
\hat{x}_{ML} = \arg\min_x (z - Kx - \mu_\epsilon)^T \Sigma_\epsilon^{-1} (z - Kx - \mu_\epsilon)
$$

Normal equations: $(K^T \Sigma_\epsilon^{-1} K) \hat{x}_{ML} = K^T \Sigma_\epsilon^{-1} (z - \mu_\epsilon)$

- Least squares: $\mu_\epsilon = 0$ and $\Sigma_\epsilon = \sigma^2 I_d$

$$
\implies \hat{x}_{ML} = \arg\min_x ||z - Kx||_2^2 = \hat{x}_{LS}
$$

- Weighted least squares: $\mu_\epsilon = 0$ and $\Sigma_\epsilon = Diag(\sigma_1^2, \ldots, \sigma_N^2)$

$$
\implies \hat{x}_{ML} = \arg\min_x \sum_n \frac{(z_n - [Kx]_n)^2}{\sigma_n^2} = \hat{x}_{WLS}
$$
Example 2: Man-Woman classification problem

5 subjects in each class were asked if they like football and statistics

<table>
<thead>
<tr>
<th></th>
<th>Women $g_1$</th>
<th></th>
<th>Men $g_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 1 0 0 0</td>
<td>1 1 1 1 1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 0 1 0 1</td>
<td>0 1 0 0 1</td>
<td></td>
</tr>
<tr>
<td>football statistics</td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Positive answers</th>
<th>Negative answers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2/5=0.4</td>
<td>3/5=0.6</td>
</tr>
<tr>
<td></td>
<td>3/5=0.6</td>
<td>2/5=0.4</td>
</tr>
<tr>
<td>football statistics</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5/5=1</td>
<td>0/5=0</td>
</tr>
<tr>
<td></td>
<td>2/5=0.4</td>
<td>3/5=0.6</td>
</tr>
</tbody>
</table>
Observations and notation

N = 10 responses \( x_n = \begin{bmatrix} x_{1n} \\ x_{2n} \end{bmatrix} \in \{0, 1\}^2 \) (2 questions)

\( x = \{x_1, \ldots, x_N\} \)

N = 10 group assignments \( g_n \in \{\text{Woman}, \text{Man}\} = \{1, 2\} \)

\( g = \{g_1, \ldots, g_N\} \)

\( x^{g_1} = \{x_n, g_n = 1\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \)

\( x^{g_2} = \{x_n, g_n = 2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \)
Model

- Independence: \( f(g) = \prod_{n=1}^{N} f(g_n) \)
- \( f(g_n = \text{woman}) = f(g_n = \text{man}) = 0.5 \)
- Conditional independence: \( f(x|g) = \prod_{n=1}^{N} f(x_n|g_n) \)
- Independence of the two questions: \( f(x_n|g_n) = f(x_{1n}|g_n) f(x_{2n}|g_n) \)
- \( \forall n = 1, \ldots, N, i = \{1, 2\}, g = \{1, 2\}, \) Independent Bernoulli distributions \((\theta^g_i \in [0, 1]):\)

\[
f(x_{in}|g_n = g) = \begin{cases} 
\theta^g_i & \text{if } x_{in} = 1 \\
1 - \theta^g_i & \text{if } x_{in} = 0 \\
0 & \text{otherwise}
\end{cases}
\]

or equivalently \( f(x_{in}|g_n = g) = (\theta^g_i)^{x_{in}} (1 - \theta^g_i)^{1-x_{in}} \)

\((\theta^g_i = 0.5 \rightarrow \text{the coin is not biased})\)
Likelihood for each group

- Learning task: Estimate \((\theta^g_1, \theta^g_2)\) given \(x^g\) \((g = \text{Woman, Man})\)
- Likelihood function:
  \[
f(x^g|\theta^g) = \prod_{n, g_n=g} f(x_{1n}|g) f(x_{2n}|g)
\]
- Log-likelihood:
  \[
  \log f(x^g|\theta^g) = \sum_{n, g_n=g} \sum_{i=1,2} x_{in} \log \theta^g_i + (1 - x_{in}) \log (1 - \theta^g_i)
  \]
- Maximization: \(\theta^g_i = \frac{\sum_{n, g_n=g} x_{in}}{N_g}\) (mean, frequencies of positive answers)

\[
\theta^1 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}, \quad \theta^2 = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}
\]
**Decision: Naive Bayes classifier**

**Sum-rule:** \( P(B) = P(B, A) + P(B, A^c) \)

**Product-rule:** \( P(A, B) = P(B|A)P(A) \)

It follows Bayes’ theorem: \( P(A|B) = \frac{P(B|A)P(A)}{P(B)} \)

with normalization \( P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) \)

**Goal:** Classify a person with \( x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) ie. \( g = ?? \)

- **Bayes’ rule:** \( f(g|x) = \frac{f(x|g)f(g)}{f(x)} = \frac{f(x|g)f(g)}{\sum_{g'} f(g')f(x|g')} \)

- Assuming \( f(\text{woman}) = f(\text{man}) = 0.5, \)

\[
\begin{align*}
  f(\text{woman}|x) & = \frac{0.4 \times 0.6}{0.4 \times 0.6 + 1 \times 0.4} = 0.375 \\
  f(\text{man}|x) & = \frac{1 \times 0.4}{0.4 \times 0.6 + 1 \times 0.4} = 0.625 = 1 - 0.375
\end{align*}
\]
Decision: Naive Bayes classifier

Goal: classify a person with \( x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

\[
\begin{align*}
  f(\text{woman}|x) &= \frac{0.6 \times 0.6}{0.6 \times 0.6 + 0 \times 0.4} = 1 \\
  f(\text{man}|x) &= \frac{0 \times 0.4}{0.6 \times 0.6 + 0 \times 0.4} = 0
\end{align*}
\]

- Conclusion: if you don't like football and like statistics, you are almost surely a woman
- Overfitting effect to the small training set
- Priors over the parameters can avoid overfitting \( \implies \text{Bayesian framework} \)
Bayesian set up
Bayesian concepts

- **Uncertainty on the parameters** $\theta$ of a model modeled through a probability distribution on $\theta$, called **prior distribution**
  The prior **encoded the information available a priori**, before observing $x$

- Inference based on the distribution of $\theta$ conditional on $x$, $f(\theta|x)$, called **posterior distribution**
Impact

- From unknown parameters to random
- **Actualization** of the information on $\theta$ by extracting the information on $\theta$ contained in the observations $x$
- Allows incorporation of imperfect information in the decision process
- Unique mathematical way to condition upon the observations (conditional perspective)
- **Penalization** factor
Three basic quantities in Bayesian inference

- Prior distribution \( f(\theta) \)
- Likelihood \( f(x|\theta) \)
- Posterior distribution \( f(\theta|x) \)

**Forward generative model:**

\[
f(\theta) \rightarrow \theta \rightarrow f(x|\theta) \rightarrow x
\]

\( \rightarrow \) involves the prior and the likelihood

**Inference is an inversion problem:**

\[
x \rightarrow f(\theta|x) \rightarrow \hat{\theta}
\]

\( \rightarrow \) involves the posterior distribution
Classification example

Assume a Beta prior over the Bernouilli parameters: $\theta^g_i \in [0, 1]$

\[ f(\theta^g_i) = \text{Beta}(\alpha, \beta) = \mathcal{B}(\alpha, \beta)^{-1}\theta^g_i^{\alpha-1}(1 - \theta^g_i)^{\beta-1} \]

\[ E[\theta^g_i] = \alpha/(\alpha+\beta), \quad \text{Mode}[\theta^g_i] = (\alpha-1)/(\alpha+\beta-2) \]

Compute the posterior distribution of $\theta^g_i$:

\[ f(\theta^g_i | x^g) = \frac{f(x^g | \theta^g_i) f(\theta^g_i)}{f(x^g)} \propto f(x^g | \theta^g_i) f(\theta^g_i) \]

\[ \log f(\theta^g_i | x^g) = cst + (A^g_i - 1) \log \theta^g_i + (B^g_i - 1) \log(1 - \theta^g_i) \]

with $A^g_i = \sum_{n, g_n = g} x_{in} + \alpha$ and $B^g_i = \sum_{n, g_n = g}(1 - x_{in}) + \beta$

so that posterior distribution:

\[ f(\theta^g_i | x^g) \propto \theta^g_i^{A^g_i-1}(1 - \theta^g_i)^{B^g_i-1} \]

\[ = \text{Beta}(A^g_i, B^g_i) \]
Classification example

<table>
<thead>
<tr>
<th></th>
<th>Women $g_1$</th>
<th>Men $g_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>football</td>
<td>1 1 0 0 0</td>
<td>1 1 1 1 1</td>
</tr>
<tr>
<td>statistics</td>
<td>1 0 1 0 1</td>
<td>0 1 0 0 1</td>
</tr>
</tbody>
</table>

Example: $\alpha = \beta = 2 \implies A^1_1 = \alpha + 2 = 4, \quad B^1_1 = \beta + 3 = 5$

$\downarrow B(4, 5)$

$\leftarrow B(2, 2)$
Bayesian inference strategies
Point estimators

**Goal:** provide an estimation of $\theta$

The two most common Bayesian estimators are:

- **Maximum a posteriori (MAP) estimator**
  \[
  \hat{\theta}_{MAP} = \arg \max_{\theta} f(\theta|x) = \arg \max_{\theta} f(x|\theta)f(\theta) = \arg \max_{\theta} \log f(x|\theta) + \log f(\theta)
  \]
  Note: if $f(\theta) = constant$ then $\hat{\theta}_{MAP} = \hat{\theta}_{MLE}$

- **Posterior Mean Estimator**
  \[
  \hat{\theta}_{PM} = E_{\theta}[\theta|x] = \int \theta f(\theta|x)d\theta
  \]
  Note: $f(\theta|x)$ requires the normalizing term $f(x) = \int f(x|\theta)f(\theta)d\theta$.

$\hat{\theta}_{MAP}$ usually easier to obtain, it involves optimization rather than integration
The Bayesian Mean Square Error (MSE) is

\[
E_{\theta,X}[||\hat{\theta} - \theta||_2^2] = \int \int ||\hat{\theta}(x) - \theta||_2^2 f(\theta, x) \, d\theta \, dx
\]

Minimum Mean Square Error (MMSE) estimator:

**Definition:** \( \hat{\theta}_{MMSE} = \arg \min_{\hat{\theta}} E_{\theta,X}[||\hat{\theta} - \theta||_2^2] \)

**Solution:** \( \hat{\theta}_{MMSE} = E_{\theta}[\theta|X] = \hat{\theta}_{PM} \)

since \( E_{\theta,X}[||\hat{\theta} - \theta||_2^2] = E_X[E_{\theta}[||\hat{\theta} - \theta||_2^2|X]] \)

and \( E_{\theta}[||\hat{\theta} - \theta||_2^2|X] \) is minimum when \( \hat{\theta} = E_{\theta}[\theta|X] \)
MAP and 0-1 Loss

The MSE quadratic cost (loss) can be replaced by a 0-1 cost

\[ E_{\theta,X}[1 - \delta_\theta(\hat{\theta})] \]

where \( 1 - \delta_\theta(\hat{\theta}) = 0 \) if \( \hat{\theta} = \theta \) (no loss) and 1 otherwise (max loss)

\[
\min E_{\theta,X}[1 - \delta_\theta(\hat{\theta})] = \max E_X[E_\theta[\delta_\theta(\hat{\theta})|X]]
\]

and \( E_\theta[\delta_\theta(\hat{\theta})|X] = p(\theta = \hat{\theta}|X) \) which is max at the MAP
Linear Minimum MSE

Assume $E[\theta] = E[X] = 0$ and consider an estimator of the form $\hat{\theta} = A^T X$

**Goal:** find matrix $A$ that minimizes the Bayesian MSE

$$MSE(A) = E_{\theta,X}[||A^T X - \theta||^2_2]$$

$$= E_{\theta,X}[\text{trace} ((A^T X - \theta)(A^T X - \theta)^T)]$$

$$= \text{trace} (E_{\theta,X}[(A^T X - \theta)(A^T X - \theta)^T])$$

$$= \text{trace} (E[\theta \theta^T] - A^T E[X \theta^T] - E[\theta X^T] A + A^T E[X X^T] A)$$

$$= \text{trace} (\Sigma_\theta - A^T \Sigma_{\theta x} - \Sigma_{\theta x} A + A^T \Sigma_x A)$$

$$\frac{\partial}{\partial A} MSE(A) = -2\Sigma_{x \theta} + 2\Sigma_x A = 0$$

$\hat{A} = \Sigma_x^{-1} \Sigma_{x \theta}$  

**Wiener-Hopf equation**

$$\hat{\theta}_{LMMSE} = \Sigma_x \Sigma_x^{-1} X$$  

**Wiener filter**

$$(\hat{\theta}_{LMMSE} = \Sigma_x \Sigma_x^{-1} (X - \mu_x) + \mu_\theta \quad \text{in the non centered case})$$
Linear Minimum MSE: Linear model example

Assume $x = K\theta + \epsilon$ where $\theta \sim \mathcal{N}(0, \sigma_\theta^2 I)$ and $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2 I)$ are independent.

Then $x \sim \mathcal{N}(0, \sigma_\theta^2 KK^T + \sigma_\epsilon^2 I)$ and

$$
\Sigma_x = \sigma_\theta^2 KK^T + \sigma_\epsilon^2 I \\
\Sigma_{\theta x} = E[X\theta^T] = E[K\theta\theta^T + \epsilon\theta^T] = K\Sigma_\theta = \sigma_\theta^2 K
$$

$$
\hat{\theta}_{LMMSE} = \sigma_\theta^2 K^T (\sigma_\theta^2 KK^T + \sigma_\epsilon^2 I)^{-1} X = K^T (KK^T + \frac{\sigma_\epsilon^2}{\sigma_\theta^2} I)^{-1} X
$$

Note: when SNR increases, $\frac{\sigma_\epsilon^2}{\sigma_\theta^2} \to 0$, $\hat{\theta}_{LMMSE} \to \hat{\theta}_{MLE} = (KK^T)^{-1} K^T X$
Classification example

**MMSE estimator:** Since $f(\theta^g_i|x^g)$ is a Beta distribution

$$E[\theta^g_i|x^g] = \frac{\sum_{n,gn=g} x_{in} + \alpha}{\alpha + \beta + N_g}$$

With $\alpha = \beta = 2$ (mode and mean at 0.5), we get

$$\theta^1 = \begin{bmatrix} 4/9 \\ 5/9 \end{bmatrix} \text{ and } \theta^2 = \begin{bmatrix} 7/9 \\ 4/9 \end{bmatrix}$$

Then for $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, it comes $f(\text{man}|x) = 8/33 = 0.242$

**MAP estimator:** using the mode of the posterior we get instead:

$$\theta^1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} \text{ and } \theta^2 = \begin{bmatrix} 6/7 \\ 3/7 \end{bmatrix}$$

and for $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, it comes $f(\text{man}|x) = 3/19 = 0.158$
Predictive distributions

Use the full posterior rather than a point estimate.

Other distributions of interest are:

**Prior predictive (marginal):**

Before we observe the data, what do we expect the distribution of observations to be?

\[ f(x) = \int f(x|\theta) f(\theta) \, d\theta \]

- What we would predict for \( x \) given no data
- Useful for assessing whether choice of prior distribution does capture prior beliefs.
Predictive distributions

**Posterior predictive**

What is the predictive distribution of a new observation $x^{new}$ given the current data $x$?

\[
f(x^{new}|x) = \int f(x^{new}, \theta|x) \, d\theta
\]

\[
= \int f(x^{new}|\theta) f(\theta|x) \, d\theta
\]

Use the assumption that $x^{new}$ is independent of $x$ given $\theta$.  

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Classification example

In each group, the posterior predictive is:

\[
f(x^{\text{new}}|x^g) = \int f(x^{\text{new}}|\theta^g) f(\theta^g|x^g) d\theta^g = f(x^{\text{new}}_{1}|x^g_1) f(x^{\text{new}}_{2}|x^g_2)
\]

\[
f(x^{\text{new}}_i|x^g_i) = \int f(x^{\text{new}}_i|\theta^g_i) f(\theta^g_i|x^g_i) d\theta^g_i
\]

\[= \frac{B(x^{\text{new}}_i + A^g_i, 1 - x^{\text{new}}_i + B^g_i)}{B(A^g_i, B^g_i)}\]

Then using Bayes’ rule:

\[
f(g^{\text{new}}|x^{\text{new}}, x, g) \propto f(x^{\text{new}}|x^{\text{new}}_{g^{\text{new}}}) f(g^{\text{new}})
\]

\[\propto f(x^{\text{new}}_{1}|x^{\text{new}}_{g^{\text{new}}_1}) f(x^{\text{new}}_{2}|x^{\text{new}}_{g^{\text{new}}_2}) \times 0.5\]

For \( x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), it comes \( f(\text{man}|x) = \frac{8}{33} = 0.242 \)
Classification example: all results for $f(\text{man}|x = [0, 1]^T)$

| Estimator           | $f(\text{man}|x)$ |
|---------------------|-------------------|
| Maximum likelihood  | 0                 |
| Bayesian MMSE       | 0.242             |
| Bayesian MAP        | 0.158             |
| Fully Bayesian      | 0.242             |
Prior distributions
From prior information to prior distributions

- All computations depends on the prior choice
- The prior is a tool summarizing available information as well as uncertainty related with this information
- The prior distribution is the key to Bayesian inference but the available prior information is usually not precise enough to lead to an exact determination

Different strategies are possible:

- Conjugate priors
- Noninformative priors
- Jeffreys prior
- Hierarchical modelling, etc.
Conjugate priors: a starting point

Specific parametric family with convenient analytical properties

**Definition:** A family $\mathcal{F}$ of probability distributions on $\theta$ is conjugate for a likelihood function $f(x|\theta)$ if, for every $\pi \in \mathcal{F}$, the posterior distribution $f(\theta|x) \propto f(x|\theta)\pi(\theta)$ also belongs to $\mathcal{F}$.

Main interest is when $\mathcal{F}$ is parametric: computing the posterior distribution reduces then to an updating of the corresponding parameters of the prior.

- The prior ”structure” on $\theta$ is propagated to the posterior (actualisation)
- **Tractability and simplicity**
- First approximations to adequate priors
Conjugate priors: Gaussian case
Exponential families

Conjugate priors are usually associated with exponential families of distributions.

**Definition:** \( C, h \) are positive functions, \( R, T \) are functions in \( \mathbb{R}^k \)

The family of distributions

\[
f(x|\theta) = C(\theta)h(x) \exp(R(\theta)T(x))
\]

is called an exponential family of dimension \( k \).

When

\[
f(x|\theta) = C(\theta)h(x) \exp(\theta x) = h(x) \exp(\theta x - \Psi(\theta))
\]

the family is said to be natural.
Exponential families

Interesting analytical properties:

- Sufficient statistics of constant dimension exist
- Include common distributions (normal, binomial, Poisson, Wishart, etc.)
- Availability of the moments:
  \[
  E_X[X | \theta] = \nabla \Psi(\theta), \quad \text{cov}(X_i, X_j) = \frac{\partial^2 \Psi}{\partial \theta_i \partial \theta_j}(\theta).
  \]
- Allow for conjugate priors
If \( f(x|\theta) = h(x) \exp(\theta x - \Psi(\theta)) \) then

\[
f(\theta|\mu, \lambda) = K(\mu, \lambda) \exp(\theta \mu - \lambda \Psi(\theta))
\]

where \( K(\mu, \lambda) \) is the normalizing constant, is conjugate for \( f(x|\theta) \).

The posterior is then \( f(\theta|\mu + x, \lambda + 1) \).

It follows an "automatic" way to derive prior from \( f(x|\theta) \) BUT \( \mu, \lambda \) have still to be specified.
Linearity of the posterior mean

\[ f(x|\theta) \text{ in the natural exponential family: } f(x|\theta) = h(x) \exp(\theta x - \Psi(\theta)) \]

\[ E_X[X] = m(\theta) = \nabla \Psi(\theta) \]

\[ f(\theta) \text{ has a conjugate prior: } f(\theta) \propto \exp(\mu x - \lambda \Psi(\theta)) \]

\[ E_\theta[m(\theta)] = \int m(\theta) f(\theta) d\theta = \frac{\mu}{\lambda} \]

If \( x_1, \ldots x_N \) i.i.d \( f(x|\theta) \) then

\[ f(\theta|x_1, \ldots, x_N) \propto f(\theta|x_1) f(x_2|\theta) \ldots f(x_N|\theta) = f(\theta|\mu + \sum_{n=1}^{N} x_n, \lambda + N) \]

\[ E_\theta[m(\theta)|x_1, \ldots, x_n] = \frac{\mu + \sum_{n=1}^{N} x_n}{\lambda + N} \]
### Common conjugate priors

<table>
<thead>
<tr>
<th>density function</th>
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<tbody>
<tr>
<td>( f(x</td>
<td>\theta) )</td>
<td>( f(\theta) )</td>
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<tr>
<td><strong>Normal</strong> ( \mathcal{N}(\theta, \sigma^2) )</td>
<td><strong>Normal</strong> ( \mathcal{N}(\mu, \tau^2) )</td>
<td><strong>Normal</strong> ( \mathcal{N}\left(\frac{\sigma^2 \mu + \tau^2 x}{\sigma^2 + \tau^2}, \left(\frac{1}{\sigma^2} + \frac{1}{\tau^2}\right)^{-1}\right) )</td>
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<tr>
<td><strong>Poisson</strong> ( \mathcal{P}(\theta) )</td>
<td><strong>Gamma</strong> ( \mathcal{G}(\alpha, \beta) )</td>
<td><strong>Gamma</strong> ( \mathcal{G}(\alpha + x, \beta + 1) )</td>
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<tr>
<td><strong>Gamma</strong> ( \mathcal{G}(\nu, \theta) )</td>
<td><strong>Gamma</strong> ( \mathcal{G}(\alpha, \beta) )</td>
<td><strong>Gamma</strong> ( \mathcal{G}(\alpha + \nu, \beta + x) )</td>
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<tr>
<td><strong>Binomial</strong> ( \text{Bin}(n, \theta) )</td>
<td><strong>Beta</strong> ( \mathcal{B}(\alpha, \beta) )</td>
<td><strong>Beta</strong> ( \mathcal{B}(\alpha + x, \beta + n - x) )</td>
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<tr>
<td><strong>Multinomial</strong> ( \mathcal{M}(\theta_1, \ldots, \theta_K) )</td>
<td><strong>Dirichlet</strong> ( \mathcal{D}(\alpha_1, \ldots, \alpha_K) )</td>
<td><strong>Dirichlet</strong> ( \mathcal{D}(\alpha_1 + x_1, \ldots, \alpha_K + x_K) )</td>
</tr>
<tr>
<td><strong>Normal</strong> ( \mathcal{N}(\mu, \frac{1}{\theta}) )</td>
<td><strong>Gamma</strong> ( \mathcal{G}(\alpha, \beta) )</td>
<td><strong>Gamma</strong> ( \mathcal{G}(\alpha + 1/2, \beta + (x - \mu)^2/2) )</td>
</tr>
</tbody>
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Non informative priors

How to encode absence of prior knowledge?

Is there such a thing as a default prior when prior information is missing?

In the absence of prior information, prior distributions solely derived from the sample distribution $f(x|\theta)$
Uniform priors (Laplace’s priors)

Equiprobability of elementary events: the same likelihood to each value of $\theta$

$$\theta \in \{\theta_1, \ldots, \theta_p\} \rightarrow f(\theta_i) = \frac{1}{p}$$

Extensions to continuous spaces:

$$f(\theta) \propto 1 \quad (= \text{constant})$$

Examples:

Location parameters: $f(x|\theta) = f(x - \theta) \rightarrow f(\theta) \propto 1$

Scale parameters: $f(x|\theta) = \frac{1}{\theta} f\left(\frac{x}{\theta}\right) \rightarrow f(\theta) \propto \frac{1}{\theta} \quad (f(\log \theta) \propto 1)$
Some drawbacks

Lack of invariance through reparameterization: \( \theta \rightarrow \eta = g(\theta) \)

\[
f(\theta) \propto 1 \quad \rightarrow \quad f(\eta) \propto \left| \frac{dg^{-1}(\eta)}{d\eta} \right| \neq \text{constant} \quad \text{(Jacobian formula)}
\]

Information is not missing anymore !!

May generate improper posterior:

\[
x \sim \mathcal{N}(\theta, \sigma^2) \quad \text{with} \quad f(\theta, \sigma^2) \propto 1
\]

Then

\[
f(\theta, \sigma^2 | x) \propto f(x | \theta) \propto \sigma^{-1} \exp\left( \frac{(x - \theta^2)^2}{2\sigma^2} \right)
\]

\[\rightarrow f(\sigma^2 | x) \propto 1 \quad \text{is improper, paradoxes occur}
\]

\[\Rightarrow \text{Invariant priors}
\]

\[\Rightarrow \text{Jeffreys’ priors as an alternative}
\]
The Jeffreys’ priors

Based on Fisher information

Univariate case:

\[ I(\theta) = E_X \left[ \left( \frac{\partial \log f(X|\theta)}{\partial \theta} \right)^2 \right] = -E_X \left[ \frac{\partial^2 \log f(X|\theta)}{\partial \theta^2} \right] \]

Multivariate case:

\[ I(\theta)_{ij} = -E_X \left[ \frac{\partial^2 \log f(X|\theta)}{\partial \theta_i \partial \theta_j} \right] \]

The Jeffreys’ prior distribution is \( f(\theta) \propto |I(\theta)|^{1/2} \)

where \(|I(\theta)|\) is the determinant of the Fisher Information matrix

Exponential family: if \( f(x|\theta) = h(x) \exp(\theta x - \Psi(\theta)) \) then

\[ I(\theta) = \nabla^2 \Psi(\theta) \text{ and } f(\theta) \propto \left( \frac{\partial^2 \Psi(\theta)}{\partial \theta_i^2} \right)^{1/2} \]
Key feature: Reparameterization invariance

Assume $f(\theta) \propto |I(\theta)|^{1/2}$ and $\eta = g(\theta)$ for a 1-to-1 mapping $g$

$$f(\eta) = f(\theta) \left| \frac{\partial \theta}{\partial \eta} \right| \propto \sqrt{|I(\theta)|} \left( \frac{\partial \theta}{\partial \eta} \right)^2$$

$$\propto \sqrt{E_X \left[ \left( \frac{\partial \log f(X|\theta)}{\partial \theta} \right)^2 \left( \frac{\partial \theta}{\partial \eta} \right)^2 \right]}$$

$$\propto \sqrt{E_X \left[ \left( \frac{\partial \log f(X|\theta)}{\partial \eta} \right)^2 \right]}$$

$$\propto |I(\eta)|^{1/2}$$
Other features

- **Information based**: $I(\theta)$ corresponds to the amount of information brought by the model on $\theta$.
- **Noninformative**: Minimize the effect of the prior which is in accordance with the model.
- **Violates the likelihood principle**
- **Usually improper**
- **May lead to incoherences in multidimensional case**
- **Have been generalized into reference priors** (Berger and Bernardo) by distinguishing between nuisance and interest parameters
Example: $x \sim \mathcal{N}(\mu, \sigma)$

- $\theta = (\mu, \sigma)$ unknown: $f(\theta) \propto 1/\sigma^2$

because

$$I(\theta) = \mathbb{E}_x \left[ \begin{pmatrix} \frac{1}{\sigma^2} & \frac{2(x - \mu)}{\sigma^3} \\ \frac{2(x - \mu)}{\sigma^3} & \frac{3(x - \mu)^2}{\sigma^4} - \frac{1}{\sigma^2} \end{pmatrix} \right] = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{pmatrix}$$

- $\theta = \mu, \sigma$ fixed: $f(\mu) \propto 1$

- $\theta = \sigma, \mu$ fixed: $f(\sigma) \propto 1/\sigma$

- $\mu$ and $\sigma$ a priori independent: $f(\theta) = f(\mu)f(\sigma) \propto 1/\sigma$
Hierarchical modelling

Consider a conjugate prior for $f(x|\theta)$: $f_1(\theta|\lambda)$

$f_1(\theta|\lambda)$ may be too restrictive and require specification of $\lambda$. $\lambda$ unknown $\rightarrow$ add a noninformative prior on $\lambda$:

$$
\begin{align*}
\lambda & \sim f_2(\lambda) \\
\theta|\lambda & \sim f_1(\theta|\lambda) \\
x|\theta & \sim f(x|\theta)
\end{align*}
$$

The prior on $\theta$ is then $f(\theta) = \int f_1(\theta|\lambda)f_2(\lambda) \, d\lambda$

- not conjugate anymore
- heavier tails (eg. Student distributions or Gaussian scale mixtures)
- Computationally flexible
Posterior distributions
Computing posterior distributions

Postersiors are not always tractable...

Observed data: $x = \{x_1, \ldots, x_N\}$ eg. a discretized signal

Hidden variables: $z = \{z_1, \ldots, z_M\}$. eg. a segmentation or a clean version of $x$

Add prior knowledge on $z$ but if the dependence structure in $z$ is too complex (eg an image), $f(z|x)$ can't be obtained analytically

Solution: "Approximate" the dependence structure

- Sampling methods (Gibbs sampler, MCMC)
- Approximations (Laplace, Variational Bayes, EP)
Conclusion

- Maximum likelihood for large training data. Risk of overfitting for small data set.

- Bayesian framework to incorporate prior information (e.g., temporal dynamics, spatial relationships) and prevent overfitting.

- MMSE and MAP provide point estimates that use prior information.

- For fully Bayesian treatment, use predictive distributions.

- If posterior distributions are not tractable, use sampling methods (e.g., MCMC) or approximate inference (e.g., Variational Bayes).
Main references
