Introduction to statistical methods in signal and image processing
Florence Forbes

To cite this version:
Florence Forbes. Introduction to statistical methods in signal and image processing. Doctoral. Peyresq, France. 2016. <cel-01423624>
Introduction to statistical methods in signal and image processing

Florence Forbes

florence.forbes@inria.fr

INRIA Mistis team & Lab. Jean Kuntzman
Université Grenoble Alpes
http://mistis.inrialpes.fr
A methodological framework for inverse problems

Linear Models: convolution (image restoration), projection (tomography), mixtures (source separation), Laplace and Fourier transform (NMR, MRI)

Inversion: instability, non-unicity or existence of the solution → Ill-posed problem

Regularization: add constraints/hypothesis on the seek solution

- Bayesian inference: \( p(z|x) \propto p(x|z)p(z) \)
- Penalized criterion minimization: \( F(z) = L(z, x) + \beta R(z) \)
Overview: Part 1- Introduction to Bayesian tools

- Introduction
- Statistical inference
  - Learning and decision
  - Maximum likelihood
- Bayesian set up
  - Prior, posterior, etc.
- Bayesian inference strategies
  - Point estimators
  - Fully Bayesian treatment
- Prior distributions
  - Conjugate priors and exponential family
  - Noninformative and Jeffreys’ priors
- Tractability of posteriors
Overview: Part 2- Probabilistic graphical models

- Directed graphs: Bayesian networks
- Conditional independence and Markov properties
- Undirected graphs: Markov random fields
- Inference and learning
- Illustration: image segmentation
Introduction
Illustration: Audio-Visual Scene Analysis

- Estimate the number of audio-visual objects
- Localize and track every object
- Determine auditory activity and visibility
Illustration: Audio-Visual Scene Analysis

- Estimate the number of audio-visual objects
- Localize and track every object
- Determine auditory activity and visibility
Illustration: Audio-Visual Scene Analysis

- Estimate the number of audio-visual objects
- Localize and track every object
- Determine auditory activity and visibility
Illustration: Audio-Visual Scene Analysis

- Estimate the number of audio-visual objects
- Localize and track every object
- Determine auditory activity and visibility
Observed Data

Right camera image:

Left camera image:

Left microphone signal:

Right microphone signal:
Visual Features Extraction

An image pair produces a set of visual observations
\[ f = \{ f_m \}_{m=1}^M \in \mathbb{R}^3; \]
\[ f = (u, v, d): u, v - \text{image coordinates, } d - \text{disparity} \]
Auditory Features Extraction

ITD = interaural time difference

An ITD detection algorithm [H. Christensen, 2007] produces for a 10ms interval of audio signals one auditory observation \( g_k \in \mathbb{R} \)
Audio-Visual Generative Model

3D visual observation space

\[ f = \{ f_m \}_{m=1}^{M} \]

3D object space

\[ \{ s_1, s_2, \ldots, s_N \} \]

1D auditory observation space

\[ g = \{ g_k \}_{k=1}^{K} \in \mathbb{R} \]
Why statistical modelling in Audio-Visual Scene Analysis?

- Observations are strongly affected by noise: detector errors, occlusions, reverberations, ambient sounds, can be accounted for with some probability distributions.

\[
P(\mathbf{f}_m \mid A_m = n; \mathbf{s}_n) = \mathcal{N}(\mathbf{f}_m; \mathcal{F}(\mathbf{s}_n), \Sigma_n);
\]

\[
P(g_k \mid B_k = n; \mathbf{s}_n) = \mathcal{N}(g_k; \mathcal{G}(\mathbf{s}_n), \Gamma_n);
\]

- Dynamically changing environment:
  can be accounted for with some prior knowledge, eg. on motion cues, trajectories are continuous, smooth, etc...

Florence Forbes
Introduction to statistical methods in signal and image processing
Statistical Model formulation

\[ s = \{ s_1, \ldots, s_n, \ldots, s_N \} \] are tying parameters

Simultaneous clustering in auditory and visual observation spaces

Model parameters: Determine \( N \) and \( s_1, \ldots, s_N \)

\[ \theta = \{ s_1, \ldots, s_N, \Sigma_1, \ldots, \Sigma_N, \Gamma_1, \ldots, \Gamma_N, \pi_1, \ldots, \pi_{N+1}, \lambda_1, \ldots, \lambda_{N+1} \} \]
Real Data Results

Meeting scenario

- Estimated speaker locations and their auditory activity for a quasi-stationary scene
- Error rates for auditory activity detection: ‘missed target’ = 0.16, ‘false alarm’ = 0.14
- Localization error: within 5cm
Real Data Results

Simple tracking scenario

- Simple dynamic scene - results on the previous frame are used to initialize the model for the next frame
- Error rates for auditory activity detection: ‘missed target’ = 0.13, ‘false alarm’ = 0.43
- Localization error: within 10cm
Real Data Results

Cocktail party scenario

- Complex dynamic scene - may fail!
- Explicit dynamic model is required!
Real Data Results

Cocktail party scenario

- Complex dynamic scene - may fail!
- Explicit dynamic model is required!
Illustration: MR Brain scan segmentation

Assign each voxel to a class (label) (among K classes)

Tissue segmentation (WM, GM, CSF)
- Masse Blanche (MB)
- Masse Grise (MG)
- Liquide Céphalo-Rachidien (LCR)

- Cortex 3D reconstruction

Structure segmentation
- Corne Frontale (LCR)
- Système Ventriculaire (LCR)
- Noyau Caudé (MG)
- Putamen (MG)
- Thalamus (MG)

- Useful for:
  - Distinguishing Cortex GM from Nuclei GM
  - Volumetric studies
  - ...

Florence Forbes
Introduction to statistical methods in signal and image processing
Tissue segmentation

- **Global** estimation of **Gaussian** intensity models for CSF, WM, GM.

- Intensity inhomogeneities are modeled by a unique bias field (e.g., Multiplicative) to be estimated.

Constrain with an atlas.
Statistical inference
Statistical inference

From a given set of observation \( x = (x_1, \ldots, x_N) \), learn a model that best describes the data

- **Probabilistic parametric model:**
  \( x = (x_1, \ldots, x_N) \) generated from a probability distribution \( f(x|\theta) \)

\[
x = (x_1, \ldots, x_N) \sim f(x|\theta)
\]

associated likelihood: \( l(\theta|x) = f(x|\theta) \) viewed as a function of \( \theta \)

- **Learning:** estimating \( \theta \)
  e.g. that maximizes \( l(\theta|x) \) (Maximum likelihood inference)
Decision

Once a model is learned, decide about:

- The occurrence of an "event",
- Classify,
- Or find the value of a variable, etc.

Example 1: Linear model

Assume $x = Kz + \epsilon$

$z =$ clean signal, $z \sim f(z|\theta)$

$\epsilon =$ noise, $\epsilon \sim f(\epsilon|\phi)$

$x =$ noisy observed signal

Goal: obtain an estimate for $z$ ($\hat{z}$)
Example 2: Classification

e.g. 2 groups of objects (people)

\[ \theta_1 \rightarrow f(x|\theta_1) \rightarrow x \in g_1 \]
\[ \theta_2 \rightarrow f(x|\theta_2) \rightarrow x \in g_2 \]

Training data: observations in \( g_1 \) and in \( g_2 \) \( \rightarrow \hat{\theta}_1, \hat{\theta}_2 \)

**Goal:** given \( x^{new} \), decide to which group it belongs

( i.e. compute \( p(g|x^{new}, \hat{\theta}_1, \hat{\theta}_2) \) )
Maximum likelihood estimation

- We observe $N$ realizations $x_1, \ldots, x_N$ of a variable $X$
- Decide on a parametric model for $X$: $f(x|\theta)$
- Estimate $\theta$ by maximizing $l(\theta|x)$ or $\log l(\theta|x)$

Example 1: Linear Gaussian model $z = Kx + \epsilon$ and $\epsilon \sim \mathcal{N}(\mu_\epsilon, \Sigma_\epsilon)$

$$\log f(z|\theta) = \log \mathcal{N}(Kx + \mu_\epsilon, \Sigma_\epsilon) \propto -(z - Kx - \mu_\epsilon)^T \Sigma_\epsilon^{-1} (z - Kx - \mu_\epsilon)$$

$$\hat{x}_{ML} = \arg\min_x (z - Kx - \mu_\epsilon)^T \Sigma_\epsilon^{-1} (z - Kx - \mu_\epsilon)$$

Normal equations: $(K^T \Sigma_\epsilon^{-1} K) \hat{x}_{ML} = K^T \Sigma_\epsilon^{-1} (z - \mu_\epsilon)$

- Least squares: $\mu_\epsilon = 0$ and $\Sigma_\epsilon = \sigma^2 \text{Id}$

$$\implies \hat{x}_{ML} = \arg\min_x ||z - Kx||_2^2 = \hat{x}_{LS}$$

- Weighted least squares: $\mu_\epsilon = 0$ and $\Sigma_\epsilon = \text{Diag}(\sigma_1^2, \ldots, \sigma_N^2)$

$$\implies \hat{x}_{ML} = \arg\min_x \sum_n \frac{(z_n - [Kx]_n)^2}{\sigma_n^2} = \hat{x}_{WLS}$$
Example 2: Man-Woman classification problem

5 subjects in each class were asked if they like football and statistics

<table>
<thead>
<tr>
<th></th>
<th>Women $g_1$</th>
<th>Men $g_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>football</td>
<td>1 1 0 0 0</td>
<td>1 1 1 1 1</td>
</tr>
<tr>
<td>statistics</td>
<td>1 0 1 0 1</td>
<td>0 1 0 0 1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Positive answers</th>
<th>Negative answers</th>
<th>Positive answers</th>
<th>Negative answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>football</td>
<td>2/5=0.4</td>
<td>3/5=0.6</td>
<td>5/5=1</td>
<td>0/5=0</td>
</tr>
<tr>
<td>statistics</td>
<td>3/5=0.6</td>
<td>2/5=0.4</td>
<td>2/5=0.4</td>
<td>3/5=0.6</td>
</tr>
</tbody>
</table>
Observations and notation

- $N = 10$ responses $x_n = \begin{bmatrix} x_{1n} \\ x_{2n} \end{bmatrix} \in \{0, 1\}^2$ (2 questions)

  $\mathbf{x} = \{x_1, \ldots, x_N\}$

- $N = 10$ group assignments $g_n \in \{\text{Woman, Man}\} (= \{1, 2\})$

  $\mathbf{g} = \{g_1, \ldots, g_N\}$

$x^{g_1} = \{x_n, g_n = 1\} = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$

$x^{g_2} = \{x_n, g_n = 2\} = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$
Model

- Independence: \( f(g) = \prod_{n=1}^{N} f(g_n) \)
- \( f(g_n = \text{woman}) = f(g_n = \text{man}) = 0.5 \)
- Conditional independence: \( f(x|g) = \prod_{n=1}^{N} f(x_n|g_n) \)
- Independence of the two questions: \( f(x_n|g_n) = f(x_{1n}|g_n) f(x_{2n}|g_n) \)
- \( \forall n = 1 \ldots N, i = \{1, 2\}, g = \{1, 2\} \), Independent Bernoulli distributions (\( \theta^g_i \in [0, 1] \)):

\[
f(x_{in}|g_n = g) = \begin{cases} 
\theta^g_i & \text{if } x_{in} = 1 \\
1 - \theta^g_i & \text{if } x_{in} = 0 \\
0 & \text{otherwise}
\end{cases}
\]

or equivalently \( f(x_{in}|g_n = g) = (\theta^g_i)^{x_{in}} (1 - \theta^g_i)^{1-x_{in}} \)

\( (\theta^g_i = 0.5 \rightarrow \text{the coin is not biased}) \)
Likelihood for each group

- Learning task: Estimate \((\theta_1^g, \theta_2^g)\) given \(x^g\) \((g = \text{Woman, Man})\)
- Likelihood function:
  \[
f(x^g | \theta^g) = \prod_{n, g_n=g} f(x_{1n} | g) f(x_{2n} | g)
\]
- Log-likelihood:
  \[
  \log f(x^g | \theta^g) = \sum_{n, g_n=g} \sum_{i=1,2} x_{in} \log \theta_i^g + (1 - x_{in}) \log(1 - \theta_i^g)
  \]
- Maximization: \(\theta_i^g = \frac{\sum_{n, g_n=g} x_{in}}{N_g}\) (mean, frequencies of positive answers)

\[
\theta^1 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}, \quad \theta^2 = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}
\]
Decision: Naive Bayes classifier

**Sum-rule:** \( P(B) = P(B, A) + P(B, A^c) \)

**Product-rule:** \( P(A, B) = P(B|A)P(A) \)

It follows Bayes’ theorem: \( P(A|B) = \frac{P(B|A)P(A)}{P(B)} \)

with normalization \( P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) \)

**Goal:** Classify a person with \( x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) ie. \( g = ?? \)

▶ **Bayes’ rule:** \( f(g|x) = \frac{f(x|g)f(g)}{f(x)} = \frac{f(x|g)f(g)}{\sum_{g'} f(g')f(x|g')} \)

▶ **Assuming** \( f(\text{woman}) = f(\text{man}) = 0.5, \)

\[
\begin{align*}
  f(\text{woman}|x) &= \frac{0.4 \times 0.6}{0.4 \times 0.6 + 1 \times 0.4} = 0.375 \\
  f(\text{man}|x) &= \frac{1 \times 0.4}{0.4 \times 0.6 + 1 \times 0.4} = 0.625 = 1 - 0.375
\end{align*}
\]
Decision: Naive Bayes classifier

Goal: classify a person with $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$f(\text{woman} | x) = \frac{0.6 \times 0.6}{0.6 \times 0.6 + 0 \times 0.4} = 1$$

$$f(\text{man} | x) = \frac{0 \times 0.4}{0.6 \times 0.6 + 0 \times 0.4} = 0$$

▶ Conclusion: if you don't like football and like statistics, you are almost surely a woman

▶ Overfitting effect to the small training set

▶ Priors over the parameters can avoid overfitting $\implies$ Bayesian framework
Bayesian set up
Bayesian concepts

- **Uncertainty on the parameters** $\theta$ of a model modeled through a probability distribution on $\theta$, called **prior distribution**
  The prior encoded the information available a priori, before observing $x$

- **Inference based on the distribution of** $\theta$ conditional on $x$, $f(\theta|x)$, called **posterior distribution**
Impact

- From unknown parameters to random
- **Actualization** of the information on $\theta$ by extracting the information on $\theta$ contained in the observations $x$
- Allows **incorporation of imperfect information** in the decision process
- Unique mathematical way to **condition upon the observations** (conditional perspective)
- **Penalization** factor
Three basic quantities in Bayesian inference

- Prior distribution \( f(\theta) \)
- Likelihood \( f(x|\theta) \)
- Posterior distribution \( f(\theta|x) \)

Forward generative model:

\[
f(\theta) \rightarrow \theta \rightarrow f(x|\theta) \rightarrow x
\]

→ involves the prior and the likelihood

Inference is an inversion problem:

\[
x \rightarrow f(\theta|x) \rightarrow \hat{\theta}
\]

→ involves the posterior distribution
Classification example

Assume a Beta prior over the Bernouilli parameters: $\theta_i^g \in [0, 1]$

$$f(\theta_i^g) = \text{Beta}(\alpha, \beta) = B(\alpha, \beta)^{-1} \theta_i^g^{\alpha-1} (1 - \theta_i^g)^{\beta-1}$$

$$E[\theta_i^g] = \alpha/(\alpha+\beta), \quad \text{Mode}[\theta_i^g] = (\alpha-1)/(\alpha+\beta-2)$$

Compute the posterior distribution of $\theta_i^g$:

$$f(\theta_i^g|x^g) = \frac{f(x^g|\theta_i^g)f(\theta_i^g)}{f(x^g)} \propto f(x^g|\theta_i^g)f(\theta_i^g)$$

$$\log f(\theta_i^g|x^g) = \text{cst} + (A_i^g - 1) \log \theta_i^g + (B_i^g - 1) \log (1 - \theta_i^g)$$

with $A_i^g = \sum_{n,g_n=g} x_{in} + \alpha$ and $B_i^g = \sum_{n,g_n=g} (1 - x_{in}) + \beta$

so that posterior distribution:

$$f(\theta_i^g|x^g) \propto \theta_i^g^{A_i^g-1} (1 - \theta_i^g)^{B_i^g-1}$$

$$= \text{Beta}(A_i^g, B_i^g)$$
Classification example

<table>
<thead>
<tr>
<th></th>
<th>Women $g_1$</th>
<th>Men $g_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>football</td>
<td>1 1 0 0 0</td>
<td>1 1 1 1 1</td>
</tr>
<tr>
<td>statistics</td>
<td>1 0 1 0 1</td>
<td>0 1 0 0 1</td>
</tr>
</tbody>
</table>

Example: $\alpha = \beta = 2 \implies A_1^1 = \alpha + 2 = 4, \quad B_1^1 = \beta + 3 = 5$

\[ \downarrow B(4, 5) \]
Bayesian inference strategies
Point estimators

**Goal:** provide an estimation of $\theta$

The two most common Bayesian estimators are:

- **Maximum a posteriori (MAP) estimator**

  $$\hat{\theta}_{MAP} = \arg\max_\theta f(\theta|x) = \arg\max_\theta f(x|\theta)f(\theta) = \arg\max_\theta \log f(x|\theta) + \log f(\theta)$$

  **Note:** if $f(\theta) = constant$ then $\hat{\theta}_{MAP} = \hat{\theta}_{MLE}$

- **Posterior Mean Estimator**

  $$\hat{\theta}_{PM} = E_\theta[\theta|x] = \int \theta f(\theta|x)d\theta$$

  **Note:** $f(\theta|x)$ requires the normalizing term $f(x) = \int f(x|\theta)f(\theta)d\theta$.

$\hat{\theta}_{MAP}$ usually easier to obtain, it involves optimization rather than integration.
The Bayesian Mean Square Error (MSE) is
\[
E_{\theta,X} [||\hat{\theta} - \theta||_2^2] = \int \int ||\hat{\theta}(x) - \theta||_2^2 f(\theta, x) \, d\theta \, dx
\]

Minimum Mean Square Error (MMSE) estimator:

**Definition:**
\[
\hat{\theta}_{MMSE} = \arg \min_{\hat{\theta}} E_{\theta,X} [||\hat{\theta} - \theta||_2^2]
\]

**Solution:**
\[
\hat{\theta}_{MMSE} = E_{\theta} [\theta | X] = \hat{\theta}_{PM}
\]

since
\[
E_{\theta,X} [||\hat{\theta} - \theta||_2^2] = E_{X} [E_{\theta} [||\hat{\theta} - \theta||_2^2 | X]]
\]
and \(E_{\theta} [||\hat{\theta} - \theta||_2^2 | X]\) is minimum when \(\hat{\theta} = E_{\theta} [\theta | X]\)
The MSE quadratic cost (loss) can be replaced by a 0-1 cost

\[ E_{\theta,X}[1 - \delta_{\theta}(\hat{\theta})] \]

where \( 1 - \delta_{\theta}(\hat{\theta}) = 0 \) if \( \hat{\theta} = \theta \) (no loss) and 1 otherwise (max loss)

\[
\min E_{\theta,X}[1 - \delta_{\theta}(\hat{\theta})] = \max E_X[E_{\theta}[\delta_{\theta}(\hat{\theta})|X]]
\]

and \( E_{\theta}[\delta_{\theta}(\hat{\theta})|X] = p(\theta = \hat{\theta}|X) \) which is max at the MAP
Linear Minimum MSE

Assume $E[\theta] = E[X] = 0$ and consider an estimator of the form $\hat{\theta} = A^T X$

**Goal:** find matrix $A$ that minimizes the Bayesian MSE

$$MSE(A) = E_{\theta,X}[||A^T X - \theta||^2]$$
$$= E_{\theta,X}[\text{trace}((A^T X - \theta)(A^T X - \theta)^T)]$$
$$= \text{trace}(E_{\theta,X}[(A^T X - \theta)(A^T X - \theta)^T])$$
$$= \text{trace}(E[\theta\theta^T] - A^T E[X\theta^T] - E[\theta X^T]A + A^T E[XX^T]A)$$
$$= \text{trace}(\Sigma_{\theta} - A^T \Sigma_{x\theta} - \Sigma_{\theta x} A + A^T \Sigma_{xx} A)$$

$$\frac{\partial}{\partial A}MSE(A) = -2\Sigma_{x\theta} + 2\Sigma_{xx} A = 0$$

$\hat{A} = \Sigma_{xx}^{-1}\Sigma_{x\theta}$ Wiener-Hopf equation

$\hat{\theta}_{LMMSE} = \Sigma_{\theta x}\Sigma_{xx}^{-1}X$ Wiener filter

$\hat{\theta}_{LMMSE} = \Sigma_{\theta x}\Sigma_{xx}^{-1}(X - \mu_x) + \mu_{\theta}$ in the non centered case
Linear Minimum MSE: Linear model example

Assume $x = K\theta + \epsilon$ where $\theta \sim \mathcal{N}(0, \sigma_\theta^2 I)$ and $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2 I)$ are independent.

Then $x \sim \mathcal{N}(0, \sigma_\theta^2 KK^T + \sigma_\epsilon^2 I)$ and

$$\begin{align*}
\Sigma_x &= \sigma_\theta^2 KK^T + \sigma_\epsilon^2 I \\
\Sigma_{\theta x} &= E[X\theta^T] = E[K\theta\theta^T + \epsilon\theta^T] = K\Sigma_{\theta} = \sigma_\theta^2 K
\end{align*}$$

$$\hat{\theta}_{LMMSE} = \sigma_\theta^2 K^T (\sigma_\theta^2 KK^T + \sigma_\epsilon^2 I)^{-1} X = K^T (KK^T + \frac{\sigma_\epsilon^2}{\sigma_\theta^2} I)^{-1} X$$

Note: when SNR increases, $\frac{\sigma_\epsilon^2}{\sigma_\theta^2} \to 0$, $\hat{\theta}_{LMMSE} \to \hat{\theta}_{MLE} = (KK^T)^{-1} K^T X$
Classification example

**MMSE estimator:** Since \( f(\theta_i^g|x^g) \) is a Beta distribution

\[
E[\theta_i^g|x^g] = \frac{\sum_{n,g_n=g} x_{in} + \alpha}{\alpha + \beta + N_g}
\]

With \( \alpha = \beta = 2 \) (mode and mean at 0.5), we get

\[
\theta^1 = \begin{bmatrix} 4/9 \\ 5/9 \end{bmatrix} \quad \text{and} \quad \theta^2 = \begin{bmatrix} 7/9 \\ 4/9 \end{bmatrix}
\]

Then for \( x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), it comes \( f(man|x) = 8/33 = 0.242 \)

**MAP estimator:** using the mode of the posterior we get instead :

\[
\theta^1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} \quad \text{and} \quad \theta^2 = \begin{bmatrix} 6/7 \\ 3/7 \end{bmatrix}
\]

and for \( x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), it comes \( f(man|x) = 3/19 = 0.158 \)
Predictive distributions

Use the full posterior rather than a point estimate.

Other distributions of interest are:

**Prior predictive (marginal):**

Before we observe the data, what do we expect the distribution of observations to be?

\[ f(x) = \int f(x|\theta) f(\theta) \, d\theta \]

- What we would predict for \( x \) given no data
- Useful for assessing whether choice of prior distribution does capture prior beliefs.
Posterior predictive

What is the predictive distribution of a new observation $x^{\text{new}}$ given the current data $x$?

$$
f(x^{\text{new}} | x) = \int f(x^{\text{new}}, \theta | x) \, d\theta
$$

$$
= \int f(x^{\text{new}} | \theta) f(\theta | x) \, d\theta
$$

Use the assumption that $x^{\text{new}}$ is independent of $x$ given $\theta$. 
Classification example

In each group, the posterior predictive is:

\[ f(x^{new}|x^g) = \int f(x^{new}|\theta^g) f(\theta^g|x^g) d\theta^g = f(x_1^{new}|x_1^g) f(x_2^{new}|x_2^g) \]

\[ f(x_i^{new}|x_i^g) = \int f(x_i^{new}|\theta_i^g) f(\theta_i^g|x_i^g) d\theta_i^g \]

\[ = \frac{B(x_i^{new} + A_i^g, 1 - x_i^{new} + B_i^g)}{B(A_i^g, B_i^g)} \]

Then using Bayes’ rule:

\[ f(g^{new}|x^{new}, x, g) \propto f(x^{new}|x^{g^{new}}) f(g^{new}) \]

\[ \propto f(x_1^{new}|x_1^{g^{new}}) f(x_2^{new}|x_2^{g^{new}}) \times 0.5 \]

For \( x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), it comes \( f(man|x) = \frac{8}{33} = 0.242 \)
Classification example: all results for \( f(man|x = [0, 1]^T) \)

| Estimator               | \( f(man|x) \) |
|-------------------------|-----------------|
| Maximum likelihood      | 0               |
| Bayesian MMSE           | 0.242           |
| Bayesian MAP            | 0.158           |
| Fully Bayesian          | 0.242           |
Prior distributions
From prior information to prior distributions

- All computations depend on the prior choice
- The prior is a tool summarizing available information as well as uncertainty related with this information
- The prior distribution is the key to Bayesian inference but the available prior information is usually not precise enough to lead to an exact determination

Different strategies are possible:

- Conjugate priors
- Noninformative priors
- Jeffreys prior
- Hierarchical modelling, etc.
Conjugate priors: a starting point

Specific parametric family with convenient analytical properties

**Definition:** A family \( \mathcal{F} \) of probability distributions on \( \theta \) is conjugate for a likelihood function \( f(x|\theta) \) if, for every \( \pi \in \mathcal{F} \), the posterior distribution \( f(\theta|x) \propto f(x|\theta)\pi(\theta) \) also belongs to \( \mathcal{F} \).

Main interest is when \( \mathcal{F} \) is parametric: computing the posterior distribution reduces then to an updating of the corresponding parameters of the prior.

- The prior "structure" on \( \theta \) is propagated to the posterior (actualisation)
- Tractability and simplicity
- First approximations to adequate priors
Conjugate priors: Gaussian case
Conjugate priors are usually associated with exponential families of distributions.

**Definition:** \( C, h \) are positive functions, \( R, T \) are functions in \( \mathbb{R}^k \)

The family of distributions

\[
f(x|\theta) = C(\theta)h(x) \exp(R(\theta)T(x))
\]

is called an exponential family of dimension \( k \).

When

\[
f(x|\theta) = C(\theta)h(x) \exp(\theta x) = h(x) \exp(\theta x - \Psi(\theta))
\]

the family is said to be natural.
Interesting analytical properties:

- Sufficient statistics of constant dimension exist
- Include common distributions (normal, binomial, Poisson, Wishart, etc.)
- Availability of the moments:
  \[ E_X[X|\theta] = \nabla \Psi(\theta), \quad \text{cov}(X_i, X_j) = \frac{\partial^2 \Psi}{\partial \theta_i \partial \theta_j}(\theta). \]
- Allow for conjugate priors
Conjugate distributions for exponential families

If \( f(x|\theta) = h(x) \exp(\theta x - \Psi(\theta)) \) then

\[
f(\theta|\mu, \lambda) = K(\mu, \lambda) \exp(\theta \mu - \lambda \Psi(\theta))
\]

where \( K(\mu, \lambda) \) is the normalizing constant, is conjugate for \( f(x|\theta) \).

The posterior is then \( f(\theta|\mu + x, \lambda + 1) \).

It follows an ”automatic” way to derive prior from \( f(x|\theta) \) BUT \( \mu, \lambda \) have still to be specified.
Linearity of the posterior mean

\[ f(x|\theta) \] in the natural exponential family: \[ f(x|\theta) = h(x) \exp(\theta x - \Psi(\theta)) \]

\[ E_X[X] = m(\theta) = \nabla \Psi(\theta) \]

\[ f(\theta) \] has a conjugate prior: \[ f(\theta) \propto \exp(\mu x - \lambda \Psi(\theta)) \]

\[ E_{\theta}[m(\theta)] = \int m(\theta)f(\theta)d\theta = \frac{\mu}{\lambda} \]

If \( x_1, \ldots x_N \) i.i.d \( f(x|\theta) \) then

\[ f(\theta|x_1, \ldots, x_N) \propto f(\theta|x_1)f(x_2|\theta) \ldots f(x_N|\theta) = f(\theta|\mu + \sum_{n=1}^{N} x_n, \lambda + N) \]

\[ E_{\theta}[m(\theta)|x_1, \ldots, x_n] = \frac{\mu + \sum_{n=1}^{N} x_n}{\lambda + N} \]
Common conjugate priors

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Distribution</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x</td>
<td>\theta)$</td>
<td>$f(\theta)$</td>
</tr>
<tr>
<td><strong>Normal</strong></td>
<td><strong>Normal</strong></td>
<td><strong>Normal</strong></td>
</tr>
<tr>
<td>$\mathcal{N}(\theta, \sigma^2)$</td>
<td>$\mathcal{N}(\mu, \tau^2)$</td>
<td>$\mathcal{N}\left(\frac{\sigma^2 \mu + \tau^2 x}{\sigma^2 + \tau^2}, \frac{1}{\sigma^2} + \frac{1}{\tau^2}\right)^{-1}$</td>
</tr>
<tr>
<td><strong>Poisson</strong></td>
<td><strong>Gamma</strong></td>
<td><strong>Gamma</strong></td>
</tr>
<tr>
<td>$\mathcal{P}(\theta)$</td>
<td>$\mathcal{G}(\alpha, \beta)$</td>
<td>$\mathcal{G}(\alpha + x, \beta + 1)$</td>
</tr>
<tr>
<td><strong>Gamma</strong></td>
<td><strong>Gamma</strong></td>
<td><strong>Gamma</strong></td>
</tr>
<tr>
<td>$\mathcal{G}(\nu, \theta)$</td>
<td>$\mathcal{G}(\alpha, \beta)$</td>
<td>$\mathcal{G}(\alpha + \nu, \beta + x)$</td>
</tr>
<tr>
<td><strong>Binomial</strong></td>
<td><strong>Beta</strong></td>
<td><strong>Beta</strong></td>
</tr>
<tr>
<td>$\text{Bin}(n, \theta)$</td>
<td>$\mathcal{B}(\alpha, \beta)$</td>
<td>$\mathcal{B}(\alpha + x, \beta + n - x)$</td>
</tr>
<tr>
<td><strong>Multinomial</strong></td>
<td><strong>Dirichlet</strong></td>
<td><strong>Dirichlet</strong></td>
</tr>
<tr>
<td>$\mathcal{M}(\theta_1, \ldots, \theta_K)$</td>
<td>$\mathcal{D}(\alpha_1, \ldots, \alpha_K)$</td>
<td>$\mathcal{D}(\alpha_1 + x_1, \ldots, \alpha_K + x_K)$</td>
</tr>
<tr>
<td><strong>Normal</strong></td>
<td><strong>Gamma</strong></td>
<td><strong>Gamma</strong></td>
</tr>
<tr>
<td>$\mathcal{N}(\mu, \frac{1}{\theta})$</td>
<td>$\mathcal{G}(\alpha, \beta)$</td>
<td>$\mathcal{G}(\alpha + 1/2, \beta + (x - \mu)^2/2)$</td>
</tr>
</tbody>
</table>
Non informative priors

How to encode absence of prior knowledge?

Is there such a thing as a default prior when prior information is missing?

In the absence of prior information, prior distributions solely derived from the sample distribution $f(x|\theta)$
Uniform priors (Laplace’s priors)

Equiprobability of elementary events: the same likelihood to each value of $\theta$

$$\theta \in \{\theta_1, \ldots, \theta_p\} \implies f(\theta_i) = \frac{1}{p}$$

Extensions to continuous spaces:

$$f(\theta) \propto 1 \quad (= \text{constant})$$

Examples:

Location parameters: $f(x|\theta) = f(x - \theta) \implies f(\theta) \propto 1$

Scale parameters: $f(x|\theta) = \frac{1}{\theta} f\left(\frac{x}{\theta}\right) \implies f(\theta) \propto \frac{1}{\theta} \quad (f(\log \theta) \propto 1)$
Some drawbacks

Lack of invariance through reparameterization: $\theta \rightarrow \eta = g(\theta)$

$$f(\theta) \propto 1 \rightarrow f(\eta) \propto \left| \frac{dg^{-1}(\eta)}{d\eta} \right| \neq \text{constant} \quad \text{(Jacobian formula)}$$

Information is not missing anymore !!

May generate improper posterior:

$$x \sim \mathcal{N}(\theta, \sigma^2) \quad \text{with} \quad f(\theta, \sigma^2) \propto 1$$

Then

$$f(\theta, \sigma^2|x) \propto f(x|\theta) \propto \sigma^{-1} \exp\left(\frac{(x - \theta^2)^2}{2\sigma^2}\right)$$

$$\Rightarrow f(\sigma^2|x) \propto 1 \quad \text{is improper, paradoxes occur}$$

$$\Rightarrow \text{Invariant priors}$$

$$\Rightarrow \text{Jeffreys’ priors as an alternative}$$
The Jeffreys’ priors

Based on Fisher information

Univariate case:

\[ I(\theta) = E_X \left[ \left( \frac{\partial \log f(X|\theta)}{\partial \theta} \right)^2 \right] = -E_X \left[ \frac{\partial^2 \log f(X|\theta)}{\partial \theta^2} \right] \]

Multivariate case:

\[ I(\theta)_{ij} = -E_X \left[ \frac{\partial^2 \log f(X|\theta)}{\partial \theta_i \partial \theta_j} \right] \]

The Jeffreys’ prior distribution is

\[ f(\theta) \propto |I(\theta)|^{1/2} \]

where \(|I(\theta)|\) is the determinant of the Fisher Information matrix

Exponential family: if \( f(x|\theta) = h(x) \exp(\theta x - \Psi(\theta)) \) then

\[ I(\theta) = \nabla^2 \Psi(\theta) \quad \text{and} \quad f(\theta) \propto \left( \frac{\partial^2 \Psi(\theta)}{\partial \theta_i^2} \right)^{1/2} \]
Key feature: Reparameterization invariance

Assume $f(\theta) \propto |I(\theta)|^{1/2}$ and $\eta = g(\theta)$ for a 1-to-1 mapping $g$

$$f(\eta) = f(\theta) \left| \frac{\partial \theta}{\partial \eta} \right| \propto \sqrt{|I(\theta)|} \left( \frac{\partial \theta}{\partial \eta} \right)^2$$

$$\propto \sqrt{E_X \left[ \left( \frac{\partial \log f(X|\theta)}{\partial \theta} \right)^2 \left( \frac{\partial \theta}{\partial \eta} \right)^2 \right]}$$

$$\propto \sqrt{E_X \left[ \left( \frac{\partial \log f(X|\theta)}{\partial \eta} \right)^2 \right]}$$

$$\propto |I(\eta)|^{1/2}$$
Other features

- **Information based**: $I(\theta)$ corresponds to the amount of information brought by the model on $\theta$.
  Noninformative: Minimize the effect of the prior which is in accordance with the model.
- Violates the likelihood principle
- Usually improper
- May lead to incoherences in multidimensional case
- Have been generalized into reference priors (Berger and Bernardo) by distinguishing between nuisance and interest parameters
Example: \( x \sim \mathcal{N}(\mu, \sigma) \)

- \( \theta = (\mu, \sigma) \) unknown: \( f(\theta) \propto 1/\sigma^2 \)

because

\[
I(\theta) = Ex \left[ \begin{pmatrix} \frac{1}{\sigma^2} & \frac{2(x - \mu)}{\sigma^3} \\ \frac{2(x - \mu)}{\sigma^3} & \frac{3(x - \mu)^2}{\sigma^4} - \frac{1}{\sigma^2} \end{pmatrix} \right] = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{pmatrix}
\]

- \( \theta = \mu, \sigma \) fixed: \( f(\mu) \propto 1 \)

- \( \theta = \sigma, \mu \) fixed: \( f(\sigma) \propto 1/\sigma \)

- \( \mu \) and \( \sigma \) a priori independent: \( f(\theta) = f(\mu)f(\sigma) \propto 1/\sigma \)
Hierarchical modelling

Consider a conjugate prior for $f(x|\theta)$: $f_1(\theta|\lambda)$

$f_1(\theta|\lambda)$ may be too restrictive and require specification of $\lambda$.

$\lambda$ unknown $\rightarrow$ add a noninformative prior on $\lambda$:

$$
\begin{align*}
\lambda & \sim f_2(\lambda) \\
\theta|\lambda & \sim f_1(\theta|\lambda) \\
x|\theta & \sim f(x|\theta)
\end{align*}
$$

The prior on $\theta$ is then $f(\theta) = \int f_1(\theta|\lambda)f_2(\lambda) \, d\lambda$

- not conjugate anymore
- heavier tails (e.g., Student distributions or Gaussian scale mixtures)
- Computationally flexible
Posterior distributions
Computing posterior distributions

Postiors are not always tractable...

Observed data: \( x = \{x_1, \ldots, x_N\} \) eg. a discretized signal

Hidden variables: \( z = \{z_1, \ldots, z_M\} \). eg. a segmentation or a clean version of \( x \)

Add prior knowledge on \( z \) but if the dependence structure in \( z \) is too complex (eg an image), \( f(z|x) \) can’t be obtained analytically

Solution: "Approximate" the dependence structure

- Sampling methods (Gibbs sampler, MCMC)
- Approximations (Laplace, Variational Bayes, EP)
Conclusion

- Maximum likelihood for large training data. Risk of overfitting for small data set.

- Bayesian framework to incorporate prior information (e.g., temporal dynamics, spatial relationships) and prevent overfitting.

- MMSE and MAP provide point estimates that use prior information.

- For fully Bayesian treatment, use predictive distributions.

- If posterior distributions are not tractable, use sampling methods (e.g., MCMC) or approximate inference (e.g., Variational Bayes).
Main references
