Introduction to statistical methods in signal and image processing
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Introduction to statistical methods in signal and image processing

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A methodological framework for inverse problems

Linear Models: convolution (image restoration), projection (tomography), mixtures (source separation), Laplace and Fourier transform (NMR, MRI)

Inversion: instability, non-unicity or existence of the solution

→ Ill-posed problem

Regularization: add constraints/hypothesis on the seek solution

- Bayesian inference: $p(z|x) \propto p(x|z)p(z)$
- Penalized criterion minimization: $F(z) = L(z, x) + \beta R(z)$
Overview: Part 1- Introduction to Bayesian tools

- Introduction
- Statistical inference
  - Learning and decision
  - Maximum likelihood
- Bayesian set up
  - Prior, posterior, etc.
- Bayesian inference strategies
  - Point estimators
  - Fully Bayesian treatment
- Prior distributions
  - Conjugate priors and exponential family
  - Noninformative and Jeffreys’ priors
- Tractability of posteriors
Overview: Part 2- Probabilistic graphical models

- Directed graphs: Bayesian networks
- Conditional independence and Markov properties
- Undirected graphs: Markov random fields
- Inference and learning
- Illustration: image segmentation
Introduction
Illustration: Audio-Visual Scene Analysis

- Estimate the number of audio-visual objects
- Localize and track every object
- Determine auditory activity and visibility
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Observed Data

Right camera image:

Left camera image:

Left microphone signal:

Right microphone signal:
Visual Features Extraction

An image pair produces a set of visual observations

\[ f = \{ f_m \}_{m=1}^{M} \in \mathbb{R}^3; \]

\[ f = (u, v, d): \ u, v \ - \text{image coordinates, } d \ - \text{disparity} \]
Auditory Features Extraction

ITD = interaural time difference

An ITD detection algorithm [H. Christensen, 2007] produces for a 10ms interval of audio signals one auditory observation $g_k \in \mathbb{R}$.
Audio-Visual Generative Model

3D visual observation space

\[ f = \{ f_m \}_{m=1}^{M} \]

3D object space

\{ s_1, s_2, \ldots, s_N \}

1D auditory observation space

\[ g = \{ g_k \}_{k=1}^{K} \in \mathbb{R} \]
Why statistical modelling in Audio-Visual Scene Analysis?

- Observations are strongly affected by noise: detector errors, occlusions, reverberations, ambient sounds, can be accounted for with some probability distributions.

\[
P(f_m | A_m = n; s_n) = \mathcal{N}(f_m; F(s_n), \Sigma_n);
\]

\[
P(g_k | B_k = n; s_n) = \mathcal{N}(g_k; G(s_n), \Gamma_n);
\]

- Dynamically changing environment:
  can be accounted for with some prior knowledge, eg. on motion cues, trajectories are continuous, smooth, etc...

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Statistical Model formulation

\[ s = \{s_1, \ldots, s_n, \ldots, s_N\} \text{ are tying parameters} \]

\[ \text{Simultaneous clustering in auditory and visual observation spaces} \]

\[ \theta = \{s_1, \ldots, s_N, \Sigma_1, \ldots, \Sigma_N, \Gamma_1, \ldots, \Gamma_N, \pi_1, \ldots, \pi_{N+1}, \lambda_1, \ldots, \lambda_{N+1}\} \]
Real Data Results

Meeting scenario

- Estimated speaker locations and their auditory activity for a quasi-stationary scene
- Error rates for auditory activity detection: ‘missed target’ = 0.16, ‘false alarm’ = 0.14
- Localization error: within 5cm
Real Data Results

Simple tracking scenario

- Simple dynamic scene - results on the previous frame are used to initialize the model for the next frame
- Error rates for auditory activity detection: ‘missed target’ = 0.13, ‘false alarm’ = 0.43
- Localization error: within 10cm
Real Data Results

Cocktail party scenario

- Complex dynamic scene - may fail!
- Explicit dynamic model is required!
Real Data Results

Cocktail party scenario

- Complex dynamic scene - may fail!
- Explicit dynamic model is required!
Assign each voxel to a class (label) (among K classes)

- Tissue segmentation (WM, GM, CSF)
  - Materie Blanche (MB)
  - Materie Grise (MG)
  - Liquide Céphalo-Rachidienn (LCR)

- Cortex 3D reconstruction

- Structure segmentation
  - Corne Frontale (LCR)
  - Noyau Caudé (MG)
  - Putamen (MG)
  - Système Ventriculaire (LCR)
  - Thalamus (MG)

- Useful for:
  - Distinguishing Cortex GM from Nuclei GM
  - Volumetric studies
  - …
Tissue segmentation

- **Global** estimation of **Gaussian** intensity models for CSF, WM, GM.

Intensity inhomogeneities are modeled by a unique bias field (e.g., Multiplicative) to be estimated.

Constrain with an atlas.
Statistical inference
Statistical inference

From a given set of observation $x = (x_1, \ldots, x_N)$, learn a model that best describes the data

- **Probabilistic parametric model:**
  
  $x = (x_1, \ldots, x_N)$ generated from a probability distribution $f(x|\theta)$

  $$x = (x_1, \ldots, x_N) \sim f(x|\theta)$$

  associated likelihood: $l(\theta|x) = f(x|\theta)$ viewed as a function of $\theta$

- **Learning:** estimating $\theta$
  
  e.g. that maximizes $l(\theta|x)$ (Maximum likelihood inference)
Decision

Once a model is learned, decide about:

- The occurrence of an "event",
- Classify,
- Or find the value of a variable, etc.

Example 1: Linear model

Assume \( x = Kz + \epsilon \)

\( z = \text{clean signal}, z \sim f(z|\theta) \)

\( \epsilon = \text{noise}, \epsilon \sim f(\epsilon|\phi) \)

\( x = \text{noisy observed signal} \)

Goal: obtain an estimate for \( z (\hat{z}) \)
Decision

Example 2: Classification

e.g. 2 groups of objects (people)

\[ \theta_1 \rightarrow f(x|\theta_1) \rightarrow x \in g_1 \]

\[ \theta_2 \rightarrow f(x|\theta_2) \rightarrow x \in g_2 \]

Training data: observations in \( g_1 \) and in \( g_2 \) \( \rightarrow \hat{\theta}_1, \hat{\theta}_2 \)

Goal: given \( x^{new} \), decide to which group it belongs

( i.e. compute \( p(g|x^{new}, \hat{\theta}_1, \hat{\theta}_2) \) )
Maximum likelihood estimation

- We observe $N$ realizations $x_1, \ldots, x_N$ of a variable $X$
- Decide on a parametric model for $X$: $f(x|\theta)$
- Estimate $\theta$ by maximizing $l(\theta|x)$ or $\log l(\theta|x)$

**Example 1: Linear Gaussian model** $z = Kx + \epsilon$ and $\epsilon \sim \mathcal{N}(\mu_\epsilon, \Sigma_\epsilon)$

$$
\log f(z|\theta) = \log \mathcal{N}(Kx + \mu_\epsilon, \Sigma_\epsilon) \propto -(z - Kx - \mu_\epsilon)^T \Sigma_\epsilon^{-1} (z - Kx - \mu_\epsilon)
$$

$$
\hat{x}_{ML} = \arg \min_x (z - Kx - \mu_\epsilon)^T \Sigma_\epsilon^{-1} (z - Kx - \mu_\epsilon)
$$

Normal equations: $(K^T \Sigma_\epsilon^{-1} K)\hat{x}_{ML} = K^T \Sigma_\epsilon^{-1} (z - \mu_\epsilon)$

- **Least squares:** $\mu_\epsilon = 0$ and $\Sigma_\epsilon = \sigma^2 \text{Id}$

$$
\implies \hat{x}_{ML} = \arg \min_x ||z - Kx||^2_2 = \hat{x}_{LS}
$$

- **Weighted least squares:** $\mu_\epsilon = 0$ and $\Sigma_\epsilon = \text{Diag}(\sigma_1^2, \ldots, \sigma_N^2)$

$$
\implies \hat{x}_{ML} = \arg \min_x \sum_n \frac{(z_n - [Kx]_n)^2}{\sigma_n^2} = \hat{x}_{WLS}
$$
Example 2: Man-Woman classification problem

5 subjects in each class were asked if they like football and statistics

<table>
<thead>
<tr>
<th></th>
<th>Women $g_1$</th>
<th>Men $g_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>football</td>
<td>1 1 0 0 0</td>
<td>1 1 1 1 1</td>
</tr>
<tr>
<td>statistics</td>
<td>1 0 1 0 1</td>
<td>0 1 0 0 1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ratio</th>
<th>Positive answers</th>
<th>Negative answers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Positive answers</td>
<td>Negative answers</td>
</tr>
<tr>
<td>football</td>
<td>2/5=0.4</td>
<td>3/5=0.6</td>
</tr>
<tr>
<td>statistics</td>
<td>3/5=0.6</td>
<td>2/5=0.4</td>
</tr>
<tr>
<td></td>
<td>5/5=1</td>
<td>0/5=0</td>
</tr>
<tr>
<td></td>
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<td>3/5=0.6</td>
</tr>
</tbody>
</table>
Observations and notation

- **$N = 10$ responses** $x_n = \begin{bmatrix} x_{1n} \\ x_{2n} \end{bmatrix} \in \{0, 1\}^2$ (2 questions)

$$x = \{x_1, \ldots, x_N\}$$

- **$N = 10$ group assignments** $g_n \in \{W	ext{oman, M}a\text{n}\} (= \{1, 2\})$

$$g = \{g_1, \ldots, g_N\}$$

$$x^{g_1} = \{x_n, g_n = 1\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$x^{g_2} = \{x_n, g_n = 2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$
Model

- Independence: \( f(g) = \prod_{n=1}^{N} f(g_n) \)

- \( f(g_n = \text{woman}) = f(g_n = \text{man}) = 0.5 \)

- Conditional independence: \( f(x|g) = \prod_{n=1}^{N} f(x_n|g_n) \)

- Independence of the two questions: \( f(x_n|g_n) = f(x_{1n}|g_n) f(x_{2n}|g_n) \)

- \( \forall n = 1 \ldots N, i = \{1, 2\}, g = \{1, 2\}, \) Independent Bernoulli distributions \( (\theta^g_i \in [0, 1]) \):

\[
f(x_{in}|g_n = g) = \begin{cases} 
\theta^g_i & \text{if } x_{in} = 1 \\
1 - \theta^g_i & \text{if } x_{in} = 0 \\
0 & \text{otherwise}
\end{cases}
\]

or equivalently \( f(x_{in}|g_n = g) = (\theta^g_i)^{x_{in}} (1 - \theta^g_i)^{1-x_{in}} \)

\( (\theta^g_i = 0.5 \rightarrow \text{the coin is not biased}) \)
Likelihood for each group

- Learning task: Estimate $(\theta^g_1, \theta^g_2)$ given $x^g$ ($g = Woman, Man$)
- Likelihood function:
  \[
  f(x^g|\theta^g) = \prod_{n, g_n=g} f(x_{1n}|g) f(x_{2n}|g)
  \]
- Log-likelihood:
  \[
  \log f(x^g|\theta^g) = \sum_{n, g_n=g} \sum_{i=1,2} x_{in} \log \theta^g_i + (1 - x_{in}) \log(1 - \theta^g_i)
  \]
- Maximization: $\theta^g_i = \frac{\sum_{n, g_n=g} x_{in}}{N_g}$ (mean, frequencies of positive answers)

\[
\theta^1 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}, \quad \theta^2 = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}
\]
Decision: Naive Bayes classifier

**Sum-rule:** \( P(B) = P(B, A) + P(B, A^c) \)

**Product-rule:** \( P(A, B) = P(B|A)P(A) \)

It follows Bayes’ theorem: \( P(A|B) = \frac{P(B|A)P(A)}{P(B)} \)

with normalization \( P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) \)

**Goal:** Classify a person with \( x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) ie. \( g = ?? \)

- **Bayes’ rule:** \( f(g|x) = \frac{f(x|g)f(g)}{f(x)} = \frac{f(x|g)f(g)}{\sum_{g'} f(g')f(x|g')} \)

- **Assuming** \( f(\text{woman}) = f(\text{man}) = 0.5, \)

\[
\begin{align*}
\quad f(\text{woman}|x) & = \frac{0.4 \times 0.6}{0.4 \times 0.6 + 1 \times 0.4} = 0.375 \\
\quad f(\text{man}|x) & = \frac{1 \times 0.4}{0.4 \times 0.6 + 1 \times 0.4} = 0.625 = 1 - 0.375
\end{align*}
\]
Decision: Naive Bayes classifier

Goal: classify a person with \( x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

\[
f(\text{woman}|x) = \frac{0.6 \times 0.6}{0.6 \times 0.6 + 0 \times 0.4} = 1
\]

\[
f(\text{man}|x) = \frac{0 \times 0.4}{0.6 \times 0.6 + 0 \times 0.4} = 0
\]

- Conclusion: if you don't like football and like statistics, you are almost surely a woman
- Overfitting effect to the small training set
- Priors over the parameters can avoid overfitting \( \Rightarrow \) Bayesian framework
Bayesian set up
Bayesian concepts

- **Uncertainty on the parameters** $\theta$ of a model modeled through a probability distribution on $\theta$, called **prior distribution**
  - The prior encoded the information available a priori, before observing $x$

- **Inference based on the distribution of** $\theta$ conditional on $x$, $f(\theta|x)$, called **posterior distribution**
Impact

- From unknown parameters to random
- **Actualization** of the information on $\theta$ by extracting the information on $\theta$ contained in the observations $x$
- Allows incorporation of imperfect information in the decision process
- Unique mathematical way to **condition upon the observations** (conditional perspective)
- **Penalization** factor
Three basic quantities in Bayesian inference

- Prior distribution $f(\theta)$
- Likelihood $f(x|\theta)$
- Posterior distribution $f(\theta|x)$

**Forward generative model:**

\[
\begin{align*}
f(\theta) &\quad \rightarrow \quad \theta \quad \rightarrow \quad f(x|\theta) \quad \rightarrow \quad x \\
\end{align*}
\]

\(\rightarrow \) involves the prior and the likelihood

**Inference is an inversion problem:**

\[
\begin{align*}
x &\quad \rightarrow \quad f(\theta|x) \quad \rightarrow \quad \hat{\theta} \\
\end{align*}
\]

\(\rightarrow \) involves the posterior distribution
Classification example

Assume a **Beta prior** over the Bernouilli parameters: $\theta_i^g \in [0, 1]$

$$f(\theta_i^g) = \text{Beta}(\alpha, \beta) = \mathcal{B}(\alpha, \beta)^{-1} \theta_i^g^{\alpha-1} (1 - \theta_i^g)^{\beta-1}$$

$$E[\theta_i^g] = \frac{\alpha}{\alpha + \beta}, \quad \text{Mode}[\theta_i^g] = \frac{(\alpha - 1)}{(\alpha + \beta - 2)}$$

Compute the posterior distribution of $\theta_i^g$:

$$f(\theta_i^g | x^g) = \frac{f(x^g | \theta_i^g) f(\theta_i^g)}{f(x^g)} \propto f(x^g | \theta_i^g) f(\theta_i^g)$$

$$\log f(\theta_i^g | x^g) = \text{cst} + (A_i^g - 1) \log \theta_i^g + (B_i^g - 1) \log(1 - \theta_i^g)$$

with $A_i^g = \sum_{n, g_n = g} x_{in} + \alpha$ and $B_i^g = \sum_{n, g_n = g} (1 - x_{in}) + \beta$

so that **posterior distribution**:

$$f(\theta_i^g | x^g) \propto \theta_i^g^{A_i^g-1} (1 - \theta_i^g)^{B_i^g-1}$$

$$= \text{Beta}(A_i^g, B_i^g)$$
Classification example

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Example: $\alpha = \beta = 2 \implies A_1 = \alpha + 2 = 4, \quad B_1 = \beta + 3 = 5$

$\downarrow B(4, 5)$

$\leftarrow B(2, 2)$
Bayesian inference strategies
Point estimators

**Goal:** provide an estimation of $\theta$

The two most common Bayesian estimators are:

- **Maximum a posteriori (MAP) estimator**

  \[
  \hat{\theta}_{MAP} = \arg \max_{\theta} f(\theta|x) = \arg \max_{\theta} f(x|\theta) f(\theta) = \arg \max_{\theta} \log f(x|\theta) + \log f(\theta)
  \]

  **Note:** if $f(\theta) = constant$ then $\hat{\theta}_{MAP} = \hat{\theta}_{MLE}$

- **Posterior Mean Estimator**

  \[
  \hat{\theta}_{PM} = E_{\theta}[\theta|x] = \int \theta f(\theta|x)d\theta
  \]

  **Note:** $f(\theta|x)$ requires the normalizing term $f(x) = \int f(x|\theta) f(\theta)d\theta$.

$\hat{\theta}_{MAP}$ usually easier to obtain, it involves optimization rather than integration.
Posterior mean and Bayesian MSE

The Bayesian Mean Square Error (MSE) is

\[ E_{\theta, X} [||\hat{\theta} - \theta||_2^2] = \int \int ||\hat{\theta}(x) - \theta||_2^2 f(\theta, x) \, d\theta \, dx \]

Minimum Mean Square Error (MMSE) estimator:

**Definition:** \( \hat{\theta}_{MMSE} = \arg \min_{\hat{\theta}} E_{\theta, X} [||\hat{\theta} - \theta||_2^2] \)

**Solution:** \( \hat{\theta}_{MMSE} = E_{\theta}[\theta|X] = \hat{\theta}_{PM} \)

since \( E_{\theta, X} [||\hat{\theta} - \theta||_2^2] = E_X[E_{\theta}[||\hat{\theta} - \theta||_2^2|X]] \)

and \( E_{\theta} [||\hat{\theta} - \theta||_2^2|X] \) is minimum when \( \hat{\theta} = E_{\theta}[\theta|X] \)
The MSE quadratic cost (loss) can be replaced by a 0-1 cost

\[ E_{\theta,X}[1 - \delta_{\theta}(\hat{\theta})] \]

where \( 1 - \delta_{\theta}(\hat{\theta}) = 0 \) if \( \hat{\theta} = \theta \) (no loss) and 1 otherwise (max loss)

\[
\min E_{\theta,X}[1 - \delta_{\theta}(\hat{\theta})] = \max E_X[E_{\theta}[\delta_{\theta}(\hat{\theta})|X]]
\]

and \( E_{\theta}[\delta_{\theta}(\hat{\theta})|X] = p(\theta = \hat{\theta}|X) \) which is max at the MAP
Linear Minimum MSE

Assume $E[\theta] = E[X] = 0$ and consider an estimator of the form $\hat{\theta} = A^T X$

**Goal:** find matrix $A$ that minimizes the Bayesian MSE

\[
MSE(A) = E_{\theta,X}[||A^T X - \theta||^2_2]
\]
\[
= E_{\theta,X}[\text{trace}((A^T X - \theta)(A^T X - \theta)^T)]
\]
\[
= \text{trace}(E_{\theta,X}[(A^T X - \theta)(A^T X - \theta)^T])
\]
\[
= \text{trace}(E[\theta \theta^T] - A^T E[X \theta^T] - E[\theta X^T]A + A^T E[X X^T]A)
\]
\[
= \text{trace}(\Sigma_{\theta} - A^T \Sigma_{x \theta} - \Sigma_{\theta x} A + A^T \Sigma_{x} A)
\]

\[
\frac{\partial}{\partial A} MSE(A) = -2\Sigma_{x \theta} + 2\Sigma_{x} A = 0
\]
\[
\hat{A} = \Sigma_{x}^{-1} \Sigma_{x \theta} \quad \text{Wiener-Hopf equation}
\]

\[
\hat{\theta}_{LMMSE} = \Sigma_{\theta x} \Sigma_{x}^{-1} X \quad \text{Wiener filter}
\]

\[
(\hat{\theta}_{LMMSE} = \Sigma_{\theta x} \Sigma_{x}^{-1} (X - \mu_x) + \mu_{\theta} \quad \text{in the non centered case})
\]
Linear Minimum MSE: Linear model example

Assume $x = K\theta + \epsilon$ where $\theta \sim \mathcal{N}(0, \sigma^2_\theta I)$ and $\epsilon \sim \mathcal{N}(0, \sigma^2_\epsilon I)$ are independent

Then $x \sim \mathcal{N}(0, \sigma^2_\theta KK^T + \sigma^2_\epsilon I)$ and

$$
\begin{align*}
\Sigma_x &= \sigma^2_\theta KK^T + \sigma^2_\epsilon I \\
\Sigma_{\theta x} &= E[\theta x^T] = E[K\theta \theta^T + \epsilon \theta^T] = K\Sigma_\theta = \sigma^2_\theta K
\end{align*}
$$

$$
\hat{\theta}_{LMMSE} = \sigma^2_\theta K^T (\sigma^2_\theta KK^T + \sigma^2_\epsilon I)^{-1} X = K^T (KK^T + \frac{\sigma^2_\epsilon}{\sigma^2_\theta} I)^{-1} X
$$

Note: when SNR increases, $\frac{\sigma^2_\epsilon}{\sigma^2_\theta} \rightarrow 0$, $\hat{\theta}_{LMMSE} \rightarrow \hat{\theta}_{MLE} = (KK^T)^{-1} K^T X$
Classification example

**MMSE estimator:** Since \(f(\theta_i^g|x^g)\) is a Beta distribution

\[
E[\theta_i^g|x^g] = \frac{\sum_{n,g} x_{in} + \alpha}{\alpha + \beta + N_g}
\]

With \(\alpha = \beta = 2\) (mode and mean at 0.5), we get

\[
\theta^1 = \begin{bmatrix}
4/9 \\
5/9
\end{bmatrix}
\text{ and } \theta^2 = \begin{bmatrix}
7/9 \\
4/9
\end{bmatrix}
\]

Then for \(x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\), it comes \(f(\text{man}|x) = 8/33 = 0.242\)

**MAP estimator:** using the mode of the posterior we get instead :

\[
\theta^1 = \begin{bmatrix}
3/7 \\
4/7
\end{bmatrix}
\text{ and } \theta^2 = \begin{bmatrix}
6/7 \\
3/7
\end{bmatrix}
\]

and for \(x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\), it comes \(f(\text{man}|x) = 3/19 = 0.158\)
Predictive distributions

Use the full posterior rather than a point estimate.

Other distributions of interest are:

**Prior predictive (marginal):**

Before we observe the data, what do we expect the distribution of observations to be?

\[ f(x) = \int f(x|\theta) f(\theta) \, d\theta \]

- What we would predict for \( x \) given no data
- Useful for assessing whether choice of prior distribution does capture prior beliefs.
Predictive distributions

**Posterior predictive**

What is the predictive distribution of a new observation $x^{new}$ given the current data $x$?

\[
f(x^{new}|x) = \int f(x^{new}, \theta|x) \, d\theta
\]

\[
= \int f(x^{new}|\theta) f(\theta|x) \, d\theta
\]

Use the assumption that $x^{new}$ is independent of $x$ given $\theta$. 
Classification example

In each group, the posterior predictive is:

\[ f(x^{new}|x^g) = \int f(x^{new}|\theta^g) f(\theta^g|x^g) d\theta^g = f(x^{new}_{1}|x^g_1)f(x^{new}_{2}|x^g_2) \]

\[ f(x^{new}_i|x^g_i) = \int f(x^{new}_i|\theta^g_i) f(\theta^g_i|x^g_i) d\theta^g_i \]

\[ = \frac{B(x^{new}_i + A^g_i, 1 - x^{new}_i + B^g_i)}{B(A^g_i, B^g_i)} \]

Then using Bayes’ rule:

\[ f(g^{new}|x^{new}, x, g) \propto f(x^{new}|x^{gnew}) f(g^{new}) \]

\[ \propto f(x^{new}_1|x^{gnew}_1)f(x^{new}_2|x^{gnew}_2) \times 0.5 \]

For \( x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), it comes \( f(man|x) = 8/33 = 0.242 \)
Classification example: all results for $f(man|x = [0, 1]^T)$

| Estimator           | $f(man|x)$ |
|---------------------|------------|
| Maximum likelihood  | 0          |
| Bayesian MMSE       | 0.242      |
| Bayesian MAP        | 0.158      |
| Fully Bayesian      | 0.242      |
Prior distributions
From prior information to prior distributions

- All computations depend on the prior choice
- The prior is a tool summarizing available information as well as uncertainty related with this information
- The prior distribution is the key to Bayesian inference but the available prior information is usually not precise enough to lead to an exact determination

Different strategies are possible:

- Conjugate priors
- Noninformative priors
- Jeffreys prior
- Hierarchical modelling, etc.
Conjugate priors: a starting point

Specific parametric family with convenient analytical properties

**Definition:** A family $\mathcal{F}$ of probability distributions on $\theta$ is conjugate for a likelihood function $f(x|\theta)$ if, for every $\pi \in \mathcal{F}$, the posterior distribution $f(\theta|x) \propto f(x|\theta)\pi(\theta)$ also belongs to $\mathcal{F}$.

Main interest is when $\mathcal{F}$ is parametric: computing the posterior distribution reduces then to an updating of the corresponding parameters of the prior.

- The prior "structure" on $\theta$ is propagated to the posterior (actualisation)
- Tractability and simplicity
- First approximations to adequate priors
Conjugate priors: Gaussian case
Conjugate priors are usually associated with exponential families of distributions.

**Definition:** $C, h$ are positive functions, $R, T$ are functions in $\mathbb{R}^k$

The family of distributions

$$f(x|\theta) = C(\theta)h(x)\exp(R(\theta)T(x))$$

is called an exponential family of dimension $k$.

When

$$f(x|\theta) = C(\theta)h(x)\exp(\theta x) = h(x)\exp(\theta x - \Psi(\theta))$$

the family is said to be natural.
Exponential families

Interesting analytical properties:

- Sufficient statistics of constant dimension exist
- Include common distributions (normal, binomial, Poisson, Wishart, etc.)
- Availability of the moments:
  \[ E_X[X|\theta] = \nabla \Psi(\theta), \quad \text{cov}(X_i, X_j) = \frac{\partial^2 \Psi}{\partial \theta_i \partial \theta_j}(\theta). \]
- Allow for conjugate priors
Conjugate distributions for exponential families

If \( f(x|\theta) = h(x) \exp(\theta x - \Psi(\theta)) \) then

\[
f(\theta|\mu, \lambda) = K(\mu, \lambda) \exp(\theta \mu - \lambda \Psi(\theta))
\]

where \( K(\mu, \lambda) \) is the normalizing constant, is **conjugate** for \( f(x|\theta) \).

The posterior is then \( f(\theta|\mu + x, \lambda + 1) \).

It follows an "automatic" way to derive prior from \( f(x|\theta) \) BUT \( \mu, \lambda \) have still to be specified.
Linearity of the posterior mean

\( f(x|\theta) \) in the natural exponential family: \( f(x|\theta) = h(x) \exp(\theta x - \Psi(\theta)) \)

\[
E_X[X] = m(\theta) = \nabla \Psi(\theta)
\]

\( f(\theta) \) has a conjugate prior: \( f(\theta) \propto \exp(\mu x - \lambda \Psi(\theta)) \)

\[
E_\theta[m(\theta)] = \int m(\theta)f(\theta)d\theta = \frac{\mu}{\lambda}
\]

If \( x_1, \ldots, x_N \) i.i.d \( f(x|\theta) \) then

\[
f(\theta|x_1, \ldots, x_N) \propto f(\theta|x_1)f(x_2|\theta) \ldots f(x_N|\theta) = f(\theta|\mu + \sum_{n=1}^{N} x_n, \lambda + N)
\]

.\[
E_\theta[m(\theta)|x_1, \ldots, x_n] = \frac{\mu + \sum_{n=1}^{N} x_n}{\lambda + N}
\]
## Common conjugate priors

| $f(x|\theta)$ | $f(\theta)$ | $f(\theta|x)$ |
|---------------|-------------|---------------|
| Normal $N(\theta, \sigma^2)$ | Normal $N(\mu, \tau^2)$ | Normal $N\left(\frac{\sigma^2 \mu + \tau^2 x}{\sigma^2 + \tau^2}, \left(\frac{1}{\sigma^2} + \frac{1}{\tau^2}\right)^{-1}\right)$ |
| Poisson $P(\theta)$ | Gamma $\mathcal{G}(\alpha, \beta)$ | Gamma $\mathcal{G}(\alpha + x, \beta + 1)$ |
| Gamma $\mathcal{G}(\nu, \theta)$ | Gamma $\mathcal{G}(\alpha, \beta)$ | Gamma $\mathcal{G}(\alpha + \nu, \beta + x)$ |
| Binomial $Bin(n, \theta)$ | Beta $\mathcal{B}(\alpha, \beta)$ | Beta $\mathcal{B}(\alpha + x, \beta + n - x)$ |
| Multinomial $\mathcal{M}(\theta_1, \ldots, \theta_K)$ | Dirichlet $\mathcal{D}(\alpha_1, \ldots, \alpha_K)$ | Dirichlet $\mathcal{D}(\alpha_1 + x_1, \ldots, \alpha_K + x_K)$ |
| Normal $\mathcal{N}(\mu, \frac{1}{\theta})$ | Gamma $\mathcal{G}(\alpha, \beta)$ | Gamma $\mathcal{G}(\alpha + 1/2, \beta + (x - \mu)^2/2)$ |
Non informative priors

How to encode absence of prior knowledge?

Is there such a thing as a default prior when prior information is missing?

In the absence of prior information, prior distributions solely derived from the sample distribution $f(x|\theta)$
Uniform priors (Laplace’s priors)

Equiprobability of elementary events: the same likelihood to each value of $\theta$

$$\theta \in \{\theta_1, \ldots, \theta_p\} \longrightarrow f(\theta_i) = \frac{1}{p}$$

Extensions to continuous spaces:

$$f(\theta) \propto 1 \quad (= \text{constant})$$

Examples:

Location parameters: \( f(x|\theta) = f(x - \theta) \longrightarrow f(\theta) \propto 1 \)

Scale parameters: \( f(x|\theta) = \frac{1}{\theta} f\left(\frac{x}{\theta}\right) \longrightarrow f(\theta) \propto \frac{1}{\theta} \quad (f(\log \theta) \propto 1) \)
Some drawbacks

Lack of invariance through reparameterization: \( \theta \rightarrow \eta = g(\theta) \)

\[
f(\theta) \propto 1 \rightarrow f(\eta) \propto \left| \frac{dg^{-1}(\eta)}{d\eta} \right| \neq \text{constant} \quad \text{(Jacobian formula)}
\]

Information is not missing anymore !!

May generate improper posterior:

\[ x \sim \mathcal{N}(\theta, \sigma^2) \quad \text{with } f(\theta, \sigma^2) \propto 1 \]

Then

\[
f(\theta, \sigma^2 | x) \propto f(x | \theta) \propto \sigma^{-1} \exp\left(\frac{(x - \theta^2)^2}{2\sigma^2}\right)
\]

\[
\implies f(\sigma^2 | x) \propto 1 \quad \text{is improper, paradoxes occur}
\]

\[\implies \text{Invariant priors} \]

\[\implies \text{Jeffreys’ priors as an alternative} \]
The Jeffreys’ priors

Based on Fisher information

Univariate case:

\[ I(\theta) = E_X \left[ \left( \frac{\partial \log f(X|\theta)}{\partial \theta} \right)^2 \right] = -E_X \left[ \frac{\partial^2 \log f(X|\theta)}{\partial \theta^2} \right] \]

Multivariate case:

\[ I(\theta)_{ij} = -E_X \left[ \frac{\partial^2 \log f(X|\theta)}{\partial \theta_i \partial \theta_j} \right] \]

The Jeffreys’ prior distribution is \( f(\theta) \propto |I(\theta)|^{1/2} \)

where \(|I(\theta)|\) is the determinant of the Fisher Information matrix

Exponential family: if \( f(x|\theta) = h(x) \exp(\theta x - \Psi(\theta)) \) then

\[ I(\theta) = \nabla^2 \Psi(\theta) \text{ and } f(\theta) \propto \left( \frac{\partial^2 \Psi(\theta)}{\partial \theta_i^2} \right)^{1/2} \]
Key feature: Reparameterization invariance

Assume $f(\theta) \propto |I(\theta)|^{1/2}$ and $\eta = g(\theta)$ for a 1-to-1 mapping $g$

\[
f(\eta) = f(\theta) \left| \frac{\partial \theta}{\partial \eta} \right| \propto \sqrt{|I(\theta)| \left( \frac{\partial \theta}{\partial \eta} \right)^2}
\]

\[
\propto \sqrt{E_X \left[ \left( \frac{\partial \log f(X|\theta)}{\partial \theta} \right)^2 \left( \frac{\partial \theta}{\partial \eta} \right)^2 \right]}
\]

\[
\propto \sqrt{E_X \left[ \left( \frac{\partial \log f(X|\theta)}{\partial \eta} \right)^2 \right]}
\]

\[
\propto |I(\eta)|^{1/2}
\]
Other features

- **Information based:** $I(\theta)$ corresponds to the amount of information brought by the model on $\theta$.
- Noninformative: Minimize the effect of the prior which is in accordance with the model.
- Violates the likelihood principle
- Usually improper
- May lead to incoherences in multidimensional case
- Have been generalized into reference priors (Berger and Bernardo) by distinguishing between nuisance and interest parameters
Example: \( x \sim \mathcal{N}(\mu, \sigma) \)

- \( \theta = (\mu, \sigma) \) unknown: \( f(\theta) \propto 1/\sigma^2 \)

because

\[
I(\theta) = E_X \left[ \begin{pmatrix} \frac{1}{\sigma^2} & \frac{2(x - \mu)}{\sigma^3} \\ \frac{2(x - \mu)}{\sigma^3} & \frac{3(x - \mu)^2}{\sigma^4} - \frac{1}{\sigma^2} \end{pmatrix} \right]
\]

\[
= \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{pmatrix}
\]

- \( \theta = \mu, \sigma \) fixed: \( f(\mu) \propto 1 \)

- \( \theta = \sigma, \mu \) fixed: \( f(\sigma) \propto 1/\sigma \)

- \( \mu \) and \( \sigma \) a priori independent: \( f(\theta) = f(\mu)f(\sigma) \propto 1/\sigma \)
Hierarchical modelling

Consider a conjugate prior for $f(x|\theta)$: $f_1(\theta|\lambda)$

$f_1(\theta|\lambda)$ may be too restrictive and require specification of $\lambda$.

$\lambda$ unknown $\rightarrow$ add a noninformative prior on $\lambda$:

$$
\begin{align*}
\lambda & \sim f_2(\lambda) \\
\theta|\lambda & \sim f_1(\theta|\lambda) \\
x|\theta & \sim f(x|\theta)
\end{align*}
$$

The prior on $\theta$ is then $f(\theta) = \int f_1(\theta|\lambda) f_2(\lambda) \, d\lambda$

- not conjugate anymore
- heavier tails (eg. Student distributions or Gaussian scale mixtures)
- Computationally flexible
Posterior distributions
Computing posterior distributions

Postersiors are not always tractable...

Observed data: \( x = \{x_1, \ldots, x_N\} \) eg. a discretized signal

Hidden variables: \( z = \{z_1, \ldots, z_M\} \). eg. a segmentation or a clean version of \( x \)

Add prior knowledge on \( z \) but if the dependence structure in \( z \) is too complex (eg an image), \( f(z|x) \) can't be obtained analytically

Solution: "Approximate" the dependence structure

- Sampling methods (Gibbs sampler, MCMC)
- Approximations (Laplace, Variational Bayes, EP)
Conclusion

- Maximum likelihood for large training data. Risk of overfitting for small data set.

- Bayesian framework to incorporate prior information (e.g., temporal dynamics, spatial relationships) and prevent overfitting.

- MMSE and MAP provide point estimates that use prior information.

- For fully Bayesian treatment, use predictive distributions.

- If posterior distributions are not tractable, use sampling methods (e.g., MCMC) or approximate inference (e.g., Variational Bayes).
Main references
