Computer Algebra for Lattice Path Combinatorics
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Computer Algebra for Lattice Path Combinatorics

The 74th Séminaire Lotharingien de Combinatoire
Ellwangen, March 23–25, 2015
Overview

1. Monday: General presentation
2. Tuesday: Guess’n’Prove
3. Wednesday: Creative telescoping
Part I: **General presentation**
Let $\mathcal{S}$ be a subset of $\mathbb{Z}^d$ (step set, or model) and $p_0 \in \mathbb{Z}^d$ (starting point).

Example: $\mathcal{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}$, $p_0 = (0,0)$
Let $\mathcal{S}$ be a subset of $\mathbb{Z}^d$ (step set, or model) and $p_0 \in \mathbb{Z}^d$ (starting point).

A path (walk) of length $n$ starting at $p_0$ is a sequence $(p_0, p_1, \ldots, p_n)$ of elements in $\mathbb{Z}^d$ such that $p_{i+1} - p_i \in \mathcal{S}$ for all $i$.

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Let $C$ be a cone of $\mathbb{R}^d$ (if $x \in C$ and $r \geq 0$ then $r \cdot x \in C$).

Example: $\mathcal{S} = \{(1, 0), (-1, 0), (1, -1), (-1, 1)\}$, $p_0 = (0, 0)$ and $C = \mathbb{R}_+^2$
General context: lattice paths confined to cones

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Example: $\mathcal{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}$, $p_0 = (0,0)$ and $C = \mathbb{R}^2_+$

Questions

- What is the number $a(n)$ of $n$-step walks contained in $C$?
- For $i \in C$, what is the number $a(n;i)$ of such walks that end at $i$?
- What about their generating series $A(t)$, resp. $A(t;x)$?
Why count walks in cones?

Many discrete objects can be encoded in that way:

- discrete mathematics (permutations, trees, words, urns, …)
- statistical physics (Ising model, …)
- probability theory (branching processes, games of chance, …)
- operations research (queueing theory, …)
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- operations research (queueing theory, …)
An old topic: The ballot problem and the reflection principle

**Ballot problem** [Bertrand, 1887]

Suppose that candidates $A$ and $B$ are running in an election. If $a$ votes are cast for $A$ and $b$ votes are cast for $B$, where $a > b$, then the probability that $A$ stays ahead of $B$ throughout the counting of the ballots is $(a - b)/(a + b)$.

**Lattice path reformulation:** given positive integers $a, b$ with $a > b$, find the number of Dyck paths starting at the origin and consisting of $a$ upsteps $\nearrow$ and $b$ downsteps $\searrow$ such that no step ends on the $x$-axis.

**Reflection principle:** Dyck paths from $(1, 1)$ to $T(a + b, a - b)$ that touch the $x$-axis $\equiv$ Dyck paths from $(1, -1)$ to $T$

Answer: good paths = paths from $(1, 1)$ to $T$ that never touch the $x$-axis

\[
\binom{a + b - 1}{a - 1} - \binom{a + b - 1}{b - 1} = \frac{a - b}{a + b} \binom{a + b}{a}
\]
An old topic: Pólya’s “promenade au hasard” / “Irrfahrt”

Motto: Drunkard: “Will I ever, ever get home again?”
Pólya (1921): “You can’t miss; just keep going and stay out of 3D!”

(Adam and Delbruck, 1968)

[Próya, 1921] The simple random walk on $\mathbb{Z}^d$ is recurrent in dimensions $d = 1, 2$ (“Alle Wege führen nach Rom”), and transient in dimension $d \geq 3$.

Über eine Aufgabe der Wahrscheinlichkeitsrechnung
betreffend die Irrfahrt im Straßennetz.
Still a topical issue

Many recent contributions:


etc.
Still a topical issue

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etc.

Specific question

Ad hoc solution

Systematic approach
Personal bias: Experimental Mathematics using Computer Algebra
Problem 6

A flea starts at (0, 0) on the infinite two-dimensional integer lattice and executes a biased random walk: At each step it hops north or south with probability 1/4, east with probability 1/4 + \(\epsilon\), and west with probability 1/4 − \(\epsilon\). The probability that the flea returns to (0, 0) sometime during its wanderings is 1/2. What is \(\epsilon\)?

**Computer algebra conjectures and proves**

\[
p(\epsilon) = 1 - \sqrt{\frac{A}{2}} \cdot _2F_1 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \right | \frac{2\sqrt{1 - 16\epsilon^2}}{A} \right)^{-1}, \text{ with } A = 1 + 8\epsilon^2 + \sqrt{1 - 16\epsilon^2}.
\]
A (very) basic cone: the full space

Rational series

If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and $C = \mathbb{R}^d$, then $A(t; x)$ is rational:

$$a(n) = |\mathcal{S}|^n \iff A(t) = \sum_{n \geq 0} a(n) t^n = \frac{1}{1 - |\mathcal{S}| t}$$

More generally:

$$A(t; x) = \frac{1}{1 - t \sum_{s \in \mathcal{S}} x^s}.$$
Also well-known: a (rational) half-space

Algebraic series [Bousquet-Mélou-Petkovšek 00]

If $S \subset \mathbb{Z}^d$ is finite and $C$ is a rational half-space, then $A(t; x)$ is algebraic, given by an explicit system of polynomial equations.
The “next” case: intersection of two half-spaces

$$f(i, j; n) = \begin{cases} 0 & \text{if } i < 0 \text{ or } j < 0 \text{ or } n < 0, \\ \sum_{i' j'} \in S} f(i - i', j - j'; n - 1) & \text{otherwise.} \end{cases}$$
The “next” case: intersection of two half-spaces

\[ (i, j) = (5, 1) \approx \]
From now on: we focus on nearest-neighbor walks in the quarter plane, i.e. walks in $\mathbb{N}^2$ starting at $(0, 0)$ and using steps in a prefixed subset $\mathcal{S}$ of 

$$\{\searrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \swarrow, \downarrow\}.$$ 

Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathcal{S} =$ 

\[ \begin{array}{c}
\searrow \\
\leftarrow \\
\nwarrow \\
\uparrow \\
\nearrow \\
\rightarrow \\
\swarrow \\
\downarrow \\
\end{array} \]
Lattice walks with small steps in the quarter plane

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$f_{n;i,j} =$ number of walks of length $n$ ending at $(i,j)$.
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$$f_{n; i, j} = \text{number of walks of length } n \text{ ending at } (i, j).$$ 

$$f_n = \sum_{i, j \geq 0} f_{n; i, j} = \text{number of total walks with length } n.$$
Generating series and combinatorial problems

- Complete generating series:

\[ F(t; x, y) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{n,i,j} x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]]. \]
Generating series and combinatorial problems

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▶ Special, combinatorially meaningful specializations:

- \( F(t; 0, 0) \) counts excursions;
- \( F(t; 1, 1) = \sum_{n \geq 0} f_n t^n \) counts walks with prescribed length;
- \( F(t; 1, 0) \) counts walks ending on the horizontal axis.
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Combinatorial questions: Given \( \mathcal{G} \), what can be said about \( F(t; x, y) \), resp. \( f_{n;i,j} \), and their variants?

- Properties of \( F \): algebraic? transcendental? D-finite?
- Explicit form: of \( F \) of \( f \)?
- Asymptotics of \( f \)?
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Our goal: Use computer algebra to give computational answers.
Small-step walks of interest

From the $2^8$ step sets $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0,0)\}$, some are:

- trivial,
- simple,
- intrinsic to the half plane,
- symmetrical.

One is left with 79 interesting distinct models.
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One is left with **79 interesting distinct models.**
The 79 models

Non-singular

Singular
The 79 models

Non-singular

Singular
Two important models: **Kreweras** and **Gessel** walks

\[ \mathcal{S} = \{\downarrow, \leftarrow, \nearrow\} \quad F_{\mathcal{S}}(t; x, y) \equiv K(t; x, y) \]

\[ \mathcal{S} = \{\uparrow, \searrow, \leftarrow, \rightarrow\} \quad F_{\mathcal{S}}(t; x, y) \equiv G(t; x, y) \]

Example: A Kreweras excursion.
“Special” models

Dyck:

Motzkin:

Pólya:

Kreweras:

Gessel:

Gouyou-Beauchamps:

King:

Exercise:
Important classes of univariate power series

- **Algebraic**: \( S(t) \in \mathbb{Q}\left[\left[ t \right] \right] \), root of a polynomial \( P(t, S(t)) = 0 \).
- **D-finite**: \( S(t) \in \mathbb{Q}\left[\left[ t \right] \right] \) satisfying a linear differential equation with polynomial coefficients.
- **Hypergeometric**: \( S(t) = \sum_{n=0}^{\infty} s_n t^n \) such that \( s_{n+1}s_n \in \mathbb{Q}(n) \).

E.g., \( _2F_1(a \mid \mid b \mid \mid c \mid \mid t) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} t^n n! \)
Important classes of univariate power series

D-finite series

Algebraic: \( S(t) \in \mathbb{Q}[[t]] \) root of a polynomial \( P \in \mathbb{Q}[t,T] \), i.e., \( P(t, S(t)) = 0 \).
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**D-finite:** $S(t) \in \mathbb{Q}[[t]]$ satisfying a linear differential equation with polynomial coefficients $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$.

**Hypergeometric:** $S(t) = \sum_{n=0}^{\infty} s_n t^n$ such that $s_{n+1}s_n \in \mathbb{Q}(n)$. E.g., $2F1(a\ b\ c\ ||\ t) = \sum_{n=0}^{\infty} \frac{(a)_n(a)_n(b)_n}{(c)_n n!} t^n$, $(a)_n = a(a+1)\cdots(a+n-1)$. 

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$$2F_1\left(\begin{array}{c}a & b \\ c \end{array} \right| t\right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a + 1) \cdots (a + n - 1).$$
Important classes of multivariate power series

D-finite series

algebraic series

$S \in \mathbb{Q}[[x,y,t]]$ is algebraic if it is the root of a $P \in \mathbb{Q}[x,y,t,T]$. 
Important classes of multivariate power series

$S \in \mathbb{Q}[[x, y, t]]$ is **algebraic** if it is the root of a $P \in \mathbb{Q}[x, y, t, T]$.

$S \in \mathbb{Q}[[x, y, t]]$ is **D-finite** if the set of all partial derivatives of $S$ spans a finite-dimensional vector space over $\mathbb{Q}(x, y, t)$.
Main results (I): algebraicity of Gessel walks

**Theorem** [Kreweras 1965; 100 pages combinatorial proof!]

\[
K(t; 0, 0) = 3F_2\left( \begin{array}{ccc} 1/3 & 2/3 & 1 \\ 3/2 & 2 & 27 t^3 \end{array} \right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.
\]

**Theorem** [Gessel’s conjecture; Kauers, Koutschan & Zeilberger 2009]

\[
G(t; 0, 0) = 3F_2\left( \begin{array}{ccc} 5/6 & 1/2 & 1 \\ 5/3 & 2 \end{array} \left| 16 t^2 \right. \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{5^n}{2^n} \frac{1}{2^n} \frac{5^n}{2^n} \frac{2^n}{4^n} \frac{2^n}{2^n} \frac{t^{2n}}{t^{2n}}.
\]

**Question:** What about \(K(t; x, y)\) and \(G(t; x, y)\)?

**Theorem** [Gessel 1986, Bousquet-Mélou 2005]

\(K(t; x, y)\) is algebraic.

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\(G(t; x, y)\) had been conjectured to be non-D-finite.

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**Theorem [Gessel 1986, Bousquet-Mélou 2005]** $K(t; x, y)$ is algebraic.

- $G(t; x, y)$ had been conjectured to be non-D-finite.

**Theorem [B. & Kauers 2010]** $G(t; x, y)$ is D-finite, even algebraic.

- Fresh news: recent human proof [B., Kurkova & Raschel 2015].
Main results (I): algebraicity of Gessel walks

Theorem [Kreweras 1965; 100 pages combinatorial proof!]

\[K(t; 0, 0) = \left[ \begin{array}{ccc} 1/3 & 2/3 & 1 \\ 3/2 & 2 & 0 \end{array} \right] 27 t^3 = \sum_{n=0}^{\infty} \frac{4^n (3^n)}{(n+1)(2n+1)} t^{3n}.\]

Theorem [Gessel’s conjecture; Kauers, Koutschan & Zeilberger 2009]

\[G(t; 0, 0) = \left[ \begin{array}{ccc} 5/6 & 1/2 & 1 \\ 5/3 & 2 & 0 \end{array} \right] 16 t^2 = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n (5/3)_n (2)_n}{(4t)^{2n}}.\]

Question: What about \(K(t; x, y)\) and \(G(t; x, y)\)?

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Theorem [B. & Kauers 2010] \(G(t; x, y)\) is D-finite, even algebraic.

- Guess’n’Prove method, using Hermite-Padé approximants

\[\rightarrow \ \text{Tuesday}\]
Main results (II): Explicit form for $G(t; x, y)$

**Theorem [B., Kauers & van Hoeij 2010]**

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \cdots$ be a root of

$$(V - 1)(1 + 3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2 + 83)t^6 + \cdots$ be a root of

$$x(V - 1)(V + 1)U^3 - 2V(3x + 5xV - 8Vt)U^2$$

$$-xV(V^2 - 24V - 9)U + 2V^2(xV - 9x - 8Vt) = 0,$$

let $W = t^2 + (y + 8)t^4 + 2(y^2 + 8y + 41)t^6 + \cdots$ be a root of

$$y(1 - V)W^3 + y(V + 3)W^2 - (V + 3)W + V - 1 = 0.$$

Then $G(t; x, y)$ is equal to

$$\frac{64(U(V+1)-2V) V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{i(y+1)(1-W)(W^2y+1)^2} = \frac{1}{tx(y+1)}.$$

▷ Computer-driven discovery and proof; no human proof yet.
Main results (II): Explicit form for $G(t; x, y)$

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\[
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- Computer-driven discovery and proof; no human proof yet.
- Proof uses guessed minimal polynomials for $G(t; x, 0) & G(t; 0, y)$
Main results (III): Conjectured D-Finite $F(t; 1, 1)$ [B. & Kauers 2009]

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Equation sizes = \{order, degree\}@\{algeq, diffeq\)

- Computerized discovery by enumeration + Hermite–Padé
- 23: Confirmed by a human proof in [B., Kurkova & Raschel 2015]
Main results (III): Conjectured D-Finite $F(t; 1, 1)$ \[B. \& Kauers 2009\]

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$A = 1 + \sqrt{2}$, $B = 1 + \sqrt{3}$, $C = 1 + \sqrt{6}$, $\lambda = 7 + 3\sqrt{6}$, $\mu = \sqrt{\frac{4\sqrt{6} - 1}{19}}$

▶ Computerized discovery by enumeration + Hermite–Padé + LLL/PSLQ.
The group of a model: the simple walk case

The characteristic polynomial \( \chi_{\mathcal{G}} := x + \frac{1}{x} + y + \frac{1}{y} \)
The characteristic polynomial $\chi_\mathcal{G} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x, y) = \left(x, \frac{1}{y}\right), \quad \phi(x, y) = \left(\frac{1}{x}, y\right),$$
The group of a model: the simple walk case

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$$\psi(x,y) = \left(x, \frac{1}{y}\right), \quad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

and thus under any element of the group

$$\langle \psi, \phi \rangle = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$
The group of a model: the general case

The polynomial \( \chi_\mathcal{G} := \sum_{(i,j) \in \mathcal{G}} x^i y^j = \sum_{i=-1}^{1} B_i(y)x^i = \sum_{j=-1}^{1} A_j(x)y^j \)
The group of a model: the general case

The polynomial \( \chi_S := \sum_{(i,j) \in S} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j \) is left invariant under

\[
\psi(x, y) = \left( x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y} \right), \quad \phi(x, y) = \left( \frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y \right),
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\[
\mathcal{G}_{\mathcal{G}} := \langle \psi, \phi \rangle.
\]
Examples of groups

Order 4,
Examples of groups

Order 4, order 6,
Examples of groups

Order 4,

order 6,

order 8,
Examples of groups

Order 4, order 6, order 8, order $\infty$. 

Alin Bostan  
Computer Algebra for Lattice Path Combinatorics
An important object: the orbit sum (OS)

The orbit sum of a model $\mathcal{G}$ is the following polynomial in $\mathbb{Q}[x, x^{-1}, y, y^{-1}]$:

$$\text{OrbitSum}(\mathcal{G}) := \sum_{\theta \in \mathcal{G}} (-1)^{\theta} \theta(xy)$$

► E.g., for the simple walk:

$$\text{OS} = x \cdot y - \frac{1}{x} \cdot y + \frac{1}{x} \cdot \frac{1}{y} - x \cdot \frac{1}{y}$$

► For 4 models, the orbit sum is zero:

E.g. for the Kreheras model:

$$\text{OS} = x \cdot y - \frac{1}{xy} \cdot y + \frac{1}{xy} \cdot x - y \cdot x + y \cdot \frac{1}{xy} - x \cdot \frac{1}{xy} = 0$$
The 79 models: finite and infinite groups

79 models
The 79 models: finite and infinite groups

79 models

23 admit a finite group
[Mishna’07]

56 have an infinite group
[Bousquet-Mélou & Mishna’10]
The 79 models: finite and infinite groups

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all \( F(t; x, y) \) D-finite

19 transcendental (OS \( \neq \) 0)
[Gessel & Zeilberger’92]
[Bousquet-Mélou’02]

4 algebraic (OS = 0)
(3 Kreweras-type + Gessel)
[BMM’10] + [B. & Kauers’10]

79 models
The 79 models: finite and infinite groups

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- all $F(t; x, y)$ D-finite

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  \hspace{1em} (OS $\neq 0$)
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  4 algebraic \hspace{1em} (OS $= 0$)
  \hspace{1em} (3 Kreweras-type + Gessel)
  \hspace{1em} [BMM’10] + [B. & Kauers’10]

→ all non-D-finite

- [Mishna & Rechnitzer’07] and
  [Melczer & Mishna’13] for 5 singular models

- [Kurkova & Raschel’13] and
  [B., Raschel & Salvy’13] for all others
The 23 models with a finite group

(i) 16 with a **vertical symmetry**, and group isomorphic to $D_2$

(ii) 5 with a **diagonal or anti-diagonal symmetry**, and group isomorphic to $D_3$

(iii) 2 with group isomorphic to $D_4$

(i): vertical symmetry; (ii)+(iii): zero drift $\sum_{s \in \mathcal{S}} s$

In red, models with $\text{OS} = 0$ and algebraic GF
Main results (IV): explicit expressions for the 19 D-finite transcendental models

**Theorem [B., Chyzak, van Hoeij, Kauers & Pech 2015]**

Let $\mathcal{G}$ be one of the 19 models with finite group $\mathcal{G}_{\mathcal{G}}$, and non-zero orbit sum. Then $F$ is expressible using iterated integrals of $\,_2F_1$ expressions.
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**Example** (King walks in the quarter plane, A025595)

\[
F(t;1,1) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot _2F_1 \left( \frac{3}{2}, \frac{3}{2} \mid \frac{16x(1+x)}{(1+4x)^2} \right) dx
\]

\[
= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots
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- Computer-driven discovery and proof; no human proof yet.
- Proof uses **creative telescoping, ODE factorization, ODE solving**.

Wednesday
Hypergeometric Series Occurring in Explicit Expressions for $F(t; 1, 1)$

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</tr>
<tr>
<td>$7$ $2F_1\left(\frac{1}{2}, \frac{1}{2} \mid \frac{1}{1} \right) w$</td>
<td>$2F_1\left(\frac{1}{2}, \frac{3}{2} \mid \frac{3}{2} \right) w$</td>
<td>$\frac{16t^2}{4t^2+1}$</td>
<td>$16$ $2F_1\left(\frac{7}{4}, \frac{9}{4} \mid \frac{3}{4} \right) w$</td>
<td>$2F_1\left(\frac{9}{4}, \frac{11}{4} \mid \frac{3}{4} \right) w$</td>
<td>$\frac{64t^3(1+t)}{(1-4t^2)^2}$</td>
</tr>
<tr>
<td>$8$ $2F_1\left(\frac{5}{4}, \frac{7}{4} \mid \frac{3}{4} \right) w$</td>
<td>$2F_1\left(\frac{7}{4}, \frac{9}{4} \mid \frac{3}{4} \right) w$</td>
<td>$\frac{64t^3(2t+1)}{(8t^2-1)^2}$</td>
<td>$19$ $2F_1\left(-\frac{1}{2}, \frac{1}{2} \mid \frac{1}{2} \right) w$</td>
<td>$2F_1\left(\frac{1}{2}, \frac{1}{2} \mid \frac{1}{2} \right) w$</td>
<td>$16t^2$</td>
</tr>
<tr>
<td>$9$ $2F_1\left(\frac{7}{4}, \frac{9}{4} \mid \frac{3}{4} \right) w$</td>
<td>$2F_1\left(\frac{7}{4}, \frac{9}{4} \mid \frac{3}{4} \right) w$</td>
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<td></td>
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</tr>
</tbody>
</table>
Main results (V): non-D-finiteness in models with an infinite group

Theorem [B., Rachel & Salvy 2013]
Let $\mathcal{S}$ be one of the 51 non-singular models with infinite group $\mathcal{G}_\mathcal{S}$. Then $F_{\mathcal{S}}(t;0,0)$, and in particular $F_{\mathcal{S}}(t;x,y)$, are non-D-finite.
Main results (V): non-D-finiteness in models with an infinite group

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Algorithmic proof. Uses Gröbner basis computations, polynomial factorization, cyclotomy testing.
Based on two ingredients: asymptotics + irrationality.

[Kurkova & Raschel 2013] Human proof that $F_{\mathcal{S}}(t; x, y)$ is non-D-finite.
No human proof yet for $F_{\mathcal{S}}(t; 0, 0)$ non-D-finite.
The 56 models with infinite group

In blue, non-singular models, solved by [B., Raschel & Salvy 2013]
In red, singular models, solved by [Melczer & Mishna 2013]
Example: the scarecrows

[B., Raschel & Salvy 2013]: $F_{\mathcal{G}}(t;0,0)$ is not D-finite for the models

\[
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\end{array}
\]

For the 1st and the 3rd, the excursions sequence $[t^n] F_{\mathcal{G}}(t;0,0)$

\[
1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots
\]

is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396 \ldots$

The irrationality of $\alpha$ prevents $F_{\mathcal{G}}(t;0,0)$ from being D-finite.
Summary: Classification of 2D non-singular walks

The Main Theorem  Let $\mathcal{G}$ be one of the 74 non-singular models. The following assertions are equivalent:

1. The full generating series $F_{\mathcal{G}}(t; x, y)$ is D-finite
2. the excursions generating series $F_{\mathcal{G}}(t; 0, 0)$ is D-finite
3. the excursions sequence $[t^n] F_{\mathcal{G}}(t; 0, 0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$
4. the group $\mathcal{G}$ is finite (and $|\mathcal{G}| = 2 \cdot \min \{ \ell \in \mathbb{N}^* \mid \frac{\ell}{\alpha+1} \in \mathbb{Z} \}$)
5. the step set $\mathcal{G}$ has either an axial symmetry, or zero drift and cardinal different from 5.
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4. the group $G_{\mathcal{S}}$ is finite (and $|G_{\mathcal{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^* \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)
5. the step set $\mathcal{S}$ has either an axial symmetry, or zero drift and cardinal different from 5.

Moreover, under (1)–(5), $F_{\mathcal{S}}(t; x, y)$ is algebraic if and only if the model $\mathcal{S}$ has positive covariance
\[
\sum_{(i,j) \in \mathcal{S}} ij - \sum_{(i,j) \in \mathcal{S}} i \cdot \sum_{(i,j) \in \mathcal{S}} j > 0,
\]
and iff it has $OS = 0$. 
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Moreover, under (1)–(5), $F_{\mathcal{S}}(t; x, y)$ is algebraic if and only if the model $\mathcal{S}$ has positive covariance $\sum_{(i,j) \in \mathcal{S}} ij - \sum_{(i,j) \in \mathcal{S}} i \cdot \sum_{(i,j) \in \mathcal{S}} j > 0$, and iff it has $\text{OS} = 0$.

In this case, $F_{\mathcal{S}}(t; x, y)$ is expressible using nested radicals. If not, $F_{\mathcal{S}}(t; x, y)$ is expressible using iterated integrals of $\binom{2}{1}$ expressions.
Main methods

(1) for proving algebraicity / D-finiteness
   (1a) Guess’n’Prove
   (1b) Creative telescoping

(2) for proving non-D-finiteness
   (2a) Infinite number of singularities, or lacunary
   (2b) Asymptotics
Main methods

(1) for proving algebraicity / D-finiteness
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   Hermite-Padé approximants
   Diagonals of rational functions

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   (2a) Infinite number of singularities, or lacunary
   (2b) Asymptotics

► All methods are algorithmic.
Summary: Walks with unit steps in $\mathbb{N}^2$

- Quadrant models: 79
  - $|G| < \infty$: 23
    - Nonzero orbit sum: 19
      - Kernel method + CT: D-finite
    - Zero orbit sum: 4
      - Guess'n'Prove: algebraic
  - $|G| = \infty$: 56
    - Asymptotics + GB
      - Not D-finite
11,074,225 distinct interesting models

3D octant models with \( \leq 6 \) steps: 20,804

- \( |G| < \infty \): 170
  - Orbit sum \( \neq 0 \): 108
    - Kernel method: D-finite
  - Orbit sum = 0: 62
    - 2D-reducible: 43
    - Not 2D-reducible: 19
    - Not D-finite?

- \( |G| = \infty ? \): 20,634

[B., Bousquet-Mélou, Kauers, Melczer 2015]

- Open question: some non-D-finite models with a finite group?
The 19 mysterious 3D-models
The 19 mysterious 3D-models
Extensions: Walks in $\mathbb{N}^2$ with longer steps

- Define (and use) a group $G$ for models with larger steps?

- **Example:** When $G = \{(0, 1), (1, -1), (-2, -1)\}$, there is an underlying group that is finite and

  \[
  xyF(t; x, y) = [x^0, y^0] \frac{(x - 2x^{-2})(y - (x - x^{-2})y^{-1})}{1 - t(xy^{-1} + y + x^{-2}y^{-1})}
  \]

  [B., Bousquet-Mélou & Melczer, in progress]

**Current status:**

- 680 models with one large step, 643 proved non D-finite, 32 of 37 have differential equations guessed.

- 5910 models with two large steps, 5754 proved non D-finite, 69 of 156 have differential equations guessed.


On 3-dimensional lattice walks confined to the positive octant, with M. Bousquet-Mélou, M. Kauers and S. Melczer. Annals of Comb., 2015.

A human proof of Gessel’s lattice path conjecture, with I. Kurkova, K. Raschel, 2015.

Explicit Differentiably Finite Generating Functions of Walks with Small Steps in the Quarter Plane, with F. Chyzak, M. van Hoeij, M. Kauers and L. Pech, 2015.
An exercise involving the model

Let $\mathcal{S} = \{N, W, SE\}$. A $\mathcal{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathcal{S}$. Show that, for any integer $n$, the following quantities are equal:

(i) the number of $\mathcal{S}$-walks of length $n$ confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$;

(ii) the number of $\mathcal{S}$-walks of length $n$ confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal $x = y$. 

For instance, for $n = 3$, this common value is 3:

(i) $(0, 0) \rightarrow (-1, 0) \rightarrow (-1, 1) \rightarrow (0, 0)$, $(0, 0) \rightarrow (0, 1) \rightarrow (-1, 1) \rightarrow (0, 0)$, and $(0, 0) \rightarrow (0, 1) \rightarrow (1, 0) \rightarrow (0, 0)$, i.e., $W–N–SE, N–W–SE, N–SE–W$.

(ii) $(0, 0) \rightarrow (0, 1) \rightarrow (1, 0) \rightarrow (0, 0)$, $(0, 0) \rightarrow (0, 1) \rightarrow (0, 2) \rightarrow (1, 1)$, and $(0, 0) \rightarrow (0, 1) \rightarrow (1, 0) \rightarrow (1, 1)$, i.e., $N–SE–W, N–N–SE, N–SE–N$. 

Alin Bostan

Computer Algebra for Lattice Path Combinatorics
Let $\mathcal{S} = \{\text{N, W, SE}\}$. A $\mathcal{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathcal{S}$. Show that, for any integer $n$, the following quantities are equal:

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For instance, for $n = 3$, this common value is 3:

(i) $(0,0) \mapsto (-1,0) \mapsto (-1,1) \mapsto (0,0)$, $(0,0) \mapsto (0,1) \mapsto (-1,1) \mapsto (0,0)$ and $(0,0) \mapsto (0,1) \mapsto (1,0) \mapsto (0,0)$, i.e., $\text{W–N–SE, N–W–SE, N–SE–W}$

(ii) $(0,0) \mapsto (0,1) \mapsto (1,0) \mapsto (0,0)$, $(0,0) \mapsto (0,1) \mapsto (0,2) \mapsto (1,1)$ and $(0,0) \mapsto (0,1) \mapsto (1,0) \mapsto (1,1)$, i.e., $\text{N–SE–W, N–N–SE, N–SE–N}$
Thanks for your attention!